Poset Ramsey Numbers for Boolean Lattices

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Abstract

A subposet Q' of a poset Q is a copy of a poset P if there is a bijection f between elements of P and Q' such that $x \leq y$ in P iff $f(x) \leq f(y)$ in Q'. For posets P, P', let the poset Ramsey number R(P, P') be the smallest N such that no matter how the elements of the Boolean lattice Q_N are colored red and blue, there is a copy of P with all red elements or a copy of P' with all blue elements. Axenovich and Walzer introduced this concept in Order (2017), where they proved $R(Q_2, Q_n) \leq 2n + 2$ and $R(Q_n, Q_m) \leq mn + n + m$, where Q_n is the Boolean lattice of dimension n. They later proved $2n \leq R(Q_n, Q_n) \leq$ $n^2 + 2n$. Walzer later proved $R(Q_n, Q_n) \leq n^2 + 1$. We provide some improved bounds for $R(Q_n, Q_m)$ for various $n, m \in \mathbb{N}$. In particular, we prove that $R(Q_n, Q_n) \leq n^2 - n + 2$, $R(Q_2, Q_n) \leq \frac{5}{3}n + 2$, and $R(Q_3, Q_n) \leq \frac{37}{16}n + \frac{39}{16}$. We also prove that $R(Q_2, Q_3) = 5$, and $R(Q_m, Q_n) \leq (m - 2 + \frac{9m - 9}{(2m - 3)(m + 1)})n + m + 3$ for all $n \geq m \geq 4$.

1 Introduction

Ramsey theory roughly says that any 2-coloring of elements in a sufficiently large discrete system contains a monochromatic system of given size. In the domain of complete graphs, the classical Ramsey theorem states that for any two graphs G and H there is a integer N_0 such that if the edges of a complete graph K_N with $N \ge N_0$ are colored in two colors then there exists either a red copy of G or a blue copy of H in K_N . The least such number N_0 is called the Ramsey number R(G, H). This theorem was proved by Ramsey [12] in 1930, but the problem of exactly determining these, "multicolor" Ramsey numbers, and k-uniform hypergraph Ramsey numbers remains open and is the subject of continuing research. For examples, see [2, 4, 5, 6, 7, 10].

In this paper, we will consider the poset Ramsey number instead of the graph Ramsey number. Given two posets (P, \leq) and (Q, \leq') , we say (P, \leq) is a *subposet* of (Q, \leq') , if there is an injective mapping $f: P \to Q$ such that for any $x, y \in P$ we have

$$x \le y$$
 if and only if $f(x) \le' f(y)$. (1)

The image f(P) is called a *copy* of P in Q. A *Boolean lattice* of dimension n, denoted Q_n , is the power set of an n-element ground set X equipped with the inclusion relation. The 2-dimension of a poset P, defined by Trotter [13] and denoted by $\dim_2(P)$, is the smallest n such that Q_n contains a copy of P.

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A poset X has Ramsey property if for any poset P there is a poset Z such that when one colors the copies of X in Z red or blue, there is a copy of P in Z such that all copies of X in this copy of P are red or all of them are blue. The general problem of determining which posets have Ramsey property was solved by Nešetřil and Rödl [11]. In this paper, X is the single-element poset. In other words, the elements of posets are colored instead of more complicated substructures.

For posets P and P', let the **poset Ramsey number** R(P, P') be the least integer Nsuch that whenever the elements of Q_N are colored in red or blue, there exists either a red copy of P or a blue copy of P'. The focus of this paper is the case where P and P' are Boolean lattices Q_m and Q_n for $m, n \in \mathbb{N}$. Axenovich and Walzer [1] give upper bound and lower bounds for $R(Q_m, Q_m)$ for various values of $m, n \in \mathbb{N}$. In particular, they prove the following.

Theorem 1. For any integers $n, m \ge 1$,

- (i) $2n \leq R(Q_n, Q_n) \leq n^2 + 2n$,
- (*ii*) $R(Q_1, Q_n) = n + 1$,
- $(iii) R(Q_2, Q_n) \le 2n + 2,$
- $(iv) \ n+m \le R(Q_n, Q_m) \le mn+n+m,$
- (v) $R(Q_2, Q_2) = 4, R(Q_3, Q_3) \in \{7, 8\},\$

(vi) A Boolean lattice $Q_{3n\log(n)}$ whose elements are colored red or blue randomly and independently with equal probability contains a monochromatic copy of Q_n asymptotically almost surely.

Walzer, in his master's thesis [14], improved the upper bound in Theorem 1, part (i) to the following.

Theorem 2. $R(Q_n, Q_n) \le n^2 + 1.$

Axenovich and Walzer also studied Ramsey numbers for Boolean algebras in [1]. A Boolean algebra \mathcal{B}_n of dimension n is a set system $\{X_0 \cup \bigcup_{i \in I} X_i : I \subseteq [n]\}$, where X_0, X_1, \ldots, X_n are pairwise disjoint sets, $X_i \neq \emptyset$ for $i = 1, \ldots, n$. Boolean algebras have a more restrictive structure than Boolean lattices. If a subset of Q_N contains a Boolean algebra of dimension n, then it contains a copy of Q_n . The converse, however, is not always true. Gunderson, Rödl, and Sidorenko [8] first considered the number $R_{Alg}(n)$, defined to be the smallest Nsuch that any red/blue coloring of subsets of [N] contains a red or a blue Boolean algebra of dimension n. Here, "contains" means subset containment in $2^{[N]}$, not containment as a subposet. The following theorem states the best known bounds on $R_{Alg}(n)$. The lower bound is given without proof by Brown, Erdős, Chung, and Graham [3], and the upper bound was proved by Axenovich and Walzer [1].

Theorem 3. There is a positive constant c such that

$$2^{cn} \le R_{Alg}(n) \le \min\{2^{2^{n+1}n\log n}, nR_h(K^n(2,\ldots,2))\}.$$

Here, $K^n(s, \ldots, s)$ is a complete *n*-uniform *n*-partite hypergraph with parts of size *s* and $R_h(K^n(2, \ldots, 2))$ is the smallest N' such that any 2-coloring of $K^n(N', \ldots, N')$ contains a monochromatic $K^n(2, \ldots, 2)$.

Gunderson, Rödl, and Sidorenko [8] also considered the number b(n,d), defined to be the maximum cardinality of a \mathcal{B}_d -free family contained in $2^{[n]}$. They proved the following bounds:

$$n^{-\frac{(1+o(1))d}{2^{d+1}-2}} \cdot 2^n \le b(n,d) \le 10^d 2^{-2^{1-d}} d^{d-2^{-d}} n^{-1/2^d} \cdot 2^n$$

Johnston, Lu, and Milans [9] later used the Lubell function to improve the upper bound to the following, where C is a constant:

$$b(n,d) \le Cn^{-1/2^d} \cdot 2^n.$$

In this paper, we improve the upper bounds on the poset Ramsery numbers $R(Q_m, Q_n)$ given by Axenovich and Walzer in [1]. In Section 3, we prove that for any integer $n \ge 1$,

Theorem 4. $R(Q_2, Q_n) \le \frac{5}{3}n + 2.$

Theorem 5. $R(Q_n, Q_n) \le n^2 - n + 2.$

Theorem 6. $R(Q_3, Q_n) \leq \frac{37}{16}n + \frac{39}{16}$.

In Section 3, for all integers $n \ge m \ge 4$, we also prove the following.

Theorem 7. $R(Q_m, Q_n) \le (m - 2 + \frac{9m - 9}{(2m - 3)(m + 1)})n + m + 3$ for all $n \ge m \ge 4$.

