# The $n$-tangle of odd $n$ qubits ${ }^{1}$ 

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#### Abstract

Coffman, Kundu and Wootters presented the 3-tangle of three qubits in [Phys. Rev. A 61, 052306 (2000)]. Wong and Christensen extended the 3 -tangle to even number of qubits, known as $n$-tangle [Phys. Rev. A 63, 044301 (2001)]. In this paper, we propose a generalization of the 3 -tangle to any odd $n$-qubit pure states and call it the $n$-tangle of odd $n$ qubits. We show that the $n$-tangle of odd $n$ qubits is invariant under permutations of the qubits, and is an entanglement monotone. The $n$-tangle of odd $n$ qubits can be considered as a natural entanglement measure of any odd $n$-qubit pure states.

Keywords: 3 -tangle, $n$-tangle of odd $n$ qubits, concurrence, residual entanglement PACS numbers: 03.67.Mn, 03.65.Ud


## 1 Introduction

Quantum entanglement is a key quantum mechanical resource in quantum computation and information, such as quantum cryptography, quantum dense coding and quantum teleportation 1. Entanglement measure, which characterizes the degree of entanglement contained in a quantum state, has been a subject under intensive research.

The entanglement of bipartite systems is well understood. The concurrence [2] is a good entanglement measure for two-qubit states and is an entanglement monotone, i.e., it is non-increasing under local quantum operations and classical communication (LOCC). Generalizations of the concurrence to higher dimensions can be found, for example, in [3, 4]. The residual entanglement, or the 3 -tangle has been constructed in terms of the concurrences as a widely accepted entanglement measure to quantify the entanglement in three-qubit pure states 5. The 3-tangle is permutationally invariant, is an entanglement monotone, and is a SLOCC (stochastic local operations and classical communication) polynomial of degree 4. Furthermore, the 3 -tangle is bounded between 0 and 1 , and it assumes value 1 for the GHZ state and vanishes for the W state [5, 6]. Several other measures have been constructed specifically for the entanglement of the three-qubit pure states [7, 8, [9]. The partial tangle, reported in [7, represents the residual two-qubit entanglement of a three-qubit pure state and reduces to the two-qubit concurrence for the W state. The $\sigma$-measure [8] and $\pi$-tangle [9] have been introduced as entanglement monotones for genuine three-qubit entanglement. Whereas the 3 -tangle vanishes for the W state, both $\sigma$-measure and $\pi$ tangle take non-zero values for the W state as well as the GHZ state. Many other entanglement measures for quantifying the entanglement of multipartite pure states have been proposed [10, 11, 12, 13, 14] (see also the review [1 and references therein). Hyperdeterminant, as a generalization of the concurrence and the 3 -tangle, has been shown to be an entanglement monotone and describes the genuine multipartite entanglement [10. The $n$-tangle is a straightforward extension of 3 -tangle to even number of qubits [11. As has been previously noted, the $n$-tangle is the square of generalization of the concurrence, is invariant under permutations, and is an entanglement monotone. Like the 3 -tangle, the $n$-tangle is equal to 1 for the GHZ state and vanishes for the W state 11. However the $n$-tangle is not residual entanglement for four or more qubits [15. It has been found that the 4 -tangle for four-qubit states can be interpreted as a type of residual entanglement similar to the interpretation of 3 -tangle for three-qubit states as the residual tangle [16]. An alternative 4 -tangle has recently been obtained by using negativity fonts and the 4 -tangle is a genuine entanglement measure of four-qubit pure states 12 . In 13], the residual entanglement of odd $n$ qubits has been proposed as an entanglement measure for odd $n$-qubit pure states and shown to be an entanglement monotone [14]. The odd $n$-tangle (although called odd $n$-tangle, it is not defined in the same way as has been done for the $n$-tangle by directly extending the definition of 3 -tangle to even $n$ qubits) has been defined by taking the average of the residual entanglement with respect to qubit $i$, which is obtained from the residual entanglement of odd $n$ qubits under transposition

[^0]on qubits 1 and $i$ [14]. It has been shown that the odd $n$-tangle is permutationally invariant, $S L$-invariant and $L U$-invariant, and is an entanglement monotone [14].

In this paper, we give an alternative formulation of the 3 -tangle. We extend the formulation in a straightforward way to any odd $n$-qubit pure states and define the $n$-tangle with respect to qubit $i$. By taking the average of the $n$-tangle with respect to qubit $i$, we define the $n$-tangle of odd $n$ qubits, which is invariant under permutations of the qubits. The extended formulation is then reduced by using simple mathematics. It turns out that the $n$-tangle with respect to qubit $i$ and the $n$-tangle of odd $n$ qubits are equal to the residual entanglement with respect to qubit $i$ and the odd $n$-tangle respectively, and consequently the former inherit the properties of the latter, like the monotonicity, invariance under $S L$ and $L U$ operations as well as the property of satisfying SLOCC equation. Moreover, the $n$-tangle with respect to qubit $i$ is a SLOCC polynomial of degree 4 . Like the 3 -tangle, the $n$-tangle of odd $n$ qubits takes value 1 for the GHZ state and vanishes for the W state. Finally we extend the $n$-tangle of odd $n$ qubits to mixed states via the convex roof construction. This work will extend our understanding of multipartite entanglement.

