General Monogamy Relations of Quantum Entanglement for Multiqubit W-class States

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Entanglement monogamy is a fundamental property of multipartite entangled states. We investigate the monogamy relations for multipartite generalized W-class states. Analytical monogamy inequalities are obtained for the concurrence of assistance, the entanglement of formation and the entanglement of assistance.

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I. INTRODUCTION

Quantum entanglement [1–6] is an essential feature of quantum mechanics that distinguishes the quantum from the classical world. It is one of the fundamental differences between quantum entanglement and classical correlations that a quantum system entangled with one of the other systems limits its entanglement with the remaining others. This restriction of entanglement shareability among multi-party systems is known as the monogamy of entanglement. The monogamy relations give rise to the structures of entanglement in the multipartite setting. For a tripartite system A, B, and C, the monogamy of an entanglement measure ε implies that the entanglement between A and BC satisfies $\varepsilon_{A|BC} \geq \varepsilon_{AB} + \varepsilon_{AC}$.

In Ref.[7, 8] the monogamy of entanglement for multiqubit W-class states has been investigated, and the monogamy relations for tangle and the squared concurrence have been proved. In this paper, we show the general monogamy relations for the x-power of concurrence of assistance, the entanglement of formation, and the entanglement of assistance for generalized multiqubit W-class states.

II. MONOGAMY OF CONCURRENCE OF ASSISTANCE

For a bipartite pure state $|\psi\rangle_{AB}$ in vector space $H_A \otimes H_B$, the concurrence is given by [9–11]

$$C(|\psi\rangle_{AB}) = \sqrt{2[1 - Tr(\rho_A^2)]},\tag{1}$$

where ρ_A is reduced density matrix by tracing over the subsystem B, $\rho_A = Tr_B(|\psi\rangle_{AB}\langle\psi|)$. The concurrence is extended to mixed states $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$, $p_i \ge 0$, $\sum_i p_i = 1$, by the convex roof construction,

$$C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \tag{2}$$

where the minimum is taken over all possible pure state decompositions of ρ_{AB} .

For a tripartite state $|\psi\rangle_{ABC}$, the concurrence of assistance (CoA) is defined by [12]

$$C_a(|\psi\rangle_{ABC}) \equiv C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),\tag{3}$$

for all possible ensemble realizations of $\rho_{AB} = Tr_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$. When $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$ is a pure state, then one has $C(|\psi\rangle_{AB}) = C_a(\rho_{AB})$.

For an N-qubit state $|\psi\rangle_{AB_1...B_{N-1}} \in H_A \otimes H_{B_1} \otimes ... \otimes H_{B_{N-1}}$, the concurrence $C(|\psi\rangle_{A|B_1...B_{N-1}})$ of the state $|\psi\rangle_{A|B_1...B_{N-1}}$, viewed as a bipartite with partitions A and $B_1B_2...B_{N-1}$, satisfies the follow inequality[13]

$$C^{\alpha}_{A|B_1B_2...B_{N-1}} \ge C^{\alpha}_{AB_1} + C^{\alpha}_{AB_2} + ... + C^{\alpha}_{AB_{N-1}}, \tag{4}$$

and

$$C^{\beta}_{A|B_{1}B_{2}...B_{N-1}} < C^{\beta}_{AB_{1}} + C^{\beta}_{AB_{2}} + ... + C^{\beta}_{AB_{N-1}},$$
(5)

where $\alpha \geq 2$, $\beta \leq 0$, $C_{AB_i} = C(\rho_{AB_i})$ is the concurrence of $\rho_{AB_i} = Tr_{B_1...B_{i-1}B_{i+1}...B_{N-1}}(\rho)$, $C_{A|B_1B_2...B_{N-1}} = C(|\psi\rangle_{A|B_1...B_{N-1}})$. Due to the monogamy of concurrence, the generalized monogamy relation based on the concurrence of assistance has been proved in Ref. [14],

$$C^{2}(|\psi\rangle_{A|B_{1}...B_{N-1}}) \leq \sum_{i=1}^{N-1} C_{a}^{2}(\rho_{AB_{i}}).$$
(6)

