# Poset Ramsey Numbers for Boolean Lattices 

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September 20, 2019


#### Abstract

A subposet $Q^{\prime}$ of a poset $Q$ is a copy of a poset $P$ if there is a bijection $f$ between elements of $P$ and $Q^{\prime}$ such that $x \leq y$ in $P$ iff $f(x) \leq f(y)$ in $Q^{\prime}$. For posets $P, P^{\prime}$, let the poset Ramsey number $R\left(P, P^{\prime}\right)$ be the smallest $N$ such that no matter how the elements of the Boolean lattice $Q_{N}$ are colored red and blue, there is a copy of $P$ with all red elements or a copy of $P^{\prime}$ with all blue elements. Axenovich and Walzer introduced this concept in Order (2017), where they proved $R\left(Q_{2}, Q_{n}\right) \leq 2 n+2$ and $R\left(Q_{n}, Q_{m}\right) \leq m n+n+m$, where $Q_{n}$ is the Boolean lattice of dimension $n$. They later proved $2 n \leq R\left(Q_{n}, Q_{n}\right) \leq$ $n^{2}+2 n$. Walzer later proved $R\left(Q_{n}, Q_{n}\right) \leq n^{2}+1$. We provide some improved bounds for $R\left(Q_{n}, Q_{m}\right)$ for various $n, m \in \mathbb{N}$. In particular, we prove that $R\left(Q_{n}, Q_{n}\right) \leq n^{2}-n+2$, $R\left(Q_{2}, Q_{n}\right) \leq \frac{5}{3} n+2$, and $R\left(Q_{3}, Q_{n}\right) \leq \frac{37}{16} n+\frac{39}{16}$. We also prove that $R\left(Q_{2}, Q_{3}\right)=5$, and $R\left(Q_{m}, Q_{n}\right) \leq\left(m-2+\frac{9 m-9}{(2 m-3)(m+1)}\right) n+m+3$ for all $n \geq m \geq 4$.


## 1 Introduction

Ramsey theory roughly says that any 2-coloring of elements in a sufficiently large discrete system contains a monochromatic system of given size. In the domain of complete graphs, the classical Ramsey theorem states that for any two graphs $G$ and $H$ there is a integer $N_{0}$ such that if the edges of a complete graph $K_{N}$ with $N \geq N_{0}$ are colored in two colors then there exists either a red copy of $G$ or a blue copy of $H$ in $K_{N}$. The least such number $N_{0}$ is called the Ramsey number $R(G, H)$. This theorem was proved by Ramsey [12] in 1930, but the problem of exactly determining these, "multicolor" Ramsey numbers, and $k$-uniform hypergraph Ramsey numbers remains open and is the subject of continuing research. For examples, see $[2,4,5,6,7,10]$.

In this paper, we will consider the poset Ramsey number instead of the graph Ramsey number. Given two posets $(P, \leq)$ and $\left(Q, \leq^{\prime}\right)$, we say $(P, \leq)$ is a subposet of $\left(Q, \leq^{\prime}\right)$, if there is an injective mapping $f: P \rightarrow Q$ such that for any $x, y \in P$ we have

$$
\begin{equation*}
x \leq y \text { if and only if } f(x) \leq^{\prime} f(y) \tag{1}
\end{equation*}
$$

The image $f(P)$ is called a copy of $P$ in $Q$. A Boolean lattice of dimension $n$, denoted $Q_{n}$, is the power set of an $n$-element ground set $X$ equipped with the inclusion relation. The 2-dimension of a poset $P$, defined by Trotter [13] and denoted by $\operatorname{dim}_{2}(P)$, is the smallest $n$ such that $Q_{n}$ contains a copy of $P$.

[^0]A poset $X$ has Ramsey property if for any poset $P$ there is a poset $Z$ such that when one colors the copies of $X$ in $Z$ red or blue, there is a copy of $P$ in $Z$ such that all copies of $X$ in this copy of $P$ are red or all of them are blue. The general problem of determining which posets have Ramsey property was solved by Nešetřil and Rödl [11]. In this paper, $X$ is the single-element poset. In other words, the elements of posets are colored instead of more complicated substructures.

For posets $P$ and $P^{\prime}$, let the poset Ramsey number $R\left(P, P^{\prime}\right)$ be the least integer $N$ such that whenever the elements of $Q_{N}$ are colored in red or blue, there exists either a red copy of $P$ or a blue copy of $P^{\prime}$. The focus of this paper is the case where $P$ and $P^{\prime}$ are Boolean lattices $Q_{m}$ and $Q_{n}$ for $m, n \in \mathbb{N}$. Axenovich and Walzer [1] give upper bound and lower bounds for $R\left(Q_{m}, Q_{m}\right)$ for various values of $m, n \in \mathbb{N}$. In particular, they prove the following.

Theorem 1. For any integers $n, m \geq 1$,
(i) $2 n \leq R\left(Q_{n}, Q_{n}\right) \leq n^{2}+2 n$,
(ii) $R\left(Q_{1}, Q_{n}\right)=n+1$,
(iii) $R\left(Q_{2}, Q_{n}\right) \leq 2 n+2$,
(iv) $n+m \leq R\left(Q_{n}, Q_{m}\right) \leq m n+n+m$,
(v) $R\left(Q_{2}, Q_{2}\right)=4, R\left(Q_{3}, Q_{3}\right) \in\{7,8\}$,
(vi) A Boolean lattice $Q_{3 n \log (n)}$ whose elements are colored red or blue randomly and independently with equal probability contains a monochromatic copy of $Q_{n}$ asymptotically almost surely.

Walzer, in his master's thesis [14], improved the upper bound in Theorem 1, part (i) to the following.

Theorem 2. $R\left(Q_{n}, Q_{n}\right) \leq n^{2}+1$.
Axenovich and Walzer also studied Ramsey numbers for Boolean algebras in [1]. A Boolean algebra $\mathcal{B}_{n}$ of dimension $n$ is a set system $\left\{X_{0} \cup \bigcup_{i \in I} X_{i}: I \subseteq[n]\right\}$, where $X_{0}, X_{1}, \ldots, X_{n}$ are pairwise disjoint sets, $X_{i} \neq \emptyset$ for $i=1, \ldots, n$. Boolean algebras have a more restrictive structure than Boolean lattices. If a subset of $Q_{N}$ contains a Boolean algebra of dimension $n$, then it contains a copy of $Q_{n}$. The converse, however, is not always true. Gunderson, Rödl, and Sidorenko [8] first considered the number $R_{\text {Alg }}(n)$, defined to be the smallest $N$ such that any red/blue coloring of subsets of $[N]$ contains a red or a blue Boolean algebra of dimesnion $n$. Here, "contains" means subset containment in $2^{[N]}$, not containment as a subposet. The following theorem states the best known bounds on $R_{\text {Alg }}(n)$. The lower bound is given without proof by Brown, Erdős, Chung, and Graham [3], and the upper bound was proved by Axenovich and Walzer [1].

Theorem 3. There is a positive constant $c$ such that

$$
2^{c n} \leq R_{A l g}(n) \leq \min \left\{2^{2^{n+1} n \log n}, n R_{h}\left(K^{n}(2, \ldots, 2)\right)\right\} .
$$

Here, $K^{n}(s, \ldots, s)$ is a complete $n$-uniform $n$-partite hypergraph with parts of size $s$ and $R_{h}\left(K^{n}(2, \ldots, 2)\right)$ is the smallest $N^{\prime}$ such that any 2 -coloring of $K^{n}\left(N^{\prime}, \ldots, N^{\prime}\right)$ contains a monochromatic $K^{n}(2, \ldots, 2)$.

Gunderson, Rödl, and Sidorenko [8] also considered the number $b(n, d)$, defined to be the maximum cardinality of a $\mathcal{B}_{d}$-free family contained in $2^{[n]}$. They proved the following bounds:

$$
n^{-\frac{(1+o(1)) d}{2^{+1}-2}} \cdot 2^{n} \leq b(n, d) \leq 10^{d} 2^{-2^{1-d}} d^{d-2^{-d}} n^{-1 / 2^{d}} \cdot 2^{n}
$$

Johnston, Lu, and Milans [9] later used the Lubell function to improve the upper bound to the following, where $C$ is a constant:

$$
b(n, d) \leq C n^{-1 / 2^{d}} \cdot 2^{n}
$$

In this paper, we improve the upper bounds on the poset Ramsery numbers $R\left(Q_{m}, Q_{n}\right)$ given by Axenovich and Walzer in [1]. In Section 3, we prove that for any integer $n \geq 1$,
Theorem 4. $R\left(Q_{2}, Q_{n}\right) \leq \frac{5}{3} n+2$.
Theorem 5. $R\left(Q_{n}, Q_{n}\right) \leq n^{2}-n+2$.
Theorem 6. $R\left(Q_{3}, Q_{n}\right) \leq \frac{37}{16} n+\frac{39}{16}$.
In Section 3, for all integers $n \geq m \geq 4$, we also prove the following.
Theorem 7. $R\left(Q_{m}, Q_{n}\right) \leq\left(m-2+\frac{9 m-9}{(2 m-3)(m+1)}\right) n+m+3$ for all $n \geq m \geq 4$.
Additionally, we are now able to identify the following previously unknown poset Ramsey number.