Additionally, we are now able to identify the following previously unknown poset Ramsey number.

Theorem 8. $R(Q_2, Q_3) = 5$.

In Section 2, we give more definitions and introduce notation. Also in Section 2, we state and prove Lemma 1, the key embedding lemma we use to prove Theorems 4, 5, 6, and 7. We prove theorems 4, 5, 6, 7, and 8 in Section 3, and we devote Section 4 to concluding remarks.

2 Notation and Key Lemma

A partially ordered set, or poset, consists of a set S together with a partial order \leq , which is a binary relation on S satisfying

Reflexive Property: $x \leq x$, for any $x \in S$.

Transitive Property: If $x \leq y$ and $y \leq z$ then $x \leq z$ for any $x, y, z \in S$.

Antisymmetric Property: If $x \leq y$ and $y \leq x$ then x = y for any $x, y \in S$.

Let [n] denote the set $\{1, 2, ..., n\}$ and $Q_n = (2^{[n]}, \subseteq)$ be the poset over the family of all subsets of [n]. The k-th level of Q_n is the set of all k-element subsets of the ground set [n], where $0 \leq k \leq n$. For any two subsets (of [n]) $S \subset T$, let $Q_{[S,T]}$ be the induced poset of Q_n over all sets F such that $S \subseteq F \subseteq T$. Let $Q_n^* := Q_n \setminus \{\emptyset, [n]\}$. Let $\hat{R}(Q_m, Q_n)$ denote the smallest N such that any red/blue coloring of Q_N^* contains either a red copy of Q_m^* or a blue copy of Q_n^* . Equivalently, $\hat{R}(Q_m, Q_n)$ is the least N such that if \emptyset and [N] are assumed to be both red and blue while the rest of Q_N is colored either red or blue, then Q_N contains either a red copy of Q_m or a blue copy of Q_n . For a subset $S \subseteq N$, let \overline{S} denote the complement set of S in [N]. When $S = \{x\}$, we simply write \overline{x} for $\overline{\{x\}}$.

The following key lemma generalizes the blob lemma of Axenovich and Walzer (see [1], Lemma 3). The special case a = b = 0 gives the blob lemma.

Lemma 1. For any nonnegative integers N, m, n, n', a, b satisfying $N \ge n' \ge n \ge a + b$ and $N \ge m$, suppose that the Boolean lattice Q_N on the ground set [N] is colored in two colors red and blue satisfying

- 1. There is an injection $i: Q_n \to Q_{n'} \subset Q_N$ with the following properties.
 - *i* maps the bottom *a*-layers of Q_n to blue sets.
 - For all sets S in the top b layers of Q_n , $i(S) \cup ([N] \setminus [n'])$ is blue.
- 2. $N \ge n' + (n+1-a-b) * m$.

Then either a blue subposet Q_n or a red subposet Q_m exists in Q_N .

Proof of Lemma 1: Let Q_N be the Boolean lattice on the ground set [N] colored red and blue with the properties listed above.

Let k = n + 1 - (a + b). Since $N \ge n' + (n + 1 - (a + b)) * m = n + k * m$, we can partition [N] like so:

$$[N] = [n'] \cup X_1 \cup X_2 \cup \dots \cup X_k$$

where $|X_i| \ge m$ for all $i \in [k]$. With this partition in mind, we create an injection f of Q_n into the blue sets of Q_N . Consider the map $f: Q_n \to Q_N$ defined by

$$f(\emptyset) = \emptyset$$

$$f(S) = i(S) \text{ for all } S \text{ with } |S| \le a$$

$$f(S) = i(S) \cup X_1^* \text{ for all } S \text{ with } |S| = a + 1$$

$$\vdots$$

$$f(S) = i(S) \cup X_1 \cup X_2 \cup \dots \cup X_j^* \text{ for all } S \text{ with } |S| = a + j$$

$$\vdots$$

$$f(S) = i(S) \cup X_1 \cup X_2 \cup \dots \cup X_k^* \text{ for all } S \text{ with } |S| = [n] - b$$

$$f(S) = i(S) \cup X_1 \cup X_2 \cup \dots X_k \text{ for all } S \text{ with } |S| \ge [n] - b + 1$$

f([n]) = [N].

Here, $i(S) \cup X_1 \cup X_2 \cup \cdots \cup X_j^*$ denotes an arbitrarily chosen blue element from the subposet with bottom element $S \cup X_1 \cup X_2 \cup \ldots X_{j-1} \cup \emptyset$ and top element $i(S) \cup X_1 \cup X_2 \cup \cdots \cup X_{j-1} \cup X_j$. If no such blue element exists, this entire subposet is red and Q_N contains a red Q_m .

If such a blue element always exists, this function is well-defined and preserves all the subset relations found in Q_n . Its image consists entirely of blue elements, so Q_N contains a blue Q_n .

3 Proof of Theorems

Proof of Theorem 4. For any integer $n \ge 2$, let $N \in \mathbb{N}$ be such that there exists a red/blue coloring of Q_N containing no red copy of Q_2 and no blue copy of Q_n . Consider such a red-blue coloring c of Q_N . Let T be a red element such that $\min\{N - |T|, |T|\} \le \min\{N - |T'|, |T'|\}$ for all red elements $T' \in Q_N$. Without loss of generality, let $N - |T| \le |T|$. Let a := N - |T|. Let S be a red element such that $|S| \le |S'|$ for all red elements $S' \in Q_{[0,T]}$. Let b := |S|.

Claim a: $|T| - |S| \le n + 1$.

Proof of Claim a: Proof by contradiction. Otherwise, suppose $|T| - |S| \ge n+2$. Let u, v be two red elements in $Q_{[S,T]}$. If u and v are incomparable, $\{S, u, v, T\}$ form a red Q_2 . So every red element in $Q_{[S,T]}$ lies on the same maximal chain. With the exception of this maximal chain, the rest of $Q_{[S,T]}$ is blue, and we can find a blue Q_n .

Claim b: $N \le 3n + 1 - 2(a + b)$.

Proof of Claim b: Otherwise, we assume $N \ge 3n + 2 - 2(a + b)$. We have $N \ge n + (n + 1 - (a + b)) * 2$. Since the bottom *a*-layers of Q_N are all colored blue, the bottom *a*-layers of $Q_{[\emptyset,[n]]}$ are all colored in blue. If we let m = 2, by Lemma 1, Q_N contains either a blue subposet Q_n or a red subposet Q_m .

From Claim a, we have

$$a + b = N - (|T| - |S|) \ge N - (n + 1).$$
⁽²⁾

Combining (2) with Claim b, we have

$$N \le 3n + 1 - 2[N - (n+1)] = 5n + 3 - 2N.$$
(3)

We get

$$N \le \frac{5n}{3} + 1,$$

which gives us the desired result.

Proof of Theorem 5. Let $n \in \mathbb{N}$. The result is known to hold for n = 1 and n = 2, so let $n \geq 3$. Let $\hat{R}(Q_n, Q_n)$ denote the smallest N such that any red-blue coloring of Q_N , where \emptyset and [N] are assumed to be both red and blue, contains either a red or blue copy of Q_n . Equivalently, any red/blue coloring of $Q_N \setminus \{\emptyset, [N]\}$ contains either a red or blue copy of Q_n^* . To prove the theorem, we first prove the following claim.

Claim c. $\hat{R}(Q_n, Q_n) \leq n^2 - n$ for all $n \geq 3$.

