Examining the dimensionality of genuine multipartite entanglement

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Entanglement in high-dimensional many-body systems plays an increasingly vital role in the foundations and applications of quantum physics. In the present paper, we introduce a theoretical concept which allows to categorize multipartite states by the number of degrees of freedom being entangled. In this regard, we derive computable and experimentally friendly criteria for arbitrary multipartite qudit systems that enable to examine in how many degrees of freedom a mixed state is genuine multipartite entangled.

Keywords: Entanglement measures, witnesses, and other characterizations – Algebraic methods – Quantum information – Foundations of quantum mechanics

I. INTRODUCTION

Ever since its discovery more than seventy years ago, quantum entanglement has been considered as the central essence of quantum theory, forcing us to rethink our view of reality, locality and causality. It impressively highlights the non-local-realistic and contextual character of nature and thereby provides insights into the very foundations of physics. Moreover, during the last two decades, it has become more and more clear that entanglement can serve as a resource for future information processing technologies, such as quantum cryptography, dense coding, quantum teleportation and quantum computing. It is even argued that entanglement plays a role in quantum phase transitions [1], ionization processes [2], high energy physics [3] and light-harvesting complexes [4].

When it comes to studying quantum phenomena in diverse systems one is regularly confronted with the problems of how to detect, characterize and quantify entanglement. With the exception of bipartite qubit systems, these problems are in general extremely hard to solve for systems of arbitrary number of parties and dimensions, i.e. multipartite qudits.

In the present paper we focus on a finer characterization of genuine multipartite entanglement [5] in multilevel systems. Genuine multipartite entangled states have been shown to be vital for fundamental tests of quantum physics [6–8] and find application in measurement-based quantum computing [9] and quantum secret sharing [10, 11]. Although, this type of entanglement is not bounded on the dimensionality of the local systems, the use of systems with more than two levels, i.e. qudits, brings with it several advantages and deeper insights. For instance, it was found that quantum correlations are more robust against decoherence the more degrees of freedom are entangled [12, 13]. Qudit entanglement also improves the security of quantum key distribution [14], and allows quantum secret sharing schemes [15], distributed protocols [16, 17] and error-correcting codes [18] which cannot be realized with qubits. It is also to be expected that quantum computers that encode more than one qubit of information in each particle will require less resources and will thus be more efficient [19, 20]. Furthermore, high-dimensional multipartite entanglement is essential for a complete understanding of quantum theory [21–25].

The aim of this paper is to provide practical and experimentally feasible criteria that allow to examine the dimensionality of genuine multipartite entanglement. The central problem is the following: Suppose we have realized a multipartite qudit scenario in the laboratory. How can we verify if a state is genuine multipartite entangled (GME) and how many degrees of freedom are involved in the entanglement?

For pure states this question is easily answered via the ranks of the reduced density matrices. However, for mixed states, as they appear in any real experiment, this is a nontrivial problem. Consider e.g. the mixed state $\rho_c = \frac{1}{2} |GHZ_3\rangle \langle GHZ_3| + \frac{1}{6} \sum_{i=0}^{2} |iii\rangle \langle iii|$, where $|GHZ_3\rangle = \frac{1}{\sqrt{3}} (|000\rangle + |111\rangle + |222\rangle)$ is a tripartite Greenberger-Horne-Zeilinger (GHZ) state entangled in three degrees of freedom. This state ρ_c

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is not truly three-dimensionally entangled since it can be decomposed into $\rho_c = \frac{1}{3} (|GHZ_{1,2}\rangle \langle GHZ_{1,2}| + |GHZ_{1,3}\rangle \langle GHZ_{1,3}| + |GHZ_{2,3}\rangle \langle GHZ_{2,3}|)$ with $|GHZ_{i,j}\rangle = \frac{1}{\sqrt{2}} (|iii\rangle + |jjj\rangle)$, which are each entangled in only two local degrees of freedom.

