

Entanglement Detection and Distillation for Arbitrary Bipartite Systems

Ming-Jing Zhao¹, Ting-Gui Zhang¹, Xianqing Li-Jost¹ and Shao-Ming Fei^{1,2}

¹ Max-Planck-Institute for Mathematics in the Sciences, Leipzig 04103, Germany

² School of Mathematical Sciences, Capital Normal University, Beijing 100048, China

Abstract We present an inequality for detecting entanglement and distillability of arbitrary dimensional bipartite systems. This inequality provides a sufficient condition of entanglement for bipartite mixed states, and a necessary and sufficient condition of entanglement for bipartite pure states. Moreover, the inequality also gives a necessary and sufficient condition for distillability.

Keywords Entanglement·Distillability·Inequality

I. INTRODUCTION

Entanglement is one of the most fascinating features of quantum theory and has numerous applications in quantum information processing [1]. As a result, various approaches have been proposed and many significant conclusions have been derived in detecting entanglement [2–13]. However there are yet no operational necessary and sufficient separability criteria for general higher dimensional quantum states. In particular, experimental detection of quantum entanglement by measuring some suitable quantum mechanical observables has practical importance. The Bell inequalities can be used to detect perfectly the entanglement of pure bipartite states [14–18]. Besides Bell inequalities, the entanglement witness could also be used for experimental detection of quantum entanglement for some special states [4, 19, 20], which could give rise to entanglement estimation [21]. The two-copy measurement of concurrence for two-qubit pure states has been already realized experimentally [22–24]. For arbitrary dimensional bipartite pure states, one method of measuring concurrence has been presented in terms of one-copy measurement [25]. Except that, uncertainty relations are also favorable in entanglement detection, for instance, one separability condition has been derived for all negative partial transpose state in experimentally accessible forms [26].

For bipartite mixed states, a necessary and sufficient inequality has been derived for detecting entanglement of two-qubit states [27]. In Ref. [28] based on a different approach,

an inequality is presented, which is both necessary and sufficient in detecting entanglement of qubit-qutrit states, and necessary for qubit-qudit states.

In this article we propose an inequality detecting entanglement for arbitrary dimensional bipartite states, which consists of local observables with measurement outcomes ± 1 . This inequality gives a necessary condition of separability for mixed states, namely, any violation of the inequality implies entanglement. It is shown that our inequality can detect entanglement of a wide class of states such as Horodecki's state, isotropic state and Werner state. For pure states, the inequality becomes both necessary and sufficient. All pure entangled states violate it. Moreover, our inequality provides also a necessary and sufficient condition for entanglement distillability, an approach to get the ideal resource from general quantum mixed states for quantum information processing [29, 30].

The paper is organized as follows. In Sec. II, we first propose an inequality in terms of the mean values of local observables. Then we prove that all separable states obey this inequality and all pure entangled states violate it. As applications some detailed examples are presented. We show then that the inequality is a necessary and sufficient condition for distillability. Comments and conclusions are given in the last section.

II. INEQUALITY FOR ARBITRARY BIPARTITE SYSTEMS

Let H_m, H_n be m, n -dimensional vector spaces with $\{|i\rangle\}_{i=0}^{m-1}$ and $\{|j\rangle\}_{j=0}^{n-1}$ the computational basis respectively. Set $\lambda_0^A = I_m$ the $m \times m$ identity matrix, $\lambda_i^A = |0\rangle\langle 0| - |i\rangle\langle i|$, $\mu_1^A = |0\rangle\langle 1| + |1\rangle\langle 0|$, and $\mu_2^A = i|0\rangle\langle 1| - i|1\rangle\langle 0|$, where $|i\rangle \in H_m, i = 1, \dots, m-1$. Set $\lambda_0^B = I_n$ the $n \times n$ identity matrix, $\lambda_i^B = |0\rangle\langle 0| - |i\rangle\langle i|$, $\mu_1^B = |0\rangle\langle 1| + |1\rangle\langle 0|$, and $\mu_2^B = i|0\rangle\langle 1| - i|1\rangle\langle 0|$, where $|i\rangle \in H_n, i = 1, \dots, n-1$. Let $A_i = U\lambda_i^A U^\dagger, i = 0, 1, \dots, m-1, A'_j = U\mu_j^A U^\dagger, j = 1, 2$, be a set of quantum mechanical observables acting on the first subsystem, with U any $m \times m$ unitary matrix. Let $B_i = V\lambda_i^B V^\dagger, i = 0, 1, \dots, n-1, B'_j = V\mu_j^B V^\dagger, j = 1, 2$, be a set of quantum mechanical observables acting on the second subsystem, with V any $n \times n$ unitary matrix. Here Roman letter i represents the imaginary unit.