Proof of Claim c. By way of contradiction, suppose there is a red-blue coloring c of Q_N (with $N = n^2 - n$) such that \emptyset and [N] are colored both red and blue while all other elements of Q_N only receive one color. Since $N = n^2 - n$, there are $n^2 - n \ge 2n$ singleton sets in the first row of Q_N . By the Pigeonhole Principle, there are n sets in the first row of Q_N with the same color. Without loss of generality, suppose at least n of these sets are blue. Then there is a subposet Q_n^* of Q_N such that level 1 of Q_n^* consists of some subset of these blue sets.

We consider an injection $i: Q_n \to Q_n^* \subset Q_N$, which maps the bottom a = 2 layers of Q_n to blue sets. Also, we also consider the top b = 1 layer of Q_N to be colored blue. By Lemma 1, since $N \ge n^2 - n = n + (n-2) * n = n + (n+1-a-b) * m$, either a blue subposet Q_n or a red subposet Q_m exists in Q_N .

Let $N = n^2 - n + 2$. Consider a Q_N , and let Q_N be colored with a coloring $c : Q_N \to \{ \text{ red, blue } \}$. We now consider the following cases.

Case 1. Sets \emptyset and [N] are the same color.

Without loss of generality, we assume both \emptyset and [N] are colored in red. If we can find two blue sets S and T with |S| = 1, |T| = N - 1, and $S \subset T$, then we can consider the $Q_{[S,T]}$. Since $|T| - |S| = N - 2 \ge n^2 - n$, by Claim c, $Q_{[S,T]}$ either a red or blue copy of $Q_n \setminus \{\emptyset, [n]\}$, which can be extended to a red or blue copy of Q_n .

If we fail to find such two blue sets S and T, there are only three subcases:

- 1. All level 1 sets are red.
- 2. All level N-1 sets are red.
- 3. There exists an element $x \in N$ such that $\{x\}$ an $[N] \setminus \{x\}$ are blue but all other sets in level 1 and level N 1 are red.

In subcase one, since $N \ge n + n(n-2)$, we can partition $[N] = [n] \cup X_1 \cup \cdots \cup X_{n-2}$ so that $|X_i| \ge n$. We map the first two layers and the last layer of Q_n into Q_N and extend this map as in the proof of Claim c to get a red copy of Q_n . Subcase two is similar. In subcase three, similar argument works for the subposet $Q_{[\emptyset,\bar{x}]}$, where $\bar{x} = [N] \setminus \{x\}$. Note that the first two layers of $Q_{[\emptyset,\bar{x}]}$ are red, while the top element \bar{x} can be treated as red since [N] is red.

Case 2. Sets \emptyset and [N] are not the same color. Without loss of generality, suppose \emptyset is red and [N] is blue.

Suppose there is a pair S, T of comparable elements, where S is blue, T is red, |S| = 1, and |T| = N - 1. Since \emptyset is red and S is blue, and [N] is red and T is blue, the poset $Q_{[S,T]}$ of dimension $n^2 - n$ can be viewed as having bottom and top elements colored both red and blue. By Claim c, $Q_{[S,T]}$ contains a red Q_n or a blue Q_n .

Otherwise, there are only four subcases:

- 1. All level 1 sets are red and all level N 1 sets are blue.
- 2. All level 1 sets are red, and there exists a red N 1-set.

- 3. All level N 1 sets are blue, and there exists a blue 1-set.
- 4. There exists an element $x \in N$ such that all level 1 sets except $\{x\}$ are red and all level N-1 sets except \bar{x} are blue.

A similar argument works for subcases 2, 3, and 4 since we can find a $Q_{[N-1]}$ so that there are three layers of one color.

In subcase 1, suppose there exists a blue set in level 2. Then we can find a blue $Q_{[N-2]}$ and a similar argument works. If there does not exist such a blue set, the bottom three layers of Q_N are red.

In this case, since

$$n^2 - n + 2 \ge n + (n - 2) * n,$$

we can partition $[N] = [n] \cup X_1 \cup \cdots \cup X_{n-2}$ so that $|X_i| \ge n$. We map the first three layers of Q_n into Q_N to get a red copy of Q_n . Applying Lemma 1 with a = 3 and b = 0, we get the desired monochromatic copy of Q_n .

In any case where $N = n^2 - n + 2$, we have shown Q_N must contain a red Q_n or a blue Q_n . It follows that $R(Q_n, Q_n) \leq n^2 - n + 2$, the desired result.

Proof of Theorem 6. For any integer $n \ge 4$, let $N \in \mathbb{N}$ be such that there exists a red/blue coloring of Q_N containing no red copy of Q_3 and no blue copy of Q_n . Consider a red-blue coloring c of Q_N . Let T be a red element such that $\min\{N - |T|, |T|\} \le \min\{N - |T'|, |T'|\}$ for all red elements $T' \in Q_N$. Without loss of generality, let $N - |T| \le |T|$. Let a := N - |T|. Let S be a red element such that $|S| \le |S'|$ for all red elements $S' \in Q_{[\emptyset,T]}$. Let b := |S|.

Let $\hat{R}(Q_3, Q_n)$ denote the smallest N such that any red/blue coloring of Q_N , where \emptyset and [N] are assumed to be both red and blue, contains either a red copy of Q_3 or a blue copy of Q_n . Equivalently, any red-blue coloring of Q_N^* contains either a red copy of Q_3^* or a blue copy of Q_n^* . To prove the theorem, we first prove the following claim.

Claim d: $\hat{R}(Q_3, Q_n) \leq \frac{7}{4}n + \frac{9}{4}$ for all n.

Proof of Claim d: By way of contradiction, suppose there is a red-blue coloring c of Q_N^* (with $N \ge \frac{7}{4}n + \frac{9}{4}$) such that it contains neither red subposet Q_3^* nor blue subposet Q_n^* .

Let $\ell = \lceil \frac{3}{8}n + \frac{5}{8} \rceil$ be a fixed integer. Consider the bottom ℓ layers of Q_N . We look for red sets A_1, A_2, A_3 with the following property.

$$\forall i \in [3], \exists x_i \in [N] \text{ such that } x_i \in A_i, \text{ but } x_i \notin A_j \quad \forall j \in [3] \setminus i.$$
(4)

We consider the following cases.

Case 1. There exist sets A_1, A_2, A_3 with property 4.

Since

$$\ell = \lceil \frac{3}{8}n + \frac{5}{8} \rceil = \lceil \frac{\frac{7}{4}n - n + \frac{5}{4}}{2} \rceil \le \lceil \frac{N - n - 1}{2} \rceil \le \frac{N - n + 1}{2}$$
$$N + 1 \ge 2\ell + n,$$
$$N + 1 \ge \ell + (\ell - 1) + n + 1$$

we are able to create an injection of Q_3 into the red sets of Q_N . Consider the map $f: Q_3 \to Q_N$ defined by

$$f(\{i\}) = A_i \text{ for all } i \in [3],$$

$$f(\{i, j\}) = X_{i,j}^* \text{ for all } \{i, j\} \subset [3].$$

Here, $X_{i,j}^*$ denotes an arbitrarily chosen red element from the subposet with bottom element $A_i \cup A_j$ and top element \bar{x}_k , where $\{i, j, k\} = [3]$. If no such red element exists, this entire *n*-dimensional subposet is blue and Q_N contains a blue Q_n .

If such a red element always exists, this function is well-defined and preserves all the subset relations found in Q_n . Its image consists entirely of red elements, so Q_N contains a red Q_3 , a contradiction.

