

# Lower bound of quantum uncertainty from extractable classical information

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The sum of entropic uncertainties for the measurement of two non-commuting observables is not always reduced by the amount of entanglement (quantum memory) between two parties, and in certain cases may be impacted by quantum correlations beyond entanglement (discord). An optimal lower bound of entropic uncertainty in the presence of any correlations may be determined by fine-graining. Here we express the uncertainty relation in a new form where the maximum possible reduction of uncertainty is shown to be given by the extractable classical information. We show that the lower bound of uncertainty matches with that using fine-graining for several examples of two-qubit pure and mixed entangled states, and also separable states with non-vanishing discord. Using our uncertainty relation we further show that even in the absence of any quantum correlations between the two parties, the sum of uncertainties may be reduced with the help of classical correlations.

*Keywords* : Uncertainty relation; Classical information; Quantum Discord

## I. INTRODUCTION

A fundamental difference from classical theory is that quantum theory limits the precision of the measurement outcomes for the measurement of two non-commuting observables. This quantum feature called uncertainty relation was first introduced by Heisenberg [1], and then extended by Robertson [2] for more general observables. The lower bound of the Heisenberg uncertainty relation is state dependent. Later, the uncertainty relation was recast in an entropic form where the uncertainty is measured by the Shannon entropy [3]. A form of the entropic uncertainty relation (EUR) introduced by Deutsch [4], was subsequently improved in the version conjectured in Ref.[5] and then proved in Ref.[6]. (See, Ref.[7] for a review of the development of EURs).

In the derivation of the above mentioned uncertainty relations, the correlation of the observed system with another system called quantum memory is not considered. Berta et al., in the Ref. [8], discussed the possibility of reduction of the lower bound of EUR in the scenario when one considers the correlation of the observed system with the quantum memory. For example, when the observed system is maximally entangled with the quantum memory, the lower bound of EUR becomes zero for the measurement of two non-commuting observables. This phenomena has been brought out in two recent experiments using respectively, pure [10] and mixed states [11]. It has been shown in Ref. [12], that the lower bound of EUR in the presence of quantum memory may be *optimized* in an experimental scenario using the *fine-grained* uncertainty relation (FUR) [13]. Recently, Coles and Piani [9] have developed the analysis in order to make the bound tighter.

In the Ref. [8], the authors showed that entanglement is the resource to reduce the uncertainty. In a subsequent work, Pati et al. [14] have claimed that quantum discord [15, 16] acts as a resource when the shared state between quantum memory and observed system is chosen from a class of states including Werner states and isotropic states. However, the above resources fail to reduce uncertainty optimally [12], in general. The motivation of the present work is to find out the physical resources responsible for the optimal reduction of entropic uncertainty, which is given operationally by using the fine-grained uncertainty relation [13]. In other words, we investigate the question as to which physical quantity is responsible for reduction of the uncertainty optimally in an experimental situation involving the measurement of two incompatible observables in the presence of shared states (correlations) between two parties.

In the present work we introduce the measure *extractable classical information* which as we show, contributes exactly to reducing the uncertainty by an amount leading to the optimal lower bound for several examples of entangled, separable as well as classical states. Here, we derive a new uncertainty relation in terms of the extractable classical

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information. We show that the lower bound of the uncertainty relation derived here is equal to the optimal lower bound obtained with the help of the fine-grained uncertainty relation for various pure and mixed states. It further follows from our relation that even in the absence of quantum correlations between the two parties, the uncertainty may be reduced with the help of classical correlations.

## II. DEFINITIONS AND MATHEMATICAL PRELIMINARIES

The entropic form of uncertainty relation, when the correlation of the observed system with quantum memory is not considered, is given by [5, 6]

$$\mathcal{H}(R) + \mathcal{H}(S) \geq \log_2 \frac{1}{c}, \quad (1)$$

where  $\mathcal{H}(k) = -\sum_i p_i^k \log_2 p_i^k$  is the Shannon entropy with  $p_i^k$  being the probability of the  $i$ -th outcome for the measurement of observable  $k \in \{R, S\}$ . The complementarity of the observables  $R$  and  $S$  is measured by the quantity  $c (= \max_{i,j} |\langle r_i | s_j \rangle|^2)$ , with  $|r_i\rangle$  and  $|s_j\rangle$  are the eigenvectors of  $R$  and  $S$ , respectively).

To discuss the EUR in the presence of quantum memory, one may consider the following game as discussed in the Ref. [8]. Bob prepares two system  $A$  and  $B$  in a bipartite quantum state  $\rho_{AB}$  and sends the system  $A$  to Alice. Now, Alice is going to measure either observable  $R$  or  $S$  on her system  $A$ . From the knowledge of the system  $B$ , Bob's task is to minimize his uncertainty about Alice's measurement outcome. Bob is able to reduce his uncertainty about Alice's measurement outcome with the help of communication from Alice regarding the choice of her measurement performed, but not its outcome. The modified form of EUR in the presence of quantum memory is given by [9]

$$\mathcal{S}(R_A|B) + \mathcal{S}(S_A|B) \geq c'(\rho_A) + \mathcal{S}(A|B) \quad (2)$$

where  $\mathcal{S}(A|B) (= \mathcal{S}(\rho_{AB}) - \mathcal{S}(\rho_B))$ , where  $\rho_B = \text{Tr}_A[\rho_{AB}]$  is called the conditional von-neumann entropy of the state  $\rho_{AB}$  and  $c'(\rho_A) = \max\{c'(\rho_A, R_A, S_A), c'(\rho_A, S_A, R_A)\}$ .  $c'(\rho_A, R_A, S_A)$  and  $c'(\rho_A, S_A, R_A)$  are defined by

$$\begin{aligned} c'(\rho_A, R_A, S_A) &= \sum_i p_i^r \log_2 \frac{1}{\max_j c_{ij}} \\ c'(\rho_A, S_A, R_A) &= \sum_j p_j^s \log_2 \frac{1}{\max_i c_{ij}}, \end{aligned} \quad (3)$$

where  $p_i^r = \langle r | \rho_A | r \rangle$  with  $\sum_i p_i^r = 1$ ,  $p_j^s = \langle s | \rho_A | s \rangle$  with  $\sum_j p_j^s = 1$  and  $c_{ij} = |\langle r_i | s_j \rangle|^2$ , i.e., overlap between eigenvector of the observables  $R$  and  $S$ . Here, the uncertainty for the measurement of the observable  $R_A$  ( $S_A$ ) on Alice's system by accessing the information stored in the quantum memory with Bob is measured by  $\mathcal{S}(R_A|B)$  ( $\mathcal{S}(S_A|B)$ ) which is the conditional von Neumann entropy of the state given by

