Monotonic Properties of Collections of Maximum Independent Sets of a Graph

Adi Jarden
Department of Mathematics and Computer Science
Ariel University, Israel
jardena@ariel.ac.il

Vadim E. Levit
Department of Mathematics and Computer Science
Ariel University, Israel
levitv@ariel.ac.il

Eugen Mandrescu
Department of Computer Science
Holon Institute of Technology, Israel
eugen_m@hit.ac.il

Abstract

Let G be a simple graph with vertex set $V(G)$. A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent. The graph G is known to be a König-Egerváry if $\alpha(G) + \mu(G) = |V(G)|$, where $\alpha(G)$ denotes the size of a maximum independent set and $\mu(G)$ is the cardinality of a maximum matching.

Let $\Omega(G)$ denote the family of all maximum independent sets, and f be the function from subcollections Γ of $\Omega(G)$ to $\mathbb{N}$ such that $f(\Gamma) = |\bigcup \Gamma| + |\bigcap \Gamma|$. Our main finding claims that f is $\triangleleft$-increasing, where the preorder $\Gamma' \triangleleft \Gamma$ means that $\bigcup \Gamma' \subseteq \bigcup \Gamma$ and $\bigcap \Gamma \subseteq \bigcap \Gamma'$. Let us say that a family $\emptyset \neq \Gamma \subseteq \Omega(G)$ is a König-Egerváry collection if $|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha(G)$. We conclude with the observation that for every graph G each subcollection of a König-Egerváry collection is König-Egerváry as well.

Keywords: maximum independent set, critical set, ker, core, corona, diadem, maximum matching, König-Egerváry graph.

1 Introduction

Throughout this paper G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X. By $G - W$ we mean either the subgraph $G[V(G) - W]$, if $W \subseteq V(G)$, or the subgraph obtained by deleting the edge set W, for $W \subseteq E(G)$. In either case, we use $G - w$, whenever $W = \{w\}$. If $A, B \subseteq V(G)$, then (A, B) stands for the set $\{ab : a \in A, b \in B, ab \in E(G)\}$.

The *neighborhood* $N(v)$ of a vertex $v \in V(G)$ is the set $\{w : w \in V(G)$ *and* $vw \in E(G)\}$; in order to avoid ambiguity, we use also $N_G(v)$ instead of $N(v)$. The *neighborhood* $N(A)$ of $A \subseteq V(G)$ is $\{v \in V(G) : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\mathrm{Ind}(G)$ we mean the family of all the independent sets of G. An independent set of maximum size is a *maximum independent set* of G, and $\alpha(G) = \max\{|S| : S \in \mathrm{Ind}(G)\}$.

Let $\Omega(G)$ denote the family of all maximum independent sets, $\mathrm{core}(G) = \bigcap\{S : S \in \Omega(G)\}$ [9], and $\mathrm{corona}(G) = \cup\{S : S \in \Omega(G)\}$ [1].

A *matching* is a set M of pairwise non-incident edges of G. If $A \subseteq V(G)$, then $M(A)$ is the set of all the vertices matched by M with vertices belonging to A. A matching of maximum cardinality, denoted $\mu(G)$, is a *maximum matching*. For every matching M, we denote the set of all vertices that M saturates by $V(M)$, and by $M(x)$ we denote the vertex y satisfying $xy \in M$.

For $X \subseteq V(G)$, the number $|X| - |N(X)|$ is the *difference* of X, denoted $d(X)$. The *critical difference* $d(G)$ is $\max\{d(X) : X \subseteq V(G)\}$. The number $\max\{d(I) : I \in \mathrm{Ind}(G)\}$ is the *critical independence difference* of G, denoted $id(G)$. Clearly, $d(G) \geq id(G)$. It was shown in [21] that $d(G) = id(G)$ holds for every graph G. If A is an independent set in G with $d(X) = id(G)$, then A is a *critical independent set* [21].

Theorem 1.1 *[2] Each critical independent set can be enlarged to a maximum independent set.*

Theorem 1.2 *[11] For a graph G, the following assertions are true:*
(i) $\ker(G) \subseteq \mathrm{core}(G)$*;*
(ii) *if A and B are critical in G, then $A \cup B$ and $A \cap B$ are critical as well;*
(iii) *G has a unique minimal independent critical set, namely,* $\ker(G)$.

It is well-known that $\alpha(G) + \mu(G) \leq |V(G)|$ holds for every graph G. Recall that if $\alpha(G) + \mu(G) = |V(G)|$, then G is a *König-Egerváry graph* [4, 20]. For example, each bipartite graph is a König-Egerváry graph as well. Various properties of König-Egerváry graphs can be found in [6, 10, 16].

A proof of a conjecture of Graffiti.pc [3] yields a new characterization of König-Egerváry graphs: these are exactly the graphs having a critical maximum independent set [8].

Theorem 1.3 *[12] For a graph G, the following assertions are equivalent:*
(i) *G is a König-Egerváry graph;*
(ii) *there exists some maximum independent set which is critical;*
(iii) *each of its maximum independent sets is critical.*

For a graph G, let us denote

$$\ker(G) = \bigcap\{A : A \textit{ is a critical independent set}\} \text{ [11]},$$

$$\mathrm{MaxCritIndep}(G) = \{S : S \textit{ is a maximum critical independent set}\}$$

$$\mathrm{diadem}(G) = \bigcup \mathrm{MaxCritIndep}(G), \text{ and } \mathrm{nucleus}(G) = \bigcap \mathrm{MaxCritIndep}(G).$$

Clearly, $\ker(G) \subseteq \text{nucleus}(G)$ holds for every graph G. In addition, by Theorem 1.1, the inclusion $\text{diadem}(G) \subseteq \text{corona}(G)$ is true for every graph G.

In [17] the following lemma was introduced.