Case 2. There exist red sets B_1, B_2, B_3 in the top ℓ layers of Q_N with the following property.

$$\forall i \in [3], \exists x_i \in [N] \text{ such that } x_i \notin B_i, \text{ but } x_i \in B_j \quad \forall j \in [3] \setminus i.$$
(5)

This case is the same is as Case 1, except everything is flipped over the middle layer(s) of Q_N . Using a similar argument, we show that Q_N contains a blue Q_n or a red Q_3 .

Case 3. There do not exist such sets A_1, A_2, A_3 or B_1, B_2, B_3 .

Suppose we are only able to find one red set A_1 . Then every set of elements of $[N] \setminus A_1$ in the first ℓ layers is blue. Note that $|A_1| \leq \ell - 1$.

Suppose we are only able to find 2 sets with property 4. Let a_3 be an arbitrarily chosen ℓ -element subset of $A_1 \cup A_2$. We claim that every set of elements of $[N] \setminus a_3$ in the first ℓ layers is blue. Suppose this is not the case, and there is a red set $X \subseteq [N] \setminus a_3$ in the first ℓ layers. Since $|A_1 \cup A_2| \leq 2(\ell - 1)$, we know $|A_1 \cup A_2 \setminus a_3| \leq \ell - 2$. Thus, there exists an $x \in X$ such that $x \notin A_1 \cup A_2$. We let x be x_3 , X be A_3 , and A_1, A_2, A_3 have property 4, a contradiction. We can eliminate at most ℓ elements from [N] and guarantee that sets formed from the remaining elements in the bottom ℓ layers are all blue.

Similarly, if we are only able to find at most two red sets with property 5, we can require the inclusion of at most ℓ elements from [N] and guarantee that sets formed in the top ℓ layers of Q_N are all blue. Since $n < n + 1 = \frac{7}{4}n + \frac{9}{4} - 2(\frac{3}{8}n + \frac{5}{8})$, we can define a mapping $i: Q_n \to Q_n^* \subset Q_N$ such that the bottom ℓ layers of Q_n map to blue elements in the bottom ℓ layers of Q_N and the top ℓ layers of Q_n map to blue elements in the top ℓ layers of Q_N .

Since

$$\ell = \lceil \frac{3}{8}n + \frac{5}{8} \rceil$$
$$2\ell = 2\lceil \frac{3}{8}n + \frac{5}{8} \rceil \ge 2(\frac{3}{8}n + \frac{5}{8}) - 1$$
$$-2\ell \le -\frac{3}{4}n - \frac{1}{4}$$

$$1 - 2\ell = -\frac{3}{4}n + \frac{3}{4}$$
$$n + 1 - 2\ell = \frac{1}{4}n + \frac{3}{4} = \frac{\frac{3}{4}n + \frac{9}{4}}{3} = \frac{\frac{7}{4}n - n + \frac{9}{4}}{3} \le \frac{N - n}{3}$$

we have

$$N \ge n + (n + 1 - 2\ell) * 3.$$

The bottom $a = \ell$ layers and the top $b = \ell$ layers of Q_n^* are blue and m = 3, by Lemma 1, Q_N contains either a blue subposet Q_n or a red subposet Q_3 .

In any case where $N \geq \frac{7}{4}n + \frac{9}{4}$ and [N] and \emptyset are colored both red and blue, we have shown that Q_N must contain a red Q_3 or a blue Q_n . It follows that $\hat{R}(Q_3, Q_n) \leq \frac{7}{4}n + \frac{9}{4}$. \Box

Suppose $a \neq 0$ and $b \neq 0$. It follows that $|T| - |S| + 1 \leq \frac{7}{4}n + \frac{9}{4}$ for all $n \in \mathbb{N}$.

Claim e: $N \le n + 3(n + 1 - (a + b)) - 1$. Proof of Claim e: Otherwise, we assume $N \ge n + 3(n + 1 - (a + b))$. Let k = n + 1 - (a + b), so $N \ge a + b + 3k$.

We can partition [N] like so:

$$[N] = [n] \cup X_1 \cup X_2 \cup \cdots \cup X_k,$$

where $|X_i| \geq 3$ for all $i \in [k]$. With this partition in mind, we define a mapping $i : Q_n \to Q_n^* \subset Q_N$, an injection of Q_n into the blue sets of Q_N . By Lemma 1, Q_N contains either a blue copy of Q_n or a red copy of Q_3 , a contradiction.

From Claim d, we have

$$a + b = N - (|T| - |S|) \ge N - (\frac{7}{4}n + \frac{5}{4}).$$
(6)

Combining (6) with Claim e, we have

$$N \le n + 3(n + 1 - (a + b)) - 1 \le n + 3(n + 1 - (N - \frac{7}{4}n - \frac{5}{4})) - 1.$$
(7)

We get

$$N \le \frac{37}{16}n + \frac{23}{16}.$$

Now suppose a = 0. We consider the remaining two cases. In each case, we assume, by way of contradiction, that $N > \frac{37}{16}n + \frac{23}{16}$.

Case 1. a = 0 and b = 0.

In this case, both \emptyset and [N] are necessarily red. If we can find two blue sets S and T with |S| = 1, |T| = N - 1, and $S \subset T$, then we can consider $Q_{[S,T]}$. In this case, since \emptyset is red and S is blue, we can consider the bottom element of $Q_{[S,T]}$ to be both red and blue. Since [N]

is red and T is blue, we can consider the top element of $Q_{[S,T]}$ to be both red and blue. By Claim d, $\hat{R}(Q_3, Q_n) + 2 \leq \frac{7}{4}n + \frac{9}{4} + 2 < \frac{37}{16}n + \frac{23}{16}$ for $n \geq 4$.

If we cannot find such sets S and T, we are left with the following three subcases:

- 1. All level 1 sets are red.
- 2. All level N-1 sets are red.
- 3. There exists an element $x \in N$ such that $\{x\}$ and \bar{x} are blue but all other sets in levels and N-1 are red.

In subcase 1, since $N > \frac{37}{16}n + \frac{23}{16} > n+3 = 3 + (3+1-1-2) * n$, we can partition $[N] = [3] \cup X$ with $|X| \ge n$. We map the first b = 2 layers and the last a = 1 layer of Q_3 into Q_N . By Lemma 1, Q_N contains either a blue Q_n or a red Q_3 . Subcase 2 is similar.

In subcase 3, a similar argument works for $Q_{[\emptyset,\bar{x}]}$. Note that the first two layers of $Q_{[\emptyset,\bar{x}]}$ are red, while the top element \bar{x} can be treated as red since [N] is red.

Case 2. a = 0 and $b \neq 0$.

In this case, \emptyset is necessarily blue and [N] is necessarily red. Suppose there is a pair S, T of comparable elements, where S is red, T is blue, |S| = 1, and |T| = N - 1. Since \emptyset is blue and S is red, and T is blue and [N] is red, the poset $Q_{[S,T]}$ of dimension at least $N-2 > \frac{37}{16}n + \frac{23}{16} - 2 > \frac{7}{4}n$ can be viewed as having bottom and top elements colored both red and blue. By Claim d, $Q_{[S,T]}$ contains a red Q_3 or a blue Q_n .

Otherwise, there are only four subcases:

- 1. All level 1 sets are blue.
- 2. All level N 1 sets are red, and there exists a red 1-set.
- 3. There exists an element $x \in N$ such that all level 1 sets except $\{x\}$ are blue and all level N 1 sets except \bar{x} are red.

In subcase 2, let S be the red 1-set. Since $N-1 > \frac{37}{16}n + \frac{23}{16} - 1 > n+3 = 3 + (3+1-1-2)*n$, we can map the first b = 1 layer and the last a = 2 layers of Q_3 into $Q_{[S,[N]]}$. By Lemma 1, Q_N contains either a blue Q_n or a red Q_3 .