Theorem 8. $R\left(Q_{2}, Q_{3}\right)=5$.
In Section 2, we give more definitions and introduce notation. Also in Section 2, we state and prove Lemma 1 , the key embedding lemma we use to prove Theorems 4, 5, 6, and 7. We prove theorems $4,5,6,7$, and 8 in Section 3, and we devote Section 4 to concluding remarks.

## 2 Notation and Key Lemma

A partially ordered set, or poset, consists of a set $S$ together with a partial order $\leq$, which is a binary relation on $S$ satisfying

Reflexive Property: $x \leq x$, for any $x \in S$.
Transitive Property: If $x \leq y$ and $y \leq z$ then $x \leq z$ for any $x, y, z \in S$.
Antisymmetric Property: If $x \leq y$ and $y \leq x$ then $x=y$ for any $x, y \in S$.

Let $[n]$ denote the set $\{1,2, \ldots, n\}$ and $Q_{n}=\left(2^{[n]}, \subseteq\right)$ be the poset over the family of all subsets of $[n]$. The $k$-th level of $Q_{n}$ is the set of all $k$-element subsets of the ground set $[n]$, where $0 \leq k \leq n$. For any two subsets (of $[n]) S \subset T$, let $Q_{[S, T]}$ be the induced poset of $Q_{n}$ over all sets $F$ such that $S \subseteq F \subseteq T$. Let $Q_{n}^{*}:=Q_{n} \backslash\{\emptyset,[n]\}$. Let $\hat{R}\left(Q_{m}, Q_{n}\right)$ denote the smallest $N$ such that any red/blue coloring of $Q_{N}^{*}$ contains either a red copy of $Q_{m}^{*}$ or a blue copy of $Q_{n}^{*}$. Equivalently, $\hat{R}\left(Q_{m}, Q_{n}\right)$ is the least $N$ such that if $\emptyset$ and $[N]$ are assumed to be both red and blue while the rest of $Q_{N}$ is colored either red or blue, then $Q_{N}$ contains either a red copy of $Q_{m}$ or a blue copy of $Q_{n}$. For a subset $S \subseteq N$, let $\bar{S}$ denote the complement set of $S$ in $[N]$. When $S=\{x\}$, we simply write $\bar{x}$ for $\overline{\{x\}}$.

The following key lemma generalizes the blob lemma of Axenovich and Walzer (see [1], Lemma 3). The special case $a=b=0$ gives the blob lemma.

Lemma 1. For any nonnegative integers $N$, m, $n$, $n^{\prime}$, $a$, b satisfying $N \geq n^{\prime} \geq n \geq a+b$ and $N \geq m$, suppose that the Boolean lattice $Q_{N}$ on the ground set $[N]$ is colored in two colors red and blue satisfying

1. There is an injection $i: Q_{n} \rightarrow Q_{n^{\prime}} \subset Q_{N}$ with the following properties.

- $i$ maps the bottom a-layers of $Q_{n}$ to blue sets.
- For all sets $S$ in the top $b$ layers of $Q_{n}, i(S) \cup\left([N] \backslash\left[n^{\prime}\right]\right)$ is blue.

2. $N \geq n^{\prime}+(n+1-a-b) * m$.

Then either a blue subposet $Q_{n}$ or a red subposet $Q_{m}$ exists in $Q_{N}$.
Proof of Lemma 1: Let $Q_{N}$ be the Boolean lattice on the ground set $[N]$ colored red and blue with the properties listed above.

Let $k=n+1-(a+b)$. Since $N \geq n^{\prime}+(n+1-(a+b)) * m=n+k * m$, we can partition $[N]$ like so:

$$
[N]=\left[n^{\prime}\right] \cup X_{1} \cup X_{2} \cup \cdots \cup X_{k}
$$

where $\left|X_{i}\right| \geq m$ for all $i \in[k]$. With this partition in mind, we create an injection $f$ of $Q_{n}$ into the blue sets of $Q_{N}$. Consider the map $f: Q_{n} \rightarrow Q_{N}$ defined by

$$
\begin{gathered}
f(\emptyset)=\emptyset \\
f(S)=i(S) \text { for all } S \text { with }|S| \leq a \\
f(S)=i(S) \cup X_{1}^{*} \text { for all } S \text { with }|S|=a+1 \\
\vdots \\
f(S)=i(S) \cup X_{1} \cup X_{2} \cup \cdots \cup X_{j}^{*} \text { for all } S \text { with }|S|=a+j \\
\vdots \\
f(S)=i(S) \cup X_{1} \cup X_{2} \cup \cdots \cup X_{k}^{*} \text { for all } S \text { with }|S|=[n]-b \\
f(S)=i(S) \cup X_{1} \cup X_{2} \cup \ldots X_{k} \text { for all } S \text { with }|S| \geq[n]-b+1
\end{gathered}
$$

$$
f([n])=[N] .
$$

Here, $i(S) \cup X_{1} \cup X_{2} \cup \cdots \cup X_{j}^{*}$ denotes an arbitrarily chosen blue element from the subposet with bottom element $S \cup X_{1} \cup X_{2} \cup \ldots X_{j-1} \cup \emptyset$ and top element $i(S) \cup X_{1} \cup X_{2} \cup \cdots \cup X_{j-1} \cup X_{j}$. If no such blue element exists, this entire subposet is red and $Q_{N}$ contains a red $Q_{m}$.

If such a blue element always exists, this function is well-defined and preserves all the subset relations found in $Q_{n}$. Its image consists entirely of blue elements, so $Q_{N}$ contains a blue $Q_{n}$.

## 3 Proof of Theorems

Proof of Theorem 4. For any integer $n \geq 2$, let $N \in \mathbb{N}$ be such that there exists a red/blue coloring of $Q_{N}$ containing no red copy of $Q_{2}$ and no blue copy of $Q_{n}$. Consider such a red-blue coloring $c$ of $Q_{N}$. Let $T$ be a red element such that $\min \{N-|T|,|T|\} \leq \min \left\{N-\left|T^{\prime}\right|,\left|T^{\prime}\right|\right\}$ for all red elements $T^{\prime} \in Q_{N}$. Without loss of generality, let $N-|T| \leq|T|$. Let $a:=N-|T|$. Let $S$ be a red element such that $|S| \leq\left|S^{\prime}\right|$ for all red elements $S^{\prime} \in Q_{[\emptyset, T]}$. Let $b:=|S|$.
Claim a: $|T|-|S| \leq n+1$.
Proof of Claim a: Proof by contradiction. Otherwise, suppose $|T|-|S| \geq n+2$. Let $u, v$ be two red elements in $Q_{[S, T]}$. If $u$ and $v$ are incomparable, $\{S, u, v, T\}$ form a red $Q_{2}$. So every red element in $Q_{[S, T]}$ lies on the same maximal chain. With the exception of this maximal chain, the rest of $Q_{[S, T]}$ is blue, and we can find a blue $Q_{n}$.
Claim b: $N \leq 3 n+1-2(a+b)$.
Proof of Claim b: Otherwise, we assume $N \geq 3 n+2-2(a+b)$. We have $N \geq n+(n+$ $1-(a+b)) * 2$. Since the bottom $a$-layers of $Q_{N}$ are all colored blue, the bottom $a$-layers of $Q_{[\emptyset,[n]]}$ are all colored in blue. If we let $m=2$, by Lemma $1, Q_{N}$ contains either a blue subposet $Q_{n}$ or a red subposet $Q_{m}$.

From Claim a, we have

$$
\begin{equation*}
a+b=N-(|T|-|S|) \geq N-(n+1) \tag{2}
\end{equation*}
$$

Combining (2) with Claim b, we have

$$
\begin{equation*}
N \leq 3 n+1-2[N-(n+1)]=5 n+3-2 N . \tag{3}
\end{equation*}
$$

We get

$$
N \leq \frac{5 n}{3}+1,
$$

which gives us the desired result.

Proof of Theorem 5. Let $n \in \mathbb{N}$. The result is known to hold for $n=1$ and $n=2$, so let $n \geq 3$. Let $\hat{R}\left(Q_{n}, Q_{n}\right)$ denote the smallest $N$ such that any red-blue coloring of $Q_{N}$, where $\emptyset$ and $[N]$ are assumed to be both red and blue, contains either a red or blue copy of $Q_{n}$. Equivalently, any red/blue coloring of $Q_{N} \backslash\{\emptyset,[N]\}$ contains either a red or blue copy of $Q_{n}^{*}$. To prove the theorem, we first prove the following claim.

