# General Monogamy Relations of Quantum Entanglement for Multiqubit W-class States 

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#### Abstract

Entanglement monogamy is a fundamental property of multipartite entangled states. We investigate the monogamy relations for multiqubit generalized W -class states. Analytical monogamy inequalities are obtained for the concurrence of assistance, the entanglement of formation and the entanglement of assistance.


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## I. INTRODUCTION

Quantum entanglement [1-6] is an essential feature of quantum mechanics that distinguishes the quantum from the classical world. It is one of the fundamental differences between quantum entanglement and classical correlations that a quantum system entangled with one of the other systems limits its entanglement with the remaining others. This restriction of entanglement shareability among multi-party systems is known as the monogamy of entanglement. The monogamy relations give rise to the structures of entanglement in the multipartite setting. For a tripartite system A, B , and C , the monogamy of an entanglement measure $\varepsilon$ implies that the entanglement between $A$ and $B C$ satisfies $\varepsilon_{A \mid B C} \geq \varepsilon_{A B}+\varepsilon_{A C}$.

In Ref. [7, 8 ] the monogamy of entanglement for multiqubit $W$-class states has been investigated, and the monogamy relations for tangle and the squared concurrence have been proved. In this paper, we show the general monogamy relations for the $x$-power of concurrence of assistance, the entanglement of formation, and the entanglement of assistance for generalized multiqubit $W$-class states.

## II. MONOGAMY OF CONCURRENCE OF ASSISTANCE

For a bipartite pure state $|\psi\rangle_{A B}$ in vector space $H_{A} \otimes H_{B}$, the concurrence is given by [9-11]

$$
\begin{equation*}
C\left(|\psi\rangle_{A B}\right)=\sqrt{2\left[1-\operatorname{Tr}\left(\rho_{A}^{2}\right)\right]} \tag{1}
\end{equation*}
$$

where $\rho_{A}$ is reduced density matrix by tracing over the subsystem $B, \rho_{A}=\operatorname{Tr}_{B}\left(|\psi\rangle_{A B}\langle\psi|\right)$. The concurrence is extended to mixed states $\rho=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|, p_{i} \geq 0, \sum_{i} p_{i}=1$, by the convex roof construction,

$$
\begin{equation*}
C\left(\rho_{A B}\right)=\min _{\left\{p_{i},\left|\psi_{i}\right\rangle\right\}} \sum_{i} p_{i} C\left(\left|\psi_{i}\right\rangle\right) \tag{2}
\end{equation*}
$$

where the minimum is taken over all possible pure state decompositions of $\rho_{A B}$.
For a tripartite state $|\psi\rangle_{A B C}$, the concurrence of assistance (CoA) is defined by 12]

$$
\begin{equation*}
C_{a}\left(|\psi\rangle_{A B C}\right) \equiv C_{a}\left(\rho_{A B}\right)=\max _{\left\{p_{i},\left|\psi_{i}\right\rangle\right\}} \sum_{i} p_{i} C\left(\left|\psi_{i}\right\rangle\right) \tag{3}
\end{equation*}
$$

for all possible ensemble realizations of $\rho_{A B}=\operatorname{Tr}_{C}\left(|\psi\rangle_{A B C}\langle\psi|\right)=\sum_{i} p_{i}\left|\psi_{i}\right\rangle_{A B}\left\langle\psi_{i}\right|$. When $\rho_{A B}=|\psi\rangle_{A B}\langle\psi|$ is a pure state, then one has $C\left(|\psi\rangle_{A B}\right)=C_{a}\left(\rho_{A B}\right)$.