In subcase 3, a similar argument works for $Q_{[x,[N]]}$. Note that the bottom layer and top two layers of $Q_{[x,[N]]}$ are red.

In subcase 1, let S be a red set such that $|S| \leq |S'|$ for all red sets S' in Q_N . Suppose S is in level ℓ . Suppose $\ell \geq n+1$. Then the bottom n+1 layers of Q_N are blue, and Q_N contains a blue copy of Q_n . Thus $\ell \leq n$.

Suppose there exists a blue N - 1- set T. Suppose $\ell \leq \lfloor \frac{9}{16}n - \frac{1}{4} \rfloor$. Since \emptyset is blue and S is red, and T is blue and [N] is red, we consider the top and bottom elements of $Q_{[S,T]}$ to be both red and blue. We have $N + 1 > \frac{37}{16}n + \frac{23}{16} + 1 > \frac{9}{16}n - \frac{1}{4} + 1 + \frac{7}{4}n \geq \ell + 1 + \frac{7}{4}n$. By Claim d, $Q_{[S,T]}$ contains a red Q_3 or a blue Q_n , so Q_N contains a red Q_3 or a blue Q_n .

Suppose there exists a blue N-1-set T, but $\ell \geq \lfloor \frac{9}{16}n + \frac{3}{4} \rfloor$. We have

$$N > \frac{37}{16}n + \frac{23}{16} > n + \left(\frac{7}{16}n - \frac{1}{4}\right) * 3 = n + \left(n + 1 - 1 - \frac{9}{16}n - \frac{1}{4}\right) * 3$$

 $\ge n + (n + 1 - 1 - \ell) * 3.$

We can map the first $b = \ell$ layers and the last a = 1 layer of Q_n into Q_N . By Lemma 1, Q_N contains either a blue Q_n or a red Q_3 .

Now suppose there is no blue N - 1-set. That is, the top 2 layers of Q_N are red. We consider $Q_{[S,[N]]}$. Since $N > \frac{37}{16}n + \frac{23}{16} \ge 2n + 3 = \ell + 3 + (3 + 1 - 1 - 2) * n$, we map the first b = 1 layer and the last a = 2 layers of Q_3 into $Q_{[S,[N]]}$. By Lemma 1, $Q_{[S,[N]]}$ contains either a blue Q_n or a red Q_3 , so Q_N contains either a blue Q_n or a red Q_3 .

Proof of Theorem 7. For any integers $m, n \in \mathbb{N}$ with $n \geq m \geq 3$, let $N \in \mathbb{N}$ be such that there exists a red/blue coloring of Q_N containing no red copy of Q_3 and no blue copy of Q_n . Consider a red-blue coloring c of Q_N . Let T be a red element such that $\min\{N - |T|, |T|\} \leq \min\{N - |T'|, |T'|\}$ for all red elements $T' \in Q_N$. Without loss of generality, let $N - |T| \leq |T|$. Let a := N - |T|. Let S be a red element such that $|S| \leq |S'|$ for all red elements $S' \in Q_{[\emptyset,T]}$. Let b := |S|.

Let $R(Q_m, Q_n)$ denote the smallest N such that any red/blue coloring of Q_N , where \emptyset and [N] are colored both red and blue, contains either a red copy of Q_m or a blue copy of Q_n . Equivalently, any red-blue coloring of Q_N^* contains either a red copy of Q_m^* or a blue copy of Q_n^* . To prove the theorem, we first prove the following claim.

Claim f: $\hat{R}(Q_m, Q_n) \leq (m-2+\frac{3}{2m-3})n+m$ for all $n \geq m \geq 4$.

Proof of Claim f: By way of contradiction, suppose there is a red-blue coloring c of Q_N^* (with $N = (m - 2 + \frac{3}{2m-3})n + m$) such that it contains neither red subposet Q_m^* nor blue subposet Q_n^* .

Let $\ell = \lceil 1 + \frac{3n}{m(2m-3)} \rceil$ be a fixed integer. Consider the bottom ℓ layers of Q_N . We look for red sets A_1, A_2, \ldots, A_m with the following property.

$$\forall i \in [m], \exists x_i \in [N] \text{ such that } x_i \in A_i, \text{ but } x_i \notin A_j \ \forall j \in [m] \backslash i.$$
(8)

We consider the following cases.

m

Case 1. There exist sets A_1, A_2, \ldots, A_m with property 8.

Since

$$\ell = \left[1 + \frac{3n}{m(2m-3)}\right]$$
$$\ell \le 2 + \frac{3n}{m(2m-3)}$$
$$\ell - 1 \le 1 + \frac{3n}{m(2m-3)}$$
$$m(\ell - 1) \le m + \frac{3n}{2m-3}$$
$$(\ell - 1) + n(m-2) + 1 \le m + \frac{3n}{2m-3} + n(m-2) + 1$$

$$m(\ell - 1) + n(m - 2) + 1 \le N + 1,$$

we are able to create an injection of Q_m into the red sets of Q_N . We can partition [N] like so:

$$[N] = [n] \cup X_1 \cup X_2 \cup \cdots \cup X_{m-1},$$

where $X_1 = \bigcup_{i=1}^m (A_i \setminus \{x_i\})$ and $|X_i| \ge n$ for all *i* with $2 \le i \le m-1$. We create an injection of Q_m into the red sets of Q_N . Consider the map $f : Q_m \to Q_N$ defined by

$$f(\emptyset) = \emptyset$$

$$f(\{i\}) = A_i \text{ for all } i \in [m]$$

$$f(\{i,j\}) = A_i \cup A_j \cup X_2^* \text{ for all } \{i,j\} \subset [m]$$

$$\vdots$$

$$f(S) = \bigcup_{i \in S} A_i \cup X_2 \cup \dots \cup X_d^* \text{ for all } S \subset [m] \text{ with } |S| = d$$

$$\vdots$$

$$f([m]) = [N].$$

Here, $\bigcup_{i \in S} A_i \cup X_2 \cup \cdots \cup X_d^*$ denotes an arbitrarily chosen red element from the subposet with bottom element $\bigcup_{i \in S} A_i \cup X_2 \cup \cdots \cup X_{d-1}$ and top element $\bigcup_{i \in S} A_i \cup X_2 \cup \cdots \cup X_d$. If no such red element exists, this entire *n*-dimensional subposet is blue and Q_N contains a blue Q_n .

If such a red element always exists, this function is well-defined and preserves all the subset relations found in Q_n . Its image consists entirely of red elements, so Q_N contains a red Q_m .

Case 2. There exist red sets B_1, B_2, \ldots, B_m in the top ℓ layers of Q_N with the following property.

$$\forall i \in [m], \exists x_i \in [N] \text{ such that } x_i \notin B_i, \text{ but } x_i \in B_j \ \forall j \in [m] \setminus i.$$
(9)

This case is the same as Case 1, except everything is flipped over the middle layer(s) of Q_N . Using a similar argument, we show that Q_N contains a blue Q_n or a red Q_3 .

Case 3. There do not exist such sets A_1, A_2, \ldots, A_m or B_1, B_2, \ldots, B_m .

