Birman-Wenzl-Murakami Algebra, Topological parameter and Berry phase

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In this paper, a 3×3 -matrix representation of Birman-Wenzl-Murakami(BWM) algebra has been presented. Based on which, unitary matrices $A(\theta, \varphi_1, \varphi_2)$ and $B(\theta, \varphi_1, \varphi_2)$ are generated via Yang-Baxterization approach. A Hamiltonian is constructed from the unitary $B(\theta, \varphi)$ matrix. Then we study Berry phase of the Yang-Baxter system, and obtain the relationship between topological parameter and Berry phase.

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I. INTRODUCTION

To the best of our knowledge, the Yang-Baxter equation (YBE) was initiated in solving the one-dimensional δ interecting models [1] and the statistical models [2]. Braid algebra and Temperley-Lieb algebra (TLA) [3] have been widely used in the construction of YBE solutions [4–9] and have been introduced to the field of quantum information, quantum computation and topological computation [10–15]. The Birman-Wenzl-Murakami (BWM) algebra [16] which contain two subalgebra (Braid algebra and TLA) was first defined and independently studied by Birman, Wenzl and Murakami. Very recently [17], S.Abramsky demonstrate the connections from knot theory to logic and computation via quantum mechanics. But, the physical meaning of the topological parameter d (describing the single loop in topology) is still unclear.

The geometrical phase [18], such as Berry phase, is an important concept in quantum mechanics [19–24]. In recent years, numerous works have been attributed to Berry phase [25], because of its possible applications to quantum computation (the so-called geometric quantum computation) [26–29]. Quantum logic gates based on geometric phases have been certified in both nuclear magnetic resonance [26] and ion trap based on quantum information architectures [30]. The Ref. [26] pointed out geometric phases have potential fault tolerance when applied to quantum information processing. In 2007, Leek, P.J. *et al.* [31] illustrated the controlled accumulation of a geometric phase, Berry phase, in a superconducting qubit.

The Ref. [32] applied TLA as a bridge to recast 4-dimensional YBE into its 2-dimensional counterpart. The 2-dimensional YBE have an important application value in topological quantum computation [33, 34]. To date, few studies have reported 3-dimensional YBE which may have potential application values in topological quantum computation. The motivation of this paper is twofold: one is that we structure 3-dimensional YBE, the other is to study the physical meaning of topological parameter d from Berry phase. This paper is organized as follows: In Sec. 2, we introduce a specialized type BWM algebra, and present a 3×3 -matrix representation of BWM algebra. In Sec. 3, we obtain unitary matrices $A(\theta, \varphi_1, \varphi_2), B(\theta, \varphi_1, \varphi_2)$ via Yang-Baxterization approach. Based on the solution, a Hamiltonian of the Yang-Baxter system is constructed, finally we study the Berry phase of this system. We end with a summary.

II. BWM ALGEBRA

As we know the Braid relations are

$$\begin{cases} b_i b_{i\pm 1} b_i = b_{i\pm 1} b_i b_{i\pm 1}, \\ b_i b_j = b_j b_i, \ |i-j| \ge 2, \end{cases}$$
(1)

where $b_i = \stackrel{1}{I} \otimes \cdots \otimes \stackrel{i-1}{I} \otimes b \otimes \stackrel{i+2}{I} \otimes \cdots$. When we just consider three tensor product space, the Braid relations becomes

$$b_{12}b_{23}b_{12} = b_{23}b_{12}b_{23},\tag{2}$$

where $b_{12} = b \otimes I$, $b_{23} = I \otimes b$, b-matrix is a $N^2 \times N^2$ matrix acted on the tensor product space $\nu \otimes \nu$, where N is the dimension of ν . It is also well known that the braid relation can be reduced to a N-dimensional braid relation $(b_{12} \to A, b_{23} \to B)$

$$ABA = BAB. \tag{3}$$

Like this reduced method, we easily obtain N-dimensional reduced BWM-algebra relations from classical BWMalgebra relations. The BWM algebra [16, 35–37] is generated by the unit I, the braid operators S_i and the TLA operators E_i and depends on two independent parameters ω and σ . Let us take the BWM relations as follows.

