Yang-Baxter \breve{R} matrix, Entanglement and Yangian

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(Detection August 2, 2021)

(Dated: August 2, 2021)

Abstract

We present a method to construct "X" form unitary Yang-Baxter \check{R} matrices, which act on the tensor product space $V_i^{j_1} \otimes V_{i+1}^{j_2}$. We can obtain a set of entangled states for $(2j_1+1)\times(2j_2+1)$ -dimensional system with these Yang-Baxter \check{R} matrices. By means of Yang-Baxter approach, a 8×8 Yang-Baxter Hamiltonian is constructed. Yangian symmetry and Yangian generators as shift operators for this Yang-Baxter system are investigated in detail.

PACS numbers: 03.65.Ud, 03.65.Fd, 02.10.Yn

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I. INTRODUCTION

Quantum entanglement[1–4], which is a bizarre of quantum theory, has been recognized as an important resource for applications in quantum information and quantum computation processing. Quantum gates[5] are represented by unitary matrices, and they are building blocks of a quantum computer. On the other hand, the topological quantum computation(TQC) also has been studied by researchers[6]. Thus quantum computation is one of the important approaches to achieve a fault-tolerant quantum computer. This proposal relies on the existence of topological states of matter, whose quasiparticle excitations are non-Abelian anyons. Thus quasiparticles obey non-Abelian braiding statistics, and quantum gate operators are implemented by braiding quasiparticles.

Recently, Kauffman *et.al.* have shown that topological entanglement and quantum entanglement have deep relations[7–9]. The authors propose that it is more fundamental to view braid matrices(or solutions to Quantum Yang-Baxter Equation[10, 11]), which can implement topological entanglement, as universal quantum gates. For example, the authors showed that the Bell matrix is nothing, but a braid matrix, and thus braid matrix local equivalent to a Control-Not(CNOT) gate[8]. This motivated a novel way to study quantum entanglement by means of Yang-Baxter approach[12–18].

The Yangian theory established by Drinfeld offer a mathematic method for the studies about the symmetry of quantum integrable models in physics[19]. Many researchers have explored the role of Yangian operators in physics[20–22]. For example, by means of Yangian, we can investigate the symmetry for the integrable systems and shift operators. But many researchers worked on complex systems, this motivated us to search a simple system with Yangian symmetry to investigate the role of Yangian operators in this system.

In Sec. II, we will present a method for constructing the "X" form Yang-Baxter \check{R} matrices, and then we will investigate the entanglement properties in Sec. III. In Sec.IV, we construct Yang-Baxter Hamiltonian with a 8×8 "X" form Yang-Baxter \check{R} matrix, then Yangian symmetry and shift operators are studied in this Yang-Baxter system.

II. THE "X" FORM YANG-BAXTER Ř MATRICES

In this paper, Yang-Baxter $\check{R}^{j_1j_2}(\theta)$ matrix and $M^{j_1j_2}$ matrix are $(2j_1+1)\times(2j_2+1)$ -dimensional matrices acting on the tensor product $V^{j_1}\otimes V^{j_2}$, where V^{j_1} and V^{j_2} are $(2j_1+1)$ and $(2j_2+1)$

dimensional vector space, respectively. As Yang-Baxter $\check{R}^{j_1j_2}(\theta)$ matrix and $M^{j_1j_2}$ matrix acting on the tensor product $V_i^{j_1} \otimes V_{i+1}^{j_2}$, we denote them by $\check{R}_i^{j_1j_2}(\theta)$ and $M_i^{j_1j_2}$, respectively. The notation $I^{j_1j_2}$ denotes $(2j_1+1)\times(2j_2+1)$ -dimension identity matrix.