II. DEFINITIONS

Let us first give a precise definition of the dimensionality of multipartite entanglement – a multipartite generalization of the Schmidt rank [26–28] complementary to the tensor rank [29] that allows to characterize multipartite qudit states. This generalization is important as the tensor rank is *not* the crucial characteristic that the many-body entangled qudit states in [15–25] have in common. This is mainly because in the multipartite case there is no simple connection between the tensor rank, *k*-separability and multilevel entanglement. For example, the states $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$, $|GHZ_3\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$ and $|\Psi_{\text{bisep.}}\rangle = |0\rangle \otimes \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ all have tensor rank 3. However, only $|GHZ_3\rangle$ is commonly regarded as multilevel-multipartite entangled. Hence, for a finer categorization of multipartite entanglement it is needed to specify novel characteristic quantities.

For a pure *n*-partite state $|\psi\rangle \in \mathcal{H} = \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_n}$ consider the set of all reduced density matrices $\{\rho_A = Tr_B(|\psi\rangle \langle \psi|)\}$ regarding all bipartitions $\gamma = \{(A|B)\}$. Evidently, iff the state $|\psi\rangle$ is fully separable then the rank of all reduced density matrices ρ_A is 1. On the other hand, the state $|\psi\rangle$ contains entanglement iff there exists a ρ_A with $\operatorname{rank}(\rho_A) > 1$. For a fixed bipartition (A|B), the dimensionality of entanglement is determined by the Schmidt rank [26–28] which equals $\operatorname{rank}(\rho_A)$. Hence, iff $\max\{\operatorname{rank}(\rho_A)\} = f$ with $f \geq 2$ then the state $|\psi\rangle$ contains *f*-dimensional entanglement. We define a state to be *f*-dimensionally genuine multipartite entangled (GME) iff it is at least *f*-dimensionally entangled with respect to all bipartitions, that is $\min\{\operatorname{rank}(\rho_A)\} = f$. This can be extended to mixed states in a natural way: A mixed state $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ is *f*-dimensionally entangled iff there exists no decomposition $\{p_i, |\psi_i\rangle\}$ into pure states $|\psi_i\rangle$ of dimensionality f_i all obeying $f_i < f$, but a decomposition into pure states satisfying $f_i \leq f$ for all *i*. A mixed state $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ is *f*-dimensionally GME iff any decomposition $\{p_i, |\psi_i\rangle\}$ of ρ contains at least one GME state $|\psi_i\rangle$ of dimensionality *f* or higher.

III. DIMENSIONALITY CRITERIA

The problem of determining the dimensionality of multipartite entanglement for a given mixed state ρ is as complex as the separability problem, since it is practically impossible to vary over all pure state decompositions of ρ . For this reason it is imperative to find computable criteria for the detection of high-dimensional genuine multipartite entanglement. First steps in this direction have recently been made by Lim *et al.* [30] and Li *et al.* [31]. However, the criterion by Lim *et al.* is only able to discriminate 2-dimensional from 3-dimensional genuine multipartite entanglement in tripartite three-level systems. The criterion by Li *et al.* applies to arbitrary dimensionality and system size, but its noise resistance is rather unsatisfactory. Hence, further progress is needed here. Recently, a framework of criteria detecting genuine multipartite entanglement was introduced in [32–35]. Although it belongs to the strongest criteria for the detection of GME states without requiring semidefinite programming [36], it does not discriminate states of different dimensionality. In the present paper, we show how this powerful framework can be extended for verifying the presence of high-dimensional genuine multipartite entanglement in arbitrary mixed states.