$$S_{i} - S_{i}^{-1} = \omega(I - E_{i}),$$

$$S_{i}S_{i\pm1}S_{i} = S_{i\pm1}S_{i}S_{i\pm1}, \ S_{i}S_{j} = S_{j}S_{i}, |i - j| \ge 2,$$

$$E_{i}E_{i\pm1}E_{i} = E_{i}, \ E_{i}E_{j} = E_{j}E_{i}, \ |i - j| \ge 2,$$

$$E_{i}S_{i} = S_{i}E_{i} = \sigma E_{i},$$

$$S_{i\pm1}S_{i}E_{i\pm1} = E_{i}S_{i\pm1}S_{i} = E_{i}E_{i\pm1},$$

$$S_{i\pm1}E_{i}S_{i\pm1} = S_{i}^{-1}E_{i\pm1}S_{i}^{-1},$$

$$E_{i\pm1}E_{i}S_{i\pm1} = E_{i\pm1}S_{i}^{-1}, \ S_{i\pm1}E_{i}E_{i\pm1} = S_{i}^{-1}E_{i\pm1},$$

$$E_{i}S_{i\pm1}E_{i} = \sigma^{-1}E_{i},$$

$$E_{i}^{2} = \left(1 - \frac{\sigma - \sigma^{-1}}{\omega}\right)E_{i},$$
(4)

where $0 \neq d = \left(1 - \frac{\sigma - \sigma^{-1}}{\omega}\right) \in \mathbb{C}$ is a topological parameter in knot theory which does not depend on the sites of the lattices.

By reducing to the N-dimensional space $(S_{12} \rightarrow A, S_{23} \rightarrow B, E_{12} \rightarrow E_A, E_{23} \rightarrow E_B)$, we have:

$$A - A^{-1} = \omega(I - E_A), \ B - B^{-1} = \omega(I - E_B),$$

$$ABA = BAB,$$

$$E_A E_B E_A = E_A, \ E_B E_A E_B = E_B,$$

$$E_A A = AE_A = \sigma E_A, \ E_B B = BE_B = \sigma E_B,$$

$$ABE_A = E_B AB = E_B E_A, \ BAE_B = E_A BA = E_A E_B,$$

$$AE_B A = B^{-1} E_A B^{-1}, \ BE_A B = A^{-1} E_B A^{-1},$$

$$E_A E_B A = E_A B^{-1}, \ E_B E_A B = E_B A^{-1},$$

$$AE_B E_A = B^{-1} E_A, \ BE_A E_B = A^{-1} E_B,$$

$$E_A BE_A = \sigma^{-1} E_A, \ E_B AE_B = \sigma^{-1} E_B,$$

$$E_A BE_A = \sigma^{-1} E_A, \ E_B AE_B = \sigma^{-1} E_B,$$

$$E_A^2 = (1 - \frac{\sigma - \sigma^{-1}}{\omega})E_A, \ E_B^2 = (1 - \frac{\sigma - \sigma^{-1}}{\omega})E_B,$$
(5)

where A, B satisfy the N-dimensional Braid relation (3), E_A, E_B satisfy the N-dimensional TLA relations

$$\begin{cases} E_A E_B E_A = E_A, \ E_B E_A E_B = E_B, \\ E_A^2 = dE_A, \ E_B^2 = dE_B, \end{cases}$$
(6)

It is interesting that Eq.(4) and Eq.(5) have the same topological parameter d.