Claim c. $\hat{R}\left(Q_{n}, Q_{n}\right) \leq n^{2}-n$ for all $n \geq 3$.
Proof of Claim c. By way of contradiction, suppose there is a red-blue coloring $c$ of $Q_{N}$ (with $N=n^{2}-n$ ) such that $\emptyset$ and $[N]$ are colored both red and blue while all other elements of $Q_{N}$ only receive one color. Since $N=n^{2}-n$, there are $n^{2}-n \geq 2 n$ singleton sets in the first row of $Q_{N}$. By the Pigeonhole Principle, there are $n$ sets in the first row of $Q_{N}$ with the same color. Without loss of generality, suppose at least $n$ of these sets are blue. Then there is a subposet $Q_{n}^{*}$ of $Q_{N}$ such that level 1 of $Q_{n}^{*}$ consists of some subset of these blue sets.

We consider an injection $i: Q_{n} \rightarrow Q_{n}^{*} \subset Q_{N}$, which maps the bottom $a=2$ layers of $Q_{n}$ to blue sets. Also, we also consider the top $b=1$ layer of $Q_{N}$ to be colored blue. By Lemma 1 , since $N \geq n^{2}-n=n+(n-2) * n=n+(n+1-a-b) * m$, either a blue subposet $Q_{n}$ or a red subposet $Q_{m}$ exists in $Q_{N}$.

Let $N=n^{2}-n+2$. Consider a $Q_{N}$, and let $Q_{N}$ be colored with a coloring $c: Q_{N} \rightarrow$ \{ red, blue \}. We now consider the following cases.

Case 1. Sets $\emptyset$ and $[N]$ are the same color.
Without loss of generality, we assume both $\emptyset$ and $[N]$ are colored in red. If we can find two blue sets $S$ and $T$ with $|S|=1,|T|=N-1$, and $S \subset T$, then we can consider the $Q_{[S, T]}$. Since $|T|-|S|=N-2 \geq n^{2}-n$, by Claim c, $Q_{[S, T]}$ either a red or blue copy of $Q_{n} \backslash\{\emptyset,[n]\}$, which can be extended to a red or blue copy of $Q_{n}$.

If we fail to find such two blue sets $S$ and $T$, there are only three subcases:

1. All level 1 sets are red.
2. All level $N-1$ sets are red.
3. There exists an element $x \in N$ such that $\{x\}$ an $[N] \backslash\{x\}$ are blue but all other sets in level 1 and level $N-1$ are red.

In subcase one, since $N \geq n+n(n-2)$, we can partition $[N]=[n] \cup X_{1} \cup \cdots \cup X_{n-2}$ so that $\left|X_{i}\right| \geq n$. We map the first two layers and the last layer of $Q_{n}$ into $Q_{N}$ and extend this map as in the proof of Claim c to get a red copy of $Q_{n}$. Subcase two is similar. In subcase three, similar argument works for the subposet $Q_{[\emptyset, \bar{x}]}$, where $\bar{x}=[N] \backslash\{x\}$. Note that the first two layers of $Q_{[\emptyset, \bar{x}]}$ are red, while the top element $\bar{x}$ can be treated as red since $[N]$ is red.

Case 2. Sets $\emptyset$ and $[N]$ are not the same color. Without loss of generality, suppose $\emptyset$ is red and $[N]$ is blue.

Suppose there is a pair $S, T$ of comparable elements, where $S$ is blue, $T$ is red, $|S|=1$, and $|T|=N-1$. Since $\emptyset$ is red and $S$ is blue, and $[N]$ is red and $T$ is blue, the poset $Q_{[S, T]}$ of dimension $n^{2}-n$ can be viewed as having bottom and top elements colored both red and blue. By Claim c, $Q_{[S, T]}$ contains a red $Q_{n}$ or a blue $Q_{n}$.

Otherwise, there are only four subcases:

1. All level 1 sets are red and all level $N-1$ sets are blue.
2. All level 1 sets are red, and there exists a red $N-1$-set.
3. All level $N-1$ sets are blue, and there exists a blue 1 -set.
4. There exists an element $x \in N$ such that all level 1 sets except $\{x\}$ are red and all level $N-1$ sets except $\bar{x}$ are blue.

A similar argument works for subcases 2,3 , and 4 since we can find a $Q_{[N-1]}$ so that there are three layers of one color.

In subcase 1, suppose there exists a blue set in level 2. Then we can find a blue $Q_{[N-2]}$ and a similar argument works. If there does not exist such a blue set, the bottom three layers of $Q_{N}$ are red.

In this case, since

$$
n^{2}-n+2 \geq n+(n-2) * n
$$

we can partition $[N]=[n] \cup X_{1} \cup \cdots \cup X_{n-2}$ so that $\left|X_{i}\right| \geq n$. We map the first three layers of $Q_{n}$ into $Q_{N}$ to get a red copy of $Q_{n}$. Applying Lemma 1 with $a=3$ and $b=0$, we get the desired monochromatic copy of $Q_{n}$.

In any case where $N=n^{2}-n+2$, we have shown $Q_{N}$ must contain a red $Q_{n}$ or a blue $Q_{n}$. It follows that $R\left(Q_{n}, Q_{n}\right) \leq n^{2}-n+2$, the desired result.

Proof of Theorem 6. For any integer $n \geq 4$, let $N \in \mathbb{N}$ be such that there exists a red/blue coloring of $Q_{N}$ containing no red copy of $Q_{3}$ and no blue copy of $Q_{n}$. Consider a red-blue coloring $c$ of $Q_{N}$. Let $T$ be a red element such that $\min \{N-|T|,|T|\} \leq \min \left\{N-\left|T^{\prime}\right|,\left|T^{\prime}\right|\right\}$ for all red elements $T^{\prime} \in Q_{N}$. Without loss of generality, let $N-|T| \leq|T|$. Let $a:=N-|T|$. Let $S$ be a red element such that $|S| \leq\left|S^{\prime}\right|$ for all red elements $S^{\prime} \in Q_{[\emptyset, T]}$. Let $b:=|S|$.

Let $\hat{R}\left(Q_{3}, Q_{n}\right)$ denote the smallest $N$ such that any red/blue coloring of $Q_{N}$, where $\emptyset$ and [ $N$ ] are assumed to be both red and blue, contains either a red copy of $Q_{3}$ or a blue copy of $Q_{n}$. Equivalently, any red-blue coloring of $Q_{N}^{*}$ contains either a red copy of $Q_{3}^{*}$ or a blue copy of $Q_{n}^{*}$. To prove the theorem, we first prove the following claim.
Claim d: $\hat{R}\left(Q_{3}, Q_{n}\right) \leq \frac{7}{4} n+\frac{9}{4}$ for all $n$.
Proof of Claim d: By way of contradiction, suppose there is a red-blue coloring cof $Q_{N}^{*}$ (with $N \geq \frac{7}{4} n+\frac{9}{4}$ ) such that it contains neither red subposet $Q_{3}^{*}$ nor blue subposet $Q_{n}^{*}$.

Let $\ell=\left\lceil\frac{3}{8} n+\frac{5}{8}\right\rceil$ be a fixed integer. Consider the bottom $\ell$ layers of $Q_{N}$. We look for red sets $A_{1}, A_{2}, A_{3}$ with the following property.

$$
\begin{equation*}
\forall i \in[3], \exists x_{i} \in[N] \text { such that } x_{i} \in A_{i}, \text { but } x_{i} \notin A_{j} \forall j \in[3] \backslash i . \tag{4}
\end{equation*}
$$

We consider the following cases.
Case 1. There exist sets $A_{1}, A_{2}, A_{3}$ with property 4.
Since

$$
\begin{gathered}
\ell=\left\lceil\frac{3}{8} n+\frac{5}{8}\right\rceil=\left\lceil\frac{\frac{7}{4} n-n+\frac{5}{4}}{2}\right\rceil \leq\left\lceil\frac{N-n-1}{2}\right\rceil \leq \frac{N-n+1}{2} \\
N+1 \geq 2 \ell+n, \\
N+1 \geq \ell+(\ell-1)+n+1
\end{gathered}
$$

we are able to create an injection of $Q_{3}$ into the red sets of $Q_{N}$. Consider the map $f: Q_{3} \rightarrow Q_{N}$ defined by

$$
\begin{gathered}
f(\{i\})=A_{i} \text { for all } i \in[3], \\
f(\{i, j\})=X_{i, j}^{*} \text { for all }\{i, j\} \subset[3] .
\end{gathered}
$$

Here, $X_{i, j}^{*}$ denotes an arbitrarily chosen red element from the subposet with bottom element $A_{i} \cup A_{j}$ and top element $\bar{x}_{k}$, where $\{i, j, k\}=[3]$. If no such red element exists, this entire $n$-dimenional subposet is blue and $Q_{N}$ contains a blue $Q_{n}$.