Let matrices $M^{j_1j_2}$ and $M^{j_2j_1}$ satisfying the following relations,

$$[M^{j_1j_2}]^2 = [M^{j_2j_1}]^2 = I^{j_1j_2}$$

$$M^{j_1j_2}_{12}M^{j_2j_1}_{23} = M^{j_2j_1}_{23}M^{j_1j_2}_{12}, \quad (i.e.[M^{j_1j_2}_{12}, M^{j_2j_1}_{23}] = 0)$$

$$M^{j_2j_1}_{12}M^{j_1j_2}_{23} = M^{j_1j_2}_{23}M^{j_2j_1}_{12}, \quad (i.e.[M^{j_2j_1}_{12}, M^{j_1j_2}_{23}] = 0).$$
(1)

In this paper, we set $[M^{j_1j_2}]^{a\alpha}_{b\beta} = [M^{j_2j_1}]^{\alpha a}_{\beta b}(-j_1 \le a, b \le j_1 \text{ and } -j_2 \le \alpha, \beta \le j_2)$ for convenience. Then two spectral-dependent Yang-Baxter \check{R} matrices via Yang-Baxterization[23–25] is obtained to be,

Here we used Tayloy expansion to derive the right hand of Eq. 2. If the matrices $M^{j_1j_2}$ and $M^{j_2j_1}$ are Hermitian matrices(*i.e.* $[M^{j_1j_2}]^{\dagger} = M^{j_1j_2}$ and $[M^{j_2j_1}]^{\dagger} = M^{j_2j_1}$), then we can verify that the matrices $\check{R}^{j_1j_2}$ and $\check{R}^{j_2j_1}$ are unitary(*i.e.* $\check{R}^{j_1j_2}(\theta)^{\dagger}\check{R}^{j_1j_2}(\theta) = \check{R}^{j_1j_2}(\theta)\check{R}^{j_1j_2}(\theta)^{\dagger} = I^{j_1j_2}$ and $\check{R}^{j_2j_1}(\theta)^{\dagger}\check{R}^{j_2j_1}(\theta) = \check{R}^{j_2j_1}(\theta)\check{R}^{j_2j_1}(\theta)^{\dagger} = I^{j_2j_1}$).

We can easily prove that $\breve{R}^{j_1j_2}(\theta)$ and $\breve{R}^{j_2j_1}(\theta)$ satisfy the following Yang-Baxter equation (YBE),

$$\breve{R}_{12}^{j_1j_2}(\theta_1)\breve{R}_{23}^{j_2j_1}(\theta_1 + \theta_2)\breve{R}_{12}^{j_1j_2}(\theta_2) = \breve{R}_{23}^{j_2j_1}(\theta_2)\breve{R}_{12}^{j_1j_2}(\theta_1 + \theta_2)\breve{R}_{23}^{j_2j_1}(\theta_1),
\breve{R}_{12}^{j_2j_1}(\theta_1)\breve{R}_{23}^{j_1j_2}(\theta_1 + \theta_2)\breve{R}_{12}^{j_2j_1}(\theta_2) = \breve{R}_{23}^{j_1j_2}(\theta_2)\breve{R}_{12}^{j_2j_1}(\theta_1 + \theta_2)\breve{R}_{23}^{j_1j_2}(\theta_1).$$
(3)

where parameters θ_1 and θ_2 are called as spectral parameters. For convenience, we take $M^{j_1j_2}$ and $M^{j_2j_1}$ as $[M^{j_2j_1}]_{\beta b}^{\alpha a} = [M^{j_1j_2}]_{b\beta}^{a\alpha} = e^{-i\varphi_{a\alpha}}\delta_{a,-b}\delta_{\alpha,-\beta}$. Considering the first equation in Eqs. 1, we set $\varphi_{a\alpha} = -\varphi_{-a-\alpha}$. Substituting $M^{j_1j_2}$ and $M^{j_2j_1}$ into the second and the third relations in Eqs. 1, we can obtain the following conditions,

$$\varphi_{a\alpha} + \varphi_{-a\alpha} = \varphi_{b\alpha} + \varphi_{-b\alpha},$$

$$\varphi_{a\alpha} + \varphi_{a-\alpha} = \varphi_{a\beta} + \varphi_{a-\beta}.$$
(4)

With this method, we can obtain high dimentional Yang-Baxter $\check{R}^{j_1j_2}$ matrices easily. By means of these Yang-Baxter $\check{R}^{j_1j_2}$ matrices, we can investigate quantum entanglement consequently.