Lemma 1.4 (Matching Lemma) *[17] If $A \in \text{Ind}(G), \Lambda \subseteq \Omega(G)$, and $|\Lambda| \geq 1$, then there exists a matching from $A - \bigcap \Lambda$ into $\bigcup \Lambda - A$.*

2 Monotonicity results

We define the following preorder, denoted $\lhd$, on the class of collections of sets.

Definition 2.1 *Let Γ, Γ' be two collections of sets. We write $\Gamma' \lhd \Gamma$ if $\bigcup \Gamma' \subseteq \bigcup \Gamma$ and $\bigcap \Gamma \subseteq \bigcap \Gamma'$.*

Example 2.2 (i) $\{\{1\}\} \ntriangleleft \{\{1,2\}\}$

(ii) $\{\{1,2\},\{2,3\}\} \ntriangleleft \{\{1,2\},\{1,3\}\}$,

(iii) $\{\{1,2\},\{2,3\}\} \lhd \{\{1,2\},\{1,3\},\{2,3\}\}$,

(iv) $\{\{1,2\},\{2,3\}\} \lhd \{\{1,2\},\{1,3\},\{2\}\}$.

(v) $\text{MaxCritIndep}(G) \lhd \Omega(G)$ *is true for every bipartite graph G, since* $\text{core}(G) \subseteq \ker(G)$ *and diadem*$(G) \subseteq \text{corona}(G)$.

Theorem 2.3 *If $\emptyset \neq \Gamma \subseteq \Omega(G)$, and $\emptyset \neq \Gamma' \subseteq \text{Ind}(G)$, then there is a matching from $\bigcap \Gamma' - \bigcap \Gamma$ into $\bigcup \Gamma - \bigcup \Gamma'$.*

Proof. Let $S = \bigcap \Gamma'$. Since S is independent, by Lemma 1.4, there is a matching M from $S - \bigcap \Gamma$ into $\bigcup \Gamma - S$. For each $x \in S - \bigcap \Gamma$, we have $M(x) \notin \bigcup \Gamma'$, because every $A \in \Gamma'$ is independent. Consequently, M is a matching from $\bigcap \Gamma' - \bigcap \Gamma$ into $\bigcup \Gamma - \bigcup \Gamma'$, as claimed. ■

Choosing $\Gamma = \Omega(G)$ in Theorem 2.3, we get the following.

Corollary 2.4 *If $\emptyset \neq \Gamma' \subseteq \text{Ind}(G)$, then there is a matching from $\bigcap \Gamma' - \text{core}(G)$ into $\text{corona}(G) - \bigcup \Gamma'$.*

Choosing $\Gamma' = \{S\} \subseteq \Omega(G)$ in Corollary 2.4, we get the following.

Corollary 2.5 *[1] For every graph G and for every $S \in \Omega(G)$, there is a matching from $S - \text{core}(G)$ into $\text{corona}(G) - S$.*

Theorem 2.6 *If $\Gamma \subseteq \Omega(G)$ and $\Gamma' \subseteq \text{Ind}(G)$ is such that $\Gamma' \lhd \Gamma$, then*

$$\left|\bigcup \Gamma'\right| + \left|\bigcap \Gamma'\right| \leq \left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right|.$$

In particular, $f : \{\Gamma : \Gamma \subseteq \Omega(G)\} \longrightarrow \mathbb{N}$, $f(\Gamma) = \left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right|$ is $\lhd$-increasing.

Proof. If $\Gamma' = \emptyset$ or $\Gamma = \emptyset$, then the inequality clearly holds. Otherwise, according to Theorem 2.3, there is a matching M from $\bigcap \Gamma' - \bigcap \Gamma$ into $\bigcup \Gamma - \bigcup \Gamma'$. Thus

$$\left| \bigcap \Gamma' - \bigcap \Gamma \right| \leq \left| \bigcup \Gamma - \bigcup \Gamma' \right|.$$

Since $\bigcap \Gamma \subseteq \bigcap \Gamma'$ and $\bigcup \Gamma' \subseteq \bigcup \Gamma$, we have

$$\left| \bigcap \Gamma' - \bigcap \Gamma \right| = \left| \bigcap \Gamma' \right| - \left| \bigcap \Gamma \right|, \text{ and } \left| \bigcup \Gamma - \bigcup \Gamma' \right| = \left| \bigcup \Gamma \right| - \left| \bigcup \Gamma' \right|,$$

which completes the proof. ■

Corollary 2.7 *If $\Gamma' \subseteq \Gamma \subseteq \Omega(G)$, then $\left| \bigcup \Gamma' \right| + \left| \bigcap \Gamma' \right| \leq \left| \bigcup \Gamma \right| + \left| \bigcap \Gamma \right|$.*

Proof. It follows immediately by Theorem 2.6, because $\Gamma' \subseteq \Gamma$ implies $\Gamma' \lhd \Gamma$. ■

Corollary 2.8 $|\mathrm{corona}(G)| + |\mathrm{core}(G)| = 2\alpha(G)$ *if and only if* $\left| \bigcup \Gamma \right| + \left| \bigcap \Gamma \right| = 2\alpha(G)$ *holds for each non-empty* $\Gamma \subseteq \Omega(G)$.

Corollary 2.9 *[17] If $\Gamma \subseteq \Omega(G), |\Gamma| \geq 1$, then $2\alpha(G) \leq \left| \bigcup \Gamma \right| + \left| \bigcap \Gamma \right|$.*

Let us consider the graphs G_1 and G_2 from Figure 1: $\mathrm{core}(G_1) = \{a, b, c, d\}$ and it is a critical set, while $\mathrm{core}(G_2) = \{x, y, z, w\}$ and it is not critical.

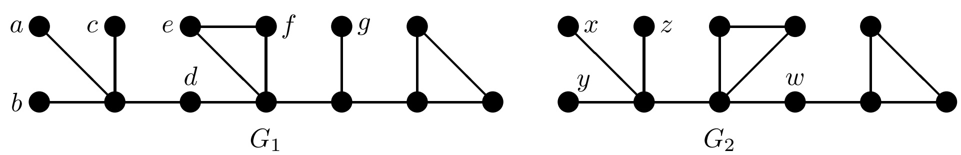

Figure 1: Both G_1 and G_2 are not König-Egerváry graphs.

