

AN INFINITE ANTICHAIN OF PLANAR TANGLEGRAMS

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ABSTRACT. Contrary to the expectation arising from the tanglegram Kuratowski theorem of É. Czabarka, L. A. Székely and S. Wagner [*SIAM J. Discrete Math.* 31(3): 1732–1750, (2017)], we construct an infinite antichain of planar tanglegrams with respect to the induced subtanglegram partial order. R.E. Tarjan, R. Laver, D.A. Spielman and M. Bóna, and possibly others, showed that the partially ordered set of finite permutations ordered by deletion of entries contains an infinite antichain, i.e. there exists an infinite collection of permutations, such that none of them contains another as a pattern. Our construction adds a twist to the construction of Spielman and Bóna [*Electr. J. Comb.* Vol. 7. N2.]

1. INTRODUCTION

Informally, a *tanglegram* is a specific kind of graph, consisting of two rooted binary trees of the same size and a perfect matching joining their leaves. Tanglegrams are drawn under specific rules, such drawings are called *tanglegram layouts*. (Formal definitions are postponed to Section 2.) The *tangle crossing number* of a tanglegram is the minimum crossing number (i.e. the minimum number of unordered crossing edge-pairs) among its layouts. The tanglegram is *planar*, if it has a layout without crossings. Tanglegrams play a major role in phylogenetics, especially in the theory of cospeciation [17]. The first binary tree is the phylogenetic tree of hosts, while the second binary tree is the phylogenetic tree of their parasites, e.g. gopher and louse [10]. The matching connects the host with its parasite. The tanglegram crossing number has been related to the number of times parasites switched hosts [10], or, working with gene trees instead of phylogenetic trees, to the number of horizontal gene transfers ([5], pp. 204–206). Tanglegrams are well-studied objects in phylogenetics and computer science (see e.g. [1, 2, 4, 6, 7, 9, 11, 13, 16, 21]).

Czabarka, Székely and Wagner [8] discovered a Kuratowski-type theorem that characterized planar tanglegrams by two excluded induced subtanglegrams. They asked

Problem 1. *Are there similar characterizations*

- (i) *for tanglegrams with tangle crossing number at most k ?*
- (ii) *for tanglegrams that have a layout without k pairwise crossing edges?*

Were the induced subtanglegram partial order a well-quasi-ordering, the answer to these questions would immediately be in the affirmative, delivering a number of algorithmic

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consequences. To be a well-quasi-ordering, there should not be an infinite antichain in the well-founded partially order.

Whether a well-founded partially ordered set has an infinite antichain has been well studied (e.g. [12, 14, 19, 20]). In particular, Kruskal’s Tree Theorem [14] would give one hope that the induced subtanglegram relation would be a well-quasi-ordering as well. However, tanglegrams, where the two trees are caterpillars, are closely related to permutations and permutation patterns (see Section 2). Laver [15], Pratt [18], Tarjan [22], and Speilman and Bóna [3] constructed infinite antichains of permutations for the partial order defined by permutation patterns.

While the antichain of permutations in [3] does not immediately yield an infinite antichain of tanglegrams (in fact, it defines a chain, as will be explained at the end of Section 3), when we turn these permutations “upside down” (i.e. in a permutation of $[n]$ we replace every entry j by $n + 1 - j$), we manage to obtain an infinite antichain of tanglegrams with respect to the induced tanglegram relation. Furthermore, the elements of the antichain are planar tanglegrams (shown in Section 4), making Problem 1 even more intriguing. An algorithmic consequence of a positive answer to Problem 1 (i) would be fixed-parameter tractability of computing the tanglegram crossing number, a result that is already known [4].

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2. DEFINITIONS AND BASIC SETUP

As customary, $[n]$ denotes the set $\{1, 2, 3, \dots, n\}$, S_n denotes the symmetric group acting on $[n]$. For $\pi \in S_n$, we use the notation $\pi = (a_1, \dots, a_n)$, if $\pi(i) = a_i$ for all $i \in [n]$.