III. THE "X" FORM K MATRICES AS QUANTUM GATES

In this section, three examples are shown to illustrate this method in detail. The case $j_1 = j_2 = 1/2$ gives us a 4×4 unitary Yang-Baxter $\check{R}^{1/2,1/2}(\theta)$ matrix. Thus we can view the $\check{R}^{1/2,1/2}(\theta)$ matrix as a quantum gate for two-qubit system. If $j_1 = 1$ and $j_2 = 1/2$, we can obtain a 6×6 Yang-Baxter $\check{R}^{1,1/2}$ matrix. This unitary $\check{R}^{1,1/2}$ can entangle quantum states in system with one qubit and one qutrit. When $j_1 = 3/2$ and $j_2 = 1/2$, a three-qubit quantum gate $\check{R}^{3/2,1/2}$ can be obtained. For quantify the entanglement of bi-particle system states, we use the negativity[26, 27] defined by,

$$N(\rho) = \frac{\|\rho^{T_B}\|_1 - 1}{d - 1}.$$
 (5)

where ρ^{T_B} is the partial transpose of a state ρ in $d \times d'(d \le d')$ quantum system, and the notation $||A||_1 = Tr \sqrt{A^{\dagger}A}$ denotes the trace norm of A. It should be noted that the negativity criterion is necessary and sufficient only for $2 \otimes 2$ and $2 \otimes 3$ quantum systems.

A. The 4×4 "X" form \breve{R} matrix

If $j_1 = j_2 = 1/2$, the equations in Eqs. (1) can be simplified as $[M^{1/2,1/2}]^2 = I^{1/2,1/2}$ and $[M_{12}^{1/2,1/2}, M_{23}^{1/2,1/2}] = 0$. Then we can obtain a matrix $M^{1/2,1/2}$ as following,

$$M^{1/2,1/2} = e^{-i(\varphi + \frac{\pi}{2})} s_1^+ s_2^+ + s_1^+ s_2^- + s_1^- s_2^+ + e^{i(\varphi + \frac{\pi}{2})} s_1^- s_2^-.$$

The Yang-Baxter $\breve{R}^{1/2,1/2}$ matrix can be obtained as follows,

$$\breve{R}^{1/2,1/2}(\theta) = e^{-i\frac{\theta}{2}M^{1/2,1/2}} = \cos\frac{\theta}{2}I^{1/2,1/2} - i\sin\frac{\theta}{2}M^{1/2,1/2},$$

or in matrix form,

$$\breve{R}^{1/2,1/2}(\theta) = \begin{pmatrix}
\cos\frac{\theta}{2} & 0 & 0 & -\sin\frac{\theta}{2}e^{-i\varphi} \\
0 & \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} & 0 \\
0 & -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 \\
\sin\frac{\theta}{2}e^{i\varphi} & 0 & 0 & \cos\frac{\theta}{2}
\end{pmatrix}.$$

In this section, we choose $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ as standard bases. Acting this Yang-Baxter $\check{R}^{1/2,1/2}$ matrix on the standard bases, we can obtain a set entangled states $\{|e_i\rangle, i = 1, 2, 3, 4\}$ £

$$\begin{pmatrix} |e_1\rangle \\ |e_2\rangle \\ |e_3\rangle \\ |e_4\rangle \end{pmatrix} = \breve{R}^{1/21/2}(\theta) \begin{pmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2}|00\rangle - \sin\frac{\theta}{2}e^{-i\varphi}|11\rangle \\ \cos\frac{\theta}{2}|01\rangle - i\sin\frac{\theta}{2}|10\rangle \\ -i\sin\frac{\theta}{2}|01\rangle + \cos\frac{\theta}{2}|10\rangle \\ \sin\frac{\theta}{2}e^{i\varphi}|00\rangle + \cos\frac{\theta}{2}|11\rangle \end{pmatrix}.$$