We define

$$\begin{aligned}
H_{U,V}^{(m,n)} &= \frac{1}{mn} \sum_{i,j} (1 - \frac{m}{2}\delta_{i1} - \frac{n}{2}\delta_{j1}) A_i \otimes B_j; \\
P_{U,V}^{(m,n)} &= \frac{1}{2m^2n^2} \sum_{i,j} (m\delta_{i1} - n\delta_{j1}) A_i \otimes B_j; \\
Q_{U,V}^{(m,n)} &= \frac{1}{16} (A'_1 \otimes B'_1 - A'_2 \otimes B'_2),
\end{aligned} \tag{1}$$

where $\delta_{kl} = 1$ if $k = l$ and is zero otherwise. According to the mean values of the operators in Eq. (1), we can construct an inequality detecting entanglement for $m \otimes n$ systems.

Theorem 1 *Any separable state ρ in $H_m \otimes H_n$ obeys the following inequality*

$$\langle H_{U,V}^{(m,n)} \rangle_\rho^2 \geq \langle P_{U,V}^{(m,n)} \rangle_\rho^2 + \langle Q_{U,V}^{(m,n)} \rangle_\rho^2 \tag{2}$$

for all $m \times m$ unitary matrix U and $n \times n$ unitary matrix V .

Proof. First we show that inequality (2) holds for all pure separable states, which is equivalent to prove that for arbitrary pure separable state ρ , the following inequality holds:

$$\langle H_{I_m, I_n}^{(m,n)} \rangle_\rho^2 \geq \langle P_{I_m, I_n}^{(m,n)} \rangle_\rho^2 + \langle Q_{I_m, I_n}^{(m,n)} \rangle_\rho^2. \tag{3}$$

Note that any pure separable state can be written as $|\xi\rangle = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_i b_j |ij\rangle$ with $\sum_{i=0}^{m-1} |a_i|^2 = \sum_{j=0}^{n-1} |b_j|^2 = 1$. Inserting this separable pure state $\rho = |\xi\rangle\langle\xi|$ into Eq. (3), one gets that

$$\langle H_{I_m, I_n}^{(m,n)} \rangle_{|\xi\rangle\langle\xi|} = \frac{1}{2} (|a_0 b_1|^2 + |a_1 b_0|^2) \geq 0. \tag{4}$$

While the right hand side of inequality (3) becomes

$$\langle P_{I_m, I_n}^{(m,n)} \rangle_{|\xi\rangle\langle\xi|}^2 + \langle Q_{I_m, I_n}^{(m,n)} \rangle_{|\xi\rangle\langle\xi|}^2 = \frac{1}{4} (|a_0 b_1|^2 - |a_1 b_0|^2)^2 + \text{Re}(a_0 a_1^* b_0 b_1^*)^2. \tag{5}$$

From Eqs. (4) and (5), it is easy to obtain that inequality (3) holds for any pure separable states.