In this paper, the A-matrix, B-matrix, E_a -matrix, E_b -matrix, A(x)-matrix and B(x)-matrix are 3×3 matrices acting on the 3-dimensional space. To the TLA relations (6), we assume E_A and E_B possess the same eigenvalues dand 0. We assume E_A is a diagonal matrix as following

$$E_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
 (7)

After tedious calculation, we obtain

$$E_B = \begin{pmatrix} \frac{d^2 - d - 1}{d} & \frac{\sqrt{d^2 - d - 1}}{d} e^{i\varphi_1} & -\frac{\sqrt{d^2 - d - 1}}{\sqrt{d}} e^{i(\varphi_1 + \varphi_2)} \\ \frac{\sqrt{d^2 - d - 1}}{d} e^{-i\varphi_1} & \frac{1}{d} & -\frac{e^{i\varphi_2}}{\sqrt{d}} \\ -\frac{\sqrt{d^2 - d - 1}}{\sqrt{d}} e^{-i(\varphi_1 + \varphi_2)} & -\frac{e^{-i\varphi_2}}{\sqrt{d}} & 1 \end{pmatrix}.$$
 (8)

It is worth to mention that $E_B = U E_A U^{-1}$, and U is a unitary transformation matrix as follows

$$U = \begin{pmatrix} \frac{1}{(d-1)d} & -\frac{\sqrt{d^2 - d - 1}}{d} e^{i\varphi_1} & -\frac{\sqrt{d^2 - d - 1}}{\sqrt{d}(d-1)} e^{i(\varphi_1 + \varphi_2)} \\ \frac{\sqrt{d^2 - d - 1}}{d} e^{-i\varphi_1} & -\frac{1}{d} & \frac{e^{i\varphi_2}}{\sqrt{d}} \\ \frac{\sqrt{d^2 - d - 1}}{\sqrt{d}(d-1)} e^{-i(\varphi_1 + \varphi_2)} & \frac{e^{-i\varphi_2}}{\sqrt{d}} & -\frac{d - 2}{d-1} \end{pmatrix},$$
(9)

where d, φ_1 and φ_2 are reals. The parameter d is the so-called topological parameter. For simplicity, we just consider the case of d > 0 in this paper.

The Ref. [35] has explored S_i have 3 different eigenvalues $(q, -q^{-1}, q^{-2})$ in the BWM-algebra (*i.e.* Eq.(4)). The same as E_A and E_B , we assume A and B have the same eigenvalues $(q, -q^{-1}, q^{-2})$. The simplest A is

$$A = \begin{pmatrix} q & 0 & 0 \\ 0 & q^{-2} & 0 \\ 0 & 0 & -q^{-1} \end{pmatrix},$$
(10)

using the unitary transformation matrix U, we have

$$B = UAU^{-1} = \begin{pmatrix} \frac{1}{q^4(d-1)d} & \frac{\sqrt{d^2 - d - 1}}{dq} e^{i\varphi_1} & -\frac{\sqrt{d^2 - d - 1}}{q^2(d-1)\sqrt{d}} e^{i(\varphi_1 + \varphi_2)} \\ \frac{\sqrt{d^2 - d - 1}}{\frac{dq}{2}} e^{-i\varphi_1} & \frac{q^2}{d} & \frac{q}{\sqrt{d}} e^{i\varphi_2} \\ -\frac{\sqrt{d^2 - d - 1}}{q^2(d-1)\sqrt{d}} e^{-i(\varphi_1 + \varphi_2)} & \frac{q}{\sqrt{d}} e^{-i\varphi_2} & \frac{d-2}{d-1} \end{pmatrix},$$
(11)

where $d = q^{-1} + 1 + q$ and the parameter q is real. The matrices A and B satisfy the braid relation (*i.e.* Eq(3)). Towards braid relation, in some models φ_i , (i = 1, 2), may have a physical significance of magnetic flux. In the paper[13], it has been shown the parameters φ_i 's are related to Berry phase.

Then we can verify that $\{I, A, E_A, B, E_B\}$ satisfy the reduced BWM-algebra (*i.e.* Eq.(5)), with $d = q^{-1} + 1 + q$. Here we have set $\omega = q - q^{-1}$ and $\sigma = q^{-2}$. It is interesting that A, B, E_A, E_B are Hermitian matrices, and have the same similar transformation $B = UAU^{-1}, E_B = UE_AU^{-1}$, where U is unitary (*i.e.* $U^{\dagger} = U^{-1}$).