The rest of the paper is organized as follows. In Section 2 we briefly review the definitions and the formulations of the concurrence, the 3 -tangle and the $n$-tangle. We then give an alternative formulation of the 3 -tangle and extend it to odd $n$ qubits. We also introduce the definitions of the $n$-tangle with respect to qubit $i$ and the $n$-tangle of odd $n$ qubits. In Section 3, we study the $n$-tangle with respect to qubit $i$ and the $n$-tangle of odd $n$ qubits in more detail and we discuss their properties. Finally, we draw our conclusion in Section 4.

## 2 The $n$-tangle of odd $n$ qubits

### 2.1 Preliminaries

The concurrence for two-qubit pure states is defined as $C(\psi)=|\langle\psi \mid \tilde{\psi}\rangle|^{2}$ 2, where $|\tilde{\psi}\rangle$ denotes the resulting state after applying the operator $\sigma_{y} \otimes \sigma_{y}$ to the complex conjugate of $|\psi\rangle$ [2], i.e. $|\tilde{\psi}\rangle=$ $\sigma_{y} \otimes \sigma_{y}\left|\psi^{*}\right\rangle$. Here the asterisk indicates complex conjugatation in the standard basis. For three-qubit pure states, the 3 -tangle $\tau_{A B C}$ (or $\tau_{123}$ ) can be calculated by means of concurrences and is given by $\tau_{A B C}=C_{A(B C)}^{2}-C_{A B}^{2}-C_{A C}^{2}$ [5], where $C_{A B}$ and $C_{A C}$ are the concurrences of the corresponding two-qubit subsytems $\rho_{A B}$ and $\rho_{A C}$, respectively, and $C_{A(B C)}^{2}=4 \operatorname{det} \rho_{A}$. Here $\rho_{A B}, \rho_{A C}$ and $\rho_{A}$ are the reduced density matrices. Let $|\psi\rangle=\sum_{i=0}^{7} a_{i}|i\rangle$, where $\sum_{i=0}^{7}\left|a_{i}\right|=1$. An expression of the 3 -tangle in terms of the coefficients for the state $|\psi\rangle$ is given by [5]

$$
\begin{equation*}
\tau_{123}=4\left|d_{1}-2 d_{2}+4 d_{3}\right| \tag{2.1}
\end{equation*}
$$

where

$$
\begin{align*}
& d_{1}=a_{0}^{2} a_{7}^{2}+a_{1}^{2} a_{6}^{2}+a_{2}^{2} a_{5}^{2}+a_{3}^{2} a_{4}^{2},  \tag{2.2}\\
& d_{2}=a_{0} a_{7} a_{3} a_{4}+a_{0} a_{7} a_{2} a_{5}+a_{0} a_{7} a_{1} a_{6}+a_{3} a_{4} a_{2} a_{5}+a_{3} a_{4} a_{1} a_{6}+a_{2} a_{5} a_{1} a_{6},  \tag{2.3}\\
& d_{3}=a_{0} a_{6} a_{5} a_{3}+a_{7} a_{1} a_{2} a_{4} . \tag{2.4}
\end{align*}
$$

A more standard form of the 3 -tangle is given as follows [5:

$$
\begin{equation*}
\tau_{123}=2\left|\sum a_{\alpha_{1} \alpha_{2} \alpha_{3}} a_{\beta_{1} \beta_{2} \beta_{3}} a_{\gamma_{1} \gamma_{2} \gamma_{3}} a_{\delta_{1} \delta_{2} \delta_{3}} \times \epsilon_{\alpha_{1} \beta_{1}} \epsilon_{\alpha_{2} \beta_{2}} \epsilon_{\gamma_{1} \delta_{1}} \epsilon_{\gamma_{2} \delta_{2}} \epsilon_{\alpha_{3} \gamma_{3}} \epsilon_{\beta_{3} \delta_{3}}\right| \tag{2.5}
\end{equation*}
$$

where the sum is over all the indices, $\alpha_{l}, \beta_{l}, \gamma_{l}$, and $\delta_{l} \in\{0,1\}, \epsilon_{00}=\epsilon_{11}=0$, and $\epsilon_{01}=-\epsilon_{10}=1$. The above formulation of the 3 -tangle is invariant under permutations of the qubits.

Let $|\psi\rangle$ be any state of $n$ qubits and $|\psi\rangle=\sum_{i=0}^{2^{n}-1} a_{i}|i\rangle$, where $\sum_{i=0}^{2^{n}-1}\left|a_{i}\right|=1$. The $n$-tangle is defined for the state $|\psi\rangle$ as follows [11]:

$$
\begin{align*}
\tau_{12 \cdots n}= & 2 \mid \sum a_{\alpha_{1} \cdots \alpha_{n}} a_{\beta_{1} \cdots \beta_{n}} a_{\gamma_{1} \cdots \gamma_{n}} a_{\delta_{1} \cdots \delta_{n}} \\
& \times \epsilon_{\alpha_{1} \beta_{1}} \epsilon_{\alpha_{2} \beta_{2}} \cdots \epsilon_{\alpha_{n-1} \beta_{n-1}} \epsilon_{\gamma_{1} \delta_{1}} \epsilon_{\gamma_{2} \delta_{2}} \cdots \epsilon_{\gamma_{n-1} \delta_{n-1}} \epsilon_{\alpha_{n} \gamma_{n}} \epsilon_{\beta_{n} \delta_{n}} \mid \tag{2.6}
\end{align*}
$$

for all even $n$ and $n=3$. However the above formula is not invariant under permutations of qubits for odd $n>3$, and therefore, the $n$-tangle remains undefined for odd $n>3$ [11.