We now prove that inequality (2) also holds for general separable mixed states,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1,$$

where $|\psi_i\rangle$ are all pure separable states. Denote

$$\begin{aligned}
c_i &= \langle H_{U,V}^{(m,n)} \rangle_{|\psi_i\rangle\langle\psi_i|}; \\
d_i &= \langle P_{U,V}^{(m,n)} \rangle_{|\psi_i\rangle\langle\psi_i|}; \\
e_i &= \langle Q_{U,V}^{(m,n)} \rangle_{|\psi_i\rangle\langle\psi_i|}.
\end{aligned}$$

Since inequality (2) holds for all pure separable states, one has $c_i^2 \geq d_i^2 + e_i^2$. Taking into account that if the inequality $c_i^2 \geq d_i^2 + e_i^2$ holds for arbitrary real numbers d_i, e_i and nonnegative $c_i, i = 1, \dots, N$, then $(\sum_{i=1}^N p_i c_i)^2 \geq (\sum_{i=1}^N p_i d_i)^2 + (\sum_{i=1}^N p_i e_i)^2$ for $0 \leq p_i \leq 1$ and $\sum_{i=1}^N p_i = 1$, we know that any mixed separable state ρ obeys inequality (2). \square

This theorem shows that any violation of this inequality implies entanglement. Hence it gives a sufficient condition of entanglement for arbitrary dimensional bipartite systems. Next we prove that for the pure state case, one has that all pure entangled states violate this inequality for some unitary U and V , so that the inequality provides a necessary and sufficient condition for entanglement of pure states.

Theorem 2 *A pure state in $H_m \otimes H_n$ is entangled if and only if it violates inequality (2) for some unitary matrices U and V .*

Proof. Here we only need to prove that any pure entangled state in $H_m \otimes H_n$ violates inequality (2) for some unitary matrices U and V . Without loss of generality, we assume $m \geq n$. By Schmidt decomposition any entangled pure state $|\phi\rangle$ can be transformed into $|\phi'\rangle$:

$$|\phi'\rangle = U^\dagger \otimes V^\dagger |\phi\rangle = \sum_{i=0}^{n-1} a_i |ii\rangle, \quad (6)$$

with $a_0 > 0, a_1 > 0$, and $a_i \geq 0$ for $i = 2, \dots, n-1$, $\sum_{i=0}^{n-1} |a_i|^2 = 1$. Here U is an $m \times m$ unitary matrix and V is an $n \times n$ unitary matrix. Then from the PPT criterion [3, 4] there exist nonzero real numbers a and b satisfying $a^2 + b^2 = 1$ and $|\psi\rangle = a|01\rangle + b|10\rangle$ such that

$$\langle (|\psi\rangle\langle\psi|)^{T_1} \rangle_{|\phi'\rangle\langle\phi'|} = \langle\psi|(|\phi'\rangle\langle\phi'|)^{T_1}|\psi\rangle < 0.$$

By expanding the partial transposed matrix $|\psi\rangle\langle\psi|^{T_1}$ according to the matrices $\{\lambda_i^A\}, \{\mu_i^A\}, \{\lambda_i^B\}$ and $\{\mu_i^B\}$ on the first and second subsystems respectively, we get

$$\begin{aligned}
& \langle (|\psi\rangle\langle\psi|)^{T_1} \rangle_{|\phi'\rangle\langle\phi'|} \\
&= \frac{1}{mn} \langle \sum_{i,j \neq 1} \lambda_i \otimes \lambda_j + \sum_{i,j} [(\frac{-m+2}{2} + \frac{m}{2}\sqrt{1-C^2})\delta_{i1} \\
&+ (\frac{-n+2}{2} - \frac{n}{2}\sqrt{1-C^2})\delta_{j1}] \lambda_i \otimes \lambda_j \rangle_{|\phi'\rangle\langle\phi'|} + \frac{1}{4} C \langle \mu_1^A \otimes \mu_1^B - \mu_2^A \otimes \mu_2^B \rangle_{|\phi'\rangle\langle\phi'|} \\
&\geq \langle H_{I_m, I_n}^{(m,n)} \rangle_{|\phi'\rangle\langle\phi'|} - \left| \sqrt{1-C^2} \langle P_{I_m, I_n}^{(m,n)} \rangle_{|\phi'\rangle\langle\phi'|} + C \langle Q_{I_m, I_n}^{(m,n)} \rangle_{|\phi'\rangle\langle\phi'|} \right| \\
&\geq \langle H_{I_m, I_n}^{(m,n)} \rangle_{|\phi'\rangle\langle\phi'|} - \left(\langle P_{I_m, I_n}^{(m,n)} \rangle_{|\phi'\rangle\langle\phi'|}^2 + \langle Q_{I_m, I_n}^{(m,n)} \rangle_{|\phi'\rangle\langle\phi'|}^2 \right)^{\frac{1}{2}} \\
&= \langle H_{U,V}^{(m,n)} \rangle_{|\phi\rangle\langle\phi|} - \left(\langle P_{U,V}^{(m,n)} \rangle_{|\phi\rangle\langle\phi|}^2 + \langle Q_{U,V}^{(m,n)} \rangle_{|\phi\rangle\langle\phi|}^2 \right)^{\frac{1}{2}},
\end{aligned} \tag{7}$$