Suppose we are only able to find at most most m-1 sets $A_1, A_2, \ldots, A_{m-1}$ with property 8. Let a_m be an arbitrarily chosen subset of $\bigcup_{i=1}^{m-1} A_i$ such that $|a_m| = (m-2)(\ell-1)+1$. We claim that every set of elements of $[N] \setminus a_m$ in the first ℓ layers is blue. Suppose this is not the case, and there is a red set $X \subseteq [N] \setminus a_m$ in the first ℓ layers. Since $|\bigcup_{i=1}^{m-1} A_i| \leq (m-1)(\ell-1)$, we know $|\bigcup_{i=1}^{m-1} A_i \setminus a_m| \leq \ell - 2$. Thus, there exists an $x \in X$ such that $x \notin \bigcup_{i=1}^{m-1} A_i$. We let x be x_m , X be A_m , and A_1, A_2, \ldots, A_m have property 8, a contradiction. We can eliminate at most $(m-2)(\ell-1)+1$ elements from [N] and guarantee that sets formed from the remaining elements in the bottom ℓ layers are all blue.

Similarly, if we are only able to find at most m-1 red sets with property 9, we can require the inclusion of at most $(m-2)(\ell-1)$ elements from [N] and guarantee that sets formed in the top ℓ layers of Q_N are all blue.

Since $n < N-2(m-2)(\ell-1)$ for all $m, n \ge 4$, we can define a mapping $i: Q_n \to Q_n^* \subset Q_N$ such that the bottom ℓ layers of Q_n map to blue elements in the bottom ℓ layers of Q_N and the top ℓ layers of Q_n map to blue elements in the top ℓ layers of Q_N .

Since

$$\begin{split} \ell &= \lceil 1 + \frac{3n}{m(2m-3)} \rceil \\ \ell &\geq 1 + \frac{3n}{m(2m-3)} \\ \ell &\geq 1 + \frac{3n}{m(2m-3)} \\ \ell - 1 &\geq \frac{3n}{m(2m-3)} \\ (m-2)(\ell-1) &\geq \frac{3n(m-2)}{m(2m-3)} \\ -2(m-2)(\ell-1) &\leq \frac{-6n(m-2)}{m(2m-3)} \\ n+1-2(m-2)(\ell-1) &\leq n+1 + \frac{-6n(m-2)}{m(2m-3)} \\ &= \frac{(m-3+\frac{3}{2m-3})n+m}{m} \\ &= \frac{(m-2+\frac{3}{2m-3})n+m-n}{m} \leq \frac{N-n}{m}, \end{split}$$

we have

$$N \ge n + (n+1 - 2(m-2)(\ell - 1)) * m.$$

The bottom $a = \ell$ layers and the top $b = \ell$ layers of Q_n^* are blue, By Lemma 1, Q_N contains either a blue subposet Q_n or a red subposet Q_m .

In any case where $N \ge (m-2+\frac{3}{2m-3})n+m$ and [N] and \emptyset are colored both red and blue, we have shown that Q_N must contain a red Q_m or a blue Q_n . It follows that $\hat{R}(Q_3, Q_n) \le (m-2+\frac{3}{2m-3})n+m$.

Suppose $a \neq 0$ and $b \neq 0$. It follows that $|T| - |S| + 1 \leq (m - 2 + \frac{3}{2m-3})n + m$ for all $n \geq m \geq 4$.

Claim g: $N \le n + m(n + 1 - (a + b)) - 1$

Proof of Claim g: Otherwise, we assume $N \ge n + m(n+1-(a+b))$. Let k = n+1-(a+b), so $N+1 \ge a+b+mk$.

We can partition [N] like so:

$$[N] = [n] \cup X_1 \cup X_2 \cup \cdots \cup X_k,$$

where $k = \frac{N-n}{m}$ and $|X_i| \ge m$ for all $i \in [k]$. With this partition in mind, we define a mapping $i: Q_n \to Q_n^* \subset Q_N$, an injection of Q_n into the blue sets of Q_N . By Lemma 1, Q_N contains either a blue copy of Q_n or a red copy of Q_m , a contradiction.

From Claim f, we have

$$a + b = N - (|T| - |S|) \ge N - ((m - 2 + \frac{3}{2m - 3})n + m - 1).$$
(10)

Combining (10) with Claim g, we have

$$N \le n + m(n + 1 - (a + b)) - 1$$

$$\le n + m(n + 1 - (N - (m - 2 + \frac{3}{2m - 3})n - m + 1)) - 1$$
(11)

We get

$$N \le (m-2 + \frac{9m-9}{(2m-3)(m+1)})n + m - 1.$$
(12)

Now suppose a = 0. We consider the remaining two cases. In each case, we assume, by way of contradiction, that $N > (m - 2 + \frac{9m - 9}{(2m - 3)(m + 1)})n + m + 2$.

Case 1. a = 0 and b = 0.

In this case, both \emptyset and [N] are necessarily red. We consider levels 1, 2, N-2, and N-1. If we can find two blue sets S and T with $|S| \leq 2$, $|T| \geq N-2$, $|T| - |S| \geq N-3$, and $S \subset T$, then we can consider $Q_{[S,T]}$. In this case, since \emptyset is red and S is blue, we can consider the bottom element of $Q_{[S,T]}$ to be both red and blue. Since [N] is red and T is blue, we can consider the top element of $Q_{[S,T]}$ to be both red and blue. By Claim f, $\hat{R}(Q_m, Q_n) + 3 \leq (m-2 + \frac{3}{2m-3})n + m + 3 < (m-2 + \frac{9m-9}{(2m-3)(m+1)})n + m + 2 < N$ for sufficiently large m and n.

If we cannot find such sets S and T, we are left with the following subcases:

- 1. All sets in levels 1 and 2 are red.
- 2. All sets in levels N 2 and N 1 are red.
- 3. All sets in levels 1 and N 1 are red.
- 4. There exist blue sets S and T with $|S| \leq 2$ and $|T| \geq N-2$, but $S \not\subset T$.

In subcase 1, since $N > (m-2 + \frac{9m-9}{(2m-3)(m+1)})n + m + 2 > m + (m-3) * n$, we can partition $[N] = [m] \cup X_1 \cup X_2 \cup \cdots \cup X_{m-3}$ with $|X_i| \ge n$ for all $i \in [m-3]$. We map the first b = 3 layers and the last a = 1 layer of Q_m into Q_N . By Lemma 1, Q_N contains either a blue Q_n or a red Q_m . Subcase 2 is similar. Subcase 3 is similar, except we map the first b = 2 layers and the last a = 2 layers of Q_m into Q_N . In subcase 4, either S is in level 1 or T is in level N - 1. Otherwise, we apply the same strategy as in subcases 1, 2, or 3. Suppose, without loss of generality, that T is in level N - 1. Then a similar argument works for $Q_{[\emptyset,T]}$. Note that the first three layers of $Q_{[\emptyset,T]}$ are red, while the top element T can be treated as red since [N] is red.

Case 2. a = 0 and $b \neq 0$.

In this case, \emptyset is necessarily blue and [N] is necessarily red. Suppose there is a pair S, T of comparable elements, where S is red, T is blue, $|S| \leq 2$, $|T| \geq N-2$, and $|T| - |S| \geq N-4$. Since \emptyset is blue and S is red, and T is blue and [N] is red, we can consider the top and bottom elements of $Q_{[S,T]}$ to be both red and blue. By Claim f, $\hat{R}(Q_m, Q_n) + 4 \leq (m-2 + \frac{3}{2m-3})n + \frac{m}{2m-3} + 4 < (m-2 + \frac{9m-9}{(2m-3)(m+1)})n + m + 2$ for sufficiently large m and n.

Otherwise, there are only four remaining subcases:

- 1. All sets in levels 1 and 2 are blue and all sets in levels N 2 and N 1 are red.
- 2. All sets in levels 1 and 2 are blue and there exists a blue set T with $|T| \ge N 2$.
- 3. All sets in levels N-2 and N-1 are red and there exists a red set S with $|S| \leq 2$.
- 4. There exists a red set S and a blue set T with $|S| \leq 2$ and $|T| \geq N-2$, but $S \not\subset T$.