In the following we study the monogamy property of the concurrence of assistance for the *n*-qubit generalized W-class states $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes ... \otimes H_{A_n}$ defined by

$$|\psi\rangle = a|000...\rangle + b_1|01...0\rangle + ... + b_n|00...1\rangle, \tag{7}$$

with $|a|^2 + \sum_{i=1}^n |b_i|^2 = 1.$

Lemma 1 For n-qubit generalized W-class states (7), we have

$$C(\rho_{A_1A_i}) = C_a(\rho_{A_1A_i}), \tag{8}$$

where $\rho_{A_1A_i} = Tr_{A_2...A_{i-1}A_{i+1}...A_n}(|\psi\rangle\langle\psi|).$

[Proof] It is direct to verify that [7], $\rho_{A_1A_i} = |x\rangle_{A_1A_i} \langle x| + |y\rangle_{A_1A_i} \langle y|$, where

$$\begin{aligned} |x\rangle_{A_1A_i} &= a|00\rangle_{A_1A_i} + b_1|10\rangle_{A_1A_i} + b_i|01\rangle_{A_1A_i}, \\ |y\rangle_{A_1A_i} &= \sqrt{\sum_{k\neq i} |b_k|^2}|00\rangle_{A_1A_i}. \end{aligned}$$

From the Hughston- Jozsa-wootters theorem Ref.[7], for any pure-state decomposition of $\rho_{A_1A_i} = \sum_{h=1}^r |\phi_h\rangle_{A_1A_i} \langle \phi_h|$, one has $|\phi_h\rangle_{A_1A_i} = u_{h1}|x\rangle_{A_1A_i} + u_{h2}|y\rangle_{A_1A_i}$ for some $r \times r$ unitary matrices u_{h1} and u_{h2} for each h. Consider the normalized state $|\phi_h\rangle_{A_1A_i} = |\phi_h\rangle_{A_1A_i}/\sqrt{p_h}$ with $p_h = |\langle \phi_h|\phi_h\rangle|$. One has the concurrence of each two-qubit pure $|\phi_h\rangle_{A_1A_i}$,

$$C^{2}(|\tilde{\phi_{h}}\rangle_{A_{1}A_{i}}) = \frac{4}{p_{h}^{2}}|u_{hi}|^{4}|b_{1}|^{2}|b_{i}|^{2}.$$

Then for the two-qubit state $\rho_{A_1A_i}$, we have

$$\sum_{h} p_h C(|\tilde{\phi_h}\rangle_{A_1A_i}) = \sum_{h} p_h \frac{2}{p_h} |u_{hi}|^2 |b_1| |b_i| = 2|b_1| |b_i|.$$

Thus we obtain

$$C(\rho_{A_1A_i}) = \min_{\{p_h, |\tilde{\phi_h}\rangle_{A_1A_i}\}} \sum_h p_h C(|\tilde{\phi_h}\rangle_{A_1A_i})$$
$$= \max_{\{p_h, |\tilde{\phi_h}\rangle_{A_1A_i}\}} \sum_h p_h C(|\tilde{\phi_h}\rangle_{A_1A_i})$$
$$= C_a(\rho_{A_1A_i}).$$

Specifically, in Ref. [8] the same result $C(\rho_{A_1A_i}) = C_a(\rho_{A_1A_i})$ has been proved for the generalized W-class states (7) with a = 0.

Theorem 1 For the n-qubit generalized W-class states $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes ... \otimes H_{A_n}$, the concurrence of assistance satisfies

$$C_a^x(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) \ge \sum_{i=1}^{m-1} C_a^x(\rho_{A_1A_{j_i}}),$$
(9)

where $x \ge 2$ and $\rho_{A_1A_{j_1}...A_{j_{m-1}}}$ is the *m*-qubit, $2 \le m \le n$, reduced density matrix of $|\psi\rangle$.