### 2.2 Alternative formulation of the 3-tangle

Here we let

$$
\begin{align*}
& \tau_{123}^{(1)}=2\left|\sum a_{\alpha_{1} \alpha_{2} \alpha_{3}} a_{\beta_{1} \beta_{2} \beta_{3}} a_{\gamma_{1} \gamma_{2} \gamma_{3}} a_{\delta_{1} \delta_{2} \delta_{3}} \times \epsilon_{\alpha_{2} \beta_{2}} \epsilon_{\alpha_{3} \beta_{3}} \epsilon_{\gamma_{2} \delta_{2}} \epsilon_{\gamma_{3} \delta_{3}} \epsilon_{\alpha_{1} \gamma_{1}} \epsilon_{\beta_{1} \delta_{1}}\right|  \tag{2.7}\\
& \tau_{123}^{(2)}=2\left|\sum a_{\alpha_{1} \alpha_{2} \alpha_{3}} a_{\beta_{1} \beta_{2} \beta_{3}} a_{\gamma_{1} \gamma_{2} \gamma_{3}} a_{\delta_{1} \delta_{2} \delta_{3}} \times \epsilon_{\alpha_{1} \beta_{1}} \epsilon_{\alpha_{3} \beta_{3}} \epsilon_{\gamma_{1} \delta_{1}} \epsilon_{\gamma_{3} \delta_{3}} \epsilon_{\alpha_{2} \gamma_{2}} \epsilon_{\beta_{2} \delta_{2}}\right|  \tag{2.8}\\
& \tau_{123}^{(3)}=2\left|\sum a_{\alpha_{1} \alpha_{2} \alpha_{3}} a_{\beta_{1} \beta_{2} \beta_{3}} a_{\gamma_{1} \gamma_{2} \gamma_{3}} a_{\delta_{1} \delta_{2} \delta_{3}} \times \epsilon_{\alpha_{1} \beta_{1}} \epsilon_{\alpha_{2} \beta_{2}} \epsilon_{\gamma_{1} \delta_{1}} \epsilon_{\gamma_{2} \delta_{2}} \epsilon_{\alpha_{3} \gamma_{3}} \epsilon_{\beta_{3} \delta_{3}}\right| \tag{2.9}
\end{align*}
$$

Inspection of Eqs. (2.5) and (2.9) reveals that $\tau_{123}=\tau_{123}^{(3)}$. Indeed, a direct calculation gives $\tau_{123}^{(1)}=$ $\tau_{123}^{(2)}=\tau_{123}^{(3)}$. Now, let us look at the formulas from a different perspective. We note that $\tau_{123}^{(2)}$ can be obtained from $\tau_{123}^{(1)}$ by taking the transposition $(1,2)$ on qubits 1 and 2. Analogously, $\tau_{123}^{(3)}$ can be obtained from $\tau_{123}^{(1)}$ by taking the transposition $(1,3)$ on qubits 1 and 3 . It turns out that we can also obtain $\tau_{123}^{(1)}=\tau_{123}^{(2)}=\tau_{123}^{(3)}$ by using the fact that the 3-tangle $\tau_{123}$ is invariant under permutations of the three qubits [5]. We may thus rewrite the 3 -tangle as follows:

$$
\begin{equation*}
\tau_{123}=\left(\tau_{123}^{(1)}+\tau_{123}^{(2)}+\tau_{123}^{(3)}\right) / 3 \tag{2.10}
\end{equation*}
$$

### 2.3 The $n$-tangle with respect to qubit $i$ and the $n$-tangle of odd $n$ qubits

We extend Eqs. (2.7)-(2.9) to any odd $n$ qubits. Let

$$
\begin{align*}
\tau_{12 \cdots n}^{(i)}= & 2\left|W_{12 \cdots n}^{(i)}\right|,  \tag{2.11}\\
W_{12 \cdots n}^{(i)}= & \sum a_{\alpha_{1} \cdots \alpha_{n}} a_{\beta_{1} \cdots \beta_{n}} a_{\gamma_{1} \cdots \gamma_{n}} a_{\delta_{1} \cdots \delta_{n}} \times \epsilon_{\alpha_{i} \gamma_{i}} \epsilon_{\beta_{i} \delta_{i}} \\
& \times \epsilon_{\alpha_{1} \beta_{1}} \cdots \epsilon_{\alpha_{i-1} \beta_{i-1}} \epsilon_{\alpha_{i+1} \beta_{i+1}} \cdots \epsilon_{\alpha_{n} \beta_{n}} \\
& \times \epsilon_{\gamma_{1} \delta_{1} \cdots \epsilon_{\gamma_{i-1} \delta_{i-1}} \epsilon_{\gamma_{i+1} \delta_{i+1}} \cdots \epsilon_{\gamma_{n} \delta_{n}},}, \tag{2.12}
\end{align*}
$$

where the sum is over all the indices and $i=1, \cdots, n$. One can verify that $\tau_{12 \cdots n}^{(i)}$ with $n \geq 5$ is invariant under any permutation of all but qubit $i$. So, we call $\tau_{12 \ldots n}^{(i)}$ the $n$-tangle with respect to qubit $i$. One can show that $\tau_{12 \cdots n}^{(1)}$ turns into $\tau_{12 \cdots n}^{(i)}$ under the transposition $(1, i)$ on qubits 1 and $i, i=2,3, \cdots, n$.

