# Concurrence for a two-qubits mixed state consisting of three pure states in the framework of $\mathrm{SU}(2)$ coherent states 

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#### Abstract

A simplified expression of concurrence for two-qubit mixed state having no more than three non-vanishing eigenvalues is obtained. Basing on $S U(2)$ coherent states, the amount of entanglement of two-qubit pure states is studied and conditions for entanglement are calculated by formulating the measure in terms of some new parameters (amplitudes of coherent states). This formalism is generalized to the case of two-qubit mixed states using the simplified expression of concurrence.


## 1 Introduction

Quantum information processing essentially depends on various quantum-mechanical phenomena, among which entanglement has been considered as one of the most crucial features. Quantum entanglement plays an important role in several fields of quantum information such as quantum teleportation [1, 2], quantum cryptography [3, 4], quantum dense coding $[5,6,7]$ and quantum computation $[8,9,10]$, etc. The fundamental question

[^0]in quantum entanglement phenomenon is: which states are entangled and which ones are not? We can find the simple answer to this question only in some cases. Then, the quantification and characterization of the amount of entanglement have attracted much attention $[11,12,13]$ in this field. For quantifying the amount of entanglement, various measures have been proposed such as concurrence [14, 15, 16, 17], negativity [18, 19, 20] and tangle [21, 22, 23], etc.
Generally, for bipartite system pure states, the entanglement measures are extensively accepted. However, in the case of mixed states, the quantification of entanglement becomes more complicated so the entanglement measures are not easy to calculate analytically and there is no general method. The entanglement measures of a mixed state is defined as the average entanglement measure of an ensemble of pure states representing the mixed state, minimized over all decompositions of the mixed state. So the problem is to find such minimization. For various particular cases, many minimizations can be done analytically [24, 25], but in general the problem is not solved mathematically and it is still in early stages with many open questions. In the case of a pair of qubits, there is a famous formula for the bipartite concurrence and the associated entanglement of formation which have been proposed by Wootters and Hill [26].
Another concept that has vital application in quantum information processing is the theory of coherent states for which preliminary concepts were presented by Schrödinger [27]. Coherent states play a crucial role in quantum physics, particularly, in quantum optics $[28,29,40,41]$ and encoding quantum information on continuous variables [30], etc. They also play an important role in mathematical physics [31], for example, they are very useful in performing stationary phase approximations to path integral [32, 33, 34]. Ones of the practical coherent states are $\mathrm{SU}(1,1)$ and $\mathrm{SU}(2)$ coherent states which are widely used in entangled nonorthogonal states studying. The entangled nonorthogonal states are very useful tools in the quantum cryptography and quantum information processing [35]. Bosonic entangled coherent state (Glauber coherent states), $\mathrm{SU}(1,1)$ and $\mathrm{SU}(2)$ coherent states are typical examples of entangled nonorthogonal states.
In quantum statistical mechanics, a macrostate of given system is characterized by a probability distribution on a certain set of microstates representing the physical properties of the system. This distribution describes the probability of finding the system in certain microstate. In Boltzmann's definition, entropy of the system is defined as a measure of the number of possible microstates. The entropy grows with this number. In this paper, we will consider a two-qubit system described by a mixed state (macro state) defined as a statistical mixture of three pure states representing microstates of the system. However, the entropy will be greater than that given in [36] and the system
reaches to equilibrium and will be more stable than that represented in [36].
Berrada et al [36] have used the simplified expression of concurrence for studying the entanglement of two-qubit mixed states having no more than two non-zero eigenvalues (i.e., mixed states defined as statistical mixture of two states) which play an important role in quantum information theory using $\mathrm{SU}(2)$ coherent states realization. This present work extends the research conducted by Berrada et al [36, 42] in the case of two-qubit mixed states consisting of three pure states.
This paper is organized as follows: In Sect. 2 the concept of arbitrary state of two- qubit is described and a expression for concurrence of a two-qubit mixed state including three orthogonal pure states is presented. Sect. 3 generalizes the findings of sect. 2 using the formalism of $\operatorname{SU}(2)$ coherent states. The conclusion and some points for further research is given in Sect. 4.

## 2 Concurrence of an arbitrary state of two-qubit system

### 2.1 The case of pure state

In this section, we give an outline of the concurrence and its quantity for pure and mixed states.
A general pure state of two-qubit system can be expressed in the standard computational basis $\{|00\rangle,|01\rangle,|10\rangle|11\rangle\}$ as

$$
\begin{equation*}
|\psi\rangle=a|00\rangle+b|01\rangle+c|10\rangle+d|11\rangle \tag{2-1}
\end{equation*}
$$

where $a, b, c$ and $d$ are complex numbers satisfying the normalization condition

$$
|a|^{2}+|b|^{2}+|c|^{2}+|d|^{2}=1
$$

The concurrence for a two-qubit state $|\psi\rangle$ may be written as

$$
\begin{align*}
C(|\psi\rangle) & =|\langle\psi \mid \tilde{\psi}\rangle|  \tag{2-2}\\
& =2|a d-b c|
\end{align*}
$$

where $|\tilde{\psi}\rangle=\left(\sigma_{y} \otimes \sigma_{y}\right)\left|\psi^{*}\right\rangle$ represents the spin-flip plus phase flip operation. $\left|\psi^{*}\right\rangle$ and $\sigma_{y}$ are the complex conjugate of $|\psi\rangle$ in the standard basis such as $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ and pauli operator in local basis $\{|0\rangle,|1\rangle\}$, respectively. The concurrence is equal to 0 for a separable state and to 1 for a maximally entangled states $\left(i . e .,|\psi\rangle=\frac{1}{\sqrt{2}}(|00\rangle \pm|11\rangle)\right.$ or $\left.|\psi\rangle=\frac{1}{\sqrt{2}}(|01\rangle \pm|01\rangle)\right)$. The relation between concurrence and entanglement of a pure state is given by

$$
\begin{equation*}
E(|\psi\rangle)=\xi(C(|\psi\rangle)) \tag{2-3}
\end{equation*}
$$

where the function $\xi$ is defined as

$$
\begin{equation*}
\xi(C)=h\left(\frac{1+\sqrt{1-C^{2}}}{2}\right) \tag{2-4}
\end{equation*}
$$

such that

$$
\begin{equation*}
h(x)=-x \log _{2} x-(1-x) \log _{2}(1-x) \tag{2-5}
\end{equation*}
$$

is the binary entropy function with argument related the concurrence given by Eq. (2-2). The entanglement of formation is a monotonously increasing function of the concurrence that ranges from 0 for a separable state to 1 for a maximally entangled states (i.e., $|\psi\rangle=\frac{1}{\sqrt{2}}(|00\rangle \pm|11\rangle)$ or $\left.|\psi\rangle=\frac{1}{\sqrt{2}}(|01\rangle \pm|10\rangle)\right)$. Therefore, one can consider concurrence directly as a measure of the entanglement.