Moreover, $\ker(G_1) = \{a, b, c\} \subset \mathrm{core}(G_1) \subset \{a, b, c, d, g\} = \mathrm{nucleus}(G_1)$, where $\mathrm{nucleus}(G_1) = A_1 \cap A_2$, and $A_1 = \{a, b, c, d, e, g\}$ and $A_2 = \{a, b, c, d, f, g\}$ are all the maximum critical independent sets of G_1. Notice that $\mathrm{diadem}(G_1) \subsetneq \mathrm{corona}(G_1)$.

Theorem 2.10 *Let G be a graph whose* $\mathrm{core}(G)$ *is a critical set. Then*
(i) $\mathrm{core}(G) \subseteq \mathrm{nucleus}(G)$;
(ii) $\mathrm{MaxCritIndep}(G) \lhd \Omega(G)$;
(iii) $|\mathrm{diadem}(G)| + |\mathrm{nucleus}(G)| \leq |\mathrm{corona}(G)| + |\mathrm{core}(G)|$;
(iv) $\mathrm{core}(G) = \mathrm{nucleus}(G)$, *if, in addition,* $\mathrm{diadem}(G) = \mathrm{corona}(G)$.

Proof. *(i)* Let $A \in \mathrm{MaxCritIndep}(G)$. According to Theorem 1.1, there exists some $S \in \Omega(G)$, such that $A \subseteq S$. Since $\mathrm{core}(G) \subseteq S$, it follows that $A \cup \mathrm{core}(G) \subseteq S$, and hence $A \cup \mathrm{core}(G)$ is independent. By Theorem 1.2, we get that $A \cup \mathrm{core}(G)$ is a critical independent set. Since $A \subseteq A \cup \mathrm{core}(G)$ and A is a maximum critical independent set, we infer that $\mathrm{core}(G) \subseteq A$. Thus, $\mathrm{core}(G) \subseteq A$ for every $A \in \mathrm{MaxCritIndep}(G)$. Therefore, $\mathrm{core}(G) \subseteq \mathrm{nucleus}(G)$.

(ii) By Part *(i)*, we know that $\mathrm{core}(G) \subseteq \mathrm{nucleus}(G)$. According to Theorem 1.1, every critical independent set is included in some maximum independent set. Hence, we deduce that $\mathrm{diadem}(G) = \bigcup \mathrm{MaxCritIndep}(G) \subseteq \bigcup \Omega(G) = \mathrm{corona}(G)$.

(iii) The inequality follows from Part *(ii)* and Theorem 2.6.

(iv) Part *(iii)* implies $|\mathrm{nucleus}(G)| \leq |\mathrm{core}(G)|$, and using now Part *(i)*, we obtain $\mathrm{core}(G) = \mathrm{nucleus}(G)$. ■

Corollary 2.11 *If* $|\Omega(G)| \leq 2$ *and* $\mathrm{diadem}(G) = \mathrm{corona}(G)$*, then* G *is a König-Egerváry graph.*

Proof. If $|\Omega(G)| = |\{S\}| = 1$, then $\mathrm{diadem}(G) = \mathrm{corona}(G) = S$, and the conclusion follows from Theorem 1.3.

Assume that $\Omega(G) = \{S_1, S_2\}$. Since $\mathrm{diadem}(G) = \mathrm{corona}(G)$, we infer that the family $\mathrm{MaxCritIndep}(G)$ contains only two maximum critical independent sets, say A_1 and A_2. By Theorem 2.10*(iv)*, we obtain $\mathrm{core}(G) = \mathrm{nucleus}(G)$. According to Theorem 2.10, we have, for instance, $A_1 \subseteq S_1$. Hence, $A_2 \subseteq S_2$, because, otherwise, $\mathrm{diadem}(G) = A_1 \cup A_2 \neq S_1 \cup S_2 = \mathrm{corona}(G)$. If there is some $x \in S_1 - A_1$, then $x \in A_2 \subseteq S_2$, because $S_1 - A_1 \subseteq S_1 \cup S_2 = A_1 \cup A_2$. Therefore, we deduce $x \in S_1 \cap S_2 = A_1 \cap A_2$, which implies $x \in A_1$, in contradiction with the assumption that $x \in S_1 - A_1$. Consequently, $A_1 = S_1$, which ensures, by Theorem 1.3, that G is a König-Egerváry graph. ■

Theorem 2.10*(i)* holds for every König-Egerváry graph, with equality, by Theorem 1.3. The same equality is satisfied by some non-König-Egerváry graphs; e.g., the graph G from Figure 2, where

$$\begin{aligned}\mathrm{core}(G) = &\{v_1, v_2, v_3, v_6, v_7, v_{10}\} \cap \{v_1, v_2, v_4, v_6, v_7, v_{10}\} \\ &\cap \{v_1, v_2, v_3, v_6, v_8, v_{10}\} \cap \{v_1, v_2, v_4, v_6, v_8, v_{10}\}.\end{aligned}$$

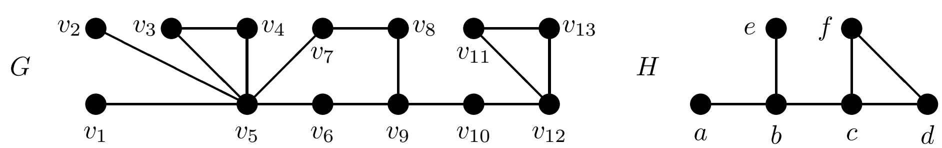

Figure 2: $\mathrm{core}(G) = \{v_1, v_2, v_6, v_{10}\}$ is a critical set, since $d(\mathrm{core}(G)) = 1 = d(G)$.