In analogy to Eq. (2.10), we define the $n$-tangle of odd $n$ qubits as follows:

$$
\begin{equation*}
\tau_{12 \cdots n}=\frac{1}{n} \sum_{i=1}^{n} \tau_{12 \cdots n}^{(i)} . \tag{2.13}
\end{equation*}
$$

It is not hard to see that $\tau_{12 \cdots n}$ is invariant under all the permutations of the qubits, and the values of $\tau_{12 \cdots n}^{(i)}$ and $\tau_{12 \cdots n}$ are bounded between 0 and 1 . Note also that when $n=3, \tau_{12 \cdots n}^{(i)}$ and $\tau_{12 \cdots n}$ become $\tau_{123}$.

### 2.4 Reduction of the formulation

We observe that it takes $3 \cdot 2^{4 n}$ multiplications to compute $\tau_{12 \cdots n}^{(i)}$ by Eqs. (2.11) and (2.12). Next we reduce the formulation of $\tau_{12 \cdots n}^{(1)}$. From Eq. (2.12), we have

$$
\begin{align*}
W_{12 \cdots n}^{(1)}= & \sum a_{\alpha_{1} \cdots \alpha_{n}} a_{\beta_{1} \cdots \beta_{n}} a_{\gamma_{1} \cdots \gamma_{n}} a_{\delta_{1} \cdots \delta_{n}} \times \epsilon_{\alpha_{1} \gamma_{1}} \epsilon_{\beta_{1} \delta_{1}} \\
& \times \epsilon_{\alpha_{2} \beta_{2}} \cdots \epsilon_{\alpha_{n} \beta_{n}} \epsilon_{\gamma_{2} \delta_{2}} \cdots \epsilon_{\gamma_{n} \delta_{n}} . \tag{2.14}
\end{align*}
$$

After some calculations, we obtain (we refer the reader to Appendix A for details)

$$
\begin{align*}
W_{12 \cdots n}^{(1)} & =2\left(P Q-T^{2}\right),  \tag{2.15}\\
\tau_{12 \cdots n}^{(1)} & =4\left|T^{2}-P Q\right|, \tag{2.16}
\end{align*}
$$

where

$$
\begin{align*}
& T=\sum_{i=0}^{2^{n-1}-1}(-1)^{N(i)} a_{i} a_{2^{n}-i-1},  \tag{2.17}\\
& P=2 \sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)} a_{2 i} a_{2^{n-1}-2 i-1},  \tag{2.18}\\
& Q=2 \sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)} a_{2^{n-1}+2 i} a_{2^{n}-2 i-1} . \tag{2.19}
\end{align*}
$$

Here $N(l)$ is the number of 1 s in the $n$-bit binary representation $l_{n-1} \ldots l_{1} l_{0}$ of $l$. We further note that it takes $\left(2^{n}+3\right)$ multiplications to compute $\tau_{12 \ldots n}^{(1)}$ using Eqs. (2.16)-(2.19). A plain calculation yields that $\tau_{12 \cdots n}^{(1)}=1$ for the $n$-qubit state $G H Z$ and $\tau_{12 \cdots n}^{(1)}=0$ for the $n$-qubit state $W$.

## 3 The $n$-tangle of odd $n$ qubits is an entanglement monotone

Let $\left|\psi^{\prime}\right\rangle$ be also any state of $n$ qubits and $\left|\psi^{\prime}\right\rangle=\sum_{i=0}^{2^{n}-1} b_{i}|i\rangle$, where $\sum_{i=0}^{2^{n}-1}\left|b_{i}\right|^{2}=1$. Two states $|\psi\rangle$ and $\left|\psi^{\prime}\right\rangle$ are SLOCC entanglement equivalent if and only if there exist invertible local operators $\alpha, \beta, \ldots$ such that [6]

$$
\begin{equation*}
\left|\psi^{\prime}\right\rangle=\underbrace{\alpha \otimes \beta \otimes \cdots}_{n}|\psi\rangle . \tag{3.1}
\end{equation*}
$$