where $C = 2ab$ is just the concurrence of the pure state $|\psi\rangle$, defined by $C(|\psi\rangle) = \sqrt{2(1 - \text{Tr} \rho_1^2)}$. ρ_1 is the reduced density matrix $\rho_1 = \text{Tr}_2(|\psi\rangle\langle\psi|)$, where Tr_2 stands for the partial trace with respect to the second subsystem. Here $A_i = U\lambda_i^A U^\dagger$, $i = 0, 1, \dots, m-1$, $A'_j = U\mu_j^A U^\dagger$, $j = 1, 2$, $B_i = V\lambda_i^B V^\dagger$, $i = 0, 1, \dots, n-1$, $B'_j = V\mu_j^B V^\dagger$, $j = 1, 2$. U and V are just the unitary matrices from the Schmidt decomposition in Eq. (6). The first inequality is due to $-|x| \leq x$ and the second one is from the Cauchy inequality. Since the left hand side of (7) is negative, we have that the pure entangled state $|\phi\rangle$ violates inequality (2) with respect to the corresponding observables. \square

Therefore any pure state $|\psi\rangle$ in $H_m \otimes H_n$ is separable if and only if inequality (2) is satisfied. Inequality (2) gives a necessary and sufficient criterion for the separability of pure states, which may be determined by experimental measurements on the local observables. For example, for pure entangled state $|\phi'\rangle$ defined above, it violates inequality (3) corresponding to $U = I_m$ and $V = I_n$ in inequality (2).

In fact for pure bipartite states there are already many Bell inequalities like Refs. [15, 18] in terms of the expectation of the local observables and Refs. [16, 17] in terms of probabilities for $d \otimes d$ systems. Inequality (2) is for arbitrary dimensional pure bipartite systems, no matter whether the dimensions of the two subsystems are the same or not. Moreover, it also provides sufficient condition for entanglement of mixed states.

Inequality (2) can be associated with nonlinear entanglement witness operators, where the violation is replaced by a negative expectation value. In fact, for given U and V , inequality (2) gives rise to a kind of nonlinear entanglement witness $W_{U,V}^{(m,n)}$:

$$\langle W_{U,V}^{(m,n)} \rangle_\rho \equiv \langle H_{U,V}^{(m,n)} \rangle_\rho - (\langle P_{U,V}^{(m,n)} \rangle_\rho^2 + \langle Q_{U,V}^{(m,n)} \rangle_\rho^2).$$

For all separable states σ , $\langle W_{U,V}^{(m,n)} \rangle_\sigma \geq 0$. If $\langle W_{U,V}^{(m,n)} \rangle_\rho < 0$ then ρ is entangled. As an entanglement witness gives rise to a surface in the state space, dividing the space into a set of entangled states and the rest, the detection of different entangled states depends on the choice of different witnesses. Here different choices of U and V detect different sets of entangled states.

Generally linear entanglement witness operators can be associated with positive maps in terms of Jamiołkowski isomorphism. And positive maps could give rise to entanglement linear witness operators, see e.g. [31]. Our inequality (2) is not linear, as many Bell-type inequalities for mixed states [27, 28]. The entanglement witness operators deduced from the inequality is also nonlinear. The direct relations between the entanglement witness operators and the positive maps are not obvious.