If such a red element always exists, this function is well-defined and preserves all the subset relations found in $Q_{n}$. Its image consists entirely of red elements, so $Q_{N}$ contains a red $Q_{3}$, a contradiction.

Case 2. There exist red sets $B_{1}, B_{2}, B_{3}$ in the top $\ell$ layers of $Q_{N}$ with the following property.

$$
\begin{equation*}
\forall i \in[3], \exists x_{i} \in[N] \text { such that } x_{i} \notin B_{i} \text {, but } x_{i} \in B_{j} \forall j \in[3] \backslash i . \tag{5}
\end{equation*}
$$

This case is the same is as Case 1, except everything is flipped over the middle layer(s) of $Q_{N}$. Using a similar argument, we show that $Q_{N}$ contains a blue $Q_{n}$ or a red $Q_{3}$.

Case 3. There do not exist such sets $A_{1}, A_{2}, A_{3}$ or $B_{1}, B_{2}, B_{3}$.
Suppose we are only able to find one red set $A_{1}$. Then every set of elements of $[N] \backslash A_{1}$ in the first $\ell$ layers is blue. Note that $\left|A_{1}\right| \leq \ell-1$.

Suppose we are only able to find 2 sets with property 4 . Let $a_{3}$ be an arbitrarily chosen $\ell$-element subset of $A_{1} \cup A_{2}$. We claim that every set of elements of $[N] \backslash a_{3}$ in the first $\ell$ layers is blue. Suppose this is not the case, and there is a red set $X \subseteq[N] \backslash a_{3}$ in the first $\ell$ layers. Since $\left|A_{1} \cup A_{2}\right| \leq 2(\ell-1)$, we know $\left|A_{1} \cup A_{2} \backslash a_{3}\right| \leq \ell-2$. Thus, there exists an $x \in X$ such that $x \notin A_{1} \cup A_{2}$. We let $x$ be $x_{3}, X$ be $A_{3}$, and $A_{1}, A_{2}, A_{3}$ have property 4 , a contradiction. We can eliminate at most $\ell$ elements from $[N]$ and guarantee that sets formed from the remaining elements in the bottom $\ell$ layers are all blue.

Similarly, if we are only able to find at most two red sets with property 5 , we can require the inclusion of at most $\ell$ elements from $[N]$ and guarantee that sets formed in the top $\ell$ layers of $Q_{N}$ are all blue. Since $n<n+1=\frac{7}{4} n+\frac{9}{4}-2\left(\frac{3}{8} n+\frac{5}{8}\right)$, we can define a mapping $i: Q_{n} \rightarrow Q_{n}^{*} \subset Q_{N}$ such that the bottom $\ell$ layers of $Q_{n}$ map to blue elements in the bottom $\ell$ layers of $Q_{N}$ and the top $\ell$ layers of $Q_{n}$ map to blue elements in the top $\ell$ layers of $Q_{N}$.

Since

$$
\begin{gathered}
\ell=\left\lceil\frac{3}{8} n+\frac{5}{8}\right\rceil \\
2 \ell=2\left\lceil\frac{3}{8} n+\frac{5}{8}\right\rceil \geq 2\left(\frac{3}{8} n+\frac{5}{8}\right)-1 \\
-2 \ell
\end{gathered}
$$

$$
\begin{gathered}
1-2 \ell=-\frac{3}{4} n+\frac{3}{4} \\
n+1-2 \ell=\frac{1}{4} n+\frac{3}{4}=\frac{\frac{3}{4} n+\frac{9}{4}}{3}=\frac{\frac{7}{4} n-n+\frac{9}{4}}{3} \leq \frac{N-n}{3},
\end{gathered}
$$

we have

$$
N \geq n+(n+1-2 \ell) * 3
$$

The bottom $a=\ell$ layers and the top $b=\ell$ layers of $Q_{n}^{*}$ are blue and $m=3$, by Lemma $1, Q_{N}$ contains either a blue subposet $Q_{n}$ or a red subposet $Q_{3}$.

In any case where $N \geq \frac{7}{4} n+\frac{9}{4}$ and $[N]$ and $\emptyset$ are colored both red and blue, we have shown that $Q_{N}$ must contain a red $Q_{3}$ or a blue $Q_{n}$. It follows that $\hat{R}\left(Q_{3}, Q_{n}\right) \leq \frac{7}{4} n+\frac{9}{4}$.

Suppose $a \neq 0$ and $b \neq 0$. It follows that $|T|-|S|+1 \leq \frac{7}{4} n+\frac{9}{4}$ for all $n \in \mathbb{N}$.
Claim e: $N \leq n+3(n+1-(a+b))-1$.
Proof of Claim e: Otherwise, we assume $N \geq n+3(n+1-(a+b))$. Let $k=n+1-(a+b)$, so $N \geq a+b+3 k$.

We can partition [ $N$ ] like so:

$$
[N]=[n] \cup X_{1} \cup X_{2} \cup \cdots \cup X_{k},
$$

where $\left|X_{i}\right| \geq 3$ for all $i \in[k]$. With this partition in mind, we define a mapping $i: Q_{n} \rightarrow$ $Q_{n}^{*} \subset Q_{N}$, an injection of $Q_{n}$ into the blue sets of $Q_{N}$. By Lemma 1, $Q_{N}$ contains either a blue copy of $Q_{n}$ or a red copy of $Q_{3}$, a contradiction.

From Claim d, we have

$$
\begin{equation*}
a+b=N-(|T|-|S|) \geq N-\left(\frac{7}{4} n+\frac{5}{4}\right) . \tag{6}
\end{equation*}
$$

Combining (6) with Claim e, we have

$$
\begin{equation*}
N \leq n+3(n+1-(a+b))-1 \leq n+3\left(n+1-\left(N-\frac{7}{4} n-\frac{5}{4}\right)\right)-1 . \tag{7}
\end{equation*}
$$

We get

$$
N \leq \frac{37}{16} n+\frac{23}{16} .
$$

Now suppose $a=0$. We consider the remaining two cases. In each case, we assume, by way of contradiction, that $N>\frac{37}{16} n+\frac{23}{16}$.

Case 1. $a=0$ and $b=0$.
In this case, both $\emptyset$ and $[N]$ are necessarily red. If we can find two blue sets $S$ and $T$ with $|S|=1,|T|=N-1$, and $S \subset T$, then we can consider $Q_{[S, T]}$. In this case, since $\emptyset$ is red and $S$ is blue, we can consider the bottom element of $Q_{[S, T]}$ to be both red and blue. Since $[N]$
is red and $T$ is blue, we can consider the top element of $Q_{[S, T]}$ to be both red and blue. By Claim d, $\hat{R}\left(Q_{3}, Q_{n}\right)+2 \leq \frac{7}{4} n+\frac{9}{4}+2<\frac{37}{16} n+\frac{23}{16}$ for $n \geq 4$.

If we cannot find such sets $S$ and $T$, we are left with the following three subcases:

1. All level 1 sets are red.
2. All level $N-1$ sets are red.
3. There exists an element $x \in N$ such that $\{x\}$ and $\bar{x}$ are blue but all other sets in levels and $N-1$ are red.

In subcase 1 , since $N>\frac{37}{16} n+\frac{23}{16}>n+3=3+(3+1-1-2) * n$, we can partition $[N]=[3] \cup X$ with $|X| \geq n$. We map the first $b=2$ layers and the last $a=1$ layer of $Q_{3}$ into $Q_{N}$. By Lemma $1, Q_{N}$ contains either a blue $Q_{n}$ or a red $Q_{3}$. Subcase 2 is similar.

In subcase 3, a similar argument works for $Q_{[\emptyset, \bar{x}]}$. Note that the first two layers of $Q_{[\emptyset, \bar{x}]}$ are red, while the top element $\bar{x}$ can be treated as red since $[N]$ is red.

Case 2. $a=0$ and $b \neq 0$.
In this case, $\emptyset$ is necessarily blue and $[N]$ is necessarily red. Suppose there is a pair $S, T$ of comparable elements, where $S$ is red, $T$ is blue, $|S|=1$, and $|T|=N-1$. Since $\emptyset$ is blue and $S$ is red, and $T$ is blue and [ $N$ ] is red, the poset $Q_{[S, T]}$ of dimension at least $N-2>\frac{37}{16} n+\frac{23}{16}-2>\frac{7}{4} n$ can be viewed as having bottom and top elements colored both red and blue. By Claim d, $Q_{[S, T]}$ contains a red $Q_{3}$ or a blue $Q_{n}$.