Let us find the entanglement degree of the above states by using negativity. For a pure two qubit state, $|\psi\rangle = a|00\rangle + b|11\rangle$ or $|\phi\rangle = a|01\rangle + b|10\rangle$, the negativity can be find to be $N(|\psi\rangle) = N(|\phi\rangle) = 2|ab|$. We can easily obtain the negativity for the above entangled states as $N(|e_i\rangle) = |\sin\theta|$, where i=1,2,3,4. With the Yang-Baxter \check{R} acting on the standard bases, we can obtain a set of entangled states, and these states possess the same entanglement degree which depends on the parameter θ . This character of the Yang-Baxter \check{R} matrices has revealed in the Refs.. For the 2-qubit quantum system, there is good entanglement measure concurrence[28, 29], $C(\rho_{12}) = Max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$. Here $\{\lambda_i\}$ denotes the eigenvalues of the matrix $\rho_{12}\sigma_1^y\sigma_2^y\rho_{12}^*\sigma_1^y\sigma_2^y$. The notations ρ_{12} and ρ_{12}^* are biqubit density matrix and its complex conjugate, correspondingly. The notations $\sigma_{1,2}^y$ are pauli matrices. We can verify that concurrence is equivalence to negativity for two-qubit "X" state(which density matrices are "X" form).

B. The 6×6 "X" form \breve{R} matrix

When $j_1 = 1$ and $j_2 = 1/2$, with the relations in Eqs. (1), we can determine two matrices $M^{1,1/2}$ and $M^{1/2,1}$. In this section, the bases for the tensor product space $V^{j_1} \otimes V^{j_2}$ are given by $\{|a\alpha\rangle: a = 1, 0, -1; \alpha = 1/2, -1/2\}$. In this case, the Eqs. (4) gives the following relation,

$$2\varphi_{0,1/2} = \varphi_{1,1/2} - \varphi_{1,-1/2}.$$
(6)

If we set $\varphi_{1,1/2} = \varphi_1$ and $\varphi_{1,-1/2} = \varphi_2$, then $\varphi_{0,1/2} = (\varphi_1 - \varphi_2)/2$. Then a 6-dimensional $M^{1,1/2}$ matrix is given as follows,

$$M^{1,1/2} = (e^{-i\varphi_1}|1, 1/2\rangle\langle -1, -1/2| + e^{-i\varphi_2}|1, -1/2\rangle\langle -1, 1/2|$$

$$+ e^{-i(\varphi_1 - \varphi_2)}|0, 1/2\rangle\langle 0, -1/2|) + H.C$$
(7)

The $M^{1,1/2}$ matrix takes the following matrix form,

$$M^{1,1/2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e^{-i\varphi_1} \\ 0 & 0 & 0 & 0 & e^{-i\varphi_2} & 0 \\ 0 & 0 & 0 & e^{-i(\varphi_1 - \varphi_2)} & 0 & 0 \\ 0 & 0 & e^{i(\varphi_1 - \varphi_2)} & 0 & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 & 0 & 0 & 0 \\ e^{i\varphi_1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
(8)

Then a 6-dimensional Yang-Baxter $\breve{R}^{1,1/2}(\theta)$ can be construct as following,

$$\check{R}^{1,1/2}(\theta) = \cos\frac{\theta}{2}I^{1,1/2} - i\sin\frac{\theta}{2}M^{1,1/2} \tag{9}$$