The residual entanglement of odd $n$ qubits for the state $|\psi\rangle$ is defined as follows 13]:

$$
\begin{equation*}
\tau(\psi)=4\left|(\overline{\mathcal{I}}(a, n))^{2}-4 \mathcal{I}^{*}(a, n-1) \mathcal{I}_{+2^{n-1}}^{*}(a, n-1)\right| \tag{3.2}
\end{equation*}
$$

where (see [13, 14)

$$
\begin{align*}
\overline{\mathcal{I}}(a, n)= & \sum_{i=0}^{2^{n-3}-1}(-1)^{N(i)}\left[\left(a_{2 i} a_{\left(2^{n}-1\right)-2 i}-a_{2 i+1} a_{\left(2^{n}-2\right)-2 i}\right)\right. \\
& \left.-\left(a_{\left(2^{n-1}-2\right)-2 i} a_{\left(2^{n-1}+1\right)+2 i}-a_{\left(2^{n-1}-1\right)-2 i} a_{2^{n-1}+2 i}\right)\right] \tag{3.3}
\end{align*}
$$

and (see [13, 14])

$$
\begin{gather*}
\mathcal{I}^{*}(a, n-1)=\sum_{i=0}^{2^{n-3}-1}(-1)^{N(i)}\left(a_{2 i} a_{\left(2^{n-1}-1\right)-2 i}-a_{2 i+1} a_{\left(2^{n-1}-2\right)-2 i}\right)  \tag{3.4}\\
\mathcal{I}_{+2^{n-1}}^{*}(a, n-1)=\sum_{i=0}^{2^{n-3}-1}(-1)^{N(i)}\left(a_{2^{n-1}+2 i} a_{\left(2^{n}-1\right)-2 i}-a_{2^{n-1}+1+2 i} a_{\left(2^{n}-2\right)-2 i}\right) . \tag{3.5}
\end{gather*}
$$

It has been also proven that if states $|\psi\rangle$ and $\left|\psi^{\prime}\right\rangle$ are SLOCC equivalent, then the following SLOCC equation holds [13]:

$$
\begin{equation*}
\tau\left(\psi^{\prime}\right)=\tau(\psi) \underbrace{|\operatorname{det}(\alpha) \operatorname{det}(\beta) \operatorname{det}(\gamma) \cdots|^{2}}_{n} . \tag{3.6}
\end{equation*}
$$

We now argue that $\tau_{12 \cdots n}^{(1)}=\tau(\psi)$. This can be seen as follows. A simple calculation shows that $\overline{\mathcal{I}}(a, n)=T$ (see (i) in Appendix A). Inspection of Eqs. (2.18) and (A21) (the reduced form of Eq. (3.4)) reveals that $\mathcal{I}^{*}(a, n-1)=P / 2$. Furthermore, inspection of Eqs. (2.19) and (A24) (the reduced form of Eq. (3.5)) reveals that $\mathcal{I}_{+2^{n-1}}^{*}(a, n-1)=Q / 2$. Substituting these results into Eq. (3.2) yields

$$
\begin{equation*}
\tau(\psi)=4\left|T^{2}-P Q\right| \tag{3.7}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\tau_{12 \cdots n}^{(1)}=\tau(\psi) . \tag{3.8}
\end{equation*}
$$

Next we recall that the residual entanglement with respect to qubit $i$ is defined as (see [14]) $\tau^{(i)}(\psi)$, which is obtained from $\tau(\psi)$ under the transposition $(1, i)$ on qubits 1 and $i$. The odd $n$-tangle is defined by taking the average of the residual entanglement with respect to qubit $i$ [14:

$$
\begin{equation*}
R(\psi)=\frac{1}{n} \sum_{i=1}^{n} \tau^{(i)}(\psi) \tag{3.9}
\end{equation*}
$$

Note that $R(\psi)$ is considered as an entanglement measure for odd $n$ qubits (14].
It follows immediately from Eq. (3.8) and the definitions of $\tau_{12 \cdots n}^{(i)}$ and $\tau^{(i)}(\psi)$ that

$$
\begin{equation*}
\tau_{12 \cdots n}^{(i)}=\tau^{(i)}(\psi), \quad i=1,2, \cdots, n . \tag{3.10}
\end{equation*}
$$

Further, Eq. (2.13), together with Eqs. (3.9) and (3.10), yields

$$
\begin{equation*}
\tau_{12 \cdots n}=R(\psi) \tag{3.11}
\end{equation*}
$$

A direct consequence of Eqs. (3.10) and (3.11) is that the $n$-tangle with respect to qubit $i$ and the $n$-tangle of odd $n$ qubits inherit the properties of the residual entanglement with respect to qubit $i$ and the odd $n$-tangle. We highlight that the $n$-tangle with respect to qubit $i$ and the $n$-tangle of odd $n$ qubits are $S L$-invariant and $L U$-invariant, and are entanglement monotones (see 14 for details).

Clearly, both $\tau_{12 \cdots n}^{(i)}$ and $\tau_{12 \cdots n}$ satisfy Eq. (3.6). The $n$-tangle with respect to qubit $i$ is called a SLOCC polynomial of degree 4 of odd $n$ qubits. It should be noted that there are no polynomial invariants of degree 2 for odd $n$ qubits [18. In view of the SLOCC equation (3.6), it is easy to see that if one of $\tau_{12 \cdots n}^{(i)}\left(\psi^{\prime}\right)$ (resp. $\left.\tau_{12 \cdots n}\left(\psi^{\prime}\right)\right)$ and $\tau_{12 \cdots n}^{(i)}(\psi)$ (resp. $\tau_{12 \cdots n}(\psi)$ ) vanishes while the other does not, then $|\psi\rangle$ and $\left|\psi^{\prime}\right\rangle$ belong to different SLOCC classes. This reveals that the $n$-tangle with respect to qubit $i$ and the $n$-tangle of odd $n$ qubits can be used for SLOCC classification.