For an $N$-qubit state $|\psi\rangle_{A B_{1} \ldots B_{N-1}} \in H_{A} \otimes H_{B_{1}} \otimes \ldots \otimes H_{B_{N-1}}$, the concurrence $C\left(|\psi\rangle_{\left.A \mid B_{1} \ldots B_{N-1}\right)}\right.$ of the state $|\psi\rangle_{A \mid B_{1} \ldots B_{N-1}}$, viewed as a bipartite with partitions $A$ and $B_{1} B_{2} \ldots B_{N-1}$, satisfies the follow inequality [13]

$$
\begin{equation*}
C_{A \mid B_{1} B_{2} \ldots B_{N-1}}^{\alpha} \geq C_{A B_{1}}^{\alpha}+C_{A B_{2}}^{\alpha}+\ldots+C_{A B_{N-1}}^{\alpha} \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{A \mid B_{1} B_{2} \ldots B_{N-1}}^{\beta}<C_{A B_{1}}^{\beta}+C_{A B_{2}}^{\beta}+\ldots+C_{A B_{N-1}}^{\beta} \tag{5}
\end{equation*}
$$

where $\alpha \geq 2, \beta \leq 0, C_{A B_{i}}=C\left(\rho_{A B_{i}}\right)$ is the concurrence of $\rho_{A B_{i}}=\operatorname{Tr}_{B_{1} \ldots B_{i-1} B_{i+1} \ldots B_{N-1}}(\rho), C_{A \mid B_{1} B_{2} \ldots B_{N-1}}=$ $C\left(|\psi\rangle_{A \mid B_{1} \ldots B_{N-1}}\right)$. Due to the monogamy of concurrence, the generalized monogamy relation based on the concurrence of assistance has been proved in Ref. [14],

$$
\begin{equation*}
C^{2}\left(|\psi\rangle_{A \mid B_{1} \ldots B_{N-1}}\right) \leq \sum_{i=1}^{N-1} C_{a}^{2}\left(\rho_{A B_{i}}\right) \tag{6}
\end{equation*}
$$

In the following we study the monogamy property of the concurrence of assistance for the $n$-qubit generalized W-class states $|\psi\rangle \in H_{A_{1}} \otimes H_{A_{2}} \otimes \ldots \otimes H_{A_{n}}$ defined by

$$
\begin{equation*}
|\psi\rangle=a|000 \ldots\rangle+b_{1}|01 \ldots 0\rangle+\ldots+b_{n}|00 \ldots 1\rangle \tag{7}
\end{equation*}
$$

with $|a|^{2}+\sum_{i=1}^{n}\left|b_{i}\right|^{2}=1$.
Lemma 1 For n-qubit generalized $W$-class states (7), we have

$$
\begin{equation*}
C\left(\rho_{A_{1} A_{i}}\right)=C_{a}\left(\rho_{A_{1} A_{i}}\right) \tag{8}
\end{equation*}
$$

where $\rho_{A_{1} A_{i}}=\operatorname{Tr}_{A_{2} \ldots A_{i-1} A_{i+1} \ldots A_{n}}(|\psi\rangle\langle\psi|)$.
[Proof] It is direct to verify that [7], $\rho_{A_{1} A_{i}}=|x\rangle_{A_{1} A_{i}}\langle x|+|y\rangle_{A_{1} A_{i}}\langle y|$, where

$$
\begin{aligned}
|x\rangle_{A_{1} A_{i}} & =a|00\rangle_{A_{1} A_{i}}+b_{1}|10\rangle_{A_{1} A_{i}}+b_{i}|01\rangle_{A_{1} A_{i}} \\
|y\rangle_{A_{1} A_{i}} & =\sqrt{\sum_{k \neq i}\left|b_{k}\right|^{2}}|00\rangle_{A_{1} A_{i}}
\end{aligned}
$$