Consider a density matrix ρ of an *n*-partite *d*-level system, i.e. a Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$. For a twofold copy $\rho^{\otimes 2}$ on $\mathcal{H}^{\otimes 2}$ we define for each bipartition (A|B) of \mathcal{H} a permutation operator \mathcal{P}_A which permutes the subsystem A with its copy A', i.e.

$$\mathcal{H}^{\otimes 2} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$$
$$\xrightarrow{\mathcal{P}_A} \mathcal{H}_{A'} \otimes \mathcal{H}_B \otimes \mathcal{H}_A \otimes \mathcal{H}_{B'}$$

For instance, for the bipartition $(\{1\}|\{2,\ldots,n\})$ and the vector $|k\rangle^{\otimes n} \otimes |l\rangle^{\otimes n} \in \mathcal{H}^{\otimes 2}$ the corresponding

operator $\mathcal{P}_{\{1\}}$ acts like

$$\mathcal{P}_{\{1\}} \ket{k}^{\otimes n} \otimes \ket{l}^{\otimes n} = \ket{l} \otimes \ket{k}^{\otimes (n-1)} \otimes \ket{k} \otimes \ket{l}^{\otimes (n-1)}$$

Using this abbreviation we introduce the quantity

$$Q_{0} = \sum_{k \neq l}^{d-1} \left(\left| \left\langle k_{n} \right| \rho \left| l_{n} \right\rangle \right| - \sum_{\gamma} \sqrt{\left\langle k, l \right| \mathcal{P}_{A}^{\dagger} \rho^{\otimes 2} \mathcal{P}_{A} \left| k, l \right\rangle} \right)$$
(1)

with $|k_n\rangle = |k\rangle^{\otimes n}$ and $|k,l\rangle = |k\rangle^{\otimes n} \otimes |l\rangle^{\otimes n}$, where the $|k\rangle, |l\rangle$ are vectors of an orthonormal basis $\{|0\rangle, \ldots, |d-1\rangle\}$ of \mathbb{C}^d and the sum runs over all bipartitions $\gamma = \{(A|B)\}$. Furthermore, we introduce the quantities $(m \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\})$

$$Q_m = \frac{1}{m} \left[\sum_{k,l=0}^{d-2} \sum_{\sigma} \left(\underbrace{|\langle \alpha^k | \rho | \beta^l \rangle|}_{\mathcal{O}_{\alpha,\beta}^{k,l}} - \sum_{\delta} \underbrace{\sqrt{\langle \alpha^k | \otimes \langle \beta^l | \mathcal{P}_{\delta}^{\dagger} \rho^{\otimes 2} \mathcal{P}_{\delta} | \alpha^k \rangle \otimes |\beta^l \rangle}_{P_{\alpha,\beta}^{k,l}} \right) - N_D \sum_{l=0}^{d-2} \sum_{\alpha} \underbrace{\langle \alpha^l | \rho | \alpha^l \rangle}_{D_{\alpha}^l} \right]$$
(2)

wherein $|\alpha^l\rangle \in \mathcal{H}$ are product vectors, where *m* of the *n* local systems contained in the set α are in the state $|l+1\rangle$ and remaining ones in $|l\rangle$, i.e. $|\alpha| = m$ and

$$|\alpha^l\rangle = \bigotimes_{i\in\alpha} |l+1\rangle_i \bigotimes_{i\notin\alpha} |l\rangle_i \quad , \tag{3}$$

and the same holds for $|\beta^l\rangle$. We have

$$\sigma = \{(\alpha, \beta) : |\alpha \cap \beta| = m - 1\}, \qquad (4)$$

$$N_D = (d-1)m(n-m-1) . (5)$$

The innermost sum depends on (α, β) and runs over [43]

$$\delta = \begin{cases} \alpha & \text{if } k = l ,\\ \{\delta \mid \delta \subset \overline{\alpha \setminus \beta}\} & \text{if } k < l ,\\ \{\delta \mid \delta \subset \overline{\beta \setminus \alpha}\} & \text{if } k > l , \end{cases}$$
(6)

where the complement (overline) is taken with respect to the set $\{1, \ldots, n\}$. Now, the main result of this paper is that if any of these functions Q_m fulfills

$$Q_m > f - 2 \qquad (\text{ where } f \in \{2, \dots, d\})$$

$$\tag{7}$$

for a given density matrix ρ , then this state is at least f-dimensionally genuine multipartite entangled.