The equality from Theorem 2.10*(iv)* may hold for some graphs where $\mathrm{diadem}(G) \neq \mathrm{corona}(G)$. For instance, the graph H from Figure 2 satisfies: $\mathrm{core}(H) = \mathrm{nucleus}(H) = \{a, e\}$, $\mathrm{corona}(H) = \{a, e, c, d, f\}$ is a critical set, but $\mathrm{diadem}(H) = \{a, e\} \neq \mathrm{corona}(H)$.

Corollary 2.12 *If* G *is a bipartite graph, then* $\ker(G) = \mathrm{core}(G) = \mathrm{nucleus}(G)$*.*

The lower bound presented in the following theorem first appeared in [17].

Theorem 2.13 *For every graph* G

$$2\alpha(G) \leq |\mathrm{corona}(G)| + |\mathrm{core}(G)| \leq 2\left(|V(G)| - \mu(G)\right).$$

Proof. Let $S \in \Omega(G)$, $\Gamma = \Omega(G)$ and $\Gamma' = \{S\}$. As $\Gamma' \subseteq \Gamma$, by Corollary 2.7, we get

$$2\alpha(G) = 2|S| \leq |\text{corona}(G)| + |\text{core}(G)|.$$

For a maximum matching M of G, let $A = \{x : \{x, M(x)\} \subseteq \text{corona}(G)\}$, and let B contain all other vertices matched by M. Hence, there is no $S \in \Omega(G)$ such that $x \in S$ and $M(x) \in S$ at the same time. Since $\text{core}(G) \subseteq S \subseteq \text{corona}(G)$ for every $S \in \Omega(G)$, we infer that $A \cap \text{core}(G) = \emptyset$. Thus $A \subseteq \text{corona}(G) - \text{core}(G)$, and, consequently,

$$|A| \leq |\text{corona}(G)| - |\text{core}(G)|.$$

On the other hand, for every $x \in B$, we have $1 \leq |\{x, M(x)\} \cap (V(G) - \text{corona}(G))|$, and this implies

$$|B| \leq 2(|V(G)| - |\text{corona}(G)|).$$

Consequently, we obtain

$$\begin{aligned} 2\mu(G) = 2|M| = |A| + |B| \leq \\ \leq |\text{corona}(G)| - |\text{core}(G)| + 2(|V(G)| - |\text{corona}(G)|) \\ = 2|V(G)| - |\text{corona}(G)| - |\text{core}(G)|, \end{aligned}$$

and this completes the proof. ■

Corollary 2.14 *If $\emptyset \neq \Gamma \subseteq \Omega(G)$, then*

$$2\alpha(G) \leq \left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right| \leq 2(|V(G)| - \mu(G)).$$

Proof. Since $\emptyset \neq \Gamma \subseteq \Omega(G)$, we have $\Gamma \lhd \Omega(G)$. Combining Corollary 2.9, Corollary 2.7 and Theorem 2.13, we infer that

$$2\alpha(G) \leq \left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right| \leq |\text{corona}(G)| + |\text{core}(G)| \leq 2(|V(G)| - \mu(G)),$$

as claimed. ■

Clearly, G is a König-Egerváry graph if and only if the lower and upper bounds in Theorem 2.13 coincide.

The graphs from Figure 3 satisfy:

$$\begin{aligned} &2\alpha(G_1) = 4 = |\text{corona}(G_1)| + |\text{core}(G_1)| < 2(|V(G_1)| - \mu(G_1)) = 6, \\ &2\alpha(G_2) = 6 < |\text{corona}(G_2)| + |\text{core}(G_2)| = 8 = 2(|V(G_2)| - \mu(G_2)) \\ &2\alpha(G_3) = 12 < |\text{corona}(G_3)| + |\text{core}(G_3)| = 13 < 2(|V(G_1)| - \mu(G_1)) = 14, \end{aligned}$$

i.e., the bounds from Theorem 2.13 are tight.

Remark 2.15 *For each $n \geq 1$, the graph K_{2n} satisfies $|\text{corona}(K_{2n})| + |\text{core}(K_{2n})| = 2n = 2(|V(K_{2n})| - \mu(K_{2n}))$.*

Remark 2.16 *Let G be the graph obtained by joining to pendant vertices to one of the vertices of K_{2n+1}. Then $|\text{corona}(G)| + |\text{core}(G)| = 2 + 2n = 2(|G| - \mu(G))$.*

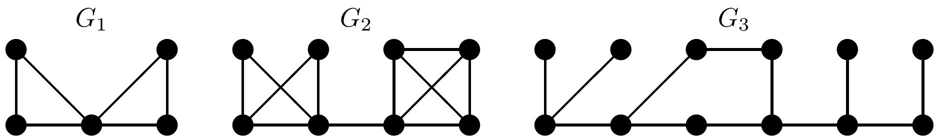

Figure 3: G_1, G_2 and G_3 are non-König-Egerváry graphs.

The graphs from Figure 4 satisfy:

$$2\alpha(G_1) = 4 = |\text{corona}(G_1)| + |\text{core}(G_1)| < 2\left(|V(G_1)| - \mu(G_1)\right) = 6,$$
$$2\alpha(G_2) = 6 < |\text{corona}(G_2)| + |\text{core}(G_2)| = 8 = 2\left(|V(G_2)| - \mu(G_2)\right)$$
$$2\alpha(G_3) = 8 < |\text{corona}(G_3)| + |\text{core}(G_3)| = 9 < 2\left(|V(G_1)| - \mu(G_1)\right) = 11,$$

i.e., the bounds from Theorem 2.13 are tight.

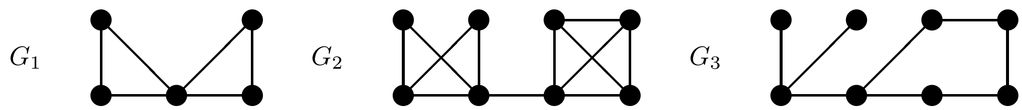

Figure 4: G_1, G_2 and G_3 are non-König-Egerváry graphs.

Corollary 2.17 *[17] If G is a König-Egerváry graph, then* $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G)$.