We exemplify the results for the GHZ state and the W state. In our previous work [19] it has been shown that $\tau(G H Z)=1$ and $\tau(W)=0$ for any $n$-qubit GHZ and W states. The above analysis directly gives rise to the conclusion that the $n$-tangle of odd $n$ qubits $\tau_{12 \cdots n}$ is equal to 1 for the GHZ state and 0 for the W state.

Finally, we extend the $n$-tangle of odd $n$ qubits to mixed states via the convex roof construction (see, e.g., the review [1]):

$$
\begin{equation*}
\tau_{12 \cdots n}(\rho)=\min \sum_{i} p_{i} \tau_{12 \cdots n}\left(\psi_{i}\right) \tag{3.12}
\end{equation*}
$$

where $p_{i} \geq 0$ and $\sum_{i} p_{i}=1$, and the minimum is taken over all possible decompositions of $\rho$ into pure states, i.e. $\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$,

## 4 Conclusion

In summary, we have proposed the $n$-tangle of odd $n$ qubits, which is a generalization of the standard form of the 3 -tangle to any odd $n$-qubit pure states. We have argued that the $n$-tangle of odd $n$ qubits is invariant under permutations of the qubits, is an entanglement monotone. The $n$-tangle of odd $n$ qubits takes value 1 for the GHZ state and vanishes for the W state. The $n$-tangle of odd $n$ qubits is considered as a natural entanglement measure of any odd $n$-qubit pure states. Finally, we have extended the $n$-tangle of odd $n$ qubits to mixed states via the convex roof construction. Our results will provide more insight into the nature of multipartite entanglement.

As is well known, two SLOCC inequivalent classes of three-qubit pure states, namely the GHZ class and the W class, can be distinguished via the 3 -tangle [6, 17]. Polynomial invariants of degree 2 have been recently exploited for SLOCC classification of four-qubit pure states 20, 21] and of the symmetric Dicke states with $l$ excitations of $n$ qubits [19]. More recently, four polynomial invariants of degree $2^{n / 2}$ of any even $n$ qubits have been presented and several different genuine entangled states inequivalent to the GHZ, the W, or the symmetric Dicke states with $l$ excitations under SLOCC have been obtained by using the polynomials [22]. Further attempts have been made to build connections between polynomial (algebraic) invariants and SLOCC classification [23, 24]. We expect the $n$-tangle of odd $n$ qubits proposed in this paper can be used for SLOCC classification of any odd $n$ qubits.

## Appendix A

We first give proofs of Eqs. (2.15)-(2.16).
Let $\bar{\alpha}_{i}$ be the complement of $\alpha_{i}$. That is, $\bar{\alpha}_{i}=0$ when $\alpha_{i}=1$. Otherwise, $\bar{\alpha}_{i}=1$. In view of that $\epsilon_{00}=\epsilon_{11}=0$ and $\epsilon_{01}=-\epsilon_{10}=1$, to compute $W_{12 \cdots n}^{(1)}$ in Eq. (2.14), we only need to consider $\beta_{i}=\bar{\alpha}_{i}$, $\delta_{i}=\bar{\gamma}_{i}, i=2, \cdots, n, \gamma_{1}=\bar{\alpha}_{1}$, and $\delta_{1}=\bar{\beta}_{1}$. Thus, Eq. (2.14) becomes

$$
\begin{equation*}
W_{12 \cdots n}^{(1)}=\sum a_{\alpha_{1} \alpha_{2} \cdots \alpha_{n}} a_{\beta_{1} \bar{\alpha}_{2} \cdots \bar{\alpha}_{n}} a_{\bar{\alpha}_{1} \gamma_{2} \cdots \gamma_{n}} a_{\bar{\beta}_{1} \bar{\gamma}_{2} \cdots \bar{\gamma}_{n}} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n} \bar{\alpha}_{n}} \epsilon_{\gamma_{2} \bar{\gamma}_{2}} \cdots \epsilon_{\gamma_{n} \bar{\gamma}_{n}} \epsilon_{\alpha_{1} \bar{\alpha}_{1}} \epsilon_{\beta_{1} \bar{\beta}_{1}} . \tag{A1}
\end{equation*}
$$

We distinguish two cases.
Case 1. $\beta_{1}=\alpha_{1}$.
In this case, $\epsilon_{\alpha_{1} \bar{\alpha}_{1}} \epsilon_{\beta_{1} \bar{\beta}_{1}}=1$. Thus, from Eq. (A1), we have

$$
\begin{equation*}
W_{12 \cdots n}^{(1)}=\sum a_{\alpha_{1} \alpha_{2} \cdots \alpha_{n}} a_{\alpha_{1} \bar{\alpha}_{2} \cdots \bar{\alpha}_{n}} a_{\bar{\alpha}_{1} \gamma_{2} \cdots \gamma_{n}} a_{\bar{\alpha}_{1} \bar{\gamma}_{2} \cdots \bar{\gamma}_{n}} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n} \bar{\alpha}_{n}} \epsilon_{\gamma_{2} \bar{\gamma}_{2}} \cdots \epsilon_{\gamma_{n} \bar{\gamma}_{n}} . \tag{A2}
\end{equation*}
$$