Subsequently, we consider the maximal violation of inequality (2). Let $F^{(m,n)}(\rho) = \max_{U,V} \{-\langle W_{U,V}^{(m,n)} \rangle_\rho, 0\}$ denote the maximal violation value with respect to a given state ρ , under all U and V . Then $F^{(m,n)}(\rho) = 0$ if ρ is separable. Additionally, $F^{(m,n)}(\rho)$ is invariant under the local unitary transformations.

From another point of view, inequality (2) can be viewed as the generalization of the main result in Ref. [27] and Ref. [28]. For $m = n = 2$, inequality (2) becomes the one in Ref. [27] which is necessary and sufficient in detecting entanglement for pure or mixed states. For $m = 2$ and $n = 3$, inequality (2) is also the necessary and sufficient condition of separability [28]. But for $m > 3$ or $n > 3$, this inequality is only necessary for separability of mixed states, but necessary and sufficient for pure states. As is shown in Ref. [28], for $m = 2$ the inequality can detect some PPT entanglement. In the following we give some examples concerning entanglement detection in terms of inequality (2).

Example 1. Horodecki's $3 \otimes 3$ state:

$$\sigma_\alpha = \frac{2}{7}|\psi^+\rangle\langle\psi^+| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-,$$

where $\sigma_+ = \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)$, $\sigma_- = \frac{1}{3}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|)$, $|\psi^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$. σ_α is separable for $2 \leq \alpha \leq 3$, bound (PPT) entangled for $3 < \alpha \leq 4$,

and free entangled for $4 < \alpha \leq 5$ [32]. Employing inequality (3) for $3 \otimes 3$ systems,

$$\begin{aligned}
& 4\langle 2I_3 \otimes I_3 - I_3 \otimes \lambda_1^B + 2I_3 \otimes \lambda_2^B - \lambda_1 \otimes I_3 - 4\lambda_1 \otimes \lambda_1^B \\
& - \lambda_1^A \otimes \lambda_2^B + 2\lambda_2^A \otimes I_n - \lambda_2^A \otimes \lambda_1^B + 2\lambda_2^A \otimes \lambda_2^B \rangle_\rho^2 \\
& \geq 36\langle I_3 \otimes \lambda_1^B - \lambda_1^A \otimes I_3 - \lambda_1^A \otimes \lambda_2^B + \lambda_2^A \otimes \lambda_1^B \rangle_\rho^2 \\
& + 81\langle \mu_1^A \otimes \mu_1^B + \mu_2^A \otimes \mu_2^B \rangle_\rho^2,
\end{aligned} \tag{8}$$

we have that the left hand side of inequality (8) is $\frac{900}{49}$ and the right hand side of inequality (8) is $\frac{36(2\alpha-5)^2}{49} + \frac{576}{49}$. Hence σ_α violates inequality (8) if and only if $\alpha > 4$.

Example 2. Isotropic states are a class of $U \otimes U^*$ invariant mixed states in $H_n \otimes H_n$ [6]:

$$\rho_{iso}(f) = \frac{1-f}{n^2-1}I_n + \frac{n^2f-1}{n^2-1}|\psi^+\rangle\langle\psi^+|,$$

with $f = \langle\psi^+|\rho_{iso}(f)|\psi^+\rangle$ satisfying $0 \leq f \leq 1$, $|\psi^+\rangle = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}|ii\rangle$. These states are shown to be separable if and only if they are PPT, i.e. $f \leq \frac{1}{n}$. Now we utilize inequality (3) with $m = n$. It can be verified that the left hand side of inequality (3) is $(\frac{1-f}{n^2-1})^2$ and the right hand side of inequality (3) is $(\frac{n^2f-1}{n^2(n^2-1)})^2$ for $\rho_{iso}(f)$. If we choose $U = V = I_n$ in inequality (2), then the violation of this inequality for this isotropic state is $-\langle W_{I_n, I_n}^{(n,n)} \rangle_{\rho_{iso}(f)} = (\frac{n^2f-1}{n^2(n^2-1)})^2 - (\frac{1-f}{n^2-1})^2$, which is positive if and only if $f > \frac{1}{n}$. Therefore, inequality (3) can detect all the entanglement of isotropic states.