Proof. Since G is a König-Egerváry graph, we have $\alpha(G) = |V(G)| - \mu(G)$, and according to Theorem 2.13, we get

$$2\alpha(G) \leq |\text{corona}(G)| + |\text{core}(G)| \leq 2\left(|V(G)| - \mu(G)\right) = 2\alpha(G),$$

and this completes the proof. ■

It is known that $|V(G)| - 1 \leq \alpha(G) + \mu(G) \leq |V(G)|$ for every unicyclic graph [13].

Theorem 2.18 *[18] If G is a unicyclic graph, then*

$$2\alpha(G) \leq |\text{corona}(G)| + |\text{core}(G)| \leq 2\alpha(G) + 1.$$

By Corollary 2.17 and Theorem 2.18 we know that every unicyclic non-König-Egerváry graph satisfies the equalities $|V(G)| - 1 = \alpha(G) + \mu(G)$ and $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G) + 1$. Consequently,

$$2\alpha(G) + 1 = 2\left(|V(G)| - \mu(G)\right) - 1 < 2\left(|V(G)| - \mu(G)\right),$$

which improves on the upper bound in Theorem 2.13, in the case of unicyclic graphs.

Corollary 2.19 *If G is a König-Egerváry graph, then* $|\text{diadem}(G)| + |\text{nucleus}(G)| = 2\alpha(G)$.

Proof. By Theorem 1.3, we have that diadem(G) = corona(G) and core(G) is critical. Combining Theorem 2.10*(iv)* and Corollary 2.17, we get the result. ■

Corollary 2.20 *If G is a König-Egerváry graph, and $\emptyset \neq \Gamma \subseteq \Omega(G)$, then*

$$\left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right| = 2\alpha(G).$$

Proof. Let $S \in \Gamma$ and define $\Gamma' = \{S\}$. Hence, we have $\Gamma' \lhd \Gamma \lhd \Omega(G)$. By Theorem 2.6 and Corollary 2.17, we obtain

$$2\alpha(G) = f(\Gamma') \leq f(\Gamma) = \left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right| \leq |\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G),$$

which clearly implies $|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha(G)$. ■

Let us notice that the converse of Corollary 2.20 is not necessarily true. For instance, the graphs G_1 and G_2 from Figure 5, clearly, both satisfy: $|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha(G)$ for every $\emptyset \neq \Gamma \subseteq \Omega(G)$, but none is a König-Egerváry graph.

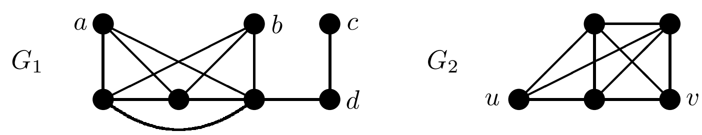

Figure 5: $\Omega(G_1) = \{\{a,b,c\},\{a,b,d\}\}$, while $\Omega(G_2) = \{u,v\}$.

3 A characterization of König-Egerváry graphs

Theorem 3.1 *If $\Gamma \subseteq \Omega(G)$ and $\left|\bigcup \Gamma\right| + \left|\bigcap \Gamma\right| = 2\alpha(G)$, then*

(i) *there is a perfect matching in $G\left[\bigcup \Gamma - \bigcap \Gamma\right]$;*

(ii) $\left|\bigcup \Gamma\right| - \left|\bigcap \Gamma\right| = 2\mu\left(G\left[\bigcup \Gamma\right]\right)$*;*

(iii) $\alpha(G\left[\bigcup \Gamma\right]) = \alpha(G)$*;*

(iv) $G\left[\bigcup \Gamma\right]$ *is a König-Egerváry graph.*

Proof. *(i)* Let $S \in \Gamma$. By Lemma 1.4, there is a matching, say M, from $S - \bigcap \Gamma$ into $\bigcup \Gamma - S$. On the other hand,

$$\begin{aligned} \left|S - \bigcap \Gamma\right| &= |S| - \left|\bigcap \Gamma\right| = \alpha(G) - \left|\bigcap \Gamma\right| \\ &= \left|\bigcup \Gamma\right| - \alpha(G) = \left|\bigcup \Gamma\right| - |S| = \left|\bigcup \Gamma - S\right|. \end{aligned}$$

Since

$$\left(S - \bigcap \Gamma\right) \cup \left(\bigcup \Gamma - S\right) = \bigcup \Gamma - \bigcap \Gamma,$$

we conclude that M is a perfect matching in $G\left[\bigcup\Gamma-\bigcap\Gamma\right]$.

(ii) By Part *(i)*, there is a perfect matching in $G\left[\bigcup\Gamma-\bigcap\Gamma\right]$. Hence,

$$2\mu\left(G\left[\bigcup\Gamma\right]\right)\geq 2\mu\left(G\left[\left|\bigcup\Gamma\right|-\left|\bigcap\Gamma\right|\right]\right)=\left|\bigcup\Gamma\right|-\left|\bigcap\Gamma\right|.$$

It remains to prove that

$$2\mu\left(G\left[\bigcup\Gamma\right]\right)\leq\left|\bigcup\Gamma\right|-\left|\bigcap\Gamma\right|.$$

Let M be a maximum matching in $G\left[\bigcup\Gamma\right]$. Since all the members of Γ are independent sets, there exists no edge xy such that $x\in\bigcap\Gamma$ and $y\in\bigcup\Gamma$. Therefore, $V(M)\cap\bigcap\Gamma=\emptyset$. Finally, we get

$$2\mu\left(G\left[\bigcup\Gamma\right]\right)=|V(M)|\leq\left|\bigcup\Gamma\right|-\left|\bigcap\Gamma\right|.$$

(iii) On the one hand, $\alpha(G\left[\bigcup\Gamma\right])\leq\alpha(G)$, because every independent set in $G\left[\bigcup\Gamma\right]$ is independent in G as well. On the other hand, if $S\in\Gamma$, then $|S|=\alpha(G)$, and S is independent in $G\left[\bigcup\Gamma\right]$. Thus $\alpha(G\left[\bigcup\Gamma\right])\geq\alpha(G)$, and consequently, we obtain $\alpha(G\left[\bigcup\Gamma\right])=\alpha(G)$.