Letting

$$
\begin{align*}
& P=\sum_{\alpha_{2} \cdots \alpha_{n}} a_{0 \alpha_{2} \cdots \alpha_{n}} a_{0 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n}} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n} \bar{\alpha}_{n}},  \tag{A3}\\
& Q=\sum_{\alpha_{2} \cdots \alpha_{n}} a_{1 \alpha_{2} \cdots \alpha_{n}} a_{1 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n}} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n} \bar{\alpha}_{n}}, \tag{A4}
\end{align*}
$$

yields

$$
\begin{equation*}
W_{12 \cdots n}^{(1)}=2 P Q . \tag{A5}
\end{equation*}
$$

Case 2. $\beta_{1}=\bar{\alpha}_{1}$.
In this case, $\epsilon_{\alpha_{1} \bar{\alpha}_{1}} \epsilon_{\beta_{1} \bar{\beta}_{1}}=-1$. Thus, from Eq. (A1), we have

$$
\begin{equation*}
W_{12 \cdots n}^{(1)}=-\sum a_{\alpha_{1} \alpha_{2} \cdots \alpha_{n}} a_{\bar{\alpha}_{1} \bar{\alpha}_{2} \cdots \bar{\alpha}_{n}} a_{\bar{\alpha}_{1} \gamma_{2} \cdots \gamma_{n}} a_{\alpha_{1} \bar{\gamma}_{2} \cdots \bar{\gamma}_{n}} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n} \bar{\alpha}_{n}} \epsilon_{\gamma_{2} \bar{\gamma}_{2}} \cdots \epsilon_{\gamma_{n} \bar{\gamma}_{n}} . \tag{A6}
\end{equation*}
$$

Let

$$
\begin{align*}
& T=\sum a_{0 \alpha_{2} \cdots \alpha_{n}} a_{1 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n}} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n} \bar{\alpha}_{n}},  \tag{A7}\\
& S=\sum a_{1 \alpha_{2} \cdots \alpha_{n}} a_{0 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n}} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n} \bar{\alpha}_{n}} . \tag{A8}
\end{align*}
$$

From that $\epsilon_{01}=-\epsilon_{10}=1, \epsilon_{\alpha_{i} \bar{\alpha}_{i}}=-\epsilon_{\bar{\alpha}_{i} \alpha_{i}}$, and therefore

$$
\begin{equation*}
S=\sum a_{0 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n}} a_{1 \alpha_{2} \cdots \alpha_{n}} \times \epsilon_{\bar{\alpha}_{2} \alpha_{2}} \cdots \epsilon_{\bar{\alpha}_{n} \alpha_{n}}=T \tag{A9}
\end{equation*}
$$

Hence

$$
\begin{equation*}
W_{12 \cdots n}^{(1)}=-2 T^{2} . \tag{A10}
\end{equation*}
$$

Eq. A10), together with Eq. (A5), yields

$$
\begin{equation*}
W_{12 \cdots n}^{(1)}=2\left(P Q-T^{2}\right) . \tag{A11}
\end{equation*}
$$

Inserting Eq. (A11) into Eq. (2.11) leads to

$$
\begin{equation*}
\tau_{12 \cdots n}^{(1)}=4\left|T^{2}-P Q\right| . \tag{A12}
\end{equation*}
$$

Next, let $\alpha_{2} \cdots \alpha_{n}$ be the binary representation of $i$. Noting that $(-1)^{N(i)}=\epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n} \bar{\alpha}_{n}}$, we may rewrite $T$ as

$$
\begin{equation*}
T=\sum_{i=0}^{2^{n-1}-1}(-1)^{N(i)} a_{i} a_{2^{n}-i-1} . \tag{A13}
\end{equation*}
$$

(i). Proof of $T=\overline{\mathcal{I}}(a, n)$

Expanding Eq. (A7), we obtain

$$
\begin{align*}
T= & \sum a_{0 \alpha_{2} \cdots \alpha_{n-1} 0} a_{1 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 1} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
& -\sum a_{0 \alpha_{2} \cdots \alpha_{n-1} 1} a_{1 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 0} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
= & \sum a_{00 \alpha_{3} \cdots \alpha_{n-1} 0} a_{11 \bar{\alpha}_{3} \cdots \bar{\alpha}_{n-1} 1} \times \epsilon_{\alpha_{3} \bar{\alpha}_{3}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
& -\sum a_{01 \alpha_{3} \cdots \alpha_{n-1} 0} a_{10 \bar{\alpha}_{3} \cdots \bar{\alpha}_{n-1} 1} \times \epsilon_{\alpha_{3} \bar{\alpha}_{3}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
& -\sum a_{00 \alpha_{3} \cdots \alpha_{n-1} 1} a_{11 \bar{\alpha}_{3} \cdots \bar{\alpha}_{n-1} 0} \times \epsilon_{\alpha_{3} \bar{\alpha}_{3}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
& +\sum a_{01 \alpha_{3} \cdots \alpha_{n-1} 1} a_{10 \bar{\alpha}_{3} \cdots \bar{\alpha}_{n-1} 0} \times \epsilon_{\alpha_{3} \bar{\alpha}_{3}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
= & \overline{\mathcal{I}}(a, n), \tag{A14}
\end{align*}
$$

where the third equality follows by letting $\alpha_{3} \cdots \alpha_{n-1}$ be the binary number of $i$ and noting that $(-1)^{N(i)}=$ $(-1)^{N\left(\alpha_{3} \cdots \alpha_{n-1}\right)}=\epsilon_{\alpha_{3} \bar{\alpha}_{3}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}}$.
(ii). Reduction of $P$