From the Hughston- Jozsa-wootters theorem Ref. [7], for any pure-state decomposition of $\rho_{A_{1} A_{i}}=\sum_{h=1}^{r}\left|\phi_{h}\right\rangle_{A_{1} A_{i}}\left\langle\phi_{h}\right|$, one has $\left|\phi_{h}\right\rangle_{A_{1} A_{i}}=u_{h 1}|x\rangle_{A_{1} A_{i}}+u_{h 2}|y\rangle_{A_{1} A_{i}}$ for some $r \times r$ unitary matrices $u_{h 1}$ and $u_{h 2}$ for each $h$. Consider the normalized state $\left|\tilde{\phi}_{h}\right\rangle_{A_{1} A_{i}}=\left|\phi_{h}\right\rangle_{A_{1} A_{i}} / \sqrt{p_{h}}$ with $p_{h}=\left|\left\langle\phi_{h} \mid \phi_{h}\right\rangle\right|$. One has the concurrence of each two-qubit pure $\left|\tilde{\phi_{h}}\right\rangle_{A_{1} A_{i}}$,

$$
C^{2}\left(\left|\tilde{\phi}_{h}\right\rangle_{A_{1} A_{i}}\right)=\frac{4}{p_{h}^{2}}\left|u_{h i}\right|^{4}\left|b_{1}\right|^{2}\left|b_{i}\right|^{2}
$$

Then for the two-qubit state $\rho_{A_{1} A_{i}}$, we have

$$
\sum_{h} p_{h} C\left(\left|\tilde{\phi}_{h}\right\rangle_{A_{1} A_{i}}\right)=\sum_{h} p_{h} \frac{2}{p_{h}}\left|u_{h i}\right|^{2}\left|b_{1}\right|\left|b_{i}\right|=2\left|b_{1}\right|\left|b_{i}\right|
$$

Thus we obtain

$$
\begin{aligned}
C\left(\rho_{A_{1} A_{i}}\right) & =\min _{\left\{p_{h},\left|\tilde{\phi}_{h}\right\rangle_{A_{1} A_{i}}\right\}} \sum_{h} p_{h} C\left(\left|\tilde{\phi}_{h}\right\rangle_{A_{1} A_{i}}\right) \\
& =\max _{\left\{p_{h},\left|\tilde{\phi}_{h}\right\rangle_{A_{1} A_{i}}\right\}} \sum_{h} p_{h} C\left(\left|\tilde{\phi}_{h}\right\rangle_{A_{1} A_{i}}\right) \\
& =C_{a}\left(\rho_{A_{1} A_{i}}\right) .
\end{aligned}
$$

Specifically, in Ref. [8] the same result $C\left(\rho_{A_{1} A_{i}}\right)=C_{a}\left(\rho_{A_{1} A_{i}}\right)$ has been proved for the generalized W-class states (7) with $a=0$.

Theorem 1 For the n-qubit generalized $W$-class states $|\psi\rangle \in H_{A_{1}} \otimes H_{A_{2}} \otimes \ldots \otimes H_{A_{n}}$, the concurrence of assistance satisfies

$$
\begin{equation*}
C_{a}^{x}\left(\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}\right) \geq \sum_{i=1}^{m-1} C_{a}^{x}\left(\rho_{A_{1} A_{j_{i}}}\right) \tag{9}
\end{equation*}
$$

where $x \geq 2$ and $\rho_{A_{1} A_{j_{1}} \ldots A_{j_{m-1}}}$ is the $m$-qubit, $2 \leq m \leq n$, reduced density matrix of $|\psi\rangle$.
[Proof] For the $n$-qubit generalized W-class state $|\psi\rangle$, according to the definitions of $C(\rho)$ and $C_{a}(\rho)$, one has $C_{a}\left(\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}\right) \geq C\left(\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}\right)$. When $x \geq 2$, we have

$$
\begin{aligned}
C_{a}^{x}\left(\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}\right) & \geq C^{x}\left(\rho_{A_{1} \mid A_{j_{1} \ldots A_{j_{m-1}}}}\right) \\
& \geq \sum_{i=1}^{m-1} C^{x}\left(\rho_{A_{1} A_{j_{i}}}\right) \\
& =\sum_{i=1}^{m-1} C_{a}^{x}\left(\rho_{A_{1} A_{j_{i}}}\right)
\end{aligned}
$$

Here we have used in the first inequality the inequality $a^{x} \geq b^{x}$ for $a \geq b>0$ and $x \geq 0$. The second inequality is due to the monogamy of concurrence (4). The last equality is due to the Lemma 1 .