$$\begin{aligned} \rho_{R_A(S_A)B} &= \sum_j (|\psi_j\rangle_{R_A(S_A)} \langle \psi_j| \otimes I) \rho_{AB} (|\psi_j\rangle_{R_A(S_A)} \langle \psi_j| \otimes I) \\ &= \sum_j p_j^{R_A(S_A)} \Pi_j^{R_A(S_A)} \otimes \rho_{B|j}^{R_A(S_A)}, \end{aligned} \quad (4)$$

where  $\Pi_j^{R_A(S_A)}$ 's are the orthogonal projectors on the eigenstate  $|\psi_j\rangle_{R_A(S_A)}$  of observable  $R_A(S_A)$ ,  $p_j^{R_A(S_A)} = \text{Tr}[(|\psi_j\rangle_{R_A(S_A)} \langle \psi_j| \otimes I) \rho_{AB} (|\psi_j\rangle_{R_A(S_A)} \langle \psi_j| \otimes I)]$ ,  $\rho_{B|j}^{R_A(S_A)} = \text{Tr}_A[(|\psi_j\rangle_{R_A(S_A)} \langle \psi_j| \otimes I) \rho_{AB} (|\psi_j\rangle_{R(S)} \langle \psi_j| \otimes I)] / p_j^{R_A(S_A)}$  and  $\rho_{AB}$  is the state of joint system 'A' and 'B'. EUR in presence of quantum memory is modified by the quantity  $\mathcal{S}(A|B)$  which measures the amount of one-way distillable entanglement [17]. For shared maximal entanglement (i.e.,  $\mathcal{S}(A|B) = -1$ ) between the system and the memory, there is no uncertainty in the measurement of incompatible observables. EUR in the presence of quantum memory has been brought out in two recent experiments using respectively, pure [10] and mixed states [11]. For experimental purposes [11], one can obtain the uncertainty relation from the inequality (2) with the help of the relation [11]

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \mathcal{S}(R_A|B) + \mathcal{S}(S_A|B), \quad (5)$$

and it is given by

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq c'(\rho_A) + \mathcal{S}(A|B), \quad (6)$$

where  $p_d^R$  ( $p_d^S$ ) is the probability of getting different outcomes when Alice and Bob measure the same observables  $R$  ( $S$ ) on their respective system. Here the lower bound of the sum of uncertainties for the shared state  $\rho_{AB}$  is given by

$$\mathcal{L}_1(\rho_{AB}) = c'(\rho_A) + \mathcal{S}(A|B) \quad (7)$$

Later, Pati et al. [14] have derived a tighter lower bound of the uncertainty relation using the state  $\rho_{R_A(S_A)B}$ . One can improve the above lower bound using inequality (2) and it is given by Eq.(4) to be

$$\mathcal{S}(R_A|B) + \mathcal{S}(S_A|B) \geq c'(\rho_A) + \mathcal{S}(A|B) + \max\{0, D_A(\rho_{AB}) - C_A^M(\rho_{AB})\}, \quad (8)$$

where the quantum discord  $D_A(\rho_{AB})$  is given by [15, 16]

$$D_A(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - C_A^M(\rho_{AB}), \quad (9)$$

with  $\mathcal{I}(\rho_{AB}) (= \mathcal{S}(\rho_A) + \mathcal{S}(\rho_B) - \mathcal{S}(\rho_{AB}))$  being the mutual information of the state  $\rho_{AB}$  which contains the total correlation present in the state  $\rho_{AB}$  shared between the system  $A$  and the system  $B$ , and the classical information  $C_A^M(\rho_{AB})$  for the shared state  $\rho_{AB}$  (when Alice measures on her system) is given by

$$C_A^M(\rho_{AB}) = \max_{\Pi^{R_A}} [\mathcal{S}(\rho_B) - \sum_j p_j^{R_A} \mathcal{S}(\rho_{B|j}^{R_A})] \quad (10)$$

In this case, the lower bound of the sum of Bob's uncertainty about Alice's measurement outcome for the measurement of observable  $R$  and  $S$  is given by

$$\mathcal{L}_2(\rho_{AB}) = c'(\rho_A) + \mathcal{S}(A|B) + \max\{0, D_A(\rho_{AB}) - C_A^M(\rho_{AB})\}, \quad (11)$$

which becomes a tighter lower bound compared to  $\mathcal{L}_1$  given by Eq.(7) for those state whose quantum discord is larger than the classical information, which is true for example, for a class of states including Werner states and isotropic states.

A new form of the uncertainty relation, *viz.*, *fine grained uncertainty relation*, was proposed by Oppenheim and Wehner [13], motivated by the realization that entropic functions provide a rather coarse way of measuring the uncertainty of a set of measurements, as they do not distinguish the uncertainty inherent in obtaining any combination of outcomes for different measurements. In particular, they considered a game according to which Alice and Bob both receive binary questions, i.e., projective spin measurements along two different directions at each side. The winning probability is given by the relation [13]

$$P^{game}(\mathcal{T}_A, \mathcal{T}_B, \rho_{AB}) = \sum_{t_A, t_B} p(t_A, t_B) \sum_{a, b} V(a, b|t_A, t_B) \langle (A_{t_A}^a \otimes B_{t_B}^b) \rangle_{\rho_{AB}} \leq P_{max}^{game} \quad (12)$$

where  $\rho_{AB}$  is a bipartite state shared by Alice and Bob, and  $\mathcal{T}_A$  and  $\mathcal{T}_B$  represent the set of measurement settings  $\{t_A\}$  and  $\{t_B\}$  chosen by Alice and Bob, respectively, with probability  $p(t_A, t_B)$ . Alice's (Bob's) question and answer are  $t_A(t_B)$  and  $a(b)$ , respectively, with  $A_{t_A}^a = \frac{[I + (-1)^a A_{t_A}]}{2}$  ( $B_{t_B}^b = \frac{[I + (-1)^b B_{t_B}]}{2}$ ) being a measurement of the observable  $A_{t_A}$  ( $B_{t_B}$ ). Here  $V(a, b|t_A, t_B)$  is some function determining the winning condition of the game, which corresponding to a special class of nonlocal retrieval games (CHSH game [13]) for which there exist only one winning answer for one of the two parties, is given by  $V(a, b|t_A, t_B) = 1$ , iff  $a \oplus b = t_A \cdot t_B$ , and 0 otherwise.  $P_{max}^{game}$  is the maximum winning probability of the game, maximized over the set of projective spin measurement settings  $\{t_A\} (\in \mathcal{T}_A)$  by Alice, the set of projective spin measurement settings  $\{t_B\} (\in \mathcal{T}_B)$  by Bob, i.e.,  $P_{max}^{game} = \max_{\mathcal{T}_A, \mathcal{T}_B, \rho_{AB}} P^{game}(\mathcal{T}_A, \mathcal{T}_B, \rho_{AB})$ . Using the maximum winning probability it is possible to discriminate between classical theory, quantum theory and no-signaling theory with the help of the degree of nonlocality [13]. A generalization to the case of tripartite systems has also been proposed [18].