[Proof] For the *n*-qubit generalized W-class state $|\psi\rangle$, according to the definitions of $C(\rho)$ and $C_a(\rho)$, one has $C_a(\rho_{A_1|A_{j_1}...A_{j_{m-1}}}) \ge C(\rho_{A_1|A_{j_1}...A_{j_{m-1}}})$. When $x \ge 2$, we have

$$C_{a}^{x}(\rho_{A_{1}|A_{j_{1}}...A_{j_{m-1}}}) \geq C^{x}(\rho_{A_{1}|A_{j_{1}}...A_{j_{m-1}}})$$
$$\geq \sum_{i=1}^{m-1} C^{x}(\rho_{A_{1}A_{j_{i}}})$$
$$= \sum_{i=1}^{m-1} C_{a}^{x}(\rho_{A_{1}A_{j_{i}}}).$$

Here we have used in the first inequality the inequality $a^x \ge b^x$ for $a \ge b > 0$ and $x \ge 0$. The second inequality is due to the monogamy of concurrence (4). The last equality is due to the Lemma 1.

Theorem 2 For the n-qubit generalized W-class state $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes ... \otimes H_{A_n}$ with $C(\rho_{A_1A_{j_i}}) \neq 0$ for $1 \leq i \leq m-1$, we have

$$C_a^y(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) < \sum_{i=1}^{m-1} C_a^y(\rho_{A_1A_{j_i}}),$$
(10)

where $y \leq 0$ and $\rho_{A_1A_{j_1}...A_{j_{m-1}}}$ is the *m*-qubit reduced density matrix as in Theorem 1.

[Proof] For $y \leq 0$, we have

$$C_a^y(\rho_{A_1|A_{j_1}...A_{j_{m-1}}}) \leq C^y(\rho_{A_1|A_{j_1}...A_{j_{m-1}}})$$

$$< \sum_{i=1}^{m-1} C^y(\rho_{A_1A_{j_i}})$$

$$= \sum_{i=1}^{m-1} C_a^y(\rho_{A_1A_{j_i}}).$$

We have used in the first inequality the relation $a^x \leq b^x$ for $a \geq b > 0$ and $x \leq 0$. The seconder inequality is due to the monogamy of concurrence (5). The last equality is due to Lemma 1.

According to (9) and (10), we can also obtain the lower bounds of $C_a(\rho_{A_1|A_{j_1}...A_{j_{m-1}}})$. As an example, consider the 5-qubit generalized W-class states (7) with $a = b_2 = \frac{1}{\sqrt{10}}, b_1 = \frac{1}{\sqrt{15}}, b_3 = \sqrt{\frac{2}{15}}, b_4 = \sqrt{\frac{3}{5}}$. We have

$$C_a(\rho_{A_1|A_2A_3}) \ge \frac{2}{\sqrt{15}} \sqrt[x]{(\frac{1}{\sqrt{10}})^x + (\sqrt{\frac{2}{15}})^x}$$

and

$$C_a(\rho_{A_1|A_2A_3A_4}) \ge \frac{2}{\sqrt{15}} \sqrt[x]{\left(\frac{1}{\sqrt{10}}\right)^x + \left(\sqrt{\frac{2}{15}}\right)^x + \sqrt{\frac{3}{5}}\right)^x}$$

with $x \ge 2$. The optimal lower bounds can be obtained by varying the parameter x, see Fig. 1, where for comparison the upper bounds are also presented by using the formula $C_a(\rho_{AB}) \le \sqrt{2(1 - Tr(\rho_A^2))}$ [15], namely, $C_a(\rho_{A_1|A_2A_3}) \le \frac{2}{\sqrt{18}}$ and $C_a(\rho_{A_1|A_2A_3A_4}) \le \frac{2}{\sqrt{18}}$. From Fig.1, one gets that the optimal lower bounds of $C_a(\rho_{A_1|A_2A_3})$ and $C_a(\rho_{A_1|A_2A_3A_4})$ are 0.249 and 0.471, respectively, attained at x = 2.