### 2.2 The case of mixed state

In the case of mixed state, the two-qubit quantum state must be represented not by a bracket but a matrix called density operator and denoted by $\rho$ in quantum mechanics. It is always to decompose $\rho$ into a mixture of the density operator of a set of pure states $\left|\psi_{i}\right\rangle$ as

$$
\begin{equation*}
\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right| \tag{2-6}
\end{equation*}
$$

where $\left\{\left|\psi_{i}\right\rangle\right\}$ are distinct normalized (not necessary orthogonal) two-qubit pure states given by

$$
\begin{equation*}
\left|\psi_{i}\right\rangle=a_{i}|00\rangle+b_{i}|01\rangle+c_{i}|10\rangle+d_{i}|11\rangle \tag{2-7}
\end{equation*}
$$

and $\left\{p_{i}\right\}$ are the corresponding probabilities (i.e., $p_{i} \geq 0$ and $\sum_{i} p_{i}=1$ ).
There is a condition for separability and inseparability of mixed states like the pure states which are mentioned above. A mixed state $\rho$ is said to be separable if it can be written as a convex sum of separable pure states, i.e., $\rho=\sum_{i} p_{i} \rho_{i}^{(A)} \otimes \rho_{i}^{(B)}$, where $\rho_{i}^{(A, B)}$ is the reduced density operator of qubit $(A, B)$, respectively, given by $\rho_{i}^{(A, B)}=\operatorname{Tr}_{(A, B)}\left(\left|\Psi_{i}\right\rangle\left\langle\Psi_{i}\right|\right)$. The state $\rho$ is entangled if it cannot be represented as a mixture of a separable pure states.
We can define the concurrence of the mixed state $\rho$ as a convex roof method which is the average concurrence of an ensemble pure states of the decomposition, minimized over all decomposition of $\rho$

$$
\begin{equation*}
C(\rho)=\inf \sum_{i} p_{i} C\left(\left|\psi_{i}\right\rangle\right) \tag{2-8}
\end{equation*}
$$

where $C\left(\left|\Psi_{i}\right\rangle\right)$ is the concurrence of the pure state $\left|\Psi_{i}\right\rangle$ given by Eq. (2-2). According to Ref. [26], Wootters and Hill have found an explicit formula of the concurrence defined as

$$
\begin{equation*}
C(\rho)=\max \left\{\lambda_{1}-\lambda_{2}-\lambda_{3}-\lambda_{4}, 0\right\} \tag{2-9}
\end{equation*}
$$

Here $\lambda_{i}$ is the square root of eigenvalues of $\rho\left(\sigma_{y} \otimes \sigma_{y}\right) \rho^{*}\left(\sigma_{y} \otimes \sigma_{y}\right)$ in decreasing order ( $\rho^{*}$ denotes the complex conjugate of $\rho$ ).
In general, for a two-qubit states with no more than two-non-zero eigenvalues which have been studied, there is an explicit formula of the square of the concurrence [36]. For a mixed state with no more than three non vanishing eigenvalues (i.e., state with three orthogonal pure states), Eq. (2-6) can be written as

$$
\begin{equation*}
\rho=\mu_{1}\left|\mu_{1}\right\rangle\left\langle\mu_{1}\right|+\mu_{2}\left|\mu_{2}\right\rangle\left\langle\mu_{2}\right|+\mu_{3}\left|\mu_{3}\right\rangle\left\langle\mu_{3}\right| \tag{2-10}
\end{equation*}
$$

where $\left|\mu_{1}\right\rangle,\left|\mu_{2}\right\rangle$ and $\left|\mu_{3}\right\rangle$ are three pure states given by

$$
\begin{align*}
\left|\mu_{1}\right\rangle & =a_{1}|00\rangle+b_{1}|01\rangle+c_{1}|10\rangle+d_{1}|11\rangle \\
\left|\mu_{2}\right\rangle & =a_{2}|00\rangle+b_{2}|01\rangle+c_{2}|10\rangle+d_{2}|11\rangle  \tag{2-11}\\
\left|\mu_{3}\right\rangle & =a_{3}|00\rangle+b_{3}|01\rangle+c_{3}|10\rangle+d_{3}|11\rangle
\end{align*}
$$

So for this mixed state, Eq. (2-8) is written as

$$
\begin{equation*}
C(\rho)=\inf \left(\mu_{1} C_{1}+\mu_{2} C_{2}+\mu_{3} C_{3} .\right. \tag{2-12}
\end{equation*}
$$

In the proposed mixed state, we obtain the minimum of the Eq.(2-12) as

$$
\begin{gather*}
C^{2}(\rho)=\left(\mu_{1}^{2} C_{1}^{2}+\mu_{2}^{2} C_{2}^{2}+\mu_{3}^{2} C_{3}^{2}\right) \\
+\frac{1}{2} \mu_{1} \mu_{2}\left|\mathbf{C}^{1}+\mathbf{C}^{2}-\mathbf{C}^{3}-\mathbf{C}^{4}\right|^{2}-\frac{1}{2} \mu_{1} \mu_{2}\left|\left(\mathbf{C}^{1}+\mathbf{C}^{2}-\mathbf{C}^{3}-\mathbf{C}^{4}\right)^{2}-4 \mathbf{c}_{1} \mathbf{c}_{2}\right| \\
+\frac{1}{2} \mu_{1} \mu_{3}\left|\mathbf{C}^{1}+\mathbf{C}^{3}-\mathbf{C}^{2}-\mathbf{C}^{4}\right|^{2}-\frac{1}{2} \mu_{1} \mu_{3}\left|\left(\mathbf{C}^{2}+\mathbf{C}^{3}-\mathbf{C}^{2}-\mathbf{C}^{4}\right)^{2}-4 \mathbf{c}_{1} \mathbf{c}_{3}\right|  \tag{2-13}\\
+\frac{1}{2} \mu_{2} \mu_{3}\left|\mathbf{C}^{1}+\mathbf{C}^{4}-\mathbf{C}^{2}-\mathbf{C}^{3}\right|^{2}-\frac{1}{2} \mu_{2} \mu_{3}\left|\left(\mathbf{C}^{1}+\mathbf{C}^{4}-\mathbf{C}^{2}-\mathbf{C}^{3}\right)^{2}-4 \mathbf{c}_{2} \mathbf{c}_{3}\right|
\end{gather*}
$$

where

$$
\begin{equation*}
C_{i}=\left|\mathbf{c}_{i}\right|=2\left|a_{i} d_{i}-b_{i} c_{i}\right| \quad(i=1,2,3) \tag{2-14}
\end{equation*}
$$

is the concurrence of the pure state $\left|\mu_{i}\right\rangle$, and

$$
\begin{align*}
& C^{1}=\left|\mathbf{C}^{1}\right|=\frac{2}{3}\left|\left(a_{1}+a_{2}+a_{3}\right)\left(d_{1}+d_{2}+d_{3}\right)-\left(b_{1}+b_{2}+b_{3}\right)\left(c_{1}+c_{2}+c_{3}\right)\right| \\
& C^{2}=\left|\mathbf{C}^{2}\right|=\frac{2}{3}\left|\left(a_{1}+a_{2}-a_{3}\right)\left(d_{1}+d_{2}-d_{3}\right)-\left(b_{1}+b_{2}-b_{3}\right)\left(c_{1}+c_{2}-c_{3}\right)\right| \\
& C^{3}=\left|\mathbf{C}^{3}\right|=\frac{2}{3}\left|\left(a_{1}-a_{2}+a_{3}\right)\left(d_{1}-d_{2}+d_{3}\right)-\left(b_{1}-b_{2}+b_{3}\right)\left(c_{1}-c_{2}+c_{3}\right)\right|  \tag{2-15}\\
& C^{4}=\left|\mathbf{C}^{4}\right|=\frac{2}{3}\left|\left(a_{1}-a_{2}-a_{3}\right)\left(d_{1}-d_{2}-d_{3}\right)-\left(b_{1}-b_{2}-b_{3}\right)\left(c_{1}-c_{2}-c_{3}\right)\right|
\end{align*}
$$