Definition 1. *A rooted tree T is a tree with a distinguished vertex called the root. Given a vertex v in a rooted tree, and a neighbor y of v , y is the parent of v , if y is on the path from v to the root; otherwise y is a child of v . The rooted tree T is binary, if every vertex has zero or two children.*

Definition 2. *For $n \geq 2$, the rooted caterpillar C_n with n leaves is the rooted binary tree, whose $n - 2$ internal vertices form a path, and the root is an endvertex of this path.*

Note that C_n has two leaves at distance $n - 1$ from the root, and for all i ($1 \leq i \leq n - 2$) it has precisely one leaf at distance i from the root. These properties characterize C_n .

Definition 3. *Given a rooted binary tree T with root r and a non-empty subset B of its leaves, the rooted binary subtree induced by B , $T[B]$, is obtained as follows: Take the smallest subtree T' of T containing all vertices of B , and designate the vertex $\rho \in V(T')$ closest to r in T as the root of T' . This rooted tree is not necessarily binary—suppress all vertices of degree 2 (except ρ) in T' to make it binary. The resulting rooted binary tree is $T[B]$.*

Definition 4. *A tanglegram of size n is an ordered triplet (T_1, T_2, M) , where T_1 and T_2 are rooted binary trees with n leaves each, and M is a perfect matching between the two leaf sets. T_1 is called the left tree and T_2 is the right tree of the tanglegram. Two tanglegrams*

are considered the same, if there is a graph isomorphism between them, which fixes the roots of the left tree and the right tree.

Definition 5. Given a tanglegram $\mathcal{T} = (T_1, T_2, M)$ and an $\emptyset \neq M' \subseteq M$, the subtanglegram induced by M' is $\mathcal{T}[M'] = (T_1[B_1], T_2[B_2], M')$, where B_i is the set of leaves in T_i matched by M' . We say that \mathcal{T}^* is an induced subtanglegram of \mathcal{T} (in notation: $\mathcal{T}^* \preceq \mathcal{T}$), if there is an $M^* \subseteq M$ such that $\mathcal{T}^* = \mathcal{T}[M^*]$.

Note that \preceq is a partial order on the set of tanglegrams, and \preceq is well-founded, i.e. it has no infinite strictly decreasing chains.

Definition 6. Given a tanglegram $\mathcal{T} = (T_1, T_2, M)$, where the root of T_i is r_i , the multiset of distance pairs, $\mathbb{D}(\mathcal{T})$, contains exactly k copies of (d_1, d_2) if and only if there exists exactly k matching edges of the form $(x_1, x_2) \in M$ such that x_i is a leaf of T_i at distance d_i from r_i .

From now on we restrict ourselves to tanglegrams, in which both the left and right trees are rooted caterpillars. Note that in this case, if two tanglegrams have the same distance pair multiset, then they are the same.

Definition 7. For $n \geq 2$, the distance labeling of the leaves of C_n is the following: for each i , $1 \leq i \leq n-2$, the leaf labeled i is the one at distance i from the root, and the two leaves at distance $n-1$ are labeled arbitrarily by $n-1$ and n .

For $n \geq 2$ and $\pi \in S_n$, the catergram \mathcal{T}_π is the tanglegram (C_n, C_n, M_π) , where M_π is defined as follows. Use the distance labeling of the leaves of both caterpillars, match the leaf on the left tree labeled i with the leaf on the right tree labeled j if and only if $\pi(i) = j$.

Note that every tanglegram, in which both the left tree and right tree are rooted caterpillars, does arise as a catergram, but the permutation that defines it is not unique.

Definition 8. Assume $n \geq 2$. Given a $\pi = (a_1, \dots, a_n) \in S_n$, we define the (not necessarily different) permutations $\widehat{\pi}$, $\widetilde{\pi}$ as

$$\widehat{\pi}(i) = \begin{cases} a_i, & \text{if } i \leq n-2 \\ a_n, & \text{if } i = n-1 \\ a_{n-1}, & \text{if } i = n \end{cases} \quad \text{and} \quad \widetilde{\pi}(i) = \begin{cases} a_i, & \text{if } a_i \notin \{n-1, n\} \\ n-1, & \text{if } a_i = n \\ n, & \text{if } a_i = n-1; \end{cases}$$

and finally let $\pi^* = (\widehat{\widetilde{\pi}})$. We define the set $\overline{\pi} = \{\pi, \widehat{\pi}, \widetilde{\pi}, \pi^*\}$.