In a recent work [12], we have shown that the lower bound of the uncertainty relation given by Eqs. (2) and (8) are not *optimal* (for the choices of their observables that maximally reduce Bob's uncertainty about Alice's measurement outcome), as illustrated by the analysis of an experiment using mixed states [11]. We have obtained the *optimal* lower bound of entropic uncertainty using fine-grained uncertainty relation [13]. Considering a situation [11] where Alice and Bob both measure the same observable on their system, we have derived a new uncertainty relation that captures the optimal lower bound for Bob's uncertainty about Alice's measurement outcomes. Our uncertainty relation is given by [12]

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \mathcal{H}(p_d^{\sigma_z}) + \mathcal{H}(p_{\text{inf}}^S), \quad (13)$$

where the lower bound given by

$$\mathcal{L}_3(\rho_{AB}) = \mathcal{H}(p_d^{\sigma_z}) + \mathcal{H}(p_{\text{inf}}^S) \quad (14)$$

is optimal if it is tight since for each pair of observables  $R, S$  there is a state for which we get the equality.  $\mathcal{L}_3(\rho_{AB})$  is obtained with the help of the fine-grained uncertainty relation [13] which gives the infimum winning probability  $p_d^{\sigma_z}$  ( $p_{\text{inf}}^S$ ) (corresponding to the minimum uncertainty) for the measurement of observable  $\sigma_z$  ( $\sigma_S = \vec{n}_S \cdot \vec{\sigma} \neq \sigma_z$  with  $\vec{n}_S \equiv \{\sin(\theta_S)\cos(\phi_S), \sin(\theta_S)\sin(\phi_S), \cos(\theta_S)\}$  being a unit vector and  $\vec{\sigma} = \{\sigma_x, \sigma_y, \sigma_z\}$  are the Pauli matrices) corresponding to the game ruled by the winning condition given by [12]

$$\begin{aligned} V(a, b) &= 1 && \text{iff } a \oplus b = 1, \\ &= 0 && \text{otherwise,} \end{aligned} \quad (15)$$

where ‘ $a$ ’ and ‘ $b$ ’ are the binary outcomes (i.e.,  $\{a, b\} \in \{0, 1\}$ ) for Alice and Bob, respectively.  $V(a, b)$  describes the experimental situation used in the Ref.[11], i.e., Alice and Bob measure the same observables and calculate the probability of getting different outcomes. In measurements and communication involving two parties, the lower bound of entropic uncertainty cannot fall below the bound (14). Using the same approach of Ref. [12], a fine-grained steering inequality has been derived [19] which provides the most optimal steering condition for two qubit systems.

### III. UNCERTAINTY RELATION USING EXTRACTABLE CLASSICAL INFORMATION

To derive the sum of uncertainties for the measurement of two incompatible observables  $R$  and  $S$ , we consider the following memory game [8]. In this game Bob prepares a particle (labeled by ‘ $A$ ’) in a particular state, say,  $\rho_A$  and sends it to Alice who measures an observable chosen from the non-commuting set  $\{R_A, S_A\}$  and communicates only the choice of the observable to Bob. Bob’s task is to reduce his uncertainty about the Alice’s measurement outcome. To win the game, Bob chooses one of the following two strategies – (i) *classical strategy*; (ii) *quantum strategy*.

*Classical strategy* : Here, Bob prepares two particles (say, 1st particle labeled by  $A$  and 2nd particles labeled by  $B$ ) in the identical state,  $\rho = \rho_A = \rho_B$ . The combined state of two particles is given by

$$\rho_{AB} = \rho_A \otimes \rho_B. \quad (16)$$

After preparation, Bob sends the 1st particle to Alice. When Alice communicates about her choice of measurement from the set of observables  $\{R_A, S_A\}$ , Bob measures the same observable on the 2nd particle possessed by himself. He infers about the Alice’s measurement outcome from his own measurement outcome. Note here that Bob keeps full information of the state of Alice since he himself has prepared it. Here, the uncertainty relation prevents Bob to know with arbitrary precision the measurement outcomes of two non-commuting observables. The EUR which gives the lower bound for the measurement of the above two non-commuting observables follows from Eq.(2) for product states and is given by [9, 20]

$$\mathcal{H}(R_B) + \mathcal{H}(S_B) \geq c'(\rho_B) + \mathcal{S}(\rho_B), \quad (17)$$

where the subscript  $B$  labels Bob’s measurement. The inequality (17) is tighter than the entropic uncertainty relation given by inequality (2), and hence, Bob can not reduce his uncertainty about Alice’s measurement outcome below the lower bound  $\mathcal{L}_0(\rho_{AB})$  given by

$$\mathcal{L}_0(\rho_{AB}) = c'(\rho_B) + \mathcal{S}(\rho_B). \quad (18)$$

Note that, the state given by Eq.(16) has zero classical correlation (i.e.,  $C_A^M = 0$ ) and zero quantum correlation (i.e.,  $D_A = 0$ ) [21]. The inequality (17) represents the entropic uncertainty relation for Bob’s measurements of two non-commuting observables  $R_B$  and  $S_B$  on his system, and pertains to the situation when there is either no correlation with the other system called quantum memory, or the correlation with the quantum memory is not considered.

*Quantum strategy* : In this strategy, Bob prepares two particles in a correlated state,  $\rho_{AB}$ , and sends the 1st particle to Alice and keeps the 2nd particle. To reduce his uncertainty further from the bound  $c'(\rho_B) + \mathcal{S}(\rho_B)$  (which is the lower bound of uncertainty corresponding to the *classical strategy*), Bob uses the correlations (quantum and/or classical) present in the state  $\rho_{AB}$ . After getting information about the choice of measurement, Bob measures the same observable as Alice’s choice. Since, Alice and Bob measure independently on their respective systems, the order of measurement, i.e., who measures first, does not affect in the considered game. Here consider Alice communicates about her choice of observable from the set  $\{R, S\}$  to Bob. First Bob measures the observable and then Alice

measures. After the measurement performed by Bob with the observable that communicated by Alice, the combined state  $\rho_{AR_B(S_B)}$  is given by

$$\rho_{AR_B(S_B)} = \sum_j p_j^{R_B(S_B)} \rho_{A|j}^{R_B(S_B)} \otimes \Pi_j^{R_B(S_B)}, \quad (19)$$

where  $\Pi_j^{R_B(S_B)} = |\psi\rangle_{R_B(S_B)}\langle\psi|$ ,  $\rho_{A|j}^{R_B(S_B)} (= \text{Tr}_B[(I \otimes \Pi_j^{R_B(S_B)})\rho_{AB}])$  is the Alice's conditional state when Bob gets  $j$ -th outcome and  $p_j^{R_B(S_B)} (= \text{Tr}[(I \otimes |\psi\rangle_{R_B(S_B)}\langle\psi|)\rho_{AB}])$  is the probability of getting  $j$ -th outcome by Bob.