are the concurrences of the pure states

$$
\begin{align*}
\left|\mu^{1}\right\rangle & =\frac{1}{\sqrt{3}}\left(\left|\mu_{1}\right\rangle+\left|\mu_{2}\right\rangle+\left|\mu_{3}\right\rangle\right) \\
\left|\mu^{2}\right\rangle & =\frac{1}{\sqrt{3}}\left(\left|\mu_{1}\right\rangle+\left|\mu_{2}\right\rangle-\left|\mu_{3}\right\rangle\right)  \tag{2-16}\\
\left|\mu^{3}\right\rangle & =\frac{1}{\sqrt{3}}\left(\left|\mu_{1}\right\rangle-\left|\mu_{2}\right\rangle+\left|\mu_{3}\right\rangle\right) \\
\left|\mu^{4}\right\rangle & =\frac{1}{\sqrt{3}}\left(\left|\mu_{1}\right\rangle-\left|\mu_{2}\right\rangle-\left|\mu_{3}\right\rangle\right)
\end{align*}
$$

respectively, where $\mathbf{C}^{i}$ and $\mathbf{c}_{i}$ are the complex concurrences of the pure states in Eq. (2-16) and Eq. (2-11), respectively. For simplicity form of Eq. (2-13), considering the following change of variable

$$
\begin{gather*}
C_{+}=\left|\mathbf{c}_{+}\right|=\left|\mathbf{C}^{1}+\mathbf{C}^{2}\right|=\frac{4}{3}\left|\left(a_{1}+a_{2}\right)\left(d_{1}+d_{2}\right)-\left(b_{1}+b_{2}\right)\left(c_{1}+c_{2}\right)+a_{3} d_{3}-b_{3} c_{3}\right| \\
C_{-}=\left|\mathbf{c}_{-}\right|=\left|\mathbf{C}^{3}+\mathbf{C}^{4}\right|=\frac{4}{3}\left|\left(a_{1}-a_{2}\right)\left(d_{1}-d_{2}\right)-\left(b_{1}-b_{2}\right)\left(c_{1}-c_{2}\right)+a_{3} d_{3}-b_{3} c_{3}\right| \\
C_{+}^{\prime}=\left|\mathbf{c}_{+}^{\prime}\right|=\left|\mathbf{C}^{1}+\mathbf{C}^{3}\right|=\frac{4}{3}\left|\left(a_{1}+a_{3}\right)\left(d_{1}+d_{3}\right)-\left(b_{1}+b_{3}\right)\left(c_{1}+c_{3}\right)+a_{2} d_{2}-b_{2} c_{2}\right| \\
C_{-}^{\prime}=\left|\mathbf{c}_{-}^{\prime}\right|=\left|\mathbf{C}^{2}+\mathbf{C}^{4}\right|=\frac{4}{3}\left|\left(a_{1}-a_{3}\right)\left(d_{1}-d_{3}\right)-\left(b_{1}-b_{3}\right)\left(c_{1}-c_{3}\right)+a_{2} d_{2}-b_{2} c_{2}\right|  \tag{2-17}\\
C_{+}^{\prime \prime}=\left|\mathbf{c}_{+}^{\prime \prime}\right|=\left|\mathbf{C}^{1}+\mathbf{C}^{4}\right|=\frac{4}{3}\left|\left(a_{2}+a_{3}\right)\left(d_{2}+d_{3}\right)-\left(b_{2}+b_{3}\right)\left(c_{2}+c_{3}\right)+a_{1} d_{1}-b_{1} c_{1}\right| \\
C_{-}^{\prime \prime}=\left|\mathbf{c}_{-}^{\prime \prime}\right|=\left|\mathbf{C}^{2}+\mathbf{C}^{3}\right|=\frac{4}{3}\left|\left(a_{2}-a_{3}\right)\left(d_{2}-d_{3}\right)-\left(b_{2}-b_{3}\right)\left(c_{2}-c_{3}\right)+a_{1} d_{1}-b_{1} c_{1}\right| .
\end{gather*}
$$

Finally, Eq. (2-13) is written as

$$
\begin{gather*}
C^{2}(\rho)=\left(\mu_{1}^{2} C_{1}^{2}+\mu_{2}^{2} C_{2}^{2}+\mu_{3}^{2} C_{3}^{2}\right) \\
+\frac{1}{2} \mu_{1} \mu_{2}\left|\mathbf{c}_{+}-\mathbf{c}_{-}\right|^{2}-\frac{1}{2} \mu_{1} \mu_{2}\left|\left(\mathbf{c}_{+}-\mathbf{c}_{-}\right)^{2}-4 \mathbf{c}_{1} \mathbf{c}_{2}\right| \\
+\frac{1}{2} \mu_{1} \mu_{3}\left|\mathbf{c}_{+}^{\prime}-\mathbf{c}_{-}^{\prime}\right|^{2}-\frac{1}{2} \mu_{1} \mu_{3}\left|\left(\mathbf{c}_{+}^{\prime}-\mathbf{c}_{-}^{\prime}\right)^{2}-4 \mathbf{c}_{1} \mathbf{c}_{3}\right|  \tag{2-18}\\
+\frac{1}{2} \mu_{2} \mu_{3}\left|\mathbf{c}_{+}^{\prime \prime}-\mathbf{c}_{-}^{\prime \prime}\right|^{2}-\frac{1}{2} \mu_{2} \mu_{3}\left|\left(\mathbf{c}_{+}^{\prime \prime}-\mathbf{c}_{-}^{\prime \prime}\right)^{2}-4 \mathbf{c}_{2} \mathbf{c}_{3}\right|
\end{gather*}
$$

This equation is the square of the concurrence for a mixed state that consists of three orthogonal pure states. Obviously, for a pure state $\mu_{1}=1, \mu_{2}=0, \mu_{3}=0$ or $\mu_{1}=0, \mu_{2}=1, \mu_{3}=0$ or $\mu_{1}=0, \mu_{2}=0, \mu_{3}=1$, our expression of concurrence backs to the definition (2-2). Also by omitting one of the pure states of this work, we could reach the same result in Ref. [36]. The advantages of this formula are: the concurrence of mixed state is expressed as a function of the concurrence of the pure states and their simple combinations, and also it can be analyzed easily.