A similar argument works for subcases 3 and 4 since we can find a $Q_{[N-2]}$ so that there are four red layers. In subcase 3, we consider $Q_{[S,[N]]}$. Since $N-2 > (m-2+\frac{9m-9}{(2m-3)(m+1)})n+m+2-2 > m+(m-3)*n$, we can map the first b=1 layer and the last a=3 layers of Q_m into $Q_{[S,[N]]}$. By Lemma 1, $Q_{[S,[N]]}$ contains either a blue Q_n or a red Q_m , so Q_N contains either a blue Q_n or a red Q_m . Subcase 4 is similar; we consider $Q_{[S,[N]]}$.

In subcase 2, we consider $Q_{[\emptyset,T]}$, a poset of dimension at least N-2. Both S and T are blue. We consider red sets of maximum and minimum cardinality in $Q_{[\emptyset,T]}$, and apply the same argument we used to get (12). Since $N-2 > (m-2 + \frac{9m-9}{(2m-3)(m+1)})n + m$, $Q_{[\emptyset,T]}$ contains either a blue Q_n or a red Q_3 , so Q_N contains either a blue Q_n or a red Q_3 .

In subcase 1, the top 3 layers of Q_N are red. Let S be a element such that $|S| \leq |T|$ for all red elements T. We consider $Q_{[S,[N]]}$. Since $N > (m-2 + \frac{9m-9}{(2m-3)(m+1)})n + m + 2 \geq (m-2)n + \frac{3n}{m(2m-3)} + 5$ for sufficiently large m and n, we have

$$N > (m-2)n + \frac{3n}{m(2m-3)} + 5 \ge (m-2)n + \left\lceil 1 + \frac{3n}{m(2m-3)} \right\rceil + 3$$
$$\ge (m-3)(n+1) + \ell + 3 = (m-3) + (m-3)n + \ell + 3$$
$$= \ell + m + (m+1-1-3) * n.$$

We map the first b = 1 layer and the last a = 3 layers of Q_3 into $Q_{[S,[N]]}$. By Lemma 1, $Q_{[S,[N]]}$ contains either a blue Q_n or a red Q_3 , so Q_N contains either a blue Q_n or a red Q_3 .

Proof of Theorem 8. Consider a coloring c of Q_4 defined by

$$c(S) = \begin{cases} \text{blue} & \text{if } |S| \text{ is even} \\ \text{red} & \text{if } |S| \text{ is odd} \end{cases}$$

for all sets S in Q_4 . This coloring of Q_4 contains no red copy of Q_2 and no blue copy of Q_3 . Thus, $R(Q_2, Q_3) > 4$. Now we need only show $R(Q_2, Q_3) \le 5$.

Consider a red-blue coloring of Q_5 containing no red Q_2 and no blue Q_3 . We consider the following cases.

Case 1. Both \emptyset and [5] are colored red.

Let u, v be two red elements in Q_5 . If u and v are incomparable, $\{\emptyset, u, v, [5]\}$ form a red Q_2 . So every red elements in Q_5 lies on the same maximal chain. With the exception of this maximal chain, the rest of Q_5 is blue, and we can find a blue Q_3 , a contradiction.

Case 2. One of \emptyset and [5] is colored red, and the other is blue.

Without loss of generality, suppose \emptyset is red and [5] is blue. Suppose there exists a red set T with |T| = 4. Without loss of generality, let T be $\{1, 2, 3, 4\}$. Consider $Q_{[\emptyset,T]}$, and let U, V be two red elements in $Q_{[\emptyset,T]}$. If U and V are incomparable, $\{\emptyset, U, V, T\}$ form a red Q_2 . So every red element in $Q_{[\emptyset,T]}$ lies on the same maximal chain. Without loss of generality, suppose this maximal chain is $\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$. Then the sets $\{4\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$ all must be blue. These sets, along with [5] form a blue Q_3 . Thus, every set in level 4 of Q_5 must be blue.

Suppose there exists two red sets S_1 and S_2 with $|S_1| = |S_2| = 1$. Then $S_1 \cup S_2$ must be blue. Moreover, every set in $Q_{[S_1 \cup S_2, [5]]}$ must be blue. Then $Q_{[S_1 \cup S_2, [5]]}$ is a blue copy of Q_3 , a contradiction. Thus, Q_5 has at most one red level 1 set.

Without loss of generality, suppose $\{1\}$ is the only red level 1 set in Q_5 . Note that $\overline{2}, \overline{3}$, and $\overline{4}$ are all blue. Consider $Q_{[\{5\},\overline{2}\cap\overline{3}]}$. If $\overline{2}\cap\overline{3} = \{1,4,5\}$ and $\{4,5\}$ are both red, then $\{\emptyset, \{1\}, \{4,5\}, \{1,4,5\}\}$ is a red copy of Q_2 . Thus, at least one of $\{4,5\}$ and $\{1,4,5\}$ is blue. Similarly, when we consider $Q_{[\{5\},\overline{2}\cap\overline{4}]}$ and $Q_{[\{5\},\overline{3}\cap\overline{4}]}$, we conclude that at least one of $\{3,5\}$ and $\{1,3,5\}$ is blue and at least one of $\{2,5\}$ and $\{1,2,5\}$ is blue. These blue sets, along with $\{5\}, \overline{2}, \overline{3}, \overline{4}$, and [5] form a blue copy of Q_3 . Thus, Q_5 has no red level 1 set.

Now, note that $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$ cannot all be blue. Otherwise,

{{1}, {1,2}, {1,3}, {1,4}, $\bar{2}, \bar{3}, \bar{4}, [5]$ } is a blue copy of Q_3 . Suppose, without loss of generality, that {1,2} is red. Consider $Q_{[\{1\},\{1,2,3\}]}$. If {2,3} and {1,2,3} are both red, then { $\emptyset, \{1,2\}, \{2,3\}, \{1,2,3\}$ } is a red copy of Q_2 . Thus, at least one of {2,3} and {1,2,3} is blue. Similarly, when we consider $Q_{[\{1\},\{1,2,4\}]}$ and $Q_{[\{1\},\{1,2,5\}]}$, we conclude that at least one of {2,4} and {1,2,4} is blue and at least one of {2,5} and {1,2,5} is blue. These blue sets, along with {1}, $\bar{3}, \bar{4}, \bar{5}$, and [5] form a blue copy of Q_3 , a contradiction.

Case 3. Both \emptyset and [5] are colored blue.

Suppose Q_5 has at most 2 red level 1 sets. In other words, Q_5 has at least 3 blue level 1 sets. Without loss of generality, suppose $\{1\}$, $\{2\}$, and $\{3\}$ are all blue. Consider $Q_{[\{1,2\},\bar{3}]}$. If every set in $Q_{[\{1,2\},\bar{3}]}$ is red, $Q_{[\{1,2\},\bar{3}]}$ is a red copy of Q_2 . Thus, there is at least one blue set in $Q_{[\{1,2\},\bar{3}]}$. Similarly, there is at least one blue set in $Q_{[\{1,3\},\bar{2}]}$ and at least one blue set in $Q_{[\{2,3\},\bar{1}]}$. These sets, along with \emptyset , $\{1\}$, $\{2\}$, $\{3\}$, and [5], form a blue copy of Q_3 . Thus, Q_5 has at least 3 red level 1 sets. By a similar argument, Q_5 also has at least 3 red level 4 sets.