Otherwise, there are only four subcases:

1. All level 1 sets are blue.
2. All level $N-1$ sets are red, and there exists a red 1 -set.
3. There exists an element $x \in N$ such that all level 1 sets except $\{x\}$ are blue and all level $N-1$ sets except $\bar{x}$ are red.
In subcase 2 , let $S$ be the red 1 -set. Since $N-1>\frac{37}{16} n+\frac{23}{16}-1>n+3=3+(3+1-1-2) * n$, we can map the first $b=1$ layer and the last $a=2$ layers of $Q_{3}$ into $Q_{[S,[N]]}$. By Lemma 1, $Q_{N}$ contains either a blue $Q_{n}$ or a red $Q_{3}$.

In subcase 3 , a similar argument works for $Q_{[x,[N]]}$. Note that the bottom layer and top two layers of $Q_{[x,[N]]}$ are red.

In subcase 1, let $S$ be a red set such that $|S| \leq\left|S^{\prime}\right|$ for all red sets $S^{\prime}$ in $Q_{N}$. Suppose $S$ is in level $\ell$. Suppose $\ell \geq n+1$. Then the bottom $n+1$ layers of $Q_{N}$ are blue, and $Q_{N}$ contains a blue copy of $Q_{n}$. Thus $\ell \leq n$.

Suppose there exists a blue $N-1$ - set $T$. Suppose $\ell \leq\left\lfloor\frac{9}{16} n-\frac{1}{4}\right\rfloor$. Since $\emptyset$ is blue and $S$ is red, and $T$ is blue and [ $N$ ] is red, we consider the top and bottom elements of $Q_{[S, T]}$ to be both red and blue. We have $N+1>\frac{37}{16} n+\frac{23}{16}+1>\frac{9}{16} n-\frac{1}{4}+1+\frac{7}{4} n \geq \ell+1+\frac{7}{4} n$. By Claim d, $Q_{[S, T]}$ contains a red $Q_{3}$ or a blue $Q_{n}$, so $Q_{N}$ contains a red $Q_{3}$ or a blue $Q_{n}$.

Suppose there exists a blue $N-1$-set $T$, but $\ell \geq\left\lfloor\frac{9}{16} n+\frac{3}{4}\right\rfloor$. We have

$$
N>\frac{37}{16} n+\frac{23}{16}>n+\left(\frac{7}{16} n-\frac{1}{4}\right) * 3=n+\left(n+1-1-\frac{9}{16} n-\frac{1}{4}\right) * 3
$$

$$
\geq n+(n+1-1-\ell) * 3
$$

We can map the first $b=\ell$ layers and the last $a=1$ layer of $Q_{n}$ into $Q_{N}$. By Lemma 1, $Q_{N}$ contains either a blue $Q_{n}$ or a red $Q_{3}$.

Now suppose there is no blue $N-1$-set. That is, the top 2 layers of $Q_{N}$ are red. We consider $Q_{[S,[N]]}$. Since $N>\frac{37}{16} n+\frac{23}{16} \geq 2 n+3=\ell+3+(3+1-1-2) * n$, we map the first $b=1$ layer and the last $a=2$ layers of $Q_{3}$ into $Q_{[S,[N]]}$. By Lemma $1, Q_{[S,[N]]}$ contains either a blue $Q_{n}$ or a red $Q_{3}$, so $Q_{N}$ contains either a blue $Q_{n}$ or a red $Q_{3}$.

Proof of Theorem 7. For any integers $m, n \in \mathbb{N}$ with $n \geq m \geq 3$, let $N \in \mathbb{N}$ be such that there exists a red/blue coloring of $Q_{N}$ containing no red copy of $Q_{3}$ and no blue copy of $Q_{n}$. Consider a red-blue coloring $c$ of $Q_{N}$. Let $T$ be a red element such that $\min \{N-|T|,|T|\} \leq$ $\min \left\{N-\left|T^{\prime}\right|,\left|T^{\prime}\right|\right\}$ for all red elements $T^{\prime} \in Q_{N}$. Without loss of generality, let $N-|T| \leq|T|$. Let $a:=N-|T|$. Let $S$ be a red element such that $|S| \leq\left|S^{\prime}\right|$ for all red elements $S^{\prime} \in Q_{[\emptyset, T]}$. Let $b:=|S|$.

Let $\hat{R}\left(Q_{m}, Q_{n}\right)$ denote the smallest $N$ such that any red/blue coloring of $Q_{N}$, where $\emptyset$ and $[N]$ are colored both red and blue, contains either a red copy of $Q_{m}$ or a blue copy of $Q_{n}$. Equivalently, any red-blue coloring of $Q_{N}^{*}$ contains either a red copy of $Q_{m}^{*}$ or a blue copy of $Q_{n}^{*}$. To prove the theorem, we first prove the following claim.

Claim f: $\hat{R}\left(Q_{m}, Q_{n}\right) \leq\left(m-2+\frac{3}{2 m-3}\right) n+m$ for all $n \geq m \geq 4$.
Proof of Claim f: By way of contradiction, suppose there is a red-blue coloring $c$ of $Q_{N}^{*}$ (with $\left.N=\left(m-2+\frac{3}{2 m-3}\right) n+m\right)$ such that it contains neither red subposet $Q_{m}^{*}$ nor blue subposet $Q_{n}^{*}$.

Let $\ell=\left\lceil 1+\frac{3 n}{m(2 m-3)}\right\rceil$ be a fixed integer. Consider the bottom $\ell$ layers of $Q_{N}$. We look for red sets $A_{1}, A_{2}, \ldots, A_{m}$ with the following property.

$$
\begin{equation*}
\forall i \in[m], \exists x_{i} \in[N] \text { such that } x_{i} \in A_{i}, \text { but } x_{i} \notin A_{j} \forall j \in[m] \backslash i \text {. } \tag{8}
\end{equation*}
$$

We consider the following cases.
Case 1. There exist sets $A_{1}, A_{2}, \ldots, A_{m}$ with property 8.
Since

$$
\begin{gathered}
\ell=\left[1+\frac{3 n}{m(2 m-3)}\right\rceil \\
\ell \leq 2+\frac{3 n}{m(2 m-3)} \\
\ell-1 \leq 1+\frac{3 n}{m(2 m-3)} \\
m(\ell-1) \leq m+\frac{3 n}{2 m-3} \\
m(\ell-1)+n(m-2)+1 \leq m+\frac{3 n}{2 m-3}+n(m-2)+1
\end{gathered}
$$

$$
m(\ell-1)+n(m-2)+1 \leq N+1
$$

we are able to create an injection of $Q_{m}$ into the red sets of $Q_{N}$. We can partition $[N]$ like so:

$$
[N]=[n] \cup X_{1} \cup X_{2} \cup \cdots \cup X_{m-1}
$$

where $X_{1}=\bigcup_{i=1}^{m}\left(A_{i} \backslash\left\{x_{i}\right\}\right)$ and $\left|X_{i}\right| \geq n$ for all $i$ with $2 \leq i \leq m-1$. We create an injection of $Q_{m}$ into the red sets of $Q_{N}$. Consider the map $f: Q_{m} \rightarrow Q_{N}$ defined by

$$
\begin{gathered}
f(\emptyset)=\emptyset \\
f(\{i\})=A_{i} \text { for all } i \in[m] \\
f(\{i, j\})=A_{i} \cup A_{j} \cup X_{2}^{*} \text { for all }\{i, j\} \subset[m] \\
\vdots \\
f(S)=\bigcup_{i \in S} A_{i} \cup X_{2} \cup \cdots \cup X_{d}^{*} \text { for all } S \subset[m] \text { with }|S|=d \\
\vdots \\
f([m])=[N]
\end{gathered}
$$

Here, $\bigcup_{i \in S} A_{i} \cup X_{2} \cup \cdots \cup X_{d}^{*}$ denotes an arbitrarily chosen red element from the subposet with bottom element $\bigcup_{i \in S} A_{i} \cup X_{2} \cup \cdots \cup X_{d-1}$ and top element $\bigcup_{i \in S} A_{i} \cup X_{2} \cup \cdots \cup X_{d}$. If no such red element exists, this entire $n$-dimenional subposet is blue and $Q_{N}$ contains a blue $Q_{n}$.

If such a red element always exists, this function is well-defined and preserves all the subset relations found in $Q_{n}$. Its image consists entirely of red elements, so $Q_{N}$ contains a red $Q_{m}$.

Case 2. There exist red sets $B_{1}, B_{2}, \ldots, B_{m}$ in the top $\ell$ layers of $Q_{N}$ with the following property.

$$
\begin{equation*}
\forall i \in[m], \exists x_{i} \in[N] \text { such that } x_{i} \notin B_{i}, \text { but } x_{i} \in B_{j} \forall j \in[m] \backslash i \tag{9}
\end{equation*}
$$

This case is the same as Case 1, except everything is flipped over the middle layer(s) of $Q_{N}$. Using a similar argument, we show that $Q_{N}$ contains a blue $Q_{n}$ or a red $Q_{3}$.