IV. PROOF

One standard strategy for detecting entanglement or distinguishing different types of entanglement is to introduce a quantity $Q(\rho)$ and to maximize it over all states of a specific type (e.g. k-separable states, states with particular Schmidt-rank, states of an entanglement class, etc.). Consequently, if for a particular state ρ this maximum is exceeded, it necessarily must be of a different kind. The main problem in deriving entanglement criteria in this way is the involved maximization. The complexity of this problem can be reduced by using quantities $Q(\rho)$ which are convex in ρ , because in this case the optimization has to be performed over all pure states only. Nevertheless, the difficulty of finding the global maximum remains.

In the present paper, a completely new approach is used. Namely, the convex quantities Q_m are constructed by incorporating the matrix elements of specific *f*-dimensionally GME states. This is done in a way such that by construction any other state of same or lower dimensionality cannot reach a certain bound. Thus, to prove that $Q_m \leq f - 1$ holds for all states which are entangled in equal or less than f degrees, no maximization has to be carried out.

First, consider the f-dimensionally genuine n-partite entangled GHZ state

$$|GHZ_f\rangle = \frac{1}{\sqrt{f}} \sum_{i=0}^{f-1} |i\rangle^{\otimes n} \quad . \tag{8}$$

In density matrix form $\rho_{fGHZ} = |GHZ_f\rangle \langle GHZ_f|$, the only nonzero elements are $\langle k|^{\otimes n} \rho_{fGHZ} |l\rangle^{\otimes n} = \frac{1}{f}$. Each term $|\langle k_n | \rho | l_n \rangle|$ in (1) singles out the absolute value of an off-diagonal element of ρ_{fGHZ} , such that

$$\sum_{k \neq l} |\langle k_n | \rho_{fGHZ} | l_n \rangle| = 2 \binom{f}{2} \frac{1}{f} = f - 1 .$$

$$\tag{9}$$

As can easily be confirmed, all terms $\sqrt{\langle k, l | \mathcal{P}_A^{\dagger} \rho_{fGHZ}^{\otimes 2} \mathcal{P}_A | k, l \rangle}$ vanish for any choice of k, l and any bipartition (A|B) as the corresponding matrix elements are all zero. Thus, we have shown that $Q_0 = f - 1$ for ρ_{fGHZ} . In addition, it was proven in [32] that

$$|\langle k_n | \rho | l_n \rangle| - \sum_{\gamma} \sqrt{\langle k, l | \mathcal{P}_A^{\dagger} \rho^{\otimes 2} \mathcal{P}_A | k, l \rangle} \le 0 , \qquad (10)$$

holds for all biseparable states. Hence, each of these terms (10) can only be larger than zero if the state ρ is genuine multipartite entangled in $|k\rangle$ and $|l\rangle$. Now, since in (1) the absolute values of all off-diagonal elements of ρ_{fGHZ} are added up, and since all terms which are subtracted from this sum are zero for ρ_{fGHZ} , it follows that $|GHZ_f\rangle$ is the only f-dimensionally GME pure state that reaches $Q_0 = f - 1$. Thus, any state that exceeds f - 1 must at least be (f + 1)-dimensionally GME, which proves (7) for m = 0.

Due to the way it is constructed, the function (1) is optimally suited to detect high-dimensional genuine multipartite entanglement in mixed states which are close to GHZ states. To show that the quantities Q_m serve their purpose (7) for m > 0, we introduce the *f*-dimensionally genuine multipartite entangled *m*-Dicke state $(m \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\})$

$$|D_f^m\rangle := \frac{1}{\sqrt{(f-1)\binom{n}{m}}} \sum_{l=0}^{f-2} \sum_{\alpha} \bigotimes_{i \in \alpha} |l+1\rangle_i \bigotimes_{i \notin \alpha} |l\rangle_i \quad , \tag{11}$$