## 3 Concurrence in the language of $\mathrm{SU}(2)$ coherent states

### 3.1 The case of pure state

The physical important of coherent states in quantum information theory is due to the fact that they are robust states which are widely used and applied for studying and solving different
problems in various quantum information processing and transmission tasks, and they are easy to generate experimentally and convenient to use. One of these states is $S U(2)$ coherent state which is one of the most important tool for analyzing entangled nonorthogonal states. By using a phase factor, a qubit can be written as follows

$$
\begin{align*}
|\theta, \varphi\rangle & =\exp \left[-\frac{\theta}{2}\left(\sigma_{+} e^{-i \varphi}-\sigma_{-} e^{i \varphi}\right)\right]|1\rangle  \tag{3-19}\\
& =\cos \frac{\theta}{2}|0\rangle+e^{i \varphi} \sin \frac{\theta}{2}|1\rangle,
\end{align*}
$$

where $\sigma_{ \pm}=\sigma_{x} \pm i \sigma_{y}$. Here $\sigma_{x}, \sigma_{y}$ and $(\theta, \varphi)$ are pauli matrices and real parameters, respectively. It can be proved that Eq. (3-19) presents exactly an $S U(2)$ coherent state (spin coherent state) of the Klauder-Peremolov [31].
The $S U(2)$ coherent state can be expressed as

$$
\begin{align*}
|\gamma, j\rangle & \equiv R(\gamma)|0, j\rangle=\exp \left[-\frac{1}{2}\left(J_{+} e^{-i \varphi}-J_{-} e^{i \varphi}\right)\right]|0, j\rangle \\
& =\left(1+|\gamma|^{2}\right)^{-j} \sum_{n=0}^{2 j}\binom{2 j}{n}^{\frac{1}{2}} \gamma^{n}|n, j\rangle, \tag{3-20}
\end{align*}
$$

where $R(\gamma)$ is the rotation operator, $J_{-}$and $J_{+}$are the raising and lowering operators of the $s u(2)$ Lie algebra, respectively. The generators of the $s u(2)$ Lie algebra, $J_{ \pm}$and $J_{z}$, satisfy the following commutation relations

$$
\begin{gather*}
{\left[J_{+}, J_{-}\right]=2 J_{z}} \\
{\left[J_{z}, J_{ \pm}\right]= \pm J_{ \pm},} \tag{3-21}
\end{gather*}
$$

and act on an irreducible unitary representation as follows

$$
\begin{align*}
J_{ \pm}|j, m\rangle= & \sqrt{(j \mp m)(j \pm m+1)}|j, m \pm 1\rangle \\
& J_{z}|j, m\rangle=m|j, m\rangle \tag{3-22}
\end{align*}
$$

The $S U(2)$ coherent state can be obtained by applying successively the raising operator on the state $|j,-j\rangle$

$$
\begin{equation*}
|\gamma, j\rangle=\frac{1}{\left(1+|\gamma|^{2}\right)^{j}} \sum_{m=-j}^{j}\left[\frac{(2 j)!}{(j+m)!(j-m)!}\right]^{\frac{1}{2}} \gamma^{j+m}|j, m\rangle \tag{3-23}
\end{equation*}
$$

A change of variable $n=j+m$ will give the form of Eq. (3-20). For a particle with spin $\frac{1}{2}$, we get

$$
\begin{align*}
\left|\gamma, \frac{1}{2}\right\rangle & =\frac{1}{\left(1+|\gamma|^{2}\right)^{\frac{1}{2}}} \sum_{n=0}^{1}\binom{1}{n} \gamma^{n}\left|n, \frac{1}{2}\right\rangle  \tag{3-24}\\
& =\frac{1}{\left(1+|\gamma|^{2}\right)^{\frac{1}{2}}}\left(\left|0, \frac{1}{2}\right\rangle+\gamma\left|1, \frac{1}{2}\right\rangle\right)
\end{align*}
$$

and for $\gamma=\tan \left(\frac{\theta}{2}\right) e^{i \varphi}$, we find that

$$
\begin{equation*}
|\gamma\rangle=\cos \frac{\theta}{2}|0\rangle+e^{i \varphi} \sin \frac{\theta}{2}|1\rangle \tag{3-25}
\end{equation*}
$$

where $\left|0, \frac{1}{2}\right\rangle \equiv|0\rangle$ and $\left|1, \frac{1}{2}\right\rangle \equiv|1\rangle$ are considered as basis states. So, representation of a qubit using a phase factor as shown in Eq. (3-19) is equivalent to a particle with spin $\frac{1}{2}$ in construction $\mathrm{SU}(2)$ coherent state.
This result shows that the treatment and transmission of the quantum information can be performed by using the $\mathrm{SU}(2)$ coherent states.
Generally, a separable pure state of two-qubit system can be written as $\left|\theta_{1}, \varphi_{1}\right\rangle \otimes\left|\theta_{2}, \varphi_{2}\right\rangle$. Considering that $\left|\theta_{1}, \varphi_{1}\right\rangle$ and $\left|\theta_{1}^{\prime}, \varphi_{1}^{\prime}\right\rangle$ are the normalized states of the qubit 1 and similarly for $\left|\theta_{2}, \varphi_{2}\right\rangle$ and $\left|\theta_{2}^{\prime}, \varphi_{2}^{\prime}\right\rangle$ of the qubit 2 , such that

$$
\begin{equation*}
\left\langle\theta_{1}, \varphi_{1} \mid \theta_{1}^{\prime}, \varphi_{1}^{\prime}\right\rangle \neq 0, \quad\left\langle\theta_{2}, \varphi_{2} \mid \theta_{2}^{\prime}, \varphi_{2}^{\prime}\right\rangle \neq 0 \tag{3-26}
\end{equation*}
$$

thus, the simplest extension of the arbitrary separable pure state to an entangled pure state of two-qubit system can be expressed by the unnormalized state

$$
\begin{equation*}
\| \psi\rangle=\cos \theta\left|\theta_{1}, \varphi_{1}\right\rangle \otimes\left|\theta_{2}, \varphi_{2}\right\rangle+e^{i \phi} \sin \theta\left|\theta_{1}^{\prime}, \varphi_{1}^{\prime}\right\rangle \otimes\left|\theta_{2}^{\prime}, \varphi_{2}^{\prime}\right\rangle \tag{3-27}
\end{equation*}
$$

Using Eq. (3-24), the unnormalized state can be written as

$$
\begin{equation*}
\| \psi\rangle=\cos \theta|\alpha\rangle \otimes|\beta\rangle+e^{i \phi} \sin \theta\left|\alpha^{\prime}\right\rangle \otimes\left|\beta^{\prime}\right\rangle \tag{3-28}
\end{equation*}
$$

where

$$
\begin{align*}
|\alpha\rangle & =\frac{1}{\sqrt{\left(1+|\alpha|^{2}\right)}}(|0\rangle+\alpha|1\rangle) \\
|\beta\rangle & =\frac{1}{\sqrt{\left(1+|\beta|^{2}\right)}}(|0\rangle+\beta|1\rangle)  \tag{3-29}\\
\left|\alpha^{\prime}\right\rangle & =\frac{1}{\sqrt{\left(1+\left|\alpha^{\prime}\right|^{2}\right)}}\left(|0\rangle+\alpha^{\prime}|1\rangle\right) \\
\left|\beta^{\prime}\right\rangle & =\frac{1}{\sqrt{\left(1+\left|\beta^{\prime}\right|^{2}\right)}}\left(|0\rangle+\beta^{\prime}|1\rangle\right)
\end{align*}
$$

are respectively the states for each qubit.
So, Eq. (3-28) becomes

$$
\begin{align*}
& \| \psi\rangle=\frac{\cos \theta}{\sqrt{\left.\left(1+|\alpha|^{2}\right)\left(1+|\beta|^{2}\right)\right)}}(|00\rangle+\beta|01\rangle+\alpha|10\rangle+\alpha \beta|11\rangle) \\
& +\frac{e^{i \phi} \sin \theta}{\sqrt{\left(1+\left|\alpha^{\prime}\right|^{2} \mid 2\left(1+\left.\left|\beta^{\prime}\right|\right|^{2}\right)\right)}}\left(|00\rangle+\beta^{\prime}|01\rangle+\alpha^{\prime}|10\rangle+\alpha^{\prime} \beta^{\prime}|11\rangle\right),  \tag{3-30}\\
& \quad=a|00\rangle+b|01\rangle+c|10\rangle+d|11\rangle
\end{align*}
$$

where

$$
\begin{gather*}
a=\lambda+\gamma \\
b=\beta \lambda+\beta^{\prime} \gamma \\
c=\alpha \lambda+\alpha^{\prime} \gamma  \tag{3-31}\\
d=\alpha \beta \lambda+\alpha^{\prime} \beta^{\prime} \gamma
\end{gather*}
$$

with

$$
\begin{equation*}
\lambda=\frac{\cos \theta}{\sqrt{\left(1+|\alpha|^{2}\right)\left(1+|\beta|^{2}\right)}}, \quad \gamma=\frac{e^{i \phi} \sin \theta}{\sqrt{\left(1+\left|\alpha^{\prime}\right|^{2}\right)\left(1+\left|\beta^{\prime}\right|^{2}\right)}} . \tag{3-32}
\end{equation*}
$$