Case 3. There do not exist such sets $A_{1}, A_{2}, \ldots, A_{m}$ or $B_{1}, B_{2}, \ldots, B_{m}$.
Suppose we are only able to find at most most $m-1$ sets $A_{1}, A_{2}, \ldots, A_{m-1}$ with property 8. Let $a_{m}$ be an arbitrarily chosen subset of $\bigcup_{i=1}^{m-1} A_{i}$ such that $\left|a_{m}\right|=(m-2)(\ell-1)+1$. We claim that every set of elements of $[N] \backslash a_{m}$ in the first $\ell$ layers is blue. Suppose this is not the case, and there is a red set $X \subseteq[N] \backslash a_{m}$ in the first $\ell$ layers. Since $\left|\bigcup_{i=1}^{m-1} A_{i}\right| \leq(m-1)(\ell-1)$,
we know $\left|\bigcup_{i=1}^{m-1} A_{i} \backslash a_{m}\right| \leq \ell-2$. Thus, there exists an $x \in X$ such that $x \notin \bigcup_{i=1}^{m-1} A_{i}$. We let $x$ be $x_{m}, X$ be $A_{m}$, and $A_{1}, A_{2}, \ldots, A_{m}$ have property 8 , a contradiction. We can eliminate at most $(m-2)(\ell-1)+1$ elements from $[N]$ and guarantee that sets formed from the remaining elements in the bottom $\ell$ layers are all blue.

Similarly, if we are only able to find at most $m-1$ red sets with property 9 , we can require the inclusion of at most $(m-2)(\ell-1)$ elements from $[N]$ and guarantee that sets formed in the top $\ell$ layers of $Q_{N}$ are all blue.

Since $n<N-2(m-2)(\ell-1)$ for all $m, n \geq 4$, we can define a mapping $i: Q_{n} \rightarrow Q_{n}^{*} \subset Q_{N}$ such that the bottom $\ell$ layers of $Q_{n}$ map to blue elements in the bottom $\ell$ layers of $Q_{N}$ and the top $\ell$ layers of $Q_{n}$ map to blue elements in the top $\ell$ layers of $Q_{N}$.

Since

$$
\begin{gathered}
\ell=\left\lceil 1+\frac{3 n}{m(2 m-3)}\right\rceil \\
\ell \geq 1+\frac{3 n}{m(2 m-3)} \\
\ell-1 \geq \frac{3 n}{m(2 m-3)} \\
(m-2)(\ell-1) \geq \frac{3 n(m-2)}{m(2 m-3)} \\
-2(m-2)(\ell-1) \leq \frac{-6 n(m-2)}{m(2 m-3)} \\
n+1-2(m-2)(\ell-1) \leq n+1+\frac{-6 n(m-2)}{m(2 m-3)} \\
=\frac{\left(m-3+\frac{3}{2 m-3}\right) n+m}{m} \\
=\frac{\left(m-2+\frac{3}{2 m-3}\right) n+m-n}{m} \leq \frac{N-n}{m},
\end{gathered}
$$

we have

$$
N \geq n+(n+1-2(m-2)(\ell-1)) * m .
$$

The bottom $a=\ell$ layers and the top $b=\ell$ layers of $Q_{n}^{*}$ are blue, By Lemma $1, Q_{N}$ contains either a blue subposet $Q_{n}$ or a red subposet $Q_{m}$.

In any case where $N \geq\left(m-2+\frac{3}{2 m-3}\right) n+m$ and $[N]$ and $\emptyset$ are colored both red and blue, we have shown that $Q_{N}$ must contain a red $Q_{m}$ or a blue $Q_{n}$. It follows that $\hat{R}\left(Q_{3}, Q_{n}\right) \leq\left(m-2+\frac{3}{2 m-3}\right) n+m$.

Suppose $a \neq 0$ and $b \neq 0$. It follows that $|T|-|S|+1 \leq\left(m-2+\frac{3}{2 m-3}\right) n+m$ for all $n \geq m \geq 4$.
Claim g: $N \leq n+m(n+1-(a+b))-1$
Proof of Claim g: Otherwise, we assume $N \geq n+m(n+1-(a+b))$. Let $k=n+1-(a+b)$, so $N+1 \geq a+b+m k$.

We can partition [ $N$ ] like so:

$$
[N]=[n] \cup X_{1} \cup X_{2} \cup \cdots \cup X_{k},
$$

where $k=\frac{N-n}{m}$ and $\left|X_{i}\right| \geq m$ for all $i \in[k]$. With this partition in mind, we define a mapping $i: Q_{n} \rightarrow Q_{n}^{*} \subset Q_{N}$, an injection of $Q_{n}$ into the blue sets of $Q_{N}$. By Lemma $1, Q_{N}$ contains either a blue copy of $Q_{n}$ or a red copy of $Q_{m}$, a contradiction.

From Claim f, we have

$$
\begin{equation*}
a+b=N-(|T|-|S|) \geq N-\left(\left(m-2+\frac{3}{2 m-3}\right) n+m-1\right) . \tag{10}
\end{equation*}
$$

Combining (10) with Claim g, we have

$$
\begin{gather*}
N \leq n+m(n+1-(a+b))-1 \\
\leq n+m\left(n+1-\left(N-\left(m-2+\frac{3}{2 m-3}\right) n-m+1\right)\right)-1 \tag{11}
\end{gather*}
$$

We get

$$
\begin{equation*}
N \leq\left(m-2+\frac{9 m-9}{(2 m-3)(m+1)}\right) n+m-1 . \tag{12}
\end{equation*}
$$

Now suppose $a=0$. We consider the remaining two cases. In each case, we assume, by way of contradiction, that $N>\left(m-2+\frac{9 m-9}{(2 m-3)(m+1)}\right) n+m+2$.

Case 1. $a=0$ and $b=0$.
In this case, both $\emptyset$ and $[N]$ are necessarily red. We consider levels $1,2, N-2$, and $N-1$. If we can find two blue sets $S$ and $T$ with $|S| \leq 2,|T| \geq N-2,|T|-|S| \geq N-3$, and $S \subset T$, then we can consider $Q_{[S, T]}$. In this case, since $\emptyset$ is red and $S$ is blue, we can consider the bottom element of $Q_{[S, T]}$ to be both red and blue. Since [ $N$ ] is red and $T$ is blue, we can consider the top element of $Q_{[S, T]}$ to be both red and blue. By Claim f, $\hat{R}\left(Q_{m}, Q_{n}\right)+3 \leq\left(m-2+\frac{3}{2 m-3}\right) n+m+3<\left(m-2+\frac{9 m-9}{(2 m-3)(m+1)}\right) n+m+2<N$ for sufficiently large $m$ and $n$.

If we cannot find such sets $S$ and $T$, we are left with the following subcases:

1. All sets in levels 1 and 2 are red.
2. All sets in levels $N-2$ and $N-1$ are red.
3. All sets in levels 1 and $N-1$ are red.
4. There exist blue sets $S$ and $T$ with $|S| \leq 2$ and $|T| \geq N-2$, but $S \not \subset T$.

In subcase 1 , since $N>\left(m-2+\frac{9 m-9}{(2 m-3)(m+1)}\right) n+m+2>m+(m-3) * n$, we can partition $[N]=[m] \cup X_{1} \cup X_{2} \cup \cdots \cup X_{m-3}$ with $\left|X_{i}\right| \geq n$ for all $i \in[m-3]$. We map the first $b=3$ layers and the last $a=1$ layer of $Q_{m}$ into $Q_{N}$. By Lemma $1, Q_{N}$ contains either a blue $Q_{n}$ or a red $Q_{m}$. Subcase 2 is similar. Subcase 3 is similar, except we map the first $b=2$ layers and the last $a=2$ layers of $Q_{m}$ into $Q_{N}$.

In subcase 4, either $S$ is in level 1 or $T$ is in level $N-1$. Otherwise, we apply the same strategy as in subcases 1,2 , or 3 . Suppose, without loss of generality, that $T$ is in level $N-1$. Then a similar argument works for $Q_{[\emptyset, T]}$. Note that the first three layers of $Q_{[\emptyset, T]}$ are red, while the top element $T$ can be treated as red since $[N]$ is red.
Case 2. $a=0$ and $b \neq 0$.
In this case, $\emptyset$ is necessarily blue and [ $N$ ] is necessarily red. Suppose there is a pair $S, T$ of comparable elements, where $S$ is red, $T$ is blue, $|S| \leq 2,|T| \geq N-2$, and $|T|-|S| \geq N-4$. Since $\emptyset$ is blue and $S$ is red, and $T$ is blue and $[N]$ is red, we can consider the top and bottom elements of $Q_{[S, T]}$ to be both red and blue. By Claim f, $\hat{R}\left(Q_{m}, Q_{n}\right)+4 \leq\left(m-2+\frac{3}{2 m-3}\right) n+$ $\frac{m}{2 m-3}+4<\left(m-2+\frac{9 m-9}{(2 m-3)(m+1)}\right) n+m+2$ for sufficiently large $m$ and $n$.