The classical information  $C_B^M(\rho_{AB})$ , given by

$$C_B^M(\rho_{AB}) = \max_{\Pi^{R_B}} [\mathcal{S}(\rho_A) - \sum_j p_j^{R_B} \mathcal{S}(\rho_{A|j}^{R_B})], \quad (20)$$

where  $\rho_A = \text{Tr}_B[\rho_{AB}]$ , gives the *maximum information* that Bob can extract on average about the Alice's system by measuring on his system when they share the state  $\rho_{AB}$ . Now, one may ask the following questions – what information can Bob extract about Alice's measurement outcomes?  $C_B^M(\rho_{AB})$  contains the information about the Alice's measurement outcomes when she measures along a particular direction which maximizes the quantity  $C_B(\rho_{AB})$  (where  $C_B(\rho_{AB})$  is taken with out maximization in Eq.(20)). When Bob gets the  $j$ -th outcome for the measurement of the observable  $R_B$  on his system, his knowledge about Alice's measurement outcomes for the measurement in the eigenbasis of  $\rho_{A|j}^{R_B}$  is given by the quantity  $\mathcal{S}(\rho_{A|j}^{R_B})$ . Since  $\mathcal{S}(\rho_A)$  is Bob's uncertainty about Alice's outcome in the absence of correlations, from the Eq.(20), it can be easily seen that  $C_B^M(\rho_{AB})$  measures the amount of Bob's uncertainty about Alice's measurement outcome reduced due to Bob's measurement. For example, for the shared Werner state  $\rho_{AB}^W$  [22] between Alice and Bob given by

$$\rho_{AB}^W = \frac{1-p}{4} I \otimes I + p |\psi^-\rangle\langle\psi^-|, \quad (21)$$

where  $I$  is the  $(2 \otimes 2)$  unitary matrix,  $|\psi^-\rangle$  is the singlet state  $(|01\rangle_{AB} - |10\rangle_{AB})/\sqrt{2}$ , and  $p$ , the mixedness parameter (lying between 0 and 1), Bob gets the maximum information about Alice's measurement outcomes given by  $\mathcal{S}(\rho_{A|j}^{R_B})$  when they measure the same observables on their respective system. Hence, classical information quantifies Bob's maximum knowledge about Alice's measurement outcome in a specific direction, say in the eigen basis of  $\rho_{A|j}^{R_B}$ .

According to our considered game, when Alice communicates her choice, say,  $R_A$  (where 'A' labels Alice's choice), Bob measures same the observable  $R_B = R_A$  on his particle (labeled by 'B'). Due to Bob's measurement, the reduced uncertainty measured by the conditional von-Neumann entropy of the state,  $\rho_{AR_B}$  given by Eq.(19) now becomes

$$\mathcal{S}(A|R_B) = \mathcal{S}(\rho_A) - C_B^R(\rho_{AB}), \quad (22)$$

where  $C_B^R(\rho_{AB}) = \mathcal{S}(\rho_A) - \sum_i p_i^{R_B} \mathcal{S}(\rho_{A|i}^{R_B})$  as obtained from the Eq.(20) without taking the maximization. This is the information obtained by Bob when he makes a measurement of the observable  $R_B$  on his system.  $C_B^R(\rho_{AB})$  gives the information about Alice's measurement outcomes when she measures in the eigenbasis of  $\rho_{A|i}^{R_B}$  on her particle. Bob's maximum information about Alice's measurement outcome in the eigenbasis  $\rho_{A|i}^{R_B}$  is given by  $C_B^M(\rho_{AB}) = \max_{R_B} C_B^R(\rho_{AB})$  which is known as the classical information where the maximization is taken over all possible observables  $R_B$ . After Bob's measurement, Alice measures the observable  $R_A$  on her particle and the combined state (19) becomes

$$\rho_{R_A, R_B(S_A, S_B)} = \sum_i p_i^{R_B(S_B)} \left( \sum_k q_{k|i}^{R_A(S_A)} \Pi_k^{R_A(S_A)} \right) \otimes \Pi_i^{R_B(S_B)}, \quad (23)$$

where  $\Pi_k^{R_A(S_A)}$  is projector corresponding to the eigenstate of observable  $R_A(S_A)$  and  $q_{k|i}^{R_A(S_A)} = \text{Tr}[\Pi_k^{R_A(S_A)} \rho_{A|i}^{R_B(S_B)}]$  is the conditional probability distribution for the measurement of observable  $R_A(S_A)$  on Alice's particle, given that Bob gets  $i$ th outcome for the measurement of the same observable  $R_B(S_B)$  on his particle. Now, Alice's reduced uncertainty for the measurement of observable  $R_A$ , i.e., more specifically, conditional entropy of the state  $\rho_{R_A, R_B}$  is given by

$$\mathcal{H}(R_A|R_B) = \mathcal{H}(R_A) - C_{A,B}^{R,R}(\rho_{AB}), \quad (24)$$

with

$$C_{A,B}^{R,R}(\rho_{AB}) = \mathcal{H}(R_A) - \sum_i p_i^{R_B} \mathcal{H}(q_i^{R_A}), \quad (25)$$

where  $\mathcal{H}(R_A)$  is the Shannon entropy of the probability distribution  $\{q_k^{R_B}\}$  corresponding to different measurement outcomes  $\{k\}$  for the measurement of observable  $R_A$  on Alice's particle and  $\mathcal{H}(q_i^{R_A})$  is the Shannon entropy of the conditional probability distribution  $\{q_{k|i}^{R_A}\}$ . We define the quantity  $C_{A,B}^{R,R}(\rho_{AB})$  as the “extractable classical information”.

Similarly, when both Alice and Bob measures the observable  $S$ , the conditional entropy of the state  $\rho_{S_A,S_B}$  (given by Eq. (23)) becomes

$$\mathcal{H}(S_A|S_B) = \mathcal{H}(S_A) - C_{A,B}^{S,S}(\rho_{AB}), \quad (26)$$

where  $C_{A,B}^{S,S}(\rho_{AB})$  is the extractable classical information for the measurement of the observable  $S$  on the both particles. Now, combining Eqs. (24) and (26), one gets

$$\mathcal{H}(R_S|R_B) + \mathcal{H}(S_A|S_B) = \mathcal{H}(R_A) + \mathcal{H}(S_A) - C_{A,B}^{R,R}(\rho_{AB}) - C_{A,B}^{S,S}(\rho_{AB}) \quad (27)$$

The sum of the first two terms on the r.h.s. of the above equation (27) represents the entropy of a single system (system  $A$ ) when there is either no correlation with the other system called quantum memory, or the correlation with the quantum memory is not considered. Hence, the sum  $\mathcal{H}(R_A) + \mathcal{H}(S_A)$  can be constrained through the inequality (17), using which we obtain

$$\mathcal{H}(R_A|R_B) + \mathcal{H}(S_A|S_B) \geq c'(\rho_A) + \mathcal{S}(\rho_A) - C_{A,B}^{R,R}(\rho_{AB}) - C_{A,B}^{S,S}(\rho_{AB}), \quad (28)$$

where  $\rho_A$  is the density state of Alice's particle. Now, using the inequality (5), Eq.(28) becomes