III. MONOGAMY OF ENTANGLEMENT OF FORMATION

The entanglement of formation of a pure state $|\psi\rangle \in H_A \otimes H_B$ is defined by

$$E(|\psi\rangle) = S(\rho_A),\tag{11}$$

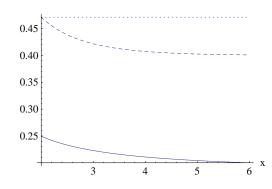


Fig. 1: Solid line is the lower bound of $C_a(\rho_{A_1|A_2A_3})$, dashed line is the lower bound of $C_a(\rho_{A_1|A_2A_3A_4})$ as functions of $x \ge 2$, and dotted line is the upper bound of $C_a(\rho_{A_1|A_2A_3})$ and $C_a(\rho_{A_1|A_2A_3A_4})$.

where $\rho_A = Tr_B(|\psi\rangle\langle\psi|)$ and $S(\rho) = Tr(\rho \log_2 \rho)$. For a bipartite mixed state $\rho_{AB} \in H_A \otimes H_B$, the entanglement of formation is given by

$$E(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \qquad (12)$$

with the infimum taking over all possible decompositions of ρ_{AB} in a mixture of pure states $\rho_{AB} = \sum_i p_i |\psi_i\rangle \langle \psi_i |$, where $p_i \ge 0$ and $\sum_i p_i = 1$.

It has been shown that the entanglement of formation does not satisfy the inequality $E_{AB} + E_{AC} \leq E_{A|BC}$ [16]. Rather it satisfies [13],

$$E^{\alpha}_{A|B_1B_2...B_{N-1}} \ge E^{\alpha}_{AB_1} + E^{\alpha}_{AB_2} + \dots + E^{\alpha}_{AB_{N-1}},\tag{13}$$

where $\alpha \geq \sqrt{2}$.

The corresponding entanglement of assistance (EoA) [17] is defined in terms of the entropy of entanglement [18] for a tripartite pure state $|\psi\rangle_{ABC}$,

$$E_a(|\psi\rangle_{ABC}) \equiv E_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \tag{14}$$

which is maximized over all possible decompositions of $\rho_{AB} = Tr_C(|\psi\rangle_{ABC}) = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $p_i \ge 0$ and $\sum_i p_i = 1$. For any N-qubit pure state $|\psi\rangle \in H_A \otimes H_{B_1} \otimes ... \otimes H_{B_{N-1}}$, it has been shown that the entanglement of assistance satisfies [13],

$$E(|\psi\rangle_{A|B_1B_2...B_{N-1}}) \le \sum_{i=1}^{N-1} E_a(\rho_{AB_i}).$$
(15)

In fact, generally we can prove the following results for the *n*-qubit generalized W-class states about the entanglement of formation and the entanglement of assistance.

Theorem 3 For the n-qubit generalized W-class states $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes ... \otimes H_{A_n}$, we have

$$E(|\psi\rangle_{A_1|A_2...A_n}) \le \sum_{i=2}^{n} E(\rho_{A_1A_i}),$$
(16)

where $\rho_{A_1A_i}$, $2 \leq i \leq n$, is the 2-qubit reduced density matrix of $|\psi\rangle$.