Proposition 9. The following facts are obvious for any $\pi = (a_1, \dots, a_n)$:

(a) We have $\mathbb{D}(\mathcal{T}_\pi) = \{(1, a_1^*), (2, a_2^*), \dots, (n-1, a_{n-1}^*), (n-1, a_n^*)\}$, where

$$a_i^* = \begin{cases} a_i, & a_i < n \\ n-1, & a_i = n. \end{cases}$$

(b) $(\widehat{\widetilde{\pi}}) = (\widehat{\widehat{\pi}})$, $\pi = (\widehat{\widehat{\pi}}) = (\widetilde{\widetilde{\pi}})$, and $\pi \notin \{\widehat{\pi}, \widetilde{\pi}\}$.

(c) $\rho \in \overline{\pi}$ iff $\overline{\rho} = \overline{\pi}$.

(d) $\widehat{\pi} = \widetilde{\pi}$ iff $\{a_{n-1}, a_n\} = \{n-1, n\}$ iff $\pi = \pi^*$; consequently $|\overline{\pi}| \in \{2, 4\}$.

(e) $\mathcal{T}_\rho = \mathcal{T}_\pi$ iff $\mathbb{D}(\mathcal{T}_\rho) = \mathbb{D}(\mathcal{T}_\pi)$ iff $\rho \in \bar{\pi}$.

Definition 10. We say that two sequences of n numbers, $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathbb{R}^n$, are order isomorphic, if for all $i, j \in [n]$, we have $a_i < a_j$ iff $b_i < b_j$. Given a $\pi \in S_n$ and a non-empty $A \subseteq [n]$, where a_1, \dots, a_k lists the elements of A in increasing order, we denote by $\pi[A]$ the permutation in $S_{|A|}$ that is order isomorphic to $(\pi(a_1), \pi(a_2), \dots, \pi(a_k))$. If $\rho \in S_m$ and $\pi \in S_n$, then we say that ρ is a pattern in π (in notation $\rho \leq \pi$), if $\pi[A] = \rho$ for some $A \subseteq [n]$.

Definition 11. Assume $\pi \in S_n$ and $\emptyset \neq A \subseteq [n]$. Then (with a slight abuse of notation) we denote by $\mathcal{T}_\pi[A]$ the induced subtanglegram $\mathcal{T}_\pi[M^*]$, where M^* is the matching containing edges of M incident upon leaves of the left tree that are labeled with elements of A .

Proposition 12. The following statements are true:

- (a) Let v be a leaf of C_n at distance i from the root r of C_n , and $y \neq v$ be another leaf that is at distance j from r . Let T be the binary tree induced by all leaves except v (so $T = C_{n-1}$) with root r^* . Then y is a leaf in T , and the distance of y from r^* is j if $j < i$, and $j - 1$ otherwise.
- (b) For any $\pi \in S_n$ and non-empty $A \subseteq [n]$, we have $\mathcal{T}_\pi[A] = \mathcal{T}_{\pi[A]}$. (This follows from (a)).
- (c) For $\rho \in S_m$ and $\pi \in S_n$, we have $\mathcal{T}_\rho \leq \mathcal{T}_\pi$ iff $\mathcal{T}_\rho = \mathcal{T}_{\pi[A]}$ for some $A \subseteq [n]$ iff $\sigma \leq \pi$ for some $\sigma \in \bar{\rho}$. (This follows from (b) and Proposition 9 (e)).

3. CONSTRUCTING THE ANTICHAIN OF TANGLEGRAMS

Definition 13. For $i \in \mathbb{Z}^+$, we set $\rho_i \in S_{[12+2i]}$ as $(\rho_i(1), \rho_i(2), \rho_i(3), \rho_i(4)) = (2, 3, 5, 1)$, $(\rho_i(9 + 2i), \rho_i(10 + 2i), \rho_i(11 + 2i), \rho_i(12 + 2i)) = (10 + 2i, 11 + 2i, 12 + 2i, 8 + 2i)$ and for $j : 5 \leq j \leq 8 + 2i$

$$\rho_i(j) = \begin{cases} j + 2, & \text{if } j \text{ is odd} \\ j - 2, & \text{if } j \text{ is even.} \end{cases}$$