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq c'(\rho_A) + \mathcal{S}(\rho_A) - C_{A,B}^{R,R}(\rho_{AB}) - C_{A,B}^{S,S}(\rho_{AB}), \quad (29)$$

where  $\mathcal{H}(p_d^\alpha)$  is the Shannon entropy of the probability distribution  $\{p_d^\alpha\}$  when Alice and Bob measure same observable  $\alpha \in \{R, S\}$  and get different outcomes. Eq.(29) represents our new uncertainty relation when both Alice and Bob measure two incompatible observables  $R$  and  $S$ . Hence, the lower bound of Bob's uncertainty about Alice's measurement outcomes is given by

$$\mathcal{L}_4(\rho_{AB}) = c'(\rho_A) + \mathcal{S}(\rho_A) - C_{A,B}^{R,R}(\rho_{AB}) - C_{A,B}^{S,S}(\rho_{AB}). \quad (30)$$

#### IV. EXAMPLES

In the following analysis we compare the bound  $\mathcal{L}_4(\rho_{AB})$  with the lower bounds through the quantum strategy obtained earlier in the literature, *viz.*,  $\mathcal{L}_1(\rho_{AB})$  (given by Eq.(7)) [8, 11], the bound  $\mathcal{L}_2(\rho_{AB})$  (given by Eq.(11) [14], the bound  $\mathcal{L}_3(\rho_{AB})$  (given by Eq.(14)) [12], as well as the bound  $\mathcal{L}_0(\rho_{AB})$  (given by Eq.(18) with  $\mathcal{S}(\rho_B) = \mathcal{S}(\text{Tr}_A \rho_{AB})$ ) obtained with the help of the classical strategy for various classes of pure and mixed entangled and separable states. We show that the lower bound given by Eq.(30) is optimal as obtained through fine-graining [12] for all the cases considered here.

**Pure entangled state :** Here we consider a pure entangled state  $\rho_{AB}^{PE}$ , given by

$$\rho_{AB}^{PE} = \sqrt{\alpha}|01\rangle_{AB} - \sqrt{1-\alpha}|10\rangle_{AB}, \quad (31)$$

where  $\alpha$  lies between 0 and 1, and the state  $\rho_{AB}^{PE}$  is maximally entangled for  $\alpha = \frac{1}{2}$ . The classical information (when Alice measures her particle) is given by

$$C_B^M(\rho_{AB}^{PE}) = \mathcal{H}(\alpha), \quad (32)$$

where  $\mathcal{H}(\alpha) = \alpha \log_2 \alpha - (1-\alpha) \log_2 (1-\alpha)$ .  $C_B^M(\rho_{AB}^{PE})$  gives the information about Alice's measurement outcome in the direction  $\{\mu \cos[\phi_S] \sin[\theta_S], \mu \sin[\phi_S] \sin[\theta_S], \frac{1-2\alpha+\cos[\theta_S]}{1+\cos[\theta_S]-2\alpha \cos[\theta_S]}\}$  (where  $\mu = \frac{2\sqrt{\alpha(1-\alpha)}}{1+\cos[\theta_S]-2\alpha \cos[\theta_S]}$ ) to Bob when he measures along  $\{\sin(\theta_S) \cos(\phi_S), \sin(\theta_S) \sin(\phi_S), \cos(\theta_S)\}$ . Let us consider that before playing the game Alice and Bob discuss about their strategy, such as, choices of the state and measurement settings. Alice chooses those settings for which Bob's uncertainty about her measurement outcome will be minimum as well as maximize the lower bound of uncertainty in the classical strategy (given by Eq.(18)), i.e.,  $c'(\rho_B) = 1$  where  $\max_i c_{ij} = \max_j c_{ij} = 1/2$ . With the help of the fine-grained uncertainty relation [12, 13], one can obtain the winning probability (corresponding

to minimum uncertainty) when Alice and Bob both measure the same observable and get different outcomes, i.e.,  $a \oplus b = 1$  [12]. The winning probability is given by

$$P^{\text{game}}(\rho_{AB}^{PE}) = \frac{1}{4}(3 + 2\sqrt{\alpha(1-\alpha)} + (1 - 2\sqrt{\alpha(1-\alpha)})\cos[2\theta_S]) \quad (33)$$

Bob's uncertainty about Alice's outcome would be minimum for the choice of observables given by

$$\begin{aligned} R &= \sigma_z & (\text{i.e., } \theta_S = 0), \\ S &= \sigma_x & (\text{i.e., } \theta_S = \frac{\pi}{2}), \end{aligned} \quad (34)$$

which leads to  $p_d^R = 1$ , and the infimum probability  $p_{\text{inf}}^S = 1/2 + \sqrt{\alpha(1-\alpha)}$ . Hence, the optimal lower bound obtained from the Eq.(14) is given by [12]

$$\mathcal{L}_3(\rho_{AB}^{PE}) = \mathcal{H}(\frac{1}{2} - \sqrt{\alpha(1-\alpha)}). \quad (35)$$

When Bob chooses the *classical strategy*, he first prepares two copies of the state  $\text{Tr}_B(\rho_{AB}^{PE})$  and send one to Alice. For the above choice of observables, Bob's uncertainty about Alice's measurement outcome is maximally reduced ( $c'(\rho_B) = 1$  by choosing the above measurement settings), and is given by the inequality (17). The lower bound (18) is given by

$$\mathcal{L}_0(\rho_{AB}^{PE}) = 1 + \mathcal{H}(\alpha). \quad (36)$$

In the *quantum strategy* using the uncertainty relations proposed in Refs. [8] and [14], Bob's uncertainty is lower bounded by ((7) and (11)), respectively, which turn out to be equal, given by

$$\mathcal{L}_1(\rho_{AB}^{PE}) = \mathcal{L}_2(\rho_{AB}^{PE}) = 1 - \mathcal{H}(\alpha). \quad (37)$$

However, in practice Bob is unable to reduce his uncertainty upto the above level, since  $\mathcal{L}_1(\rho_{AB}^{PE})$  is not the optimal lower bound. The main reason is that Bob only extracts the information  $C_{A,B}^{\sigma_z, \sigma_z}(\rho_{AB}^{PE})$  ( $C_{A,B}^{\sigma_x, \sigma_x}(\rho_{AB}^{PE})$ ) given by  $\mathcal{H}(\alpha)$  ( $1 - \mathcal{H}(\frac{1}{2} - \sqrt{\alpha(1-\alpha)})$ ) when both of them measure the same spin observables  $R = \sigma_z$  ( $S = \sigma_x$ ) on their respective particle. Hence, the lower bound (given by Eq.(30)) of Bob's uncertainty is given by

$$\mathcal{L}_4(\rho_{AB}^{PE}) = \mathcal{H}(\frac{1}{2} - \sqrt{\alpha(1-\alpha)}). \quad (38)$$

From the Eqs.(35) and (38), it is clear that the quantities  $C_{A,B}^{\sigma_z, \sigma_z}(\rho_{AB}^{PE})$  and  $C_{A,B}^{\sigma_x, \sigma_x}(\rho_{AB}^{PE})$  are responsible for reducing Bob's uncertainty about Alice's measurement outcome optimally. This explains in terms of physical resources why the lower bound  $\mathcal{L}_1(\rho_{AB}^{PE})$  ( $\leq \mathcal{L}_3(\rho_{AB}^{PE})$ ) is not experimentally reachable, whereas the lower bound  $\mathcal{L}_3(\rho_{AB}^{PE})$  given by fine-graining is indeed attainable.