Finally, the normalized pure state can be expressed in standard computational basis $\{|00\rangle,|01\rangle$, $|10\rangle,|11\rangle\}$ as

$$
\begin{equation*}
|\psi\rangle=\frac{1}{\sqrt{N}}(a|00\rangle+b|01\rangle+c|10\rangle+d|11\rangle) \tag{3-33}
\end{equation*}
$$

with

$$
\begin{equation*}
N=\langle\psi \mid \psi\rangle=|a|^{2}+|b|^{2}+|c|^{2}+|d|^{2} \tag{3-34}
\end{equation*}
$$

The concurrence of the state (3-33) is given in terms of the amplitudes of coherent states as

$$
\begin{equation*}
C(|\psi\rangle)=|\langle\psi \mid \tilde{\psi}\rangle|=2\left|\frac{\lambda \gamma}{N}\left(\alpha-\alpha^{\prime}\right)\left(\beta-\beta^{\prime}\right)\right| . \tag{3-35}
\end{equation*}
$$

The minimum of the concurrence $(C(|\psi\rangle)=0$, i.e., the state is separable) is attained in either of the following situations: $\alpha=\alpha^{\prime}$ or $\beta=\beta^{\prime}$ or $\lambda=0$ or $\gamma=0$. Furthermore, its maximum is satisfied when $C(|\psi\rangle)=1$ which corresponds to maximally entangled states.

### 3.2 The case of mixed state

The realization that entanglement is a resource for a number of useful tasks in quantum information is led to a tremendous interest in its properties, quantification, and in method by which it can be produced. In this way, we use a helpful method basing on spin coherent states for quantifying the entanglement of two-qubit mixed consisting of three pure states.

Let us consider a class of mixed states defined as a statistical mixture of three pure states of two-qubit system

$$
\begin{equation*}
\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right| \quad(i=1,2,3) \tag{3-36}
\end{equation*}
$$

where

$$
\begin{equation*}
\left|\psi_{i}\right\rangle=\frac{1}{\sqrt{N_{i}}}\left(a_{i}|00\rangle+b_{i}|01\rangle+c_{i}|10\rangle+d_{i}|11\rangle\right) \tag{3-37}
\end{equation*}
$$

represents the pure states of two-qubit system, with

$$
\begin{gather*}
a_{i}=\lambda_{i}+\gamma_{i} \\
b_{i}=\beta_{i} \lambda_{i}+\beta_{i}^{\prime} \gamma_{i} \\
c_{i}=\alpha_{i} \lambda_{i}+\alpha_{i}^{\prime} \gamma_{i}  \tag{3-38}\\
d_{i}=\alpha_{i} \beta_{i} \lambda_{i}+\alpha_{i}^{\prime} \beta_{i}^{\prime} \gamma_{i}
\end{gather*}
$$

and

$$
\begin{equation*}
N_{i}=\left\langle\psi_{i} \mid \psi_{i}\right\rangle=\left|a_{i}\right|^{2}+\left|b_{i}\right|^{2}+\left|c_{i}\right|^{2}+\left|d_{i}\right|^{2} \tag{3-39}
\end{equation*}
$$

The expression of the concurrence of the mixed state with three orthogonal states given by Eq. (2-18) can be directly generalized to the case of mixed state with three nonorthogonal states [38]. So, in our case we find that

$$
\begin{gather*}
C^{2}(\rho)=\left(p_{1}^{2} C_{1}^{2}+p_{2}^{2} C_{2}^{2}+p_{3}^{2} C_{3}^{2}\right) \\
+\frac{1}{2} p_{1} p_{2}\left|\mathbf{c}_{+}-\mathbf{c}_{-}\right|^{2}-\frac{1}{2} p_{1} p_{2}\left|\left(\mathbf{c}_{+}-\mathbf{c}_{-}\right)^{2}-4 \mathbf{c}_{1} \mathbf{c}_{2}\right| \\
+\frac{1}{2} p_{1} p_{3}\left|\mathbf{c}_{+}^{\prime}-\mathbf{c}_{-}^{\prime}\right|^{2}-\frac{1}{2} p_{1} p_{3}\left|\left(\mathbf{c}_{+}^{\prime}-\mathbf{c}_{-}^{\prime}\right)^{2}-4 \mathbf{c}_{1} \mathbf{c}_{3}\right|  \tag{3-40}\\
+\frac{1}{2} p_{2} p_{3}\left|\mathbf{c}_{+}^{\prime \prime}-\mathbf{c}_{-}^{\prime \prime}\right|^{2}-\frac{1}{2} p_{2} p_{3}\left|\left(\mathbf{c}_{+}^{\prime \prime}-\mathbf{c}_{-}^{\prime \prime}\right)^{2}-4 \mathbf{c}_{2} \mathbf{c}_{3}\right|,
\end{gather*}
$$

where

$$
\begin{equation*}
C_{i}=\left|\mathbf{c}_{i}\right|=2\left|\frac{\lambda_{i} \gamma_{i}}{N_{i}}\left(\alpha_{i}-\alpha_{i}^{\prime}\right)\left(\beta_{i}-\beta_{i}^{\prime}\right)\right| \tag{3-41}
\end{equation*}
$$