So for example, the first two permutations in our sequence will be

$$\begin{aligned} \rho_1 &= (2, 3, 5, 1, 7, 4, 9, 6, 11, 8, 12, 13, 14, 10) \\ \rho_2 &= (2, 3, 5, 1, 7, 4, 9, 6, 11, 8, 13, 10, 14, 15, 16, 12). \end{aligned}$$

Spielman and Bóna [3] showed that if π_i is ρ_i turned “upside down”, then $\{\pi_i : i \in \mathbb{Z}^+\}$ is an antichain for the pattern partial order of permutations. We are now ready to show our result:

Theorem 14. $\{\mathcal{T}_{\rho_i} : i \in \mathbb{Z}^+\}$ is an antichain with respect to the relation \preceq .

Proof. In the proof we will use the fact that for any k and any $\gamma \in \bar{\rho}_k$, the permutation γ has exactly two entries that are preceded by at least 3 larger elements: the entry 1 and the entry $8 + 2i$; moreover, if $\gamma \in \{\rho_k, \widehat{\rho}_k\}$ then $8 + 2k$ is preceded by exactly 4 larger elements, but these 4 elements are not order isomorphic in ρ_k and $\widehat{\rho}_k$.

By Proposition 12 (c), it is sufficient to show that for any $i < j$ and for any $\sigma \in \overline{\rho_i}$, $\sigma \not\leq \rho_j$. By our starting remark, if $\sigma < \rho_j$, then the entries 1 and $8 + 2i$ in σ should map to the entries 1 and $8 + 2j$ in ρ_j , and the preceding larger elements must map to preceding larger entries; consequently $\widetilde{\rho_i} \not\leq \rho_j$. As $8 + 2j$ is the last entry of ρ_j , but not of $\widehat{\rho_i}$ or ρ_i^* (unless $\rho_i^* = \rho_i$), we get that $\widehat{\rho_i} \not\leq \rho_j$ and $\rho_i^* \not\leq \rho_j$. So what remains to be shown is $\rho_i \not\leq \rho_j$, which was essentially stated and proved in [3], but for completeness, we include a (somewhat different) proof here.

Suppose for contrary that $\rho_i < \rho_j$, i.e. entries of ρ_i map to entries of ρ_j in an order preserving fashion. By our earlier remarks, the first 4 elements of ρ_i must map to the first 4 elements of ρ_j and the last 6 elements of ρ_i must map to the last 6 elements of ρ_j , so we must map the sequence $(7, 4, 9, 6, \dots, 7 + 2i, 4 + 2i)$ to $(7, 4, 9, 6, \dots, 7 + 2j, 4 + 2j)$ by leaving out $2(j - i) \geq 2$ elements.

Let x be an entry of the contiguous subsequence $(7, 4, 9, 6, \dots, 7 + 2k, 4 + 2k)$ of ρ_k . If x is even, then there are no entries that appear after x in ρ_k that are smaller than x , and x is preceded by the entry $x + 1$. If x is odd, then there are exactly two entries in ρ_k that follow x and are smaller than x , and they are both even.

Let x now be the first entry that is erased from ρ_j . The entries before x in ρ_i are mapped to the same entries, respectively, in ρ_j , and the entry x in ρ_i is mapped to a different entry that appears after x in ρ_j .

If x is even, then, as the entry $x + 1$ is before x in ρ_i , x must map to an entry smaller than $x + 1$ but is after x in ρ_j . As such an entry does not exist, x must be odd.

As x is odd, it is immediately followed by the even entry $x - 3$ in both ρ_i and ρ_j , and preceded by the entry $x - 2$, which was not erased from ρ_j . As entry $x - 2$ in ρ_i maps to entry $x - 2$ in ρ_j , and entry x in ρ_i maps to an entry after x in ρ_j , it follows that entry $x - 3$ in ρ_i must map to an entry that is after $x - 3$ in ρ_j and is smaller than $x - 3$. Since such an entry does not exist, $\rho_i \not\leq \rho_j$. \square \square