Expanding Eq. (A3), we obtain

$$
\begin{align*}
P= & \sum a_{0 \alpha_{2} \cdots \alpha_{n-1} 0} a_{0 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 1} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
& -\sum a_{0 \alpha_{2} \cdots \alpha_{n-1} 1} a_{0 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 0} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
= & 2 \sum a_{0 \alpha_{2} \cdots \alpha_{n-1} 0} a_{0 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 1} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
= & 2 \sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)} a_{2 i} a_{2^{n-1}-2 i-1}, \tag{A15}
\end{align*}
$$

where the second equality follows from

$$
\begin{align*}
& \sum a_{0 \alpha_{2} \cdots \alpha_{n-1} 1} a_{0 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 0} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
& =-\sum a_{0 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 0} a_{0 \alpha_{2} \cdots \alpha_{n-1} 1} \times \epsilon_{\bar{\alpha}_{2} \alpha_{2}} \cdots \epsilon_{\bar{\alpha}_{n-1} \alpha_{n-1}}  \tag{A16}\\
& =-\sum a_{0 \alpha_{2} \cdots \alpha_{n-1} 0} a_{0 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 1} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}}, \tag{A17}
\end{align*}
$$

and the third equality follows by letting $\alpha_{2} \cdots \alpha_{n-1}$ be the binary number of $i$ and noting that $(-1)^{N(i)}=$ $\epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}}$.
(iii). Reduction of $Q$

Eq. (A4) gives, by analogy with Eq. (A15),

$$
\begin{align*}
Q= & \sum a_{1 \alpha_{2} \cdots \alpha_{n-1} 0} a_{1 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 1} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
& -\sum a_{1 \alpha_{2} \cdots \alpha_{n-1} 1} a_{\bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 0} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
= & 2 \sum a_{1 \alpha_{2} \cdots \alpha_{n-1} 0} a_{1 \bar{\alpha}_{2} \cdots \bar{\alpha}_{n-1} 1} \times \epsilon_{\alpha_{2} \bar{\alpha}_{2}} \cdots \epsilon_{\alpha_{n-1} \bar{\alpha}_{n-1}} \\
= & 2 \sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)} a_{2^{n-1}+2 i} a_{2^{n}-2 i-1} . \tag{A18}
\end{align*}
$$

(iv). Reduction of $\mathcal{I}^{*}(a, n-1)$

By Eq. (3.4), we have

$$
\begin{equation*}
\mathcal{I}^{*}(a, n-1)=\sum_{i=0}^{2^{n-3}-1}(-1)^{N(i)} a_{2 i} a_{\left(2^{n-1}-1\right)-2 i}-\sum_{i=0}^{2^{n-3}-1}(-1)^{N(i)} a_{2 i+1} a_{\left(2^{n-1}-2\right)-2 i} \tag{A19}
\end{equation*}
$$

Let $k=2^{n-2}-1-i$. Then $N(k)+N(i)=n-2$, and hence $(-1)^{N(i)}=-(-1)^{N(k)}$, and

$$
\begin{equation*}
\sum_{i=0}^{2^{n-3}-1}(-1)^{N(i)} a_{2 i+1} a_{\left(2^{n-1}-2\right)-2 i}=-\sum_{k=2^{n-3}}^{2^{n-2}-1}(-1)^{N(k)} a_{2 k} a_{\left(2^{n-1}-1\right)-2 k} \tag{A20}
\end{equation*}
$$

This leads to

$$
\begin{equation*}
\mathcal{I}^{*}(a, n-1)=\sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)} a_{2 i} a_{\left(2^{n-1}-1\right)-2 i} \tag{A21}
\end{equation*}
$$

(v). Reduction of $\mathcal{I}_{+2^{n-1}}^{*}(a, n-1)$

By Eq. (3.5), we have

$$
\begin{equation*}
\mathcal{I}_{+2^{n-1}}^{*}(a, n-1)=\sum_{i=0}^{2^{n-3}-1}(-1)^{N(i)} a_{2^{n-1}+2 i} a_{\left(2^{n}-1\right)-2 i}-\sum_{i=0}^{2^{n-3}-1}(-1)^{N(i)} a_{2^{n-1}+1+2 i} a_{\left(2^{n}-2\right)-2 i} \tag{A22}
\end{equation*}
$$

Letting $k=2^{n-2}-1-i$, we have

$$
\begin{equation*}
\sum_{i=0}^{2^{n-3}-1}(-1)^{N(i)} a_{2^{n-1}+1+2 i} a_{\left(2^{n}-2\right)-2 i}=-\sum_{k=2^{n-3}}^{2^{n-2}-1}(-1)^{N(k)} a_{2^{n-1}+2 k} a_{\left(2^{n}-1\right)-2 k} \tag{A23}
\end{equation*}
$$

This leads to

$$
\begin{equation*}
\mathcal{I}_{+2^{n-1}}^{*}(a, n-1)=\sum_{i=0}^{2^{n-2}-1}(-1)^{N(i)} a_{2^{n-1}+2 i} a_{\left(2^{n}-1\right)-2 i} \tag{A24}
\end{equation*}
$$

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