When $\breve{R}^{1,1/2}(\theta)$ act on the standard basis(product states),

$$\begin{pmatrix}
|e_{1}\rangle \\
|e_{2}\rangle \\
|e_{3}\rangle \\
|e_{4}\rangle \\
|e_{5}\rangle \\
|e_{6}\rangle
\end{pmatrix} = \check{R}^{1,1/2}(\theta) \begin{pmatrix}
|1, 1/2\rangle \\
|1, -1/2\rangle \\
|0, 1/2\rangle \\
|0, -1/2\rangle \\
|-1, 1/2\rangle \\
|-1, -1/2\rangle
\end{pmatrix}$$
(10)

Then we obtain six entangled states,

$$|e_{1}\rangle = \cos\frac{\theta}{2}|1, 1/2\rangle - i\sin\frac{\theta}{2}e^{-i\varphi_{1}}| - 1, -1/2\rangle$$

$$|e_{2}\rangle = \cos\frac{\theta}{2}|1, -1/2\rangle - i\sin\frac{\theta}{2}e^{-i\varphi_{2}}| - 1, 1/2\rangle$$

$$|e_{3}\rangle = \cos\frac{\theta}{2}|0, 1/2\rangle - i\sin\frac{\theta}{2}e^{-i(\varphi_{1}-\varphi_{2})}|0, -1/2\rangle$$

$$|e_{4}\rangle = -i\sin\frac{\theta}{2}e^{i(\varphi_{1}-\varphi_{2})}|0, 1/2\rangle + \cos\frac{\theta}{2}|0, -1/2\rangle$$

$$|e_{5}\rangle = -i\sin\frac{\theta}{2}e^{i\varphi_{2}}|1, -1/2\rangle + \cos\frac{\theta}{2}|-1, 1/2\rangle$$

$$|e_{6}\rangle = -i\sin\frac{\theta}{2}e^{i\varphi_{1}}|1, 1/2\rangle + \cos\frac{\theta}{2}|-1, -1/2\rangle$$

$$(11)$$

Using the formula of negativity, we can obtain the entanglement degree for the eigenstates of this Yang-Baxter system as $N(|e_i\rangle) = |\sin\theta|$. These eigenstates possess the same degree of entanglement.

C. The 8×8 Yang-Baxter system

When $j_1 = 3/2$ and $j_2 = 1/2$, we can obtain a 8×8 $M^{3/2,1/2}$ matrix which satisfying the relations Eqs.(1). For the following convenience, we introduce the notation $\{|i\rangle; i = 1, 2 \cdots 8\}$ to denote the

standard three-qubit basis.

$$M^{3/2,1/2} = i(e^{-i\varphi_1}s_1^+s_2^+s_3^+ + e^{-i\varphi_2}s_1^+s_2^+s_3^- + e^{-i\varphi_3}s_1^+s_2^-s_3^+ + e^{-i\varphi_4}s_1^+s_2^-s_3^-)$$

$$- i(e^{i\varphi_4}s_1^-s_2^+s_3^+ + e^{i\varphi_3}s_1^-s_2^+s_3^- + e^{i\varphi_2}s_1^-s_2^-s_3^+ + e^{i\varphi_1}s_1^-s_2^-s_3^-)$$

If parameters φ_i 's satisfy the relation $\varphi_1 + \varphi_4 = \varphi_2 + \varphi_3$, then the $M^{\frac{3}{2}\frac{1}{2}}$ satisfy the relations in Eqs.(1). Then we can obtain a 8×8 unitary Yang-Baxter \check{R} -matrix,

$$\check{R}^{3/2,1/2}(\theta) = \cos\frac{\theta}{2}I^{3/2,1/2} - i\sin\frac{\theta}{2}M^{3/2,1/2}$$