is the concurrence of the pure state $\left|\psi_{i}\right\rangle$,

$$
\begin{gather*}
C_{ \pm}=\left|\mathbf{c}_{ \pm}\right|=\frac{4}{3} \left\lvert\, \frac{\lambda_{1} \gamma_{1}}{N_{1}}\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\left(\beta_{1}-\beta_{1}^{\prime}\right)+\frac{\lambda_{2} \gamma_{2}}{N_{2}}\left(\alpha_{2}-\alpha_{2}^{\prime}\right)\left(\beta_{2}-\beta_{2}^{\prime}\right)+\frac{\lambda_{3} \gamma_{3}}{N_{3}}\left(\alpha_{3}-\alpha_{3}^{\prime}\right)\left(\beta_{3}-\beta_{3}^{\prime}\right)\right. \\
\pm \frac{1}{\sqrt{N_{1} N_{2}}}\left(\lambda_{1} \lambda_{2}\left(\alpha_{1}-\alpha_{2}\right)\left(\beta_{1}-\beta_{2}\right)+\lambda_{1} \gamma_{2}\left(\alpha_{1}-\alpha_{2}^{\prime}\right)\left(\beta_{1}-\beta_{2}^{\prime}\right)+\lambda_{2} \gamma_{1}\left(\alpha_{1}^{\prime}-\alpha_{2}\right)\left(\beta_{1}^{\prime}-\beta_{2}\right)\right. \\
\left.\quad+\gamma_{1} \gamma_{2}\left(\alpha_{1}^{\prime}-\alpha_{2}^{\prime}\right)\left(\beta_{1}^{\prime}-\beta_{2}^{\prime}\right)\right) \mid \\
\begin{array}{c}
C_{ \pm}^{\prime}=\left|\mathbf{c}_{ \pm}^{\prime}\right|=\frac{4}{3} \left\lvert\, \frac{\lambda_{1} \gamma_{1}}{N_{1}}\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\left(\beta_{1}-\beta_{1}^{\prime}\right)+\frac{\lambda_{2} \gamma_{2}}{N_{2}}\left(\alpha_{2}-\alpha_{2}^{\prime}\right)\left(\beta_{2}-\beta_{2}^{\prime}\right)+\frac{\lambda_{3} \gamma_{3}}{N_{3}}\left(\alpha_{3}-\alpha_{3}^{\prime}\right)\left(\beta_{3}-\beta_{3}^{\prime}\right)\right. \\
\pm \frac{1}{\sqrt{N_{1} N_{3}}}\left(\lambda_{1} \lambda_{3}\left(\alpha_{1}-\alpha_{3}\right)\left(\beta_{1}-\beta_{3}\right)+\lambda_{1} \gamma_{3}\left(\alpha_{1}-\alpha_{3}^{\prime}\right)\left(\beta_{1}-\beta_{3}^{\prime}\right)+\lambda_{3} \gamma_{1}\left(\alpha_{1}^{\prime}-\alpha_{3}\right)\left(\beta_{1}^{\prime}-\beta_{3}\right)\right. \\
\\
\left.\quad+\gamma_{1} \gamma_{3}\left(\alpha_{1}^{\prime}-\alpha_{3}^{\prime}\right)\left(\beta_{1}^{\prime}-\beta_{3}^{\prime}\right)\right) \mid
\end{array} \tag{3-42}
\end{gather*}
$$

and

$$
\begin{gather*}
C_{ \pm}^{\prime \prime}=\left|\mathbf{c}_{ \pm}^{\prime \prime}\right|=\frac{4}{3} \left\lvert\, \frac{\lambda_{1} \gamma_{1}}{N_{1}}\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\left(\beta_{1}-\beta_{1}^{\prime}\right)+\frac{\lambda_{2} \gamma_{2}}{N_{2}}\left(\alpha_{2}-\alpha_{2}^{\prime}\right)\left(\beta_{2}-\beta_{2}^{\prime}\right)+\frac{\lambda_{3} \gamma_{3}}{N_{3}}\left(\alpha_{3}-\alpha_{3}^{\prime}\right)\left(\beta_{3}-\beta_{3}^{\prime}\right)\right. \\
\pm \frac{1}{\sqrt{N_{2} N_{3}}}\left(\lambda_{2} \lambda_{3}\left(\alpha_{2}-\alpha_{3}\right)\left(\beta_{2}-\beta_{3}\right)+\lambda_{2} \gamma_{3}\left(\alpha_{2}-\alpha_{3}^{\prime}\right)\left(\beta_{2}-\beta_{3}^{\prime}\right)+\lambda_{3} \gamma_{2}\left(\alpha_{2}^{\prime}-\alpha_{3}\right)\left(\beta_{2}^{\prime}-\beta_{3}\right)\right. \\
\left.+\gamma_{2} \gamma_{3}\left(\alpha_{2}^{\prime}-\alpha_{3}^{\prime}\right)\left(\beta_{2}^{\prime}-\beta_{3}^{\prime}\right)\right) \mid . \tag{3-44}
\end{gather*}
$$

The simplified expression of concurrence of the mixed state reveals some general important features:
A. The concurrence has an upper and lower bound expressed as

$$
\begin{equation*}
\left(p_{1} C_{1}-p_{2} C_{2}-p_{3} C_{3}\right)^{2} \leq C^{2}(\rho) \leq\left(p_{1} C_{1}+p_{2} C_{2}+p_{3} C_{3}\right)^{2} \tag{3-45}
\end{equation*}
$$

where

$$
\begin{gather*}
\left(p_{1} C_{1}-p_{2} C_{2}-p_{3} C_{3}\right)^{2}=4\left(p_{1}\left|\frac{\lambda_{1} \gamma_{1}}{N_{1}}\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\left(\beta_{1}-\beta_{1}^{\prime}\right)\right|\right. \\
\quad-p_{2}\left|\frac{\lambda_{2} \gamma_{2}}{N_{2}}\left(\alpha_{2}-\alpha_{2}^{\prime}\right)\left(\beta_{2}-\beta_{2}^{\prime}\right)\right|  \tag{3-46}\\
\left.-p_{3}\left|\frac{\lambda_{3} \gamma_{3}}{N_{3}}\left(\alpha_{3}-\alpha_{3}^{\prime}\right)\left(\beta_{3}-\beta_{3}^{\prime}\right)\right|\right)^{2}
\end{gather*}
$$

and

$$
\begin{gather*}
\left(p_{1} C_{1}+p_{2} C_{2}+p_{3} C_{3}\right)^{2}=4\left(p_{1}\left|\frac{\lambda_{1} \gamma_{1}}{N_{1}}\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\left(\beta_{1}-\beta_{1}^{\prime}\right)\right|\right. \\
+p_{2}\left|\frac{\lambda_{2} \gamma_{2}}{N_{2}}\left(\alpha_{2}-\alpha_{2}^{\prime}\right)\left(\beta_{2}-\beta_{2}^{\prime}\right)\right|  \tag{3-47}\\
\left.+p_{3}\left|\frac{\lambda_{3} \gamma_{3}}{N_{3}}\left(\alpha_{3}-\alpha_{3}^{\prime}\right)\left(\beta_{3}-\beta_{3}^{\prime}\right)\right|\right)^{2}
\end{gather*}
$$

are the lower and upper bounds of concurrence, respectively.
B. If the three pure states consisting the mixed state have only real components (i.e., $a_{i}, b_{i}, c_{i}$ and $d_{i}$ are real numbers), then:

- for

$$
\begin{align*}
0 & \leq 4 \mathbf{c}_{1} \mathbf{c}_{2} \leq\left(\mathbf{c}_{+}-\mathbf{c}_{-}\right)^{2} \\
0 & \leq 4 \mathbf{c}_{1} \mathbf{c}_{3} \leq\left(\mathbf{c}_{+}^{\prime}-\mathbf{c}_{-}^{\prime}\right)^{2}  \tag{3-48}\\
0 & \leq 4 \mathbf{c}_{2} \mathbf{c}_{3} \leq\left(\mathbf{c}_{+}^{\prime \prime}-\mathbf{c}_{-}^{\prime \prime}\right)^{2}
\end{align*}
$$

the concurrence reaches the upper bound

$$
\begin{equation*}
C^{2}(\rho)=4\left(p_{1}\left|\frac{\lambda_{1} \gamma_{1}}{N_{1}}\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\left(\beta_{1}-\beta_{1}^{\prime}\right)\right|+p_{2}\left|\frac{\lambda_{2} \gamma_{2}}{N_{2}}\left(\alpha_{2}-\alpha_{2}^{\prime}\right)\left(\beta_{2}-\beta_{2}^{\prime}\right)\right|+p_{3}\left|\frac{\lambda_{3} \gamma_{3}}{N_{3}}\left(\alpha_{3}-\alpha_{3}^{\prime}\right)\left(\beta_{3}-\beta_{3}^{\prime}\right)\right|\right)^{2} \tag{3-49}
\end{equation*}
$$