Example 3. Werner states are a class of $U \otimes U$ invariant mixed states in $H_n \otimes H_n$ [33]:

$$\rho_{wer}(f) = \frac{n-f}{n^3-n}I_n + \frac{nf-1}{n^3-n}\tilde{V},$$

where $\tilde{V} = \sum_{i,j=0}^{n-1}|ij\rangle\langle ji|$ and $f = \langle\psi^+|\rho_{wer}(f)|\psi^+\rangle$, $-1 \leq f \leq 1$. These states are separable if and only if they are PPT, i.e. $f \geq 0$. According to inequality (2), we choose $U = |0\rangle\langle 1| + |0\rangle\langle 1| + \sum_{i=2}^{n-1}|i\rangle\langle i|$ and $V = I_n$, then the violation of this inequality is $-\langle W_{U,V}^{(n,n)} \rangle_{\rho_{wer}(f)} = (\frac{nf-1}{n^3-n})^2 - (\frac{f+1}{n(n+1)})^2$, which is positive if and only if $f < \frac{2-n}{2n-1}$ (See FIG. 1). Obviously, when $n = 3$, Werner state violates this inequality if $f < -0.2$.

In the following, we study the relations between inequality (2) and entanglement distillability.

Theorem 3 *A state $\rho \in H_m \otimes H_n$ is distillable if and only if there exist two projectors A and B mapping high dimensional spaces to two dimensional ones such that the restriction of the state of N copies of ρ to such $2 \otimes 2$ subspace violates inequality (2).*

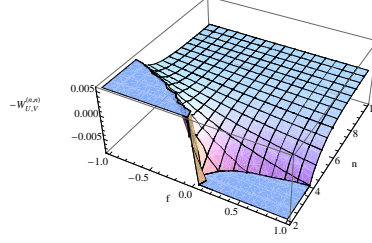


FIG. 1: The violation value of Werner state with respect to inequality (2) under $U = |0\rangle\langle 1| + |0\rangle\langle 1| + \sum_{i=2}^{n-1} |i\rangle\langle i|$ and $V = I_n$.

Proof. Any state ρ is distillable if and only if there exist two projectors A and B mapping high dimensional spaces to two dimensional ones such that $A \otimes B \rho^{\otimes N} A \otimes B$ is entangled [30]. However, $A \otimes B \rho^{\otimes N} A \otimes B$ is two-qubit entangled if and only if there exist $m^N \times m^N$ unitary matrix U_0 , $n^N \times n^N$ unitary matrix V_0 , and nonzero real numbers a and b , $a^2 + b^2 = 1$, such that $U_0 \otimes V_0(a|01\rangle + b|10\rangle)$ is the eigenvector of $A \otimes B(\rho^{\otimes N})^{T_1} A \otimes B$ with respect to a negative eigenvalue. Taking into account that inequality (2) is the necessary and sufficient condition for two-qubit entanglement [28], we have that ρ is distillable if and only if one finds two projectors A and B mapping high dimensional spaces to two dimensional ones such that the restriction of the state of N copies of ρ to such $2 \otimes 2$ subspace violates inequality (2). \square

Some necessary conditions for distillability have been proposed in Refs. [6, 18, 34]. The distillability condition we obtained here is both necessary and sufficient and could be verified experimentally in principle.

Example 4. For non-PPT (NPT) entangled state

$$\begin{aligned} \varrho_1 = & \frac{p}{6}(|00\rangle\langle 00| + |01\rangle\langle 01| + |02\rangle\langle 02| + |10\rangle\langle 10| + |11\rangle\langle 11| + |12\rangle\langle 12|) - \frac{p}{6}(|00\rangle\langle 12| \\ & + |01\rangle\langle 12| + |12\rangle\langle 00| + |12\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|) + \frac{1-p}{2}(|22\rangle\langle 22| + |33\rangle\langle 33|), \end{aligned}$$

Ref. [35] has proved that the distillability of this state can not be detected by reduction criterion [6] and the criterion in Ref. [34]. The distillability of this state can not be recognized either by the result in Ref. [18]. However, setting $A = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $B = (|0\rangle + |1\rangle)(\langle 0| + \langle 1|) + |2\rangle\langle 2|$, one can verify that $A \otimes B \varrho_1 A \otimes B$ violates inequality (2) with $m = 2$, $n = 3$ and $U = I_2$, $V = |1\rangle\langle 2| + \frac{1}{\sqrt{2}}|0\rangle(\langle 0| + \langle 1|) + \frac{1}{\sqrt{2}}|2\rangle(\langle 0| - \langle 1|)$. Therefore our inequality gives a better recognition of distillability for this case.