We can verify that the Yang-Baxter $\check{R}^{3/2,1/2}(\theta)$ matrix is unitary(*i.e.* $\check{R}(\theta)^{\dagger}\check{R}(\theta) = \check{R}(\theta)\check{R}(\theta)^{\dagger} = I$). Let $H_0 = s_1^3 \otimes I_2 \otimes I_3$. With this Yang-Baxter \check{R} -matrix and this simple Hamiltonian, we can derive a hamiltonian as $H = \check{R}(\theta)^{\dagger}H_0\check{R}(\theta) = \sum_{i=1}^4 \mathbf{B}_i \cdot \mathbf{S}_i$, where $\mathbf{B}_i = (sin\theta cos\varphi_i, sin\theta sin\varphi_i, cos\theta)$ and

$$S_{1}^{+} = |1\rangle\langle 8|, \ S_{1}^{-} = |8\rangle\langle 1|, \ S_{1}^{3} = \frac{1}{2}(|1\rangle\langle 1| - |8\rangle\langle 8|);$$

$$S_{2}^{+} = |2\rangle\langle 7|, \ S_{2}^{-} = |7\rangle\langle 2|, \ S_{2}^{3} = \frac{1}{2}(|2\rangle\langle 2| - |7\rangle\langle 7|);$$

$$S_{3}^{+} = |3\rangle\langle 6|, \ S_{3}^{-} = |6\rangle\langle 3|, \ S_{3}^{3} = \frac{1}{2}(|3\rangle\langle 3| - |6\rangle\langle 6|);$$

$$S_{4}^{+} = |4\rangle\langle 5|, \ S_{4}^{-} = |5\rangle\langle 4|, \ S_{4}^{3} = \frac{1}{2}(|4\rangle\langle 4| - |5\rangle\langle 5|).$$

After some algebra, we can obtain the eigenvalues $\{E_i^{\alpha}\}$ and eigenvectors $\{|e_i^{\alpha}\rangle\}$ ($\alpha = +, -; i = 1, 2, 3, 4$) for Hamiltonian H as following,

$$E_i^+ = -E_i^- = 1/2,$$

and corresponding eigenvectors,

$$\begin{split} |e_1^+\rangle &= cos\frac{\theta}{2}|1\rangle + sin\frac{\theta}{2}e^{i\varphi_1}|8\rangle, \ |e_1^-\rangle &= -sin\frac{\theta}{2}e^{-i\varphi_1}|1\rangle + cos\frac{\theta}{2}|8\rangle; \\ |e_2^+\rangle &= cos\frac{\theta}{2}|2\rangle + sin\frac{\theta}{2}e^{i\varphi_2}|7\rangle, \ |e_2^-\rangle &= -sin\frac{\theta}{2}e^{-i\varphi_2}|2\rangle + cos\frac{\theta}{2}|7\rangle; \\ |e_3^+\rangle &= cos\frac{\theta}{2}|3\rangle + sin\frac{\theta}{2}e^{i\varphi_3}|6\rangle, \ |e_3^-\rangle &= -sin\frac{\theta}{2}e^{-i\varphi_3}|3\rangle + cos\frac{\theta}{2}|6\rangle; \\ |e_4^+\rangle &= cos\frac{\theta}{2}|4\rangle + sin\frac{\theta}{2}e^{i\varphi_4}|5\rangle, \ |e_4^-\rangle &= -sin\frac{\theta}{2}e^{-i\varphi_4}|4\rangle + cos\frac{\theta}{2}|5\rangle. \end{split}$$

In fact, the Hamiltonian H can be recast as following,

$$H = \sum_{i=1}^{4} (|e_i^+\rangle \langle e_i^+| - |e_i^-\rangle \langle e_i^-|)$$
 (12)