III. YANG-BAXTERIZATION, HAMILTONIAN, BERRY PHASE

In this section, A Hamiltonian is constructed from the unitary $B(\theta, \varphi)$ matrix. Then we study the Berry phase of the Yang-Baxter system, and obtain the relationship between the topological parameter and the Berry phase. We first explain the basic formula of YBE. The Yang-Baxter matrix \check{R} is a $N^2 \times N^2$ matrix acting on the tensor product space $\nu \otimes \nu$, where N is the dimension of ν . Such a matrix \check{R} satisfies the relativistic YBE[32] as follows

$$\check{R}_{12}(u)\check{R}_{23}(\frac{u+v}{1+\beta^2 uv})\check{R}_{12}(v) = \check{R}_{23}(v)\check{R}_{12}(\frac{u+v}{1+\beta^2 uv})\check{R}_{23}(u).$$
(12)

In this paper, we focus on 3-dimensional space. The reduced relativistic YBE reads

$$A(u)B(\frac{u+v}{1+\beta^2 uv})A(v) = B(v)A(\frac{u+v}{1+\beta^2 uv})B(u).$$
(13)

Let the unitary Yang-Baxter matrix take the form

$$\begin{cases}
A(u) = \rho(u)(I + F(u)E_A) \\
B(u) = \rho(u)(I + F(u)E_B).
\end{cases}$$
(14)

Following Xue *et al.* [38], we obtain

$$\begin{cases} F(u) = \frac{e^{-2i\theta} - 1}{d}, \\ e^{-2i\theta} = \frac{\beta^2 u^2 + 2i\varepsilon\beta u\sqrt{d^2/(4 - d^2)} + 1}{\beta^2 u^2 - 2i\varepsilon\beta u\sqrt{d^2/(4 - d^2)} + 1}, \end{cases}$$
(15)

where the new parameter θ is real. Let $\rho(u) = e^{i\theta}$. The Yang-Baxter matrix can be rewritten in the following form

$$\begin{cases} A(\theta, \varphi_1, \varphi_2) = e^{i\theta}I - f(\theta)E_A, \\ B(\theta, \varphi_1, \varphi_2) = e^{i\theta}I - f(\theta)E_B, \end{cases}$$
(16)

where $f(\theta) = 2i \sin \theta / d$.

The Yang-Baxter matrix depends on three parameters: the first is θ (θ is time-independent); the others are φ_i , (i = 1, 2) contained in the matrix E. In physics the parameter φ_1 and φ_2 are flux which depends on time t. Usually take $\varphi_i = \omega_i t$, (i = 1, 2) and ω_i are the frequency. Operators $A(\theta, \varphi_1, \varphi_2)$ and $B(\theta, \varphi_1, \varphi_2)$, satisfying $B(\theta, \varphi_1, \varphi_2) = UA(\theta, \varphi_1, \varphi_2)U^{-1}$, are unitary operators $(A(\theta, \varphi_1, \varphi_2)^{\dagger} = A(\theta, \varphi_1, \varphi_2)^{-1}, B(\theta, \varphi_1, \varphi_2)^{\dagger} = B(\theta, \varphi_1, \varphi_2)^{-1})$.

To simplify the following discussion, we will restrict attention to the case $\varphi_1 = -\varphi_2 = \varphi$. Following Ge *et al.* [13], we can obtain Yang Baxter Hamiltonian through the Schrödinger evolution of the states

$$\hat{H} = i\hbar \frac{\partial B(\theta,\varphi)}{\partial t} B^{\dagger}(\theta,\varphi), \qquad (17)$$

where φ be time dependent as $\varphi = \omega t$ and θ be time independent.