Theorem 2 For the n-qubit generalized $W$-class state $|\psi\rangle \in H_{A_{1}} \otimes H_{A_{2}} \otimes \ldots \otimes H_{A_{n}}$ with $C\left(\rho_{A 1 A_{j_{i}}}\right) \neq 0$ for $1 \leq i \leq$ $m-1$, we have

$$
\begin{equation*}
C_{a}^{y}\left(\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}\right)<\sum_{i=1}^{m-1} C_{a}^{y}\left(\rho_{A_{1} A_{j_{i}}}\right) \tag{10}
\end{equation*}
$$

where $y \leq 0$ and $\rho_{A_{1} A_{j_{1}} \ldots A_{j_{m-1}}}$ is the m-qubit reduced density matrix as in Theorem 1 .
[Proof] For $y \leq 0$, we have

$$
\begin{aligned}
C_{a}^{y}\left(\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}\right) & \leq C^{y}\left(\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}\right) \\
& <\sum_{i=1}^{m-1} C^{y}\left(\rho_{A_{1} A_{j_{i}}}\right) \\
& =\sum_{i=1}^{m-1} C_{a}^{y}\left(\rho_{A_{1} A_{j_{i}}}\right)
\end{aligned}
$$

We have used in the first inequality the relation $a^{x} \leq b^{x}$ for $a \geq b>0$ and $x \leq 0$. The seconder inequality is due to the monogamy of concurrence (5). The last equality is due to Lemma 1.

According to (9) and (10), we can also obtain the lower bounds of $C_{a}\left(\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}\right)$. As an example, consider the 5 -qubit generalized $W$-class states (7) with $a=b_{2}=\frac{1}{\sqrt{10}}, b_{1}=\frac{1}{\sqrt{15}}, b_{3}=\sqrt{\frac{2}{15}}, b_{4}=\sqrt{\frac{3}{5}}$. We have

$$
C_{a}\left(\rho_{A_{1} \mid A_{2} A_{3}}\right) \geq \frac{2}{\sqrt{15}} \sqrt[x]{\left(\frac{1}{\sqrt{10}}\right)^{x}+\left(\sqrt{\frac{2}{15}}\right)^{x}}
$$

and

$$
C_{a}\left(\rho_{A_{1} \mid A_{2} A_{3} A_{4}}\right) \geq \frac{2}{\sqrt{15}} \sqrt[x]{\left.\left(\frac{1}{\sqrt{10}}\right)^{x}+\left(\sqrt{\frac{2}{15}}\right)^{x}+\sqrt{\frac{3}{5}}\right)^{x}}
$$

with $x \geq 2$. The optimal lower bounds can be obtained by varying the parameter $x$, see Fig. 1 , where for comparison the upper bounds are also presented by using the formula $C_{a}\left(\rho_{A B}\right) \leq \sqrt{2\left(1-\operatorname{Tr}\left(\rho_{A}^{2}\right)\right)}$ 15], namely, $C_{a}\left(\rho_{A_{1} \mid A_{2} A_{3}}\right) \leq \frac{2}{\sqrt{18}}$ and $C_{a}\left(\rho_{A_{1} \mid A_{2} A_{3} A_{4}}\right) \leq \frac{2}{\sqrt{18}}$. From Fig.1, one gets that the optimal lower bounds of $C_{a}\left(\rho_{A_{1} \mid A_{2} A_{3}}\right)$ and $C_{a}\left(\rho_{A_{1} \mid A_{2} A_{3} A_{4}}\right)$ are 0.249 and 0.471 , respectively, attained at $x=2$.