[Proof] For the *n*-qubit generalized W-class states $|\psi\rangle$, we have

$$E(|\psi\rangle_{A_{1}|A_{2}...A_{n}}) = f\left(C^{2}(|\psi\rangle_{A_{1}|A_{2}...A_{n}})\right)$$
$$= f\left(\sum_{i=2}^{n} C^{2}(\rho_{A_{1}A_{i}})\right)$$
$$\leq \sum_{i=2}^{n} f(C^{2}(\rho_{A_{1}A_{i}}))$$
$$= \sum_{i=2}^{n} E(\rho_{A_{1}A_{i}}),$$

where for simplify, we have denoted $f(x) = h(\frac{1+\sqrt{1-x}}{2})$ with $h(x) = -x \log_2(x) - (1-x) \log_2(1-x)$. We have used in the first and last equalities that the entanglement of formation obeys the relation $E(\rho) = f(C^2(\rho))$ for a bipartite $2 \otimes D, D \geq 2$, quantum state ρ [19]. The second equality is due to the fact that $C^2(|\psi\rangle_{A_1...A_n}) = \sum_{i=2}^n C^2(\rho_{A_1A_i})$. The inequality is due to the fact $f(x+y) \leq f(x) + f(y)$.

As for the entanglement of assistance, we have the following conclusion.

Theorem 4 For the n-qubit generalized W-class states $|\psi\rangle \in H_{A_1} \otimes H_{A_2} \otimes ... \otimes H_{A_n}$, we have

$$E(\rho_{A_1|A_{j_1}\dots A_{j_{m-1}}}) \le \sum_{i=1}^{m-1} E_a(\rho_{A_1A_{j_i}}),$$
(17)

where $\rho_{A_1|A_{j_1}...A_{j_{m-1}}}$ is the m-qubit reduced density matrix of $|\psi\rangle$, $2 \le m \le n$.

[Proof] From the lemma 2 of Ref.[7], one has $\rho_{A_1|A_{j_1}...A_{j_{m-1}}}$ of $|\psi\rangle$ is a mixture of a generalized W class state and vacuum. Then, we have

$$E(\rho_{A_{1}|A_{j_{1}}...A_{j_{m-1}}}) \leq \sum_{h} p_{h} E(|\psi\rangle_{A_{1}|A_{j_{1}}...A_{j_{m-1}}}^{h})$$

$$\leq \sum_{h} p_{h} \sum_{i=1}^{m-1} E(\rho_{A_{1}A_{j_{i}}}^{h})$$

$$= \sum_{i=1}^{m-1} \left[\sum_{h} p_{h} E(\rho_{A_{1}A_{j_{i}}}^{h})\right]$$

$$\leq \sum_{i=1}^{m-1} \left[\sum_{h} p_{h} \left(\sum_{j} q_{j} E(|\psi_{j}\rangle_{A_{1}A_{j_{i}}}^{h} \langle \psi_{j}|)\right)\right]$$

$$= \sum_{i=1}^{m-1} \sum_{hj} p_{h} q_{j} E(|\psi_{j}\rangle_{A_{1}A_{j_{i}}}^{h} \langle \psi_{j}|).$$

We obtain the first inequality by noting that $|\psi\rangle_{A_1|A_{j_1}...A_{j_{m-1}}}^h$ is a generalized W class state or vacuum[7]. When $|\psi\rangle_{A_1|A_{j_1}...A_{j_{m-1}}}^h$ is a generalized W class state, then we have $E(|\psi\rangle_{A_1|A_{j_1}...A_{j_{m-1}}}^h) \leq \sum_{i=1}^{m-1} E(\rho_{A_1A_{j_i}}^h)$; When $|\psi\rangle_{A_1|A_{j_1}...A_{j_{m-1}}}^h$ is a vacuum, then we have $E(|\psi\rangle_{A_1|A_{j_1}...A_{j_{m-1}}}^h) = 0 \leq \sum_{i=1}^{m-1} E(\rho_{A_1A_{j_i}}^h)$. The second inequality is due to the definition of the entanglement of formation (12) for mixed quantum states. Since $\sum_{hj} p_h q_j = 1$ and $\sum_{hj} p_h q_j |\psi_j\rangle_{A_1A_{j_i}}^h \langle\psi_j|$ is a pure decomposition of $\rho_{A_1A_{j_i}}$, we have (17).