We remark here that in the infinite antichain of permutations $\{\pi_i : i \in \mathbb{Z}^+\}$ of [3], π_i is our ρ_i is turned “upside down”. For example,

$$\begin{aligned}\pi_1 &= (13, 12, 10, 14, 8, 11, 6, 9, 4, 7, 3, 2, 1, 5) \\ \pi_2 &= (15, 14, 12, 16, 10, 13, 8, 11, 6, 9, 4, 7, 3, 2, 1, 5).\end{aligned}$$

One can easily check that for $A = [16] \setminus \{2, 4\}$ we get $\widetilde{\pi}_1 = \pi_2[A]$, showing that $\mathcal{T}_{\pi_1} \leq \mathcal{T}_{\pi_2}$. Moreover, for every $i \in \mathbb{Z}^+$, setting $A_i = [14 + 2i] \setminus \{2, 4\}$, we observe that $\widetilde{\pi}_i = \pi_{i+1}[A_i]$, showing that

Proposition 15. $\{\mathcal{T}_{\pi_i} : i \in \mathbb{Z}^+\}$ is an infinite chain in the induced subtanglegram partial order.

This is why we had to put a twist on the construction of [3].

4. PLANARITY OF THE TANGLEGRAMS IN THE ANTICHAIN

Lastly, we show that the tanglegrams \mathcal{T}_{ρ_i} are planar. For this we need to define layouts first.

Definition 16. A plane binary tree is a rooted binary tree, in which the children of internal vertices are specified as left and right children. A plane binary tree is easy to draw on one side of a line, without edge crossings, such that only the leaves of the tree are on the line. We will say that the plane binary tree P is a plane tree of the rooted binary tree T , if P is isomorphic to T as a graph.

Note that if we label all vertices of a rooted binary tree with n leaves, then there are 2^{n-1} labeled plane trees whose underlying labeled graph is this labeled rooted binary tree.

Definition 17. A layout (L, R, M) of the tanglegram $\mathcal{T} = (T_1, T_2, M)$ is given by a left plane binary tree L isomorphic to T_1 , drawn in the halfplane $x \leq 0$, having its leaves on the line $x = 0$, a right plane binary tree R isomorphic to T_2 drawn in the halfplane $x \geq 1$, having its leaves on the line $x = 1$, and the perfect matching M between their leaves drawn in straight line segments. (See Figure 1.)

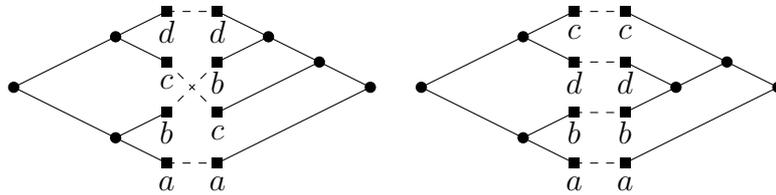


FIGURE 1. Two layouts of the same tanglegram. The leaf labels help showing that the two tanglegrams are identical.

Definition 18. A tanglegram is planar if it has a layout without crossing edges.

Theorem 19 (Czabarka, Székely, Wagner [8]). Every non-planar tanglegram contains one of the two tanglegrams in Figure 2 as an induced subtanglegram.

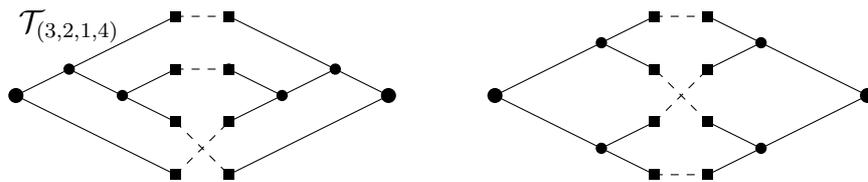


FIGURE 2. The two tanglegrams excluded from planar tanglegrams. The tanglegram on the left is the catergram $\mathcal{T}_{(3,2,1,4)}$, but the tanglegram on the right is not a catergram, as the trees are not caterpillars.

Now we are ready to show:

Proposition 20. For every $i \in \mathbb{Z}^+$ the catergram \mathcal{T}_{ρ_i} is planar.