(iv) Using the hypothesis and Part *(ii)*, we deduce that

$$2\alpha(G)-\left|\bigcap\Gamma\right|=\left|\bigcup\Gamma\right|=\left|\bigcap\Gamma\right|+2\mu\left(G\left[\bigcup\Gamma\right]\right),$$

which, by our assumption and Part *(iii)*, implies

$$2\left|\bigcup\Gamma\right|=2\mu\left(G\left[\bigcup\Gamma\right]\right)+2\alpha(G)=2\mu\left(G\left[\bigcup\Gamma\right]\right)+2\alpha(G\left[\bigcup\Gamma\right]),$$

i.e., $G\left[\bigcup\Gamma\right]$ is a König-Egerváry graph. ■

In particular, if we take $\Gamma=\Omega(G)$ in Theorem 3.1, we get the following.

Corollary 3.2 *If* $|\text{corona}(G)|+|\text{core}(G)|=2\alpha(G)$, *then* $G[\text{corona}(G)]$ *is a König-Egerváry graph.*

Notice that the equality $|\text{corona}(G)|+|\text{core}(G)|=2\alpha(G)$ is not enough to infer that G is a König-Egerváry graph, e.g., see the graph G_1 from Figure 6, that has: $\alpha(G_1)=3$, $\text{core}(G_1)=\{d\}$, $\text{corona}(G_1)=\{a,b,d,f,g\}$,.

Corollary 3.3 *If G is a König-Egerváry graph, and* $\emptyset\neq\Gamma\subseteq\Omega(G)$, *then*

(i) $\alpha(G\left[\bigcup\Gamma\right])=\alpha(G)$ *and* $\Gamma\subseteq\Omega(G\left[\bigcup\Gamma\right])$*;*

(ii) $\text{corona}(G\left[\bigcup\Gamma\right])=\bigcup\Gamma$ *and* $\text{core}(G\left[\bigcup\Gamma\right])=\bigcap\Gamma$.

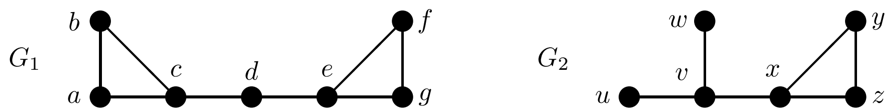

Figure 6: $\alpha(G_1) = 3$, $\mathrm{core}(G_1) = \{d\}$, $\mathrm{corona}(G_1) = \{a, b, d, f, g\}$, while $\mathrm{core}(G_2) = \{u, w\}$ and $V(G_2) - \mathrm{corona}(G_2) = \{v\}$.

Proof. *(i)* It is true by Corollary 2.20 and Theorem 3.1*(iii)*.

(ii) Since $V\left(G\left[\bigcup\Gamma\right]\right) = \bigcup\Gamma$, we have $\mathrm{corona}(G\left[\bigcup\Gamma\right]) \subseteq \bigcup\Gamma$. But by Part *(i)*, $\bigcup\Gamma \subseteq \mathrm{corona}(G\left[\bigcup\Gamma\right])$.

By Part *(i)*, $\mathrm{core}(G\left[\bigcup\Gamma\right]) = \bigcap\Omega(G\left[\bigcup\Gamma\right]) \subseteq \bigcap\Gamma$, so it is enough to prove that $\left|\mathrm{core}(G\left[\bigcup\Gamma\right])\right| = \left|\bigcap\Gamma\right|$. According to Theorem 3.1*(iv)*, $G\left[\bigcup\Gamma\right]$ is König-Egerváry. Therefore, using Corollary 2.17, we get

$$\begin{aligned} 2\alpha\left(G\left[\bigcup\Gamma\right]\right) &= \left|\mathrm{corona}(G\left[\bigcup\Gamma\right])\right| + \left|\mathrm{core}(G\left[\bigcup\Gamma\right])\right| \\ &= \left|\bigcup\Gamma\right| + \left|\mathrm{core}(G\left[\bigcup\Gamma\right])\right|. \end{aligned}$$

Since, by Corollary 2.20, we have that $\left|\bigcup\Gamma\right| + \left|\bigcap\Gamma\right| = 2\alpha(G)$, we finally obtain the equality $\left|\mathrm{core}(G\left[\bigcup\Gamma\right])\right| = \left|\bigcap\Gamma\right|$, as claimed. $\blacksquare$

The following proposition shows that a characterization of König-Egerváry graphs cannot relate only to the maximum independent sets.

Proposition 3.4 *For every König-Egerváry graph $G \notin \{K_1, K_2\}$, there is a non-König-Egerváry graph G', such that G is an induced subgraph of G' and $\Omega(G') = \Omega(G)$.*

Proof. Let $n = |V(G)|$, and K_{n+1} be a complete graph, such that $V(G) \cap V(K_{n+1}) = \emptyset$. We define G' as the graph having:

$$\begin{aligned} &V(G') = V(G) \cup V(K_{n+1}) \\ &E(G') = E(G) \cup E(K_{n+1}) \cup \{xy : x \in V(G), y \in V(K_{n+1})\}. \end{aligned}$$

Clearly, $\alpha(G') = \alpha(G)$, $|V(G')| = 2n + 1$, and $\mu(G') = n$. Hence we get

$$\alpha(G') + \mu(G') = \alpha(G) + n < 2n + 1 = |V(G')|.$$

Therefore, G' is not a König-Egerváry graph, while G is an induced subgraph of G' that clearly satisfies $\Omega(G') = \Omega(G)$. $\blacksquare$

Let us mention that the difference $|V(G) - \mathrm{corona}(G)| - |\mathrm{core}(G)|$ may reach any positive integer. For instance, $G = K_n - e, n \geq 4$.