III. COMMENTS AND CONCLUSIONS

We have provided an inequality in terms of the expectation values of local observables, each with two possible measurement outcomes, for detecting entanglement and distillability for arbitrary dimensional bipartite systems. Any violation of this inequality implies that the state being measured is entangled. Moreover, all pure entangled states violate the inequality, namely the inequality is both necessary and sufficient in detecting entanglement for bipartite pure states. As examples we have analyzed the entanglement detection of the Horodecki's state, isotropic state and Werner state. It has been shown that the inequality can detect considerable mixed entangled states. Above all, the inequality is a necessary and sufficient condition of distillability. The results may be used in experimental entanglement detection and distillability verification.

In addition, since the dimensions of the two subsystems in inequality (2) are arbitrary, this inequality could be also used to detect entanglement in multipartite systems: if a multipartite state is bipartite separable under some partition, then it fulfills inequality (2) under the corresponding bipartite partition. A multipartite pure state is bipartite separable if and only if it fulfills inequality (2) for all unitary operators U and V under this bipartite partition, for which the two subsystems may have different dimensions.

Acknowledgments S. M. Fei acknowledges the support from NSFC under No. 11275131.

-
- [1] Nielsen, M.A., Chuang, I.L. Quantum Computation and Quantum Information (Cambridge University Press, Cambridge) (2000)
 - [2] Gühne, O., Tóth, G.: Entanglement detection. Phys. Rep. **474**, 1 (2009)
 - [3] Peres, A.: Separability Criterion for Density Matrices. Phys. Rev. Lett. **77**, 1413 (1996)
 - [4] Horodecki, M., Horodecki, P., Horodecki, R.: Separability of mixed states: necessary and sufficient conditions. Phys. Lett. A **223**, 1 (1996)
 - [5] Lewenstein, M., Sanpera, A.: Separability and Entanglement of Composite Quantum Systems. Phys. Rev. Lett. **80**, 2261 (1998)
 - [6] Horodecki, M., Horodecki, P.: Reduction criterion of separability and limits for a class of distillation protocols. Phys. Rev. A **59**, 4206 (1999)

- [7] Nielsen, M.A., Kempe, J.: Separable States Are More Disordered Globally than Locally. *Phys. Rev. Lett.* **86**, 5184 (2001)
- [8] Horodecki, P.: Separability criterion and inseparable mixed states with positive partial transposition. *Phys. Lett. A* **232**, 333 (1997)
- [9] Chen, K., Wu, L.A.: A matrix realignment method for recognizing entanglement. *Quantum Inf. Comput.* **3**, 193 (2003)
- [10] Rudolph, O.: Further Results on the Cross Norm Criterion for Separability. *Quantum Inf. Proc.* **3**, 219 (2005)
- [11] Cavalcanti, D., Cunha, M.O.T.: Estimating entanglement of unknown states. *App. Phys. Lett.* **89**, 084102 (2006)
- [12] Brandao, F.G.S.L., Vianna, R.O.: Robust semidefinite programming approach to the separability problem. *Phys. Rev. A* **70**, 062309 (2004)
- [13] Wiesniak, M., Vedral, V., Brukner, C.: Magnetic susceptibility as a macroscopic entanglement witness. *New J. Phys.* **7**, 258 (2005)
- [14] Gisin, N.: Bell's inequality holds for all non-product states. *Phys. Lett. A* **154**, 201 (1991)
- [15] Gisin, N., Peres, A.: Maximal violation of Bell's inequality for arbitrarily large spin. *Phys. Lett. A* **162**, 15 (1992)
- [16] Collins, D., Gisin, N., Linden, N., Massar, S., Popescu, S.: Bell Inequalities for Arbitrarily High-Dimensional Systems. *Phys. Rev. Lett.* **88**, 040404 (2002)
- [17] Chen, J.L., Deng, D.L., Hu, M.G.: Gisin's theorem for two d-dimensional systems based on the Collins-Gisin-Linden-Masser-Popescu inequality. *Phys. Rev. A* **77**, 060306 (2008)
- [18] Li, M., Fei, S.M.: Gisin's Theorem for Arbitrary Dimensional Multipartite States. *Phys. Rev. Lett.* **104**, 240502 (2010)
- [19] Hyllus, P., Gühne, O., Bruß, D., Lewenstein, M.: Relations between entanglement witnesses and Bell inequalities. *Phys. Rev. A* **72**, 012321 (2005)
- [20] Lewenstein, M., Kraus, B., Horodecki, P., Cirac, J.I.: Characterization of separable states and entanglement witnesses. *Phys. Rev. A* **63**, 044304 (2001)
- [21] Eisert, J., Brandão, F.G.S.L., Audenaert, K.M.R.: Quantitative entanglement witnesses. *New J. Phys.* **9**, 46 (2007)
- [22] Mintert, F., Kuś, M., Buchleitner, A.: Concurrence of Mixed Multipartite Quantum States. *Phys. Rev. Lett.* **95**, 260502 (2005)