Proof. Since any leaf-induced subtree of a rooted caterpillar is another caterpillar, Theorem 19 yields that \mathcal{T}_{ρ_i} is not planar iff it contains an induced $\mathcal{T}_{(3,2,1,4)}$. By Proposition 12 (c) this happens precisely when one of $(3, 2, 1, 4)$, $(4, 2, 1, 3)$, $(3, 2, 4, 1)$, $(4, 2, 3, 1)$ is a pattern of ρ_i . As ρ_i does not contain a decreasing subsequence of length 3, $(3, 2, 1, 4)$ and $(4, 2, 1, 3)$ are not among its patterns. The last entry of the remaining $(3, 2, 4, 1)$ and $(4, 2, 3, 1)$ has three larger elements preceding it, and the first two elements are in decreasing order. If they are patterns of ρ_i , then 1 must map to either 1 or $8 + 2i$. If 1 maps to 1, then the other three elements must map to the sequence $(2, 3, 5)$, and if 1 maps to $8 + 2i$, then the remaining three elements must map to a subsequence of $(9 + 2i, 10 + 2i, 11 + 2i, 12 + 2i)$. As both of these are increasing, $(3, 2, 4, 1)$ and $(4, 2, 3, 1)$ are not patterns of ρ_i . \square \square

Just having a proof that \mathcal{T}_{ρ_i} is planar is somewhat unsatisfactory; one naturally wants to see a planar layout of of this catergram.

First note that given a plane tree P of any rooted binary tree T with n unique labeled leaves, the drawing of P gives an ordering (ℓ_1, \dots, ℓ_n) of the labels by the order they appear on their line in the drawing. Moreover, if v is an internal vertex of T , then the set of leaves that are descendants of v , i.e. the leaves separated by v from the root, must appear in a contiguous block of (ℓ_1, \dots, ℓ_n) . It is easy to see that if (ℓ_1, \dots, ℓ_n) is an ordering of the leaf labels such that for every internal vertex v of T the leaves that are descendants of v appear in a continuous block of (ℓ_1, \dots, ℓ_n) , then there is precisely one plane tree P of T that puts the leaves in the order (ℓ_1, \dots, ℓ_n) on its line of leaves.

If v is an internal vertex of the caterpillar C_n whose leaves are labeled according to our distance convention, then there is an $i \in [n]$ such that the set of leaves that are descendants of v are exactly the leaves labeled with entries that are at least i . Therefore a permutation $(\ell_1, \dots, \ell_n) \in S_n$ arises from a plane tree of C_n precisely when for every $i \in [n]$, the entries bigger than i appear only one side (left or right) of i in (ℓ_1, \dots, ℓ_n) .

Definition 21. *Given a rooted binary tree T on n leaves, which are labeled by the elements of $[n]$, we call a permutation $(\ell_1, \dots, \ell_n) \in S_n$ consistent with T , if for every internal vertex v of T , then the set of leaves that are descendants of v appear in a contiguous segment of (ℓ_1, \dots, ℓ_n) . A permutation (ℓ_1, \dots, ℓ_n) is cater-good, if it is consistent with the distance labeled caterpillar C_n (see Definition 7), i.e. for every $i \in [n]$, the entries bigger than i appear only one side (left or right) of i in (ℓ_1, \dots, ℓ_n) .*

Proposition 22. *The following facts are obvious:*

- (a) *The tanglegram (T_1, T_2, M) , where the leaves of T_1 and T_2 are labeled, is planar iff there are permutations $\pi_1 = (a_1, \dots, a_n)$ and $\pi_2 = (b_1, \dots, b_n)$ of the leaf labels of T_i , such that π_i is consistent with T_i for $i = 1, 2$, and $M = \{a_i b_i : i \in [n]\}$.*
- (b) *The catergram T_σ is planar iff there is cater-good a permutation (a_1, \dots, a_n) such that $(\sigma(a_1), \dots, \sigma(a_n))$ is also cater-good. A planar layout is obtained by these permutation, putting leaves in their order on the lines $x = 0$ and $x = 1$.*
- (c) *If a permutation (c_1, \dots, c_n) of $[n]$ is unimodal, then it is cater-good.*