**Werner State :** For the class of Werner State  $\rho_{AB}^W$ , given by Eq.(21), the classical information is given by

$$C_B^M(\rho_{AB}^W) = 1 - \mathcal{H}(\frac{1-p}{2}). \quad (39)$$

$C_B^M(\rho_{AB}^W)$  gives Bob the information about the measurement outcome of Alice when they measure same observables. The quantum discord of the state  $\rho_{AB}^W$  is given by

$$D_B(\rho_{AB}^W) = \mathcal{I}(\rho_W) - C_B^M, \quad (40)$$

where  $\mathcal{I}(\rho_{AB}^W) = 2 + 3\frac{1-p}{4}\log_2 \frac{1-p}{4} + \frac{1+3p}{4}\log_2 \frac{1+3p}{4}$  is the mutual information of  $\rho_{AB}^W$ .

In the *Classical strategy*, for the choice observables given by Eq.(34) (which minimize Bob's uncertainty optimally [12]), Bob's uncertainty is lower bounded by (18)

$$\mathcal{L}_0(\rho_{AB}^W) = 2, \quad (41)$$

where  $\rho_A^W = \text{Tr}_B[\rho_{AB}^W] = \frac{I}{2}$ . When Bob uses the *quantum strategy* [8, 11], his uncertainty (given by Eq.(6)) is bounded by

$$\mathcal{L}_1(\rho_{AB}^W) = 2 - \mathcal{I}(\rho_{AB}^W), \quad (42)$$

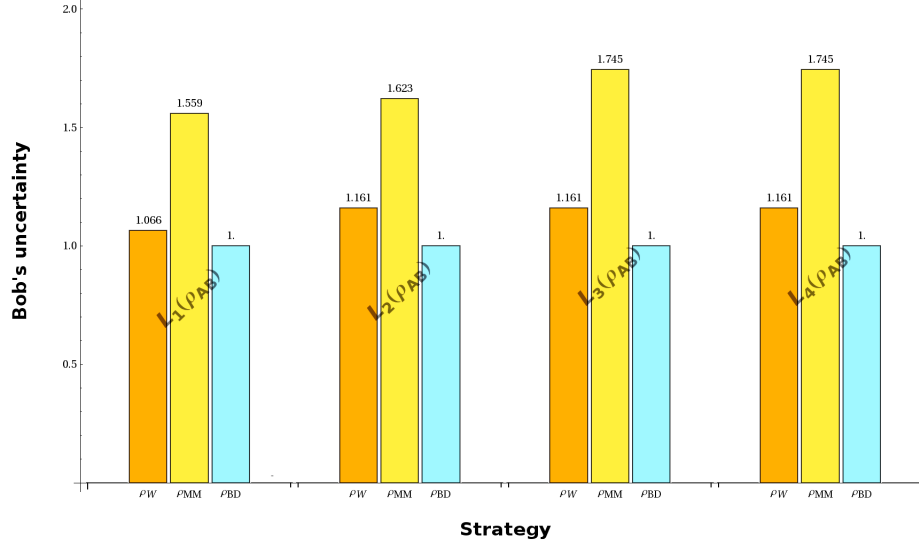


FIG. 1: A comparison of the different lower bounds for the (i) Werner state with  $p = 0.723$ , (ii) the state with maximally mixed marginals with the ci's given by  $c_x = 0.5$ ,  $c_y = -0.2$ , and  $c_z = -0.3$ , and (iii) the Bell diagonal state with  $p = 0.5$ .

where for the state  $\rho_{AB}^W$ ,  $\mathcal{S}(A|B) = 1 - \mathcal{I}(\rho_{AB}^W)$ . The improved lower bound (11) given by Pati et al.[14] for the Werner class of states turns out to be

$$\mathcal{L}_2(\rho_{AB}^W) = 2 - 2C_B^M(\rho_{AB}^W) = 2\mathcal{H}\left(\frac{1-p}{2}\right). \quad (43)$$

Note that Bob is able to gain his knowledge about Alice's measurement outcomes by an amount  $C_B^M(\rho_{AB}^W)$  when both Alice and Bob measure the same observables  $R = \sigma_z$  ( $S = \sigma_x$ ) on their respective particles. Hence, Bob's uncertainty (given by Eq.(29)) is lower bounded by

$$\mathcal{L}_3(\rho_{AB}^W) = 2 - 2C_B^M(\rho_{AB}^W) = 2\mathcal{H}\left(\frac{1-p}{2}\right). \quad (44)$$

Now, using fine-graining the optimal lower bound for Bob's uncertainty is given by [12]

$$\mathcal{L}_4(\rho_{AB}^W) = 2\mathcal{H}\left(\frac{1-p}{2}\right), \quad (45)$$

Thus, for the Werner class of states, Bob can actually minimize his uncertainty about Alice's measurement outcome upto  $2\mathcal{H}(\frac{1-p}{2})$  (given by Eqs.(43), (44) and (45)). The lower bound  $\mathcal{L}_1(\rho_{AB}^W) (\leq \mathcal{L}_3(\rho_{AB}^W))$  is not experimentally reachable.

**Bell diagonal state :** The Bell diagonal state, used in Ref.[11] is given by

$$\rho_{AB}^{BD} = p\rho_2 + (1-p)\rho_S, \quad (46)$$

where  $\rho_2$  is the density matrix of the state  $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$ . The classical information of the state  $\rho_{AB}^{BD}$  is given by

$$C_B^M(\rho_{AB}^{BD}) = 1. \quad (47)$$

Here  $C_B^M(\rho_{AB}^{BD})$  gives Bob the information about Alice's measurement outcome for the measurement along  $\{\sin(\theta_S)\cos(\phi_S), -\sin(\theta_S)\sin(\phi_S), \cos(\theta_S)\}$  from his measurement outcome along the direction  $\{\sin(\theta_S)\cos(\phi_S), \sin(\theta_S)\sin(\phi_S), \cos(\theta_S)\}$ . The quantum discord of  $\rho_{AB}^{BD}$  is given by

$$D_B(\rho_{AB}^{BD}) = 1 - \mathcal{H}(p), \quad (48)$$

where  $\mathcal{I}(\rho_{AB}^{BD}) (= 2 - \mathcal{H}(p))$  is the mutual information of the state  $\rho_{AB}^{BD}$ . From the Eqs.(47) and (48), it is clear that for the state  $\rho_{AB}^{BD}$ ,  $C_B^M(\rho_{AB}^{BD}) \geq D_B(\rho_{AB}^{BD})$ .