Consider the state $|\psi\rangle$ in a three-qubit Hilbert space $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Its coefficients with respect to a basis of product states (the 'computational basis') are $\psi_i = \langle i|\psi\rangle$, $i \in \{0, 1 \cdots 8\}$. An important measure for the entanglement in pure three-qubit states is the three-tangle (or residual

tangle) introduced in Ref.[30]. The three-tangle of $|\psi\rangle$ is a so-called polynomial invariant and can be written in terms of the coefficients ψ_i as

$$\tau_{3}(\psi) = 4|d_{1} - 2d_{2} + 4d_{3}|$$

$$d_{1} = \psi_{1}^{2}\psi_{8}^{2} + \psi_{2}^{2}\psi_{7}^{2} + \psi_{3}^{2}\psi_{6}^{2} + \psi_{5}^{2}\psi_{4}^{2}$$

$$d_{2} = \psi_{1}\psi_{8}\psi_{4}\psi_{5} + \psi_{1}\psi_{8}\psi_{6}\psi_{3} + \psi_{1}\psi_{8}\psi_{7}\psi_{2}$$

$$+ \psi_{4}\psi_{5}\psi_{6}\psi_{3} + \psi_{4}\psi_{5}\psi_{7}\psi_{2} + \psi_{6}\psi_{3}\psi_{7}\psi_{2}$$

$$d_{3} = \psi_{1}\psi_{7}\psi_{6}\psi_{4} + \psi_{8}\psi_{2}\psi_{3}\psi_{5} .$$
(13)

Then we can obtain three-tangle for the Eigenstates are as following,

$$\tau_3(|e_i^{\alpha}\rangle) = \sin^2\theta.$$

By using the definition of concurrence we can obtain the

$$C_{AB}(|e_i^{\alpha}\rangle) = C_{AC}(|e_i^{\alpha}\rangle) = C_{BC}(|e_i^{\alpha}\rangle) = 0$$

where i = 1, 2, 3, 4 and $\alpha = +, -$. When the parameter $\theta = \pi/2$, $\tau_3(|e_i^{\alpha}\rangle) = 1$ and $C_{XY}(|e_i^{\alpha}\rangle) = 0$ (XY = AB, BC, AC). Then we can say these eigenstates are GHZ type states.

IV. YANGIAN SYMMETRY AND SHIFT OPERATORS

In the Sec.III, we construct a Hamiltonian(*i.e.* Eq.(12)) with the Yang-Baxter $\check{R}^{3/2,1/2}$ matrix. As is known to all, the Yangian is a very important tool to study symmetry and shift operators. Motivated this, we will investigate the symmetry to this Yang-Baxter Hamiltonian and Yangian generators as shift operators in detail.

In fact, with the eigenvectors $\{|e_i^{\alpha}\rangle\}$ we can construct a special Yangian Y(sl(2)) realization $\{I_{\pm}, I_3\}$ and $\{F_{\pm}, F_3\}$ as following,

$$I_{+} = |e_{1}^{+}\rangle\langle e_{2}^{+}| + |e_{3}^{+}\rangle\langle e_{4}^{+}| + |e_{1}^{-}\rangle\langle e_{2}^{-}| + |e_{3}^{-}\rangle\langle e_{4}^{-}|$$

$$I_{-} = |e_{2}^{+}\rangle\langle e_{1}^{+}| + |e_{4}^{+}\rangle\langle e_{3}^{+}| + |e_{2}^{-}\rangle\langle e_{1}^{-}| + |e_{4}^{-}\rangle\langle e_{3}^{-}|$$

$$I_{3} = \frac{1}{2}[(|e_{1}^{+}\rangle\langle e_{1}^{+}| + |e_{3}^{+}\rangle\langle e_{3}^{+}| + |e_{1}^{-}\rangle\langle e_{1}^{-}| + |e_{3}^{-}\rangle\langle e_{3}^{-}|)$$

$$- (|e_{2}^{+}\rangle\langle e_{2}^{+}| + |e_{4}^{+}\rangle\langle e_{4}^{+}|) + |e_{2}^{-}\rangle\langle e_{2}^{-}| + |e_{4}^{-}\rangle\langle e_{4}^{-}|)],$$