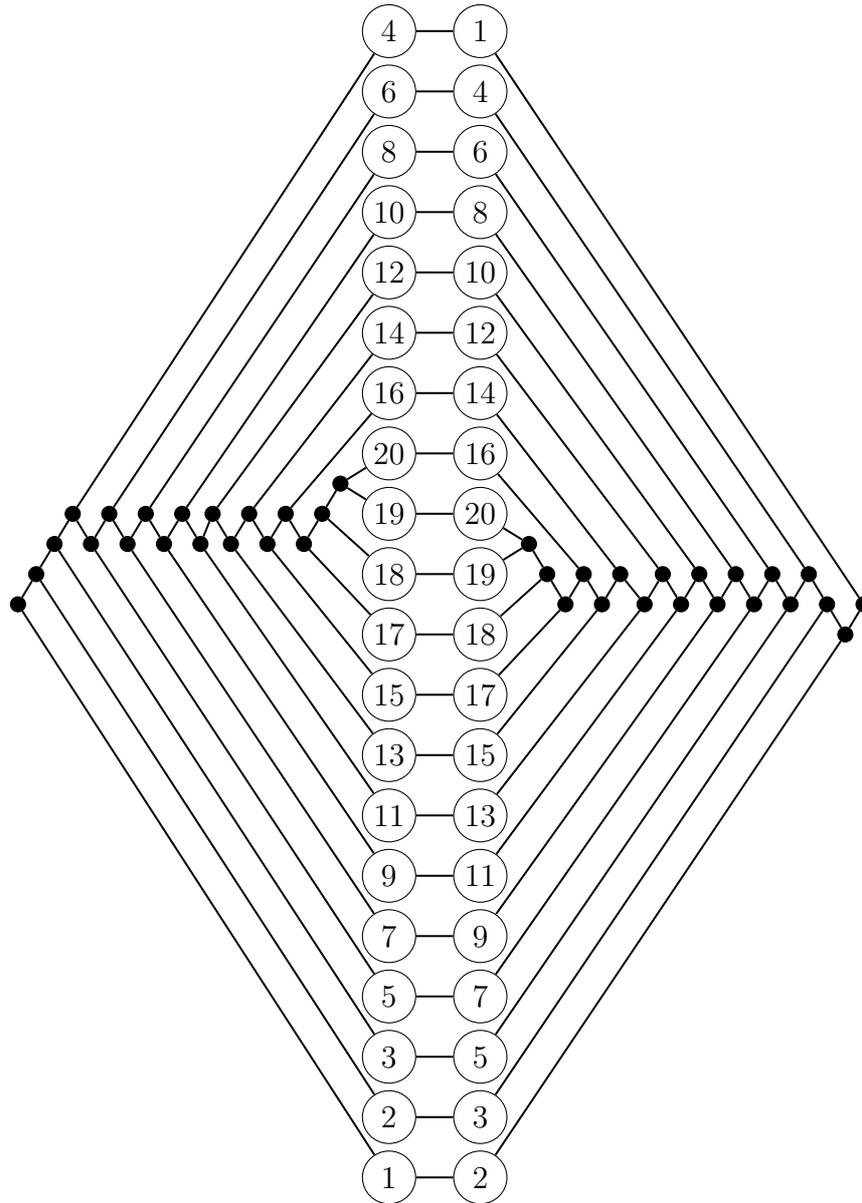


FIGURE 3. A planar drawing of \mathcal{T}_{ρ_4} as described in Proposition 22 (d).

- (d) For every $i \in \mathbb{Z}^+$, a planar drawing of \mathcal{T}_{ρ_i} is given by the permutation (a_1, \dots, a_{12+2i}) where $a_1 = 4$, $(a_1, a_2, a_3) = (1, 2, 3)$, $(a_{8+i}, a_{9+i}, a_{10+i}, a_{11+i}) = (9 + 2i, 10 + 2i, 11 + 2i, 12 + 2i)$, and for $j \in [4 + i]$, $a_{3+j} = 3 + 2j$ and $a_{12+2i-j} = 4 + 2j$.

Note that the permutation $(a_1 = 1, \dots, a_{12+2i})$ in (d) is unimodal, and consequently so is $(\rho_i(a_1), \dots, \rho_i(a_{12+2i})) = (a_2, a_3, \dots, a_{12+2i}, 1)$. Figure 3 gives the planar drawing of \mathcal{T}_{ρ_4} determined by the permutation given in this Proposition.

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