In the *classical strategy* for the choice of above observables, Bob's uncertainty is bounded by

$$\mathcal{L}_0(\rho_{AB}^{BD}) = 2. \quad (49)$$

In the *quantum strategy*, theoretically Bob's uncertainty (obtained using Eq.(2) and (8)) is lower bounded by

$$\mathcal{L}_1(\rho_{AB}^{BD}) = \mathcal{L}_2(\rho_{AB}^{BD}) = \mathcal{H}(p), \quad (50)$$

where  $S(A|B) = \mathcal{H}(p) - 1$ . With the measurement on his system of an observable communicated by Alice, Bob extracts the classical information by an amount  $C_{A,B}^{\sigma_z, \sigma_z}(\rho_{AB}^{BD}) = 1 - \mathcal{H}(p)$  ( $C_{A,B}^{\sigma_y, \sigma_y}(\rho_{AB}^{BD}) = 1$ ) for the spin measurement along  $z$ - direction ( $y$ - direction). Hence, Bob's uncertainty is lower bounded by

$$\mathcal{L}_4(\rho_{AB}^{BD}) = \mathcal{H}(p). \quad (51)$$

In this case the optimal lower bound for Bob's uncertainty about Alice's measurement outcome given by fine-graining [12] also turns out to be

$$\mathcal{L}_3(\rho_{AB}^{BD}) = \mathcal{H}(p). \quad (52)$$

Here the lower bound predicted by [8, 14] is optimal. Eqs.(51) and (52) show that the extractable classical information  $C_{A,B}^{\sigma_z, \sigma_z}(\rho_{AB}^{BD}) = 1 - \mathcal{H}(p)$ ,  $C_{A,B}^{\sigma_y, \sigma_y}(\rho_{AB}^{BD}) = 1$  is responsible for reducing Bob's uncertainty optimally.

**Maximally mixed marginal state :** The maximally mixed marginal state  $\rho_{AB}^{MM}$  is given by

$$\rho_{AB}^{MM} = \frac{1}{4}(\mathcal{I} + \sum_{i=x,y,z} c_i \sigma_i \otimes \sigma_i). \quad (53)$$

where the coefficients  $c_i$ 's ( $i \in \{x, y, z\}$ ) are constrained by the eigenvalues  $\lambda_i \in [0, 1]$  of  $\rho_{AB}^{MM}$  given by

$$\lambda_0 = \frac{1 - c_x - c_y - c_z}{4}, \quad \lambda_1 = \frac{1 - c_x + c_y + c_z}{4}, \quad \lambda_2 = \frac{1 + c_x - c_y + c_z}{4}, \quad \lambda_3 = \frac{1 + c_x + c_y - c_z}{4}. \quad (54)$$

The mutual information of the state  $\rho_{AB}^{MM}$  is given by

$$\mathcal{I}(\rho_{AB}^{MM}) = 2 + \sum_{j=0}^3 \lambda_j \log 2[\lambda_j]. \quad (55)$$

The classical information of the state is given by [23]

$$C_B^M(\rho_{AB}^{MM}) = \frac{1 - c_M}{2} \log 2[1 - c_M] + \frac{1 + c_M}{2} \log 2[1 + c_M], \quad (56)$$

where  $c_M = \max[|c_x|, |c_y|, |c_z|]$ , and the quantum discord of the state  $\rho_{AB}^{MM}$  is given by

$$D_B(\rho_{AB}^{MM}) = \mathcal{I}(\rho_{AB}^{MM}) - C_B^M(\rho_{AB}^{MM}). \quad (57)$$

As usual, before playing the game, Alice and Bob discuss the measurement settings (i.e., strategy for the game) for the shared state  $\rho_{AB}^{MM}$ . To optimize the uncertainty, Bob takes the help of the fine-grained uncertainty relation (FUR) [12]. Here, the winning probability when Alice and Bob both measure the observable  $S$  is given by

$$P_S^{\text{game}} = \frac{1}{2}(1 - c_x \sin^2[\theta_S] \cos^2[\phi_S] - c_y \sin^2[\theta_S] \sin^2[\phi_S] - c_z \cos^2[\theta_S]). \quad (58)$$

The measurement settings may be chosen such that the quantity  $P_S^{\text{game}}$  is maximized. For the measurement setting  $\sigma_z$  (i.e.,  $\theta_S = 0$ ),  $P_{\sigma_z}^{\text{game}}$  will be maximum when  $c_z - c_x < 0$ . Here we consider  $c_x = 0.5$ ,  $c_y = -0.2$ , and  $c_z = -0.3$ , and for this choices, the observable  $R = \sigma_z$  and  $S = \sigma_x$  minimizes Bob's uncertainty [12]. For the above choice the optimal lower bound of Bob's uncertainty is lower bounded by [12]

$$\mathcal{L}_3(\rho_{AB}^{MM}) \approx 1.745. \quad (59)$$

When Bob chooses the classical strategy, for the above choice of observable his uncertainty given by (17) is lower bounded by

$$\mathcal{L}_0(\rho_{AB}^{MM}) = 2. \quad (60)$$

Employing the quantum strategy, Bob's uncertainty (6) is lower bounded by

$$\mathcal{L}_1(\rho_{AB}^{MM}) \approx 1.5589, \quad (61)$$

whereas, the bound [14] is given by

$$\mathcal{L}_2(\rho_{AB}^{MM}) \approx 1.6226, \quad (62)$$

where the classical information  $C_B^M(\rho_{AB}^{MM}) = 0.1887$  (obtained from Eq.(56) using our choice of  $c_i$ 's) and the quantum discord  $D_B(\rho_{AB}^{MM}) = 0.2524$  (obtained from Eq.(57)), tightens Berta's lower bound given by Eq.(61)[8].