- for

$$
\begin{align*}
& 4 \mathbf{c}_{1} \mathbf{c}_{2} \geq\left(\mathbf{c}_{+}-\mathbf{c}_{-}\right)^{2} \geq 0 \\
& 4 \mathbf{c}_{1} \mathbf{c}_{3} \geq\left(\mathbf{c}_{+}^{\prime}-\mathbf{c}_{-}^{\prime}\right)^{2} \geq 0  \tag{3-50}\\
& 4 \mathbf{c}_{2} \mathbf{c}_{3} \geq\left(\mathbf{c}_{+}^{\prime \prime}-\mathbf{c}_{-}^{\prime \prime}\right)^{2} \geq 0
\end{align*}
$$

the concurrence is as follows

$$
\begin{gather*}
C^{2}(\rho)=4\left(p_{1}\left|\frac{\lambda_{1} \gamma_{1}}{N_{1}}\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\left(\beta_{1}-\beta_{1}^{\prime}\right)\right|-p_{2}\left|\frac{\lambda_{2} \gamma_{2}}{N_{2}}\left(\alpha_{2}-\alpha_{2}^{\prime}\right)\left(\beta_{2}-\beta_{2}^{\prime}\right)\right|-p_{3}\left|\frac{\lambda_{3} \gamma_{3}}{N_{3}}\left(\alpha_{3}-\alpha_{3}^{\prime}\right)\left(\beta_{3}-\beta_{3}^{\prime}\right)\right|\right)^{2} \\
+L_{\alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2}}^{\alpha_{1}^{\prime}, \alpha_{2}^{\prime}, \beta_{1}^{\prime}, \beta_{2}^{\prime}}+M_{\alpha_{1}, \alpha_{3}, \beta_{1}, \beta_{3}}^{\alpha_{1}^{\prime}, \alpha_{3}^{\prime}, \beta_{1}^{\prime}, \beta_{3}^{\prime}}+N_{\alpha_{2}, \alpha_{3}, \beta_{2}, \beta_{3}}^{\alpha_{2}^{\prime}, \alpha_{3}^{\prime}, \beta_{3}^{\prime}, \beta_{3}^{\prime}} \tag{3-51}
\end{gather*}
$$

where

$$
\begin{gather*}
L_{\alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2}}^{\alpha_{1}^{\prime},,_{2}^{\prime}, \beta_{1}^{\prime}, \beta_{2}^{\prime}}=\frac{64 p_{1} p_{2}}{9 N_{1} N_{2}}\left(\lambda_{1} \lambda_{2}\left(\alpha_{1}-\alpha_{2}\right)\left(\beta_{1}-\beta_{2}\right)+\lambda_{1} \gamma_{2}\left(\alpha_{1}-\alpha_{2}^{\prime}\right)\left(\beta_{1}-\beta_{2}^{\prime}\right)+\lambda_{2} \gamma_{1}\left(\alpha_{1}^{\prime}-\alpha_{2}\right)\left(\beta_{1}^{\prime}-\beta_{2}\right)\right. \\
\left.+\gamma_{1} \gamma_{2}\left(\alpha_{1}^{\prime}-\alpha_{2}^{\prime}\right)\left(\beta_{1}^{\prime}-\beta_{2}^{\prime}\right)\right)^{\frac{1}{2}} \\
M_{\alpha_{1}, \alpha_{3}, \beta_{1}, \beta_{3}}^{\alpha_{1}^{\prime}, \alpha_{3}^{\prime}, \beta_{1}^{\prime}, \beta_{3}^{\prime}}=\frac{64 p_{1} p_{3}}{9 N_{1} N_{3}}\left(\lambda_{1} \lambda_{3}\left(\alpha_{1}-\alpha_{3}\right)\left(\beta_{1}-\beta_{3}\right)+\lambda_{1} \gamma_{3}\left(\alpha_{1}-\alpha_{3}^{\prime}\right)\left(\beta_{1}-\beta_{3}^{\prime}\right)+\lambda_{3} \gamma_{1}\left(\alpha_{1}^{\prime}-\alpha_{3}\right)\left(\beta_{1}^{\prime}-52\right)\right. \\
\left.+\beta_{3}\right) \\
\left.\gamma_{3}\left(\alpha_{1}^{\prime}-\alpha_{3}^{\prime}\right)\left(\beta_{1}^{\prime}-\beta_{3}^{\prime}\right)\right)^{\frac{1}{2}} \\
N_{\alpha_{2}, \alpha_{3}, \beta_{2}, \beta_{3}}^{\alpha_{2}^{\prime}, \alpha_{3}^{\prime}, \beta_{2}^{\prime}, \beta_{3}^{\prime}}=\frac{64 p_{2} p_{3}}{9 N_{2} N_{3}}\left(\lambda_{2} \lambda_{3}\left(\alpha_{2}-\alpha_{3}\right)\left(\beta_{2}-\beta_{3}\right)+\lambda_{2} \gamma_{3}\left(\alpha_{2}-\alpha_{3}^{\prime}\right)\left(\beta_{2}-\beta_{3}^{\prime}\right)+\lambda_{3} \gamma_{2}\left(\alpha_{2}^{\prime}-\alpha_{3}\right)\left(\beta_{2}^{\prime}-\beta_{3}\right)\right.  \tag{3-54}\\
\left.+\gamma_{2} \gamma_{3}\left(\alpha_{2}^{\prime}-\alpha_{3}^{\prime}\right)\left(\beta_{2}^{\prime}-\beta_{3}^{\prime}\right)\right)^{\frac{1}{2}}
\end{gather*}
$$

- for

$$
\begin{align*}
\mathbf{c}_{1} \mathbf{c}_{2} & \leq 0 \\
\mathbf{c}_{1} \mathbf{c}_{3} & \leq 0  \tag{3-55}\\
\mathbf{c}_{2} \mathbf{c}_{3} & \leq 0,
\end{align*}
$$

the concurrence is equivalent to lower bound

$$
\begin{equation*}
C^{2}(\rho)=4\left(p_{1}\left|\frac{\lambda_{1} \gamma_{1}}{N_{1}}\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\left(\beta_{1}-\beta_{1}^{\prime}\right)\right|-p_{2}\left|\frac{\lambda_{2} \gamma_{2}}{N_{2}}\left(\alpha_{2}-\alpha_{2}^{\prime}\right)\left(\beta_{2}-\beta_{2}^{\prime}\right)\right|-p_{3}\left|\frac{\lambda_{3} \gamma_{3}}{N_{3}}\left(\alpha_{3}-\alpha_{3}^{\prime}\right)\left(\beta_{3}-\beta_{3}^{\prime}\right)\right|\right)^{2} \tag{3-56}
\end{equation*}
$$