Theorem 3.5 *For a graph G the following assertions are equivalent:*

(i) *G is a König-Egerváry graph;*

(ii) *for every* $S_1, S_2 \in \Omega(G)$ *there is a matching from* $V(G) - S_1 \cup S_2$ *into* $S_1 \cap S_2$*;*

(iii) *there exist* $S_1, S_2 \in \Omega(G)$*, such that there is a matching from* $V(G) - S_1 \cup S_2$ *into* $S_1 \cap S_2$*.*

Proof. *(i)* $\Rightarrow$ *(ii)* Suppose that G is a König-Egerváry graph, and let $H = G[S_1 \cup S_2]$ for some arbitrary $S_1, S_2 \in \Omega(G)$. Since $|S_1 \cup S_2| + |S_1 \cap S_2| = 2\alpha(G)$, Theorem 3.1*(ii),(iv)* ensures that $\alpha(G) = \alpha(H)$ and H is also a König-Egerváry graph. Thus

$$|V(G)| - \mu(G) = \alpha(G) = \alpha(H) = |V(H)| - \mu(H),$$

and, consequently,

$$|V(G)| - |V(H)| = \mu(G) - \mu(H).$$

Let M be a maximum matching of G. Applying Theorem 3.1*(ii)* with $\Gamma = \{S_1, S_2\}$, we obtain

$$|S_1 \cup S_2| - |S_1 \cap S_2| = 2\mu(H).$$

Hence,

$$\begin{gathered}|V(M) - S_1 \cap S_2| \leq |V(G) - S_1 \cap S_2| = |V(G) - S_1 \cup S_2| + |S_1 \cup S_2 - S_1 \cap S_2| \\ = (|V(G)| - |V(H)|) + 2\mu(H) = (\mu(G) - \mu(H)) + 2\mu(H) = \mu(G) + \mu(H).\end{gathered}$$

Therefore, $|V(M) - S_1 \cap S_2| \leq \mu(G) + \mu(H)$. But

$$2\mu(G) = |V(M)| = |V(M) \cap S_1 \cap S_2| + |V(M) - S_1 \cap S_2|.$$

Thus

$$\begin{gathered}|V(M) \cap S_1 \cap S_2| \geq 2\mu(G) - (\mu(G) + \mu(H)) = \\ = \mu(G) - \mu(H) = |V(G)| - |V(H)| = |V(G) - S_1 \cup S_2|.\end{gathered}$$

Clearly, $M(y) \in V(G) - S_1 \cup S_2$ for every $y \in V(M) \cap S_1 \cap S_2$. In other words, M induces an injective mapping, say M_1, from $V(M) \cap S_1 \cap S_2$ into $V(G) - S_1 \cup S_2$. Since $|V(M) \cap S_1 \cap S_2| \geq |V(G) - S_1 \cup S_2|$, we conclude that M_1 is a bijection. Therefore, M_1^{-1} is a matching from $V(G) - S_1 \cup S_2$ into $V(M) \cap S_1 \cap S_2$. Hence, M_1^{-1} is a matching from $V(G) - S_1 \cup S_2$ into $S_1 \cap S_2$.

(ii) $\Rightarrow$ *(iii)* It is clear.

(iii) $\Rightarrow$ *(i)* Suppose that there exist two sets $S_1, S_2 \in \Omega(G)$, such that there is a matching, say M_1, from $V(G) - S_1 \cup S_2$ into $S_1 \cap S_2$. Let $H = G[S_1 \cup S_2]$. In general, $\mu(G) \leq \mu(G - v) + 1$ for every vertex $v \in V(G)$. Consequently, we have $|V(G)| - |V(H)| \geq \mu(G) - \mu(H)$, because H is a subgraph of G

Let M_2 be a maximum matching in $G[S_1 \cup S_2 - S_1 \cap S_2]$. Since there are no edges connecting $S_1 \cup S_2 - S_1 \cap S_2$ and $S_1 \cap S_2$, we infer that $M_1 \cup M_2$ is a matching in G. Consequently, we obtain

$$\begin{gathered}\mu(G) \geq |M_1| + |M_2| = |V(G) - S_1 \cup S_2| + \mu(G[S_1 \cup S_2 - S_1 \cap S_2]) = \\ = |V(G)| - |S_1 \cup S_2| + \mu(G[S_1 \cup S_2]) = |V(G)| - |V(H)| + \mu(H).\end{gathered}$$

Hence, $|V(G)| - |V(H)| = \mu(G) - \mu(H)$.

By Theorem 3.1*(iii),(iv)*, we infer that H is a König-Egerváry graph, and $\alpha(H) = \alpha(G)$. Therefore,

$$|V(G)| - \mu(G) = |V(H)| - \mu(H) = \alpha(H) = \alpha(G).$$

Thus $|V(G)| = \alpha(G) + \mu(G)$, which means that G is a König-Egerváry graph as well. ■

The conditions *(ii)* or *(iii)* from Theorem 3.5 are not equivalent when we take more than two maximum independent sets.

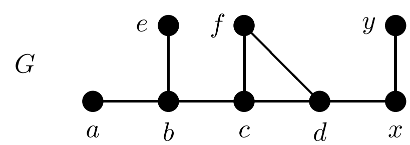

Figure 7: core$(G) = \{a, e\}$ is a critical set.

For instance, consider the graph G in Figure 7 and

$$S_1 = \{a, e, f, x\}, S_2 = \{a, e, c, x\}, S_3 = \{a, e, d, y\}, S_4 = \{a, e, f, y\}, S_5 = \{a, e, c, y\}.$$

There is a matching from $V(G) - S_1 \cup S_2 \cup S_3 = \{b\}$ into $S_1 \cap S_2 \cap S_3 = \{a, e\}$, but there is no matching from $V(G) - S_1 \cup S_4 \cup S_5 = \{b, d\}$ into $S_1 \cap S_4 \cap S_5 = \{a, e\}$. Notice that G is not a König-Egerváry graph.