When both Alice and Bob measure same observable,  $\sigma_z$  ( $\sigma_x$ ) on their respective system, Bob extracts the information given by  $C_{A,B}^{\sigma_z, \sigma_z}(\rho_{AB}^{MM}) = 0.0659$  ( $C_{A,B}^{\sigma_x, \sigma_x}(\rho_{AB}^{MM}) = 0.1887$ ) for the above choice of  $c_i$ 's. Now, using Eq.(29) Bob's uncertainty is lower bounded by

$$\mathcal{L}_4(\rho_{AB}^{MM}) \approx 1.745, \quad (63)$$

which is equal to the optimal lower bound obtained using fine-grained uncertainty relation [12]. One sees that though in this case,  $\mathcal{L}_2(\rho_{AB}^{MM})$  tightens the bound  $\mathcal{L}_1(\rho_{AB}^{MM})$ , it is not possible for either of them to be realized in practice since they are not optimal. Fig.1 depicts the main result of the paper, *viz.*, the optimal lower bound obtained through the quantum strategy where the concept of extractable classical information is applied. For the three classes of the states depicted, one sees that the result

$$\mathcal{L}_1 \leq \mathcal{L}_2 \leq (\mathcal{L}_3 = \mathcal{L}_4) \quad (64)$$

holds.

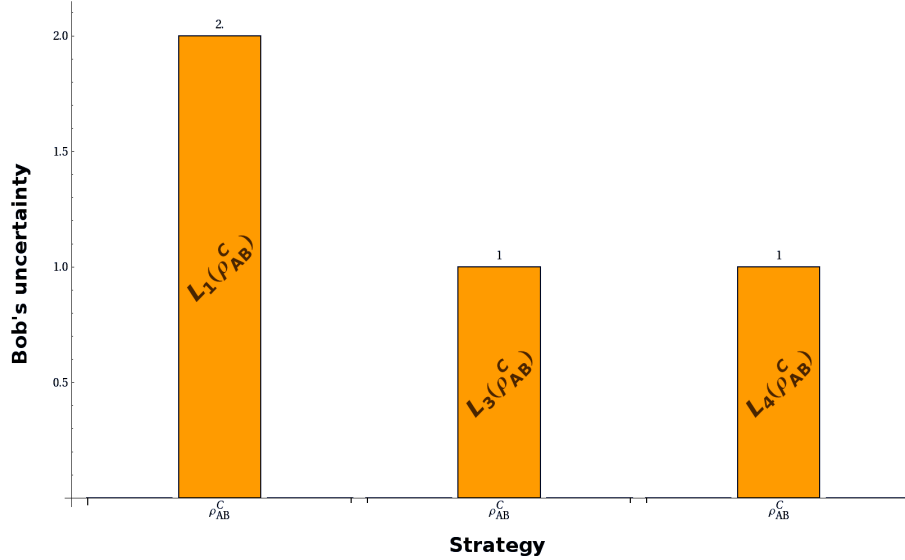


FIG. 2: A comparison of the different lower bounds for the shared classical state choosing  $p=0.5$ .

**Classical state :** Now, we consider classical state  $\rho_{AB}^C$ , given by

$$\rho_{AB}^C = p|00\rangle\langle 00| + (1-p)|11\rangle\langle 11|. \quad (65)$$

The state,  $\rho_{AB}^C$  is a zero discord state [21], i.e.,

$$D_B(\rho_{AB}^C) = 0. \quad (66)$$

The classical information of the state  $\rho_{AB}^C$  is given by

$$C_B^M(\rho_{AB}^C) = \mathcal{H}(p). \quad (67)$$

$C_B^M(\rho_{AB}^C)$  gives the information about Alice's measurement outcome for the measurement of observable  $\sigma_z$  to Bob, when Bob measures the same observable  $\sigma_z$ . The winning probability of the game characterized by winning condition  $a \oplus b = 1$  [12] is

$$P^{\text{game}}(\rho_{AB}^C) = \frac{\sin^2[\theta_S]}{2}. \quad (68)$$

Hence, the choices (for Alice) of the set of observables  $\{R, S\}$  (which minimize Bob's uncertainty about Alice's outcome) are given by Eq.(34).

In this case, when Bob chooses the *classical strategy*, his uncertainty (given in Eq.(2)) for the choices of settings given by Eq.(34) is lower bounded by an amount

$$\mathcal{L}_0(\rho_{AB}^C) = 1 + \mathcal{H}(p). \quad (69)$$

When Bob applies the *quantum strategy* [8, 11, 14] his uncertainty is lower bounded by

$$\mathcal{L}_1(\rho_{AB}^C) = \mathcal{L}_2(\rho_{AB}^C) = 1. \quad (70)$$

For the state  $\rho_{AB}^C$ , Bob's extractable classical information (given by Eq.25) is  $C_{A,B}^{\sigma_z, \sigma_z}(\rho_{AB}^{MM}) = \mathcal{H}(p)$  ( $C_{A,B}^{\sigma_x, \sigma_x}(\rho_{AB}^{MM}) = 0$ ) when both of them measure the same observable  $R = \sigma_z$  ( $S = \sigma_x$ ) on their respective particles. Hence, the lower bound given by Eq.(30) becomes

$$\mathcal{L}_4\rho_{AB}^C = 1. \quad (71)$$

Finally, the optimal lower bound given by Eq.(14) is also

$$\mathcal{L}_3\rho_{AB}^C = 1. \quad (72)$$

Hence in this case,  $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_3 = \mathcal{L}_4 = 1 < \mathcal{L}_0$ . We thus observe that even purely classical correlations can play a role in reducing the uncertainty using a shared bipartite state when the quantum strategy is employed. This result is displayed in Fig.2.

## V. CONCLUSIONS

To summarize, in the present work we derive a new form of the uncertainty relation which enables to reduce Bob's uncertainty maximally about Alice's measurement outcome while they choose same observable with the help of a shared state and communication between the two. Using examples of several classes of pure and mixed states of two qubits possessing quantum and classical correlations, we show that the lower bound of the uncertainty relation derived here is equal to the optimal lower bound (in the sense that Bob's uncertainty is reduced maximally) obtained with the help of the fine-grained uncertainty relation [12, 13]. We identify as the extractable classical information the physical quantity that is responsible for maximally reducing the uncertainty for the measurement of two non-commuting observables. Thus, the uncertainty relation presented here provides an explanation in terms of physical resources as to how the uncertainty of measurement of two incompatible observables may be reduced maximally using shared correlations and classical communication. Our analysis further explains how the uncertainty may be reduced using the quantum strategy even in the absence of quantum correlations when the two parties share just a classically correlated state.

Before concluding, it will be worthwhile to stress that in the present work we have reformulated the uncertainty relation in the presence of quantum memory. The lower bound of uncertainty is derived here using an approach that is different from the fine-graining employed earlier in the context of memory [12] and steering [19]. Though it turns out that for several important and widely considered examples of two-qubit states the bound derived here and that using fine-graining turn out to be numerically equivalent, there is as yet no proof of formal equivalence between the two derived bounds using two *a priori* different concepts of classical information and fine-graining, respectively. Further investigation into this issue is called for in order to clarify whether the connection between fine-graining and extractable classical information (the validity of the last equality of our relation (64)) would hold true for other classes of two-qubit states, or could even be extended to the case of higher dimensional systems. Finally, it will be interesting to extend our present analysis in the light of other recent improvements in the entropic uncertainty relations [24].

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