C. When

$$
\begin{align*}
\mathbf{c}_{+} & =\mathbf{c}_{-} \\
\mathbf{c}_{+}^{\prime} & =\mathbf{c}_{-}^{\prime}  \tag{3-57}\\
\mathbf{c}_{+}^{\prime \prime} & =\mathbf{c}_{-}^{\prime \prime}
\end{align*}
$$

the concurrence reaches again the lower bound

$$
\begin{equation*}
C^{2}(\rho)=4\left(p_{1}\left|\frac{\lambda_{1} \gamma_{1}}{N_{1}}\left(\alpha_{1}-\alpha_{1}^{\prime}\right)\left(\beta_{1}-\beta_{1}^{\prime}\right)\right|-p_{2}\left|\frac{\lambda_{2} \gamma_{2}}{N_{2}}\left(\alpha_{2}-\alpha_{2}^{\prime}\right)\left(\beta_{2}-\beta_{2}^{\prime}\right)\right|-p_{3}\left|\frac{\lambda_{3} \gamma_{3}}{N_{3}}\left(\alpha_{3}-\alpha_{3}^{\prime}\right)\left(\beta_{3}-\beta_{3}^{\prime}\right)\right|\right)^{2} \tag{3-58}
\end{equation*}
$$

We remark that we can obtain the concurrence for many other cases.
D. We take the case where two pure states of mixed state are separable (i.e, $C_{1}=C_{2}=0$, $C_{1}=C_{3}=0$ or $C_{2}=C_{3}=0$ ). Then, the concurrence of the mixed state becomes as

$$
\begin{equation*}
C^{2}(\rho)=4 p_{i}^{2}\left|\frac{\lambda_{i} \gamma_{i}}{N_{i}}\left(\alpha_{i}-\alpha_{i}^{\prime}\right)\left(\beta_{i}-\beta_{i}^{\prime}\right)\right|^{2} \tag{3-59}
\end{equation*}
$$

( $i=1$ for $C_{2}=C_{3}=0 ; i=2$ for $C_{1}=C_{3}=0$ and $i=3$ for $C_{1}=C_{2}=0$ ).
We notice that from the result obtained in section (3.1), the pure states $\left|\psi_{i}\right\rangle(i=1,2,3)$ are separable when $\alpha_{i}=\alpha_{i}^{\prime}$ or $\beta_{i}=\beta_{i}^{\prime}$ or $\lambda_{i}=0$ or $\gamma_{i}=0$.
The above equation shows that the pure state $\left|\psi_{1}\right\rangle$ (respectively $\left|\psi_{2}\right\rangle$ or $\left|\psi_{3}\right\rangle$ ) and its probability $p_{1}$ (respectively $p_{2}$ or $p_{3}$ ) contain the vital information about the entanglement of two-qubit mixed state.
Without loss of generality, we consider the simple case where $\alpha_{i}=\beta_{i}$ and $\alpha_{i}^{\prime}=\beta_{i}^{\prime}$, then the concurrence is simplified as [37]

$$
\begin{gather*}
C^{2}(\rho)=p_{i}^{2} \frac{\left(\alpha_{i}-\alpha_{i}^{\prime}\right)^{4}}{\left(2 \alpha_{i}^{2} \alpha_{i}^{\prime 2}+\alpha_{i}^{\prime 2}+2 \alpha_{i} \alpha_{i}^{\prime}+\alpha_{i}^{2}+2\right)^{2}} \\
=\left(\frac{p_{i}}{1+2 X_{i}}\right)^{2} \tag{3-60}
\end{gather*}
$$

where

$$
\begin{equation*}
X_{i}=\left(\frac{\alpha_{i} \alpha_{i}^{\prime}+1}{\alpha_{i}-\alpha_{i}^{\prime}}\right)^{2} \in[0, \infty[. \tag{3-61}
\end{equation*}
$$

Now we discuss two important limit cases:

1. $X_{i}=0\left(\alpha_{i}=\frac{-1}{\alpha_{i}^{\prime}}\right)$, i.e., the state $\left|\psi_{i}\right\rangle$ is a Bell state, the two-qubit state

$$
\rho=p_{1}\left|\psi_{1}\right\rangle\left\langle\psi_{1}\right|+p_{2}\left|\psi_{2}\right\rangle\left\langle\psi_{2}\right|+p_{3}\left|\psi_{3}\right\rangle\left\langle\psi_{3}\right|
$$

is a statistical mixture of a Bell state and two separable pure states which corresponds to $C^{2}(\rho)=p_{i}^{2}$. These states represent an important class of mixed states which are widely used and applied in different quantum information processing and transmission tasks and have some applications.
2. $X_{i} \rightarrow \infty\left(\alpha_{i}=\alpha_{i}^{\prime}\right)$, i.e., the state $\left|\psi_{i}\right\rangle$ is separable, the state

$$
\rho=p_{1}\left|\psi_{1}\right\rangle\left\langle\psi_{1}\right|+p_{2}\left|\psi_{2}\right\rangle\left\langle\psi_{2}\right|+p_{3}\left|\psi_{3}\right\rangle\left\langle\psi_{3}\right|
$$

is a classical mixture of three separable pure states which corresponds to separable mixed state $C(\rho)=0$.
An important case is merit notifying: the completely mixed state for which the density operator is simply $\rho=\frac{I}{d}$ where d is dimension of the Hilbert space (in our case $\rho=\frac{I}{4}$ ), this corresponds to $p_{1}=p_{2}=p_{3}=\frac{1}{3}$. Plotting $C^{2}(\rho)$ as a function of X and P and also $C^{2}(\rho)$ as a function of $\alpha$ and $\alpha^{\prime}$ (Fig1. (a) and (b)), we see that the maximal value of the concurrence $C(\rho)$ is $\frac{1}{3}$ rather than 1 as for the pure state. This concept can be explained by the fact that in a completely mixed state, the concurrence is equally shared by the subsystems.
By comparing these plots with those indicated in Ref [36], we notice that square of concurrence is reduced from 0.25 to 0.11 (or in other words) by increasing the number of two-qubit pure states from two to three.

## 4 Conclusion

In this paper, we have studied the entanglement of two-qubit states, our approach was to write the measure in terms of the amplitudes of $S U(2)$ coherent states. As a measure of entanglement, we have used the concurrence. We expressed it as a function of the amplitudes of coherent states and we have given the sufficient conditions for the minimal and maximal of entanglement in the case of two-qubit nonorthogonal pure states. By determining a simplified
expression of concurrence for a two-qubit mixed state having no more than three non-zero eigenvalues, we have generalized the formalism of two-qubit nonorthogonal pure states to the case of a class of mixed states. However, we have studied the behavior of the square of the two-qubit mixed state concurrence where their conditions were depending on both of the amplitudes and corresponding probabilities.
By studying a simple case, we found that the concurrence of the mixed state cannot be higher than the probability of one of the qubits. Furthermore, for the completely mixed state it cannot exceed one third.
In this way, it is shown that the $S U(2)$ coherent states are useful elements to determine and measure the entanglement of two-qubit states and their use is not only of the theoretical purpose but also of some practical importance having in mind their experimental accessibility [39]. The two-qubit nonorthogonal states are expected to have more applications in quantum information theory. Throughout the paper we have only considered the bipartite entanglement. The more difficult task is to quantify the genuine multipartite entanglement. In this context, We intend to use and generalize the result to the case of various qubits and consider possible applications in quantum information.

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## Figure Captions

Fig1: (a) Schema of $C^{2}(\rho)$ as a function of X and P. (b) Schema of $C^{2}(\rho)$ as a function of $\alpha$ and $\alpha^{\prime}$ for $P_{1}=P_{2}=P_{3}=\frac{1}{3},-5 \leq \alpha \leq 5$ and $-5 \leq \alpha^{\prime} \leq 5$.


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