and

$$\begin{split} F_{+} &= 2\alpha(|e_{1}^{+}\rangle\langle e_{4}^{+}| + \beta|e_{3}^{+}\rangle\langle e_{2}^{+}|) + 2\gamma(|e_{1}^{-}\rangle\langle e_{4}^{-}| + \delta|e_{3}^{-}\rangle\langle e_{2}^{-}|) \\ F_{-} &= 2\alpha(\beta|e_{4}^{+}\rangle\langle e_{1}^{+}| + |e_{2}^{+}\rangle\langle e_{3}^{+}|) + 2\gamma(\delta|e_{4}^{-}\rangle\langle e_{1}^{-}| + |e_{2}^{-}\rangle\langle e_{3}^{-}|) \\ F_{3} &= \alpha(|e_{1}^{+}\rangle\langle e_{3}^{+}| - |e_{2}^{+}\rangle\langle e_{4}^{+}| + \beta|e_{3}^{+}\rangle\langle e_{1}^{+}| - \beta|e_{4}^{+}\rangle\langle e_{2}^{+}|) \\ &+ \gamma(|e_{1}^{-}\rangle\langle e_{3}^{-}| - |e_{2}^{-}\rangle\langle e_{4}^{-}| + \delta|e_{3}^{-}\rangle\langle e_{1}^{-}| - \delta|e_{4}^{-}\rangle\langle e_{2}^{-}|). \end{split}$$

It is not difficulty to verify that $\{I_{\pm}, I_3\}$ and $\{F_{\pm}, F_3\}$ satisfy the following Yanigian Y(sl(2)) relations,

$$[I_3, I_{\pm}] = \pm I_{\pm}, \quad [I_+, I_-] = 2I_3$$

 $[I_3, F_{\pm}] = [F_3, I_{\pm}] = \pm F_{\pm}, \quad [I_{\pm}, F_{\mp}] = \pm 2F_3$
 $[I_3, F_3] = [I_+, F_+] = 0,$

and

$$[F_3, [F_+, F_-]] = 0, \quad [F_\pm, [F_3, F_\pm]] = 0$$

 $[F_\pm, [F_\pm, F_\mp]] \pm 2[F_3, [F_3, F_\pm]] = 0.$

We can verify that the Hamiltonian and Yangian operators satisfy the following relation,

$$[H, Y_{\alpha}] = 0,$$

where Y = I, F and $\alpha = \pm, 3$. That is to say this Hamiltonian possess a Yangian Y(sl(2)) symmetry.

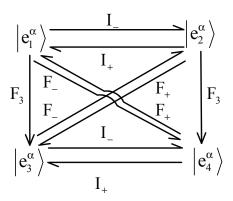


FIG. 1: The states transfer graph for the Yang-Baxter Hamiltonian($\alpha = \pm$).

This maybe the simplest Hamiltonian with Yangian Y(sl(2)) symmetry. In quantum physics, the

Yangian generators can be used to construct shift operators. Then we will construct shift operators for this Yang-Baxter Hamiltonian. When the Yangian operators $\{I_{\pm}, I_3\}$ and $\{F_{\pm}, F_3\}$ act on the eigenstates of this Yang-Baxter Hamiltonian, we can obtain a state transfer graph in Fig.(1).

V. SUMMARY

In this paper, we construct a set of $(2j_1 + 1) \times (2j_2 + 1)$ -dimensional "X" form Yang-Baxter $\check{R}^{j_1j_2}(\theta)$. We investigated this set unitary Yang-Baxter $\check{R}^{j_1j_2}(\theta)$ as quantum gate in quantum computation processing. When these "X" form Yang-Baxter $\check{R}^{j_1j_2}(\theta)$ matrices act on standard bases, we can obtain a set of entangled states, which possess the same degree of quantum entanglement. We also construct a Yang-Baxter Hamiltonian with Yangian Y(sl(2)) symmetry. And Yangian generators can be viewed as shift operators.

Acknowledgments

This work was supported by NSF of China (Grants No. 10875026) and the Fundamental Research Funds for the Central Universities(Grants No. 09SSXT026)

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