For convenience, we introduce the Gell-Mann matrices $I_{\lambda}[39]$, a basis for su(3) algebra. Such matrices satisfy $[I_{\lambda}, I_{\mu}] = i f_{\lambda\mu\nu} I_{\nu}, (\lambda, \mu, \nu = 1, 2, ..., 8)$, where $f_{\lambda\mu\nu}$ are the structure constants of su(3). We denote $I_{\pm} = I_1 \pm i I_2$, $V_{\pm} = I_4 \mp i I_5, U_{\pm} = I_6 \pm i I_7$ and $Y = \frac{2}{\sqrt{3}} I_8$. Let

$$\begin{cases} S_{+} = \zeta (-i(d^{2} - d - 1)^{1/2}(e^{-i\theta} + 2i\sin\theta d^{-2})I_{+} + id^{1/2}(e^{-i\theta} + 2i\sin\theta d^{-2})U_{-}), \\ S_{-} = \zeta (i(d^{2} - d - 1)^{1/2}(e^{i\theta} - 2i\sin\theta d^{-2})I_{-} - id^{1/2}(e^{i\theta} - 2i\sin\theta d^{-2})U_{+}), \\ S_{3} = \frac{1}{2} [(1 + d - d^{2})(1 - d^{2})^{-1}(\frac{I}{3} + \frac{Y}{2} + I_{3}) - (\frac{I}{3} + \frac{Y}{2} - I_{3}) - d(1 - d^{2})^{-1}(\frac{I}{3} - Y) \\ + d^{1/2}(d^{2} - d - 1)^{1/2}(1 - d^{2})^{-1}(V_{-} + V_{+})], \end{cases}$$
(18)

where $\zeta = \frac{d^2}{\sqrt{(d^2-1)(d^4-4(d^2-1)\sin^2\theta)}}$. These operators satisfy the su(2) algebra relations $([S_+, S_-] = 2S_3, [S_3, S_{\pm}] = \pm S_{\pm}, (S_{\pm})^2 = 0, S_{\pm} = S_1 \pm iS_2).$

In terms of the operators (18), the Hamiltonian Eq.(17) can be recast as following

$$\hat{H} = -4\omega\hbar\sin\theta(d^2 - 1)^{1/2}d^{-2}(\sin\alpha\cos\beta S_1 + \sin\alpha\sin\beta S_2 + \cos\alpha S_3).$$
(19)

Its eigenvalues are $E_0 = 0, E_{\pm} = \mp \omega \hbar \cos \alpha$, where $\cos \alpha = \frac{2 \sin \theta \sqrt{d^2 - 1}}{d^2}, \beta = \varphi$, here $d \ge 1$. By the way, its Casimir operator is $\kappa = \frac{1}{2}(S_+S_- + S_-S_+) + S_3^2$. It is easy to find the eigenvalues of κ are $\frac{1}{2}(\frac{1}{2} + 1) = \frac{3}{4}$ and 0(0 + 1) = 0, which correspond to spin-1/2 and spin-0. According to the definition of Berry phase, when $\varphi(t)$ evolves adiabatically from 0 to 2π , the corresponding Berry phase is

$$\gamma_{\alpha} = i \int_{0}^{T} \langle \Psi_{\alpha} | \frac{\partial}{\partial t} | \Psi_{\alpha} \rangle dt.$$
⁽²⁰⁾

Noting that Hamiltonian returns to its original form after the time $T = 2\pi/\omega$, we easily obtain the corresponding Berry phases of this Yang-baxter system

$$\begin{cases} \gamma_0 = 0, \\ \gamma_{\pm} = \pm \pi (1 - \cos \alpha) = \pm \frac{\Omega}{2}, \end{cases}$$
(21)

where $\Omega = 2\pi(1 - \cos \alpha)$ is the solid angle enclosed by the loop on the Bloch sphere. The system also equals to spin-1/2 system and spin-0 system. Substituting $\cos \alpha = \frac{2\sin\theta\sqrt{d^2-1}}{d^2}$ into Eq.(21), we obtain $\gamma_{\pm} = \pm \pi (1 - \frac{2\sin\theta\sqrt{d^2-1}}{d^2})$. Substituting θ with $\frac{\pi}{2} - \theta$, we rewrite Berry phase as follows

$$\gamma_{\pm} = \pm \pi (1 - \frac{2\cos\theta\sqrt{d^2 - 1}}{d^2}).$$
(22)