where the inner sum runs over all α with $|\alpha| = m$ (see also [33]). Note that this includes a generalization $|W_f\rangle = |D_f^1\rangle$ of the prominent W state for qudits[44]. First, observe that for any fixed choice of k = l in Q_m , (2) reduces to the inequalities from Refs. [32, 33], i.e. in this case it is proven that $Q_m \leq 0$ holds for all biseparable states. On the other hand, by summing over all k and l in Q_m we add up the absolute value of specific off-diagonal elements $O_{\alpha,\beta}^{k,l}$ of the m-Dicke state $|D_f^m\rangle$. From these off-diagonal elements (determined by the proper set σ) there are corresponding diagonal elements. For a subset of all those off-diagonal elements this suffices (as with the inequality based on Q_0), however, for some there are no corresponding diagonal elements belonging to a biseparable state. In order to guarantee that $Q_m \leq 0$ for all biseparable states one also needs to subtract the corresponding diagonal elements of the Dicke state (labeled $D_{\alpha}^l)$. Counting the cardinality of this subset is a purely combinatorial problem (similar to [33]) resulting in the factor N_D . By construction, this guarantees for fixed dimensionality, the maximal value of Q_m for the corresponding *Dicke state*, as in this case the sum of the off-diagonal elements is maximal, whereas all $P_{\alpha,\beta}^{k,l}$ are zero. By scaling this maximum with the constant $\frac{1}{m}$ we can unify all quantities Q_m in one consistent framework, i.e. the only f-dimensionally GME pure state that can attain $Q_m = f - 1$ is $|D_f^m\rangle$.

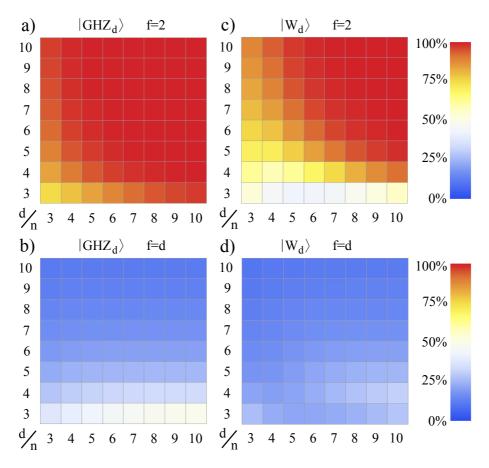


FIG. 1: (Color online) The noise resistance for the states $|GHZ_d\rangle$ and $|W_d\rangle$. The scale indicates up to which value of white noise a state is detected to be f-dimensionally genuine multipartite entangled (GME). a) up to the illustrated thresholds for p, the state $\rho = \frac{p}{dn} \mathbb{1} + (1-p) |GHZ_d\rangle \langle GHZ_d|$ is detected to be GME, i.e. $Q_0 > 0$. b) for noise p below the illustrated thresholds the state ρ is detected by $Q_0 > d-2$ to be truly d-dimensionally GME. c) and d) illustrate these thresholds for the state $\rho = \frac{p}{dn} \mathbb{1} + (1-p) |W_d\rangle \langle W_d|$ using $Q_1(\rho) > f - 2$.

V. DETECTION STRENGTH

The introduced criteria allow to examine the dimensionality of multipartite entanglement in a noisy environment. Fig. 1 shows the robustness for the states $|GHZ_d\rangle$ and $|W_d\rangle$ in the presence of white noise. We compared the illustrated thresholds with the thresholds of entanglement witnesses that follow from the fidelity of a state (see [27, 31]), i.e. a state is *d*-dimensionally GME if $\langle GHZ_d | \rho | GHZ_d \rangle > \frac{d-1}{d}$ or $\langle W_d | \rho | W_d \rangle > \frac{n(d-1)-1}{n(d-1)}$, respectively. Here, we found that our criteria are strictly stronger for all d > 2 and n > 2 – specifically, they outperform the noise robustness of previously known criteria [31] for GHZ-like states. E.g., the tripartite state

$$\rho = (1-p) \left| GHZ_3 \right\rangle \left\langle GHZ_3 \right| + p \frac{1}{27} \mathbb{1} , \qquad (12)$$