IV. CONCLUSIONS AND REMARKS

Entanglement monogamy is a fundamental property of multipartite entangled states. We have shown the monogamy for the x-power of concurrence of assistance $C_a(\rho_{A_1|A_{j_i}...A_{j_{m-1}}})$ of the m-qubit reduced density matrices, $2 \le m \le$

n, for the *n*-qubit generalized W-class states. The monogamy relations for the entanglement of formation and the entanglement of assistance the monogamy relation for the *n*-qubit generalized W-class states have been also investigated. These relations give rise to the restrictions of entanglement distribution among the qubits in generalized W-class states.

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- F. Mintert, M. Kuś, and A. Buchleitner, Concurrence of Mixed Bipartite Quantum States in Arbitrary Dimensions, Phys. Rev. Lett. 92, 167902 (2004).
- [2] K. Chen, S. Albeverio, and S. M. Fei, Concurrence of Arbitrary Dimensional Bipartite Quantum States, Phys. Rev. Lett. 95, 040504 (2005).
- [3] H. P. Breuer, Separability criteria and bounds for entanglement measures, J. Phys. A: Math. Gen. 39, 11847 (2006).
- [4] H. P. Breuer, Optimal Entanglement Criterion for Mixed Quantum States, Phys. Rev. Lett. 97, 080501 (2006).
- [5] J. I. de Vicente, Lower bounds on concurrence and separability conditions, Phys. Rev. A 75, 052320 (2007).
- [6] C. J. Zhang, Y. S. Zhang, S. Zhang, and G. C. Guo, Optimal entanglement witnesses based on local orthogonal observables , Phys. Rev. A 76, 012334 (2007).
- [7] J. S. Kim, Strong monogamy of quantum entanglement for multiqubit W-class states, Phys. Rev. A 90, 062306 (2014).
- [8] J. S. Kim, and B. C. Sanders, Generalized W-class state and its monogamy relation, J. Phys. A 41, 495301 (2008).
- [9] A. Uhlmann, Fidelity and concurrence of conjugated states, Phys. Rev. A 62, 032307 (2000).
- [10] P. Rungta, V. Bužek, C. M. Caves, M. Hillery, and G. J. Milburn, Universal state inversion and concurrence in arbitrary dimensions, Phys. Rev. A 64, 042315 (2001).
- [11] S. Albeverio, S. M. Fei, A note on invariants and entanglements ,J Opt B: Quantum Semiclass Opt. 3, 223 (2001).
- [12] C. S. Yu, and H. S. Song, Entanglement monogamy of tripartite quantum states , Phys. Rev. A 77, 032329 (2008).
- [13] X. N. Zhu, S. M. Fei, Entanglement monogamy relations of qubit systems , Phys. Rev. A 90, 024304 (2014).
- [14] G. Goura, S. Bandyopadhyayb, and B. C. Sandersc, Dual monogamy inequality for entanglement, J. Math. Phys. 48, 012108 (2007).
- [15] Z. G. Li, S. M. Fei, S. Albeverio, and W. M. Liu, Bound of entanglement of assistance and monogamy constraints, Phys. Rev. A 80, 034301 (2009).
- [16] V. Coffman, J. Kundu, and W. K. Wootters, Distributed entanglement, Phys. Rev. A 61, 052306 (2000).
- [17] O. Cohen, Unlocking Hidden Entanglement with Classical Information, Phys. Rev. Lett. 80, 2493 (1998).
- [18] G. Gour, D. A. Meyer, and B. C. Sanders, Deterministic entanglement of assistance and monogamy constraints , Phys. Rev. A 72, 042329 (2005).
- [19] Y. K. Bai, Y. F. Xu, and Z. D. Wang, Hierarchical monogamy relations for the squared entanglement of formation in multipartite systems, Phys. Rev. A 90, 062343 (2014).