It is worth mentioning that in some papers[13], the Berry phase $\gamma_{\pm} = \pm \pi (1 - \cos \theta)$ of Yang-Baxter system only depends on the spectral parameter θ . It is interesting that in our paper, the Berry phases(22) not only depends on the spectral parameter θ , but also depends on the topological parameter d. The Berry phase (Eq.(22)) reduce to $\gamma_{\pm} = \pm \pi (1 - \cos \theta)$ if $d = \sqrt{2}$. The FIG. 1, which corresponds to the Berry phase γ_{\pm} . The FIG. 1(a) illustrate the

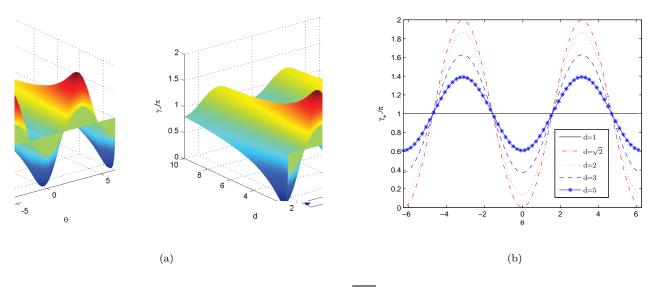


FIG. 1. (The left figure shows the Berry phase $\gamma_{+} = \pi (1 - \frac{2 \cos \theta \sqrt{d^2 - 1}}{d^2})$ versus the the topological parameter d and the spectral parameter θ . The right figure, the sectional drawings have also provided with the same values of parameters. d = 1 (solid line), $d = \sqrt{2}$ (dot-dashed line), d = 2 (dotted line), d = 3 (dashed line), d = 5 (star line).)

Berry phases Eq.(22) versus the spectral parameter θ and the topological parameter d. The FIG. 1(b) illustrate the Berry phase γ_+ versus the spectral parameter θ , when d choice specific values. It is demonstrated that the Berry phase γ_+ is maximum (minimum) at parameter $\theta = (2n + 1)\pi$ ($\theta = 2n\pi$), for a given definit value of d. Where nis integer. The maximum (minimum) of γ_+ is $\pi(1 + \frac{2\sqrt{d^2-1}}{d^2})$ ($\pi(1 - \frac{2\sqrt{d^2-1}}{d^2})$). As d increase, the maximum of γ_+ decreases, the minimum of γ_+ increases. The Berry phase γ_+ tend to a constant value π , with d approaches infinity.

IV. SUMMARY

In this paper we presented BWM-algebra (A, B, E_A, E_B) and solution of YBE $(A(\theta, \varphi_1, \varphi_2), B(\theta, \varphi_1, \varphi_2))$ in 3-dimensional representation which satisify $B = UAU^{-1}$, $E_B = UE_AU^{-1}$ and $B(\theta, \varphi_1, \varphi_2) = UA(\theta, \varphi_1, \varphi_2)U^{-1}$. The evolution of the Yang-Baxter system is explored by constructing a Hamiltonian from the unitary $B(\theta, \varphi)$ matrix. We study the Berry phase of the Yang-Baxter system, and obtain the relationship between topological parameter and Berry phase $\gamma_{\pm} = \pm \pi (1 - \frac{2\cos\theta\sqrt{d^2-1}}{d^2})$. Then we compare the Berry phase of Yang-Baxter system in Ref.[13] with us, and find the topological parameter d plays a deformation role in the Berry phase. We have been discussing in this paper is still an open problem that will require a deal of further investigations.

V. ACKNOWLEDGMENTS

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