Let S_1, S_2, S_3 be 3 red level 1 sets, and let T_1, T_2, T_3 be 3 level 4 sets. We consider the following subcases.

Subcase 3.1 At least one of S_1, S_2 , and S_3 is a subset of T_1, T_2 , and T_3 .

Without loss of generality, let $S_1 = \{1\}$ be red and a subset of $T_1 = \overline{3} = \{1, 2, 4, 5\}$, $T_2 = \overline{4} = \{1, 2, 3, 5\}$, and $T_3 = \overline{5} = \{1, 2, 3, 4\}$, all of which are red. Note that no two of $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{1, 2, 5\}$ can be red without creating a red copy of Q_2 . Also, no two of $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$ can be red without creating a red copy of Q_2 .

Suppose $\{1, 2\}$ is red, which means $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$ must all be blue, and $\{1, 4, 5\}$, $\{1, 3, 5\}$, and $\{1, 3, 4\}$ must all be blue. These 6 sets, along with \emptyset and [5], form a blue copy of Q_3 , a contradiction.

Suppose exactly one of $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{1, 2, 5\}$ is red. Without loss of generality, suppose $\{1, 2, 3\}$ is red. Neither $\{1, 4\}$ nor $\{1, 5\}$ can be red without creating a red copy of Q_2 with $\{1\}$, $\{1, 2, 3\}$, and $\{1, 2, 3, 4\}$. Suppose $\{1, 3\}$ is red, which means $\{1, 2\}$, $\{1, 4\}$, and $\{1, 5\}$ must all be blue. Then $\{1, 4, 5\}$ must be red. If $\{1, 3, 4, 5\}$ is red, it forms a red copy of Q_2 with $\{1\}$, $\{1, 3\}$, and $\{1, 4, 5\}$. If $\{1, 3, 4, 5\}$ is blue, it forms a blue copy of Q_3 with \emptyset , $\{1, 2\}$, $\{1, 4\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, and [5]. Thus, Q_5 contains a red copy of Q_2 or a blue copy of Q_3 , a contradiction.

Suppose $\{1, 2, 3\}$ is red and none of $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$ are red. Then $\{1, 4, 5\}$ must be red, and $\{1, 3, 4\}$ and $\{1, 3, 5\}$ must be blue. Then $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, and $\{3, 5\}$ must be blue, and $\{2, 4, 5\}$ must be red. Then $\{4\}$, $\{5\}$, and $\{4, 5\}$ must be blue. Then $\{4\}$, $\{5\}$, $\{1, 2\}$, $\{4, 5\}$, $\{1, 2, 4\}$, and $\{1, 2, 5\}$, along with \emptyset and [5], form a blue copy of Q_3 , a contradiction.

Now suppose none of $\{1, 2, 3\}$, $\{1, 2, 4\}$, or $\{1, 2, 5\}$ are red. Again, no two of $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$ and $\{1, 5\}$ are red. Suppose one of $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$ is red. Without loss of generality, suppose $\{1, 3\}$ is red. Then $\{1, 4, 5\}$ is must be red, and $\{1, 3, 4, 5\}$ must be blue. Then $\{1, 2\}$, $\{1, 4\}$, $\{1, 5\}$, $\{1, 2, 3\}$, $\{1, 2, 5\}$, and $\{1, 3, 4, 5\}$, along with \emptyset and [5], form a blue copy of Q_3 , a contradiction.

Suppose none of $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, or $\{1,5\}$ are red. Then $\{1,4,5\}$, $\{1,3,4\}$, and $\{1,3,5\}$ must all be red, and $\{1,3,4,5\}$ must be blue. Then $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, and $\{1,3,4,5\}$, along with \emptyset and [5], form a blue copy of Q_3 , a contradiction.

In any case where at least one of S_1 , S_2 , and S_3 is a subset of T_1 , T_2 , and T_3 , Q_5 contains a red copy of Q_2 or a blue copy of Q_3 .

Subcase 3.2 None of S_1, S_2 , and S_3 is a subset of T_1, T_2 , and T_3 .

Without loss of generality, let $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$, $T_1 = \overline{1} = \{2, 3, 4, 5\}$, $T_2 = \overline{2} = \{1, 3, 4, 5\}$, and $T_3 = \overline{3} = \{1, 2, 4, 5\}$ all be red. Certainly, if every level 2 set and every level 3 set is blue, or if one or both of $\{4, 5\}$ and $\{1, 2, 3\}$ are the only red sets, then Q_5 contains a blue copy of Q_3 .

Suppose one of $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ is red. Without loss of generality, suppose $\{1, 2\}$ is red. Then $\{1, 4\}$, $\{1, 5\}$, $\{2, 4\}$, $\{2, 5\}$, $\{1, 4, 5\}$, and $\{2, 4, 5\}$ must all be blue. Suppose either $\{1, 2, 3, 4\}$ or $\{1, 2, 3, 5\}$ is red. Without loss of generality, suppose $\{1, 2, 3, 4\}$ is red. Then $\{1, 3\}$, $\{2, 3\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$ must all be blue, and $\{1, 2, 3, 5\}$ must be red. Then $\{1, 3, 5\}$ and $\{2, 3, 5\}$ must be blue. The sets $\{1, 4\}$, $\{1, 5\}$, $\{1, 3\}$, $\{1, 4, 5\}$, $\{1, 3, 4\}$, and $\{1, 3, 5\}$, along with \emptyset and [5], form a blue copy of Q_3 , a contradiction.

Now suppose $\{1,2\}$ is red and $\{1,2,3,4\}$ and $\{1,2,3,5\}$ are both blue. Then $\{1,3\}$ must be red, and $\{1,2,3\}$ must be blue. Then $\{1,4\}$, $\{1,5\}$, $\{1,2,3\}$, $\{1,4,5\}$, $\{1,2,3,4\}$, and

 $\{1, 2, 3, 5\}$, along with \emptyset and [5], form a blue copy of Q_3 , a contradiction. The argument is similar if any one of $\{1, 4, 5\}$, $\{2, 4, 5\}$, and $\{3, 4, 5\}$ is red.

Suppose any level 2 set other than $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, or $\{4, 5\}$ is red. Without loss of generality, suppose $\{1, 4\}$ is red. Then $\{1, 2\}$, $\{1, 3\}$, $\{1, 5\}$, $\{1, 2, 5\}$, and $\{1, 3, 5\}$ are all blue. Then $\{1, 2, 3\}$ must be red, and $\{1, 2, 3, 4\}$ must be blue. Then $\{1, 2\}$, $\{1, 3\}$, $\{1, 5\}$, $\{1, 2, 5\}$, and $\{1, 3, 5\}$ are all blue. $\{1, 3, 5\}$, and $\{1, 2, 3, 4\}$, along with \emptyset and [5], form a blue copy of Q_3 , a contradiction. The argument is similar if any level 3 set other than $\{1, 4, 5\}$, $\{2, 4, 5\}$, $\{3, 4, 5\}$, or $\{1, 2, 3\}$ is red.

In any case where none of S_1 , S_2 , and S_3 is a subset of T_1 , T_2 , and T_3 , Q_5 contains a red copy of Q_2 or a blue copy of Q_3 .

4 Concluding Remarks

There remains a significant gap between our upper bounds and the best known lower bounds given by Axenovich and Walzer. We believe the true values of $R(Q_m, Q_n)$ for sufficiently large m and n are significantly less than our upper bounds. Assuming, without loss of generality, that $n \ge m$, we make the following conjecture for sufficiently large m and n.

Conjecture 1. $R(Q_m, Q_n) = o(n^2).$

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