Theorem 3.6 *G is a König-Egerváry graph if and only if the following conditions hold:*
(i) $|\mathrm{corona}(G)| + |\mathrm{core}(G)| = 2\alpha(G)$,
(ii) *there is a matching from* $V(G) - \mathrm{corona}(G)$ *into* $\mathrm{core}(G)$.

Proof. Let $H = G[\mathrm{corona}(G)]$. Theorem 3.1*(ii)* implies that $\alpha(G) = \alpha(H)$.

Suppose that G is a König-Egerváry graph. Condition *(i)* holds by Corollary 2.17.

By Condition *(i)* and Theorem 3.1*(iv)*, H is also a König-Egerváry graph. Thus

$$|V(G)| - \mu(G) = \alpha(G) = \alpha(H) = |V(H)| - \mu(H),$$

and, consequently,

$$|V(G)| - |V(H)| = \mu(G) - \mu(H).$$

Let M be a maximum matching of G. Now, applying Theorem 3.1*(ii)* with $\Gamma = \Omega(G)$, we obtain

$$|\mathrm{corona}(G)| - |\mathrm{core}(G)| = 2\mu(H).$$

Hence,

$$\begin{aligned}|V(M) - \mathrm{core}(G)| &\leq |V(G) - \mathrm{core}(G)| = |V(G) - \mathrm{corona}(G)| + |\mathrm{corona}(G) - \mathrm{core}(G)| \\ &= (|V(G)| - |V(H)|) + 2\mu(H) = (\mu(G) - \mu(H)) + 2\mu(H) = \mu(G) + \mu(H).\end{aligned}$$

Therefore, $|V(M) - \mathrm{core}(G)| \leq \mu(G) + \mu(H)$. But

$$2\mu(G) = |V(M)| = |V(M) \cap \mathrm{core}(G)| + |V(M) - \mathrm{core}(G)|.$$

Thus

$$\begin{aligned}|V(M) \cap \mathrm{core}(G)| &\geq 2\mu(G) - (\mu(G) + \mu(H)) = \\ &= \mu(G) - \mu(H) = |V(G)| - |V(H)| = |V(G) - \mathrm{corona}(G)|.\end{aligned}$$

Clearly, $M(y) \in V(G) - \mathrm{corona}(G)$ for every $y \in V(M) \cap \mathrm{core}(G)$. In other words, M induces an injective mapping, say M_1, from $V(M) \cap \mathrm{core}(G)$ into $V(G) - \mathrm{corona}(G)$. Since $|V(M) \cap \mathrm{core}(G)| \geq |V(G) - \mathrm{corona}(G)|$, we conclude that M_1 is a bijection. Therefore, M_1^{-1} is a matching from $V(G) - \mathrm{corona}(G)$ into $V(M) \cap \mathrm{core}(G)$. This completes the proof of Condition *(ii)*.

Now, suppose that Conditions *(i)* and *(ii)* hold.

In general, $\mu(G) \leq \mu(G - v) + 1$ for every vertex $v \in V(G)$. Consequently, if H is a subgraph of G, then $|V(G)| - |V(H)| \geq \mu(G) - \mu(H)$.

Condition *(ii)* implies that there exists a matching in G comprised of a matching M_1 from $V(G) - \mathrm{corona}(G)$ into $\mathrm{core}(G)$, and a maximum matching M_2 of $G[\mathrm{corona}(G) - \mathrm{core}(G)]$. Since there are no edges connecting $\mathrm{corona}(G) - \mathrm{core}(G)$ and $\mathrm{core}(G)$, we obtain

$$\begin{aligned}\mu(G) &\geq |M_1| + |M_2| = |V(G) - \mathrm{corona}(G)| + \mu(G[\mathrm{corona}(G) - \mathrm{core}(G)]) = \\ &= |V(G)| - |\mathrm{corona}(G)| + \mu(G[\mathrm{corona}(G)]) = |V(G)| - |V(H)| + \mu(H).\end{aligned}$$

Hence, $|V(G)| - |V(H)| = \mu(G) - \mu(H)$.

Condition *(i)* together with Theorem 3.1*(iii),(iv)* ensure that H is a König-Egerváry graph, and $\alpha(H) = \alpha(G)$. Therefore,

$$|V(G)| - \mu(G) = |V(H)| - \mu(H) = \alpha(H) = \alpha(G).$$

Thus $|V(G)| = \alpha(G) + \mu(G)$, which means that G is a König-Egerváry graph as well. ■

Remark 3.7 *The graphs G_1 and G_2 in Figure 6 show that none of Conditions* (i) *or* (ii) *from Theorem 3.6 is enough to infer that G is a König-Egerváry graph.*

Corollary 3.8 *If G is a König-Egerváry graph then $|V(G) - \mathrm{corona}(G)| \leq |\mathrm{core}(G)|$.*

4 Conclusions

In this paper we focus on interconnections between unions and intersections of maximum independents sets of a graph. Let us say that a family $\emptyset \neq \Gamma \subseteq \Omega(G)$ is a König-Egerváry collection if $|\bigcup \Gamma| + |\bigcap \Gamma| = 2\alpha(G)$. The set of all König-Egerváry collections is denoted as $\Im(G) = \Im(\Omega(G))$. One of the main findings of this paper can be interpreted as the claim that $\Im(G)$ is an abstract simplicial complex for every graph. In other words, every subcollection of a König-Egerváry collection is König-Egerváry as well. We incline to think that $\Im(G)$ is a new important invariant of a graph, which may be compared with the nerve of the family of all maximum independent sets.

Being more specific, we propose the following.

Problem 4.1 *Characterize graphs enjoying* $\mathrm{core}(G) = \mathrm{nucleus}(G)$.

Problem 4.2 *Characterize graphs satisfying*

$$|\mathrm{corona}(G)| + |\mathrm{core}(G)| = 2\left(|V(G)| - \mu(G)\right).$$

Conjecture 4.3 *If $|\mathrm{diadem}(G)| + |\mathrm{nucleus}(G)| = 2\alpha(G)$, then G is a König-Egerváry graph.*

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