Otherwise, there are only four remaining subcases:

1. All sets in levels 1 and 2 are blue and all sets in levels $N-2$ and $N-1$ are red.
2. All sets in levels 1 and 2 are blue and there exists a blue set $T$ with $|T| \geq N-2$.
3. All sets in levels $N-2$ and $N-1$ are red and there exists a red set $S$ with $|S| \leq 2$.
4. There exists a red set $S$ and a blue set $T$ with $|S| \leq 2$ and $|T| \geq N-2$, but $S \not \subset T$.

A similar argument works for subcases 3 and 4 since we can find a $Q_{[N-2]}$ so that there are four red layers. In subcase 3 , we consider $Q_{[S,[N]]}$. Since $N-2>\left(m-2+\frac{9 m-9}{(2 m-3)(m+1)}\right) n+$ $m+2-2>m+(m-3) * n$, we can map the first $b=1$ layer and the last $a=3$ layers of $Q_{m}$ into $Q_{[S,[N]]}$. By Lemma $1, Q_{[S,[N]]}$ contains either a blue $Q_{n}$ or a red $Q_{m}$, so $Q_{N}$ contains either a blue $Q_{n}$ or a red $Q_{m}$. Subcase 4 is similar; we consider $Q_{[S,[N]]}$.

In subcase 2 , we consider $Q_{[\emptyset, T]}$, a poset of dimension at least $N-2$. Both $S$ and $T$ are blue. We consider red sets of maximum and minimum cardinality in $Q_{[\emptyset, T]}$, and apply the same argument we used to get (12). Since $N-2>\left(m-2+\frac{9 m-9}{(2 m-3)(m+1)}\right) n+m, Q_{[\emptyset, T]}$ contains either a blue $Q_{n}$ or a red $Q_{3}$, so $Q_{N}$ contains either a blue $Q_{n}$ or a red $Q_{3}$.

In subcase 1 , the top 3 layers of $Q_{N}$ are red. Let $S$ be a element such that $|S| \leq|T|$ for all red elements $T$. We consider $Q_{[S,[N]]}$. Since $N>\left(m-2+\frac{9 m-9}{(2 m-3)(m+1)}\right) n+m+2 \geq$ $(m-2) n+\frac{3 n}{m(2 m-3)}+5$ for sufficiently large $m$ and $n$, we have

$$
\begin{gathered}
N>(m-2) n+\frac{3 n}{m(2 m-3)}+5 \geq(m-2) n+\left\lceil 1+\frac{3 n}{m(2 m-3)}\right\rceil+3 \\
\geq(m-3)(n+1)+\ell+3=(m-3)+(m-3) n+\ell+3 \\
=\ell+m+(m+1-1-3) * n .
\end{gathered}
$$

We map the first $b=1$ layer and the last $a=3$ layers of $Q_{3}$ into $Q_{[S,[N]]}$. By Lemma 1, $Q_{[S,[N]]}$ contains either a blue $Q_{n}$ or a red $Q_{3}$, so $Q_{N}$ contains either a blue $Q_{n}$ or a red $Q_{3}$.

Proof of Theorem 8. Consider a coloring $c$ of $Q_{4}$ defined by

$$
c(S)= \begin{cases}\text { blue } & \text { if }|S| \text { is even } \\ \text { red } & \text { if }|S| \text { is odd }\end{cases}
$$

for all sets $S$ in $Q_{4}$. This coloring of $Q_{4}$ contains no red copy of $Q_{2}$ and no blue copy of $Q_{3}$. Thus, $R\left(Q_{2}, Q_{3}\right)>4$. Now we need only show $R\left(Q_{2}, Q_{3}\right) \leq 5$.

Consider a red-blue coloring of $Q_{5}$ containing no red $Q_{2}$ and no blue $Q_{3}$. We consider the following cases.

Case 1. Both $\emptyset$ and [5] are colored red.
Let $u, v$ be two red elements in $Q_{5}$. If $u$ and $v$ are incomparable, $\{\emptyset, u, v,[5]\}$ form a red $Q_{2}$. So every red elements in $Q_{5}$ lies on the same maximal chain. With the exception of this maximal chain, the rest of $Q_{5}$ is blue, and we can find a blue $Q_{3}$, a contradiction.

Case 2. One of $\emptyset$ and [5] is colored red, and the other is blue.
Without loss of generality, suppose $\emptyset$ is red and [5] is blue. Suppose there exists a red set $T$ with $|T|=4$. Without loss of generality, let $T$ be $\{1,2,3,4\}$. Consider $Q_{[\emptyset, T]}$, and let $U, V$ be two red elements in $Q_{[\emptyset, T]}$. If $U$ and $V$ are incomparable, $\{\emptyset, U, V, T\}$ form a red $Q_{2}$. So every red element in $Q_{[\emptyset, T]}$ lies on the same maximal chain. Without loss of generality, suppose this maximal chain is $\{\emptyset,\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\}\}$. Then the sets $\{4\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$ all must be blue. These sets, along with [5] form a blue $Q_{3}$. Thus, every set in level 4 of $Q_{5}$ must be blue.

Suppose there exists two red sets $S_{1}$ and $S_{2}$ with $\left|S_{1}\right|=\left|S_{2}\right|=1$. Then $S_{1} \cup S_{2}$ must be blue. Moreover, every set in $Q_{\left[S_{1} \cup S_{2},[5]\right]}$ must be blue. Then $Q_{\left[S_{1} \cup S_{2},[5]\right]}$ is a blue copy of $Q_{3}$, a contradiction. Thus, $Q_{5}$ has at most one red level 1 set.

Without loss of generality, suppose $\{1\}$ is the only red level 1 set in $Q_{5}$. Note that $\overline{2}, \overline{3}$, and $\overline{4}$ are all blue. Consider $Q_{[\{5\}, \overline{2} n \overline{3}]}$. If $\overline{2} \cap \overline{3}=\{1,4,5\}$ and $\{4,5\}$ are both red, then $\{\emptyset,\{1\},\{4,5\},\{1,4,5\}\}$ is a red copy of $Q_{2}$. Thus, at least one of $\{4,5\}$ and $\{1,4,5\}$ is blue. Similarly, when we consider $Q_{[\{5\}, \overline{2} \cap \overline{4}]}$ and $Q_{[\{5\}, \overline{3} \cap \overline{4}]}$, we conclude that at least one of $\{3,5\}$ and $\{1,3,5\}$ is blue and at least one of $\{2,5\}$ and $\{1,2,5\}$ is blue. These blue sets, along with $\{5\}, \overline{2}, \overline{3}, \overline{4}$, and [5] form a blue copy of $Q_{3}$. Thus, $Q_{5}$ has no red level 1 set.

Now, note that $\{1,2\},\{1,3\}$, and $\{1,4\}$ cannot all be blue. Otherwise,
$\{\{1\},\{1,2\},\{1,3\},\{1,4\}, \overline{2}, \overline{3}, \overline{4},[5]\}$ is a blue copy of $Q_{3}$. Suppose, without loss of generality, that $\{1,2\}$ is red. Consider $Q_{[\{1\},\{1,2,3\}]}$. If $\{2,3\}$ and $\{1,2,3\}$ are both red, then $\{\emptyset,\{1,2\},\{2,3\},\{1,2,3\}\}$ is a red copy of $Q_{2}$. Thus, at least one of $\{2,3\}$ and $\{1,2,3\}$ is blue. Similarly, when we consider $Q_{[\{1\},\{1,2,4\}]}$ and $Q_{[\{1\},\{1,2,5\}]}$, we conclude that at least one of $\{2,4\}$ and $\{1,2,4\}$ is blue and at least one of $\{2,5\}$ and $\{1,2,5\}$ is blue. These blue sets, along with $\{1\}, \overline{3}, \overline{4}, \overline{5}$, and [5] form a blue copy of $Q_{3}$, a contradiction.