is detected to be 3-dimensionally GME by $Q_0 > 1$ for $0 \le p < 0.375$, whereas $\langle GHZ_3 | \rho | GHZ_3 \rangle > \frac{2}{3}$ merely detects the range $0 \le p < 0.346$. For

$$\rho = (1-p) |W_3\rangle \langle W_3| + p \frac{1}{27} \mathbb{1} , \qquad (13)$$

the difference is even more significant: Using $Q_1 > 1$, we detect the range $0 \le p < 0.265$ in comparison to $0 \le p < 0.173$ following from $\langle W_3 | \rho | W_3 \rangle > \frac{5}{6}$. A further example of the detection strength is given

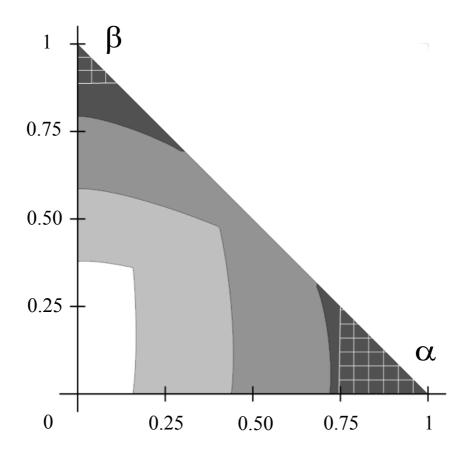


FIG. 2: Illustration of the detection strength of $Q_0, Q_1 > f - 2$ for the tripartite four-level state $\rho = \alpha |GHZ_4\rangle \langle GHZ_4| + \beta |W_4\rangle \langle W_4| + \frac{1-\alpha-\beta}{64} \mathbb{1}$. The dark gray region is detected to be 4-dimensionally GME (In comparison, fidelity-based criteria merely detect the white meshed part of this region to be 4-dimensionally GME). The middle gray region is detected to be at least 3-dimensionally GME, and the light gray region is detected to be at least 2-dimensionally GME.

in Fig. 2. As can be seen therein, the region of states detected by our criteria is considerably larger than the region revealed by fidelity-based witnesses. Finally, let us stress that the quantities Q_m are by construction optimally suited to detect f-dimensional GME states which are close to GHZ, W and Dicke states. If instead an unclassified input state is given one can improve the detection by maximizing the outcome of Q_m over local-unitary transformations. An appropriate optimization scheme can be found in [38–40].

VI. CONCLUSION

Creating high-dimensional multipartite entangled states is one of the current challenges in experiments on quantum physics. In the present paper, we gave a precise mathematical characterization of such states and provided criteria for the dimensionality of genuine multipartite entanglement applicable to arbitrary multi-qudit systems. These criteria are easily computable since they do not rely on semidefinite programming or eigenvalue computations, but only on functions of density matrix elements. They are also advantageous in experiments, as they are rather robust against noise and to apply them it is not necessary to determine the entire density matrix of the system under consideration. In detail, due to the fact that the quantities Q_m only involve the matrix elements of *GHZ*, *W* and *Dicke states*, it is merely needed to determine these few entries of the density matrix, which can always be achieved via local measurements and corresponding correlations (see also the discussion in e.g. [32–35]). Consequently, they can be experimentally implemented with a reduced number of local observables, since the number of measurements for a full quantum state tomography scales exponentially in the number of parties n, i.e. is of the order $\mathcal{O}(d^{2n})$ [41], whereas the number of density matrix elements that occur in Q_m is only of the order $\mathcal{O}(d^{2n}_m)$), that is polynomial in n (Note that the notation in terms of two-fold copies of a state is only a matter of compactness, i.e. in experiments it is not necessarily needed to have two copies at a time). Finally, it is noteworthy that our results are even promising to be closely related to measures of genuine multipartite entanglement, as e.g. for multipartite qubits the quantity Q_0 yields a strong lower bound on the gme-concurrence [42].

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