- [23] Walborn, S.P., Ribeiro, P.H.S., Davidovich, L., Mintert, F., Buchleitner, A.: Experimental determination of entanglement with a single measurement. *Nature* **440**, 1022 (2006)
- [24] Walborn, S.P., Ribeiro, P.H.S., Davidovich, L., Mintert, F., Buchleitner, A.: Experimental determination of entanglement by a projective measurement. *Phys. Rev. A* **75**, 032338 (2007)
- [25] Fei, S.M., Zhao, M.J., Chen, K., Wang, Z.X.: Experimental determination of entanglement for arbitrary pure states. *Phys. Rev. A* **80**, 032320 (2009)
- [26] Nha, H., Zubairy, M.S.: Uncertainty Inequalities as Entanglement Criteria for Negative Partial-Transpose States. *Phys. Rev. Lett.* **101**, 130402 (2008)
- [27] Yu, S., Pan, J.W., Chen, Z.B., Zhang, Y.D.: Comprehensive Test of Entanglement for Two-Level Systems via the Indeterminacy Relationship. *Phys. Rev. Lett.* **91**, 217903 (2003)
- [28] Zhao, M.J., Ma, T., Fei, S.M., Wang, Z.X.: Inequalities detecting quantum entanglement for $2 \otimes d$ systems. *Phys. Rev. A* **83**, 052120 (2011)
- [29] Bennett, C.H., Brassard, G., Popescu, S., Schumacher, B., Smolin, J.A., Wootters, W.K.: Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels. *Phys. Rev. Lett.* **76**, 722 (1996)
- [30] Horodecki, M., Horodecki, P., Horodecki, R.: Mixed-State Entanglement and Distillation: Is there a “Bound” Entanglement in Nature?. *Phys. Rev. Lett.* **80**, 5239 (1998)
- [31] Qi, X., Hou, J.: Characterization of optimal entanglement witnesses, *Phys. Rev. A* **85**, 022334 (2012)
- [32] Horodecki, P., Horodecki, M., Horodecki, R.: Bound Entanglement Can Be Activated. *Phys. Rev. Lett.* **82**, 1056 (1999)
- [33] Werner, R.F.: Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Phys. Rev. A* **40**, 4277 (1989)
- [34] Ou, Y.C., Fan, H., Fei, S.M.: Proper monogamy inequality for arbitrary pure quantum states. *Phys. Rev. A* **78**, 012311 (2008)
- [35] Zhao, M.J., Zhu, X.N., Fei, S.M., Li-Jost, X.: Lower bound on concurrence and distillation for arbitrary-dimensional bipartite quantum states. *Phys. Rev. A* **84**, 062322 (2011)