Case 3. Both $\emptyset$ and [5] are colored blue.
Suppose $Q_{5}$ has at most 2 red level 1 sets. In other words, $Q_{5}$ has at least 3 blue level 1 sets. Without loss of generality, suppose $\{1\},\{2\}$, and $\{3\}$ are all blue. Consider $Q_{[\{1,2\}, \overline{3}]}$. If every set in $Q_{[\{1,2\}, \overline{3}]}$ is red, $Q_{[\{1,2\}, \overline{3}]}$ is a red copy of $Q_{2}$. Thus, there is at least one blue set in $Q_{[\{1,2\}, \overline{3}]}$. Similarly, there is at least one blue set in $Q_{[\{1,3\}, \overline{2}]}$ and at least one blue set in $Q_{[\{2,3\}, \overline{1}]}$. These sets, along with $\emptyset,\{1\},\{2\},\{3\}$, and [5], form a blue copy of $Q_{3}$. Thus, $Q_{5}$ has at least 3 red level 1 sets. By a similar argument, $Q_{5}$ also has at least 3 red level 4 sets.

Let $S_{1}, S_{2}, S_{3}$ be 3 red level 1 sets, and let $T_{1}, T_{2}, T_{3}$ be 3 level 4 sets. We consider the following subcases.

Subcase 3.1 At least one of $S_{1}, S_{2}$, and $S_{3}$ is a subset of $T_{1}, T_{2}$, and $T_{3}$.
Without loss of generality, let $S_{1}=\{1\}$ be red and a subset of $T_{1}=\overline{3}=\{1,2,4,5\}$, $T_{2}=\overline{4}=\{1,2,3,5\}$, and $T_{3}=\overline{5}=\{1,2,3,4\}$, all of which are red. Note that no two of $\{1,2,3\},\{1,2,4\}$, and $\{1,2,5\}$ can be red without creating a red copy of $Q_{2}$. Also, no two of $\{1,2\},\{1,3\},\{1,4\}$, and $\{1,5\}$ can be red without creating a red copy of $Q_{2}$.

Suppose $\{1,2\}$ is red, which means $\{1,3\},\{1,4\}$, and $\{1,5\}$ must all be blue, and $\{1,4,5\}$, $\{1,3,5\}$, and $\{1,3,4\}$ must all be blue. These 6 sets, along with $\emptyset$ and [5], form a blue copy of $Q_{3}$, a contradiction.

Suppose exactly one of $\{1,2,3\},\{1,2,4\}$, and $\{1,2,5\}$ is red. Without loss of generality, suppose $\{1,2,3\}$ is red. Neither $\{1,4\}$ nor $\{1,5\}$ can be red without creating a red copy of $Q_{2}$ with $\{1\},\{1,2,3\}$, and $\{1,2,3,4\}$. Suppose $\{1,3\}$ is red, which means $\{1,2\},\{1,4\}$, and $\{1,5\}$ must all be blue. Then $\{1,4,5\}$ must be red. If $\{1,3,4,5\}$ is red, it forms a red copy of $Q_{2}$ with $\{1\},\{1,3\}$, and $\{1,4,5\}$. If $\{1,3,4,5\}$ is blue, it forms a blue copy of $Q_{3}$ with $\emptyset$, $\{1,2\},\{1,4\},\{1,5\},\{1,2,4\},\{1,2,5\}$, and [5]. Thus, $Q_{5}$ contains a red copy of $Q_{2}$ or a blue copy of $Q_{3}$, a contradiction.

Suppose $\{1,2,3\}$ is red and none of $\{1,3\},\{1,4\}$, and $\{1,5\}$ are red. Then $\{1,4,5\}$ must be red, and $\{1,3,4\}$ and $\{1,3,5\}$ must be blue. Then $\{2,4\},\{2,5\},\{3,4\}$, and $\{3,5\}$ must be blue, and $\{2,4,5\}$ must be red. Then $\{4\},\{5\}$, and $\{4,5\}$ must be blue. Then $\{4\}$, $\{5\},\{1,2\},\{4,5\},\{1,2,4\}$, and $\{1,2,5\}$, along with $\emptyset$ and [5], form a blue copy of $Q_{3}$, a contradiction.

Now suppose none of $\{1,2,3\},\{1,2,4\}$, or $\{1,2,5\}$ are red. Again, no two of $\{1,2\},\{1,3\}$, $\{1,4\}$ and $\{1,5\}$ are red. Suppose one of $\{1,3\},\{1,4\}$, and $\{1,5\}$ is red. Without loss of generality, suppose $\{1,3\}$ is red. Then $\{1,4,5\}$ is must be red, and $\{1,3,4,5\}$ must be blue. Then $\{1,2\},\{1,4\},\{1,5\},\{1,2,3\},\{1,2,5\}$, and $\{1,3,4,5\}$, along with $\emptyset$ and [5], form a blue copy of $Q_{3}$, a contradiction.

Suppose none of $\{1,2\},\{1,3\},\{1,4\}$, or $\{1,5\}$ are red. Then $\{1,4,5\},\{1,3,4\}$, and $\{1,3,5\}$ must all be red, and $\{1,3,4,5\}$ must be blue. Then $\{1,2\},\{1,3\},\{1,4\},\{1,2,3\}$, $\{1,2,4\}$, and $\{1,3,4,5\}$, along with $\emptyset$ and [5], form a blue copy of $Q_{3}$, a contradiction.

In any case where at least one of $S_{1}, S_{2}$, and $S_{3}$ is a subset of $T_{1}, T_{2}$, and $T_{3}, Q_{5}$ contains a red copy of $Q_{2}$ or a blue copy of $Q_{3}$.

Subcase 3.2 None of $S_{1}, S_{2}$, and $S_{3}$ is a subset of $T_{1}, T_{2}$, and $T_{3}$.
Without loss of generality, let $S_{1}=\{1\}, S_{2}=\{2\}, S_{3}=\{3\}, T_{1}=\overline{1}=\{2,3,4,5\}$, $T_{2}=\overline{2}=\{1,3,4,5\}$, and $T_{3}=\overline{3}=\{1,2,4,5\}$ all be red. Certainly, if every level 2 set and every level 3 set is blue, or if one or both of $\{4,5\}$ and $\{1,2,3\}$ are the only red sets, then $Q_{5}$ contains a blue copy of $Q_{3}$.

Suppose one of $\{1,2\},\{1,3\}$ and $\{2,3\}$ is red. Without loss of generality, suppose $\{1,2\}$ is red. Then $\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{1,4,5\}$, and $\{2,4,5\}$ must all be blue. Suppose either $\{1,2,3,4\}$ or $\{1,2,3,5\}$ is red. Without loss of generality, suppose $\{1,2,3,4\}$ is red. Then $\{1,3\},\{2,3\},\{1,3,4\}$, and $\{2,3,4\}$ must all be blue, and $\{1,2,3,5\}$ must be red. Then $\{1,3,5\}$ and $\{2,3,5\}$ must be blue. The sets $\{1,4\},\{1,5\},\{1,3\},\{1,4,5\},\{1,3,4\}$, and $\{1,3,5\}$, along with $\emptyset$ and [5], form a blue copy of $Q_{3}$, a contradiction.

Now suppose $\{1,2\}$ is red and $\{1,2,3,4\}$ and $\{1,2,3,5\}$ are both blue. Then $\{1,3\}$ must be red, and $\{1,2,3\}$ must be blue. Then $\{1,4\},\{1,5\},\{1,2,3\},\{1,4,5\},\{1,2,3,4\}$, and
$\{1,2,3,5\}$, along with $\emptyset$ and [5], form a blue copy of $Q_{3}$, a contradiction. The argument is similar if any one of $\{1,4,5\},\{2,4,5\}$, and $\{3,4,5\}$ is red.

Suppose any level 2 set other than $\{1,2\},\{1,3\},\{2,3\}$, or $\{4,5\}$ is red. Without loss of generality, suppose $\{1,4\}$ is red. Then $\{1,2\},\{1,3\},\{1,5\},\{1,2,5\}$, and $\{1,3,5\}$ are all blue. Then $\{1,2,3\}$ must be red, and $\{1,2,3,4\}$ must be blue. Then $\{1,2\},\{1,3\},\{1,5\},\{1,2,5\}$, $\{1,3,5\}$, and $\{1,2,3,4\}$, along with $\emptyset$ and [5], form a blue copy of $Q_{3}$, a contradiction. The argument is similar if any level 3 set other than $\{1,4,5\},\{2,4,5\},\{3,4,5\}$, or $\{1,2,3\}$ is red.

In any case where none of $S_{1}, S_{2}$, and $S_{3}$ is a subset of $T_{1}, T_{2}$, and $T_{3}, Q_{5}$ contains a red copy of $Q_{2}$ or a blue copy of $Q_{3}$.

## 4 Concluding Remarks

There remains a significant gap between our upper bounds and the best known lower bounds given by Axenovich and Walzer. We believe the true values of $R\left(Q_{m}, Q_{n}\right)$ for sufficiently large $m$ and $n$ are significantly less than our upper bounds. Assuming, without loss of generality, that $n \geq m$, we make the following conjecture for sufficiently large $m$ and $n$.
Conjecture 1. $R\left(Q_{m}, Q_{n}\right)=o\left(n^{2}\right)$.

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