## III. MONOGAMY OF ENTANGLEMENT OF FORMATION

The entanglement of formation of a pure state $|\psi\rangle \in H_{A} \otimes H_{B}$ is defined by

$$
\begin{equation*}
E(|\psi\rangle)=S\left(\rho_{A}\right) \tag{11}
\end{equation*}
$$



Fig. 1: Solid line is the lower bound of $C_{a}\left(\rho_{A_{1} \mid A_{2} A_{3}}\right)$, dashed line is the lower bound of $C_{a}\left(\rho_{A_{1} \mid A_{2} A_{3} A_{4}}\right)$ as functions of $x \geq 2$, and dotted line is the upper bound of $C_{a}\left(\rho_{A_{1} \mid A_{2} A_{3}}\right)$ and $C_{a}\left(\rho_{A_{1} \mid A_{2} A_{3} A_{4}}\right)$.
where $\rho_{A}=\operatorname{Tr}_{B}(|\psi\rangle\langle\psi|)$ and $S(\rho)=\operatorname{Tr}\left(\rho \log _{2} \rho\right)$. For a bipartite mixed state $\rho_{A B} \in H_{A} \otimes H_{B}$, the entanglement of formation is given by

$$
\begin{equation*}
E\left(\rho_{A B}\right)=\min _{\left\{p_{i},\left|\psi_{i}\right\rangle\right\}} \sum_{i} p_{i} E\left(\left|\psi_{i}\right\rangle\right) \tag{12}
\end{equation*}
$$

with the infimum taking over all possible decompositions of $\rho_{A B}$ in a mixture of pure states $\rho_{A B}=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$, where $p_{i} \geq 0$ and $\sum_{i} p_{i}=1$.

It has been shown that the entanglement of formation does not satisfy the inequality $E_{A B}+E_{A C} \leq E_{A \mid B C}$ [16]. Rather it satisfies [13],

$$
\begin{equation*}
E_{A \mid B_{1} B_{2} \ldots B_{N-1}}^{\alpha} \geq E_{A B_{1}}^{\alpha}+E_{A B_{2}}^{\alpha}+\ldots+E_{A B_{N-1}}^{\alpha} \tag{13}
\end{equation*}
$$

where $\alpha \geq \sqrt{2}$.
The corresponding entanglement of assistance (EoA) 17] is defined in terms of the entropy of entanglement [18] for a tripartite pure state $|\psi\rangle_{A B C}$,

$$
\begin{equation*}
E_{a}\left(|\psi\rangle_{A B C}\right) \equiv E_{a}\left(\rho_{A B}\right)=\max _{\left\{p_{i},\left|\psi_{i}\right\rangle\right\}} \sum_{i} p_{i} E\left(\left|\psi_{i}\right\rangle\right) \tag{14}
\end{equation*}
$$

which is maximized over all possible decompositions of $\rho_{A B}=\operatorname{Tr}_{C}\left(|\psi\rangle_{A B C}\right)=\sum_{i} p_{i}\left|\psi_{i}\right\rangle\left\langle\psi_{i}\right|$, with $p_{i} \geq 0$ and $\sum_{i} p_{i}=1$. For any $N$-qubit pure state $|\psi\rangle \in H_{A} \otimes H_{B_{1}} \otimes \ldots \otimes H_{B_{N-1}}$, it has been shown that the entanglement of assistance satisfies 13],

$$
\begin{equation*}
E\left(|\psi\rangle_{A \mid B_{1} B_{2} \ldots B_{N-1}}\right) \leq \sum_{i=1}^{N-1} E_{a}\left(\rho_{A B_{i}}\right) \tag{15}
\end{equation*}
$$

In fact, generally we can prove the following results for the $n$-qubit generalized W -class states about the entanglement of formation and the entanglement of assistance.

Theorem 3 For the n-qubit generalized $W$-class states $|\psi\rangle \in H_{A_{1}} \otimes H_{A_{2}} \otimes \ldots \otimes H_{A_{n}}$, we have

$$
\begin{equation*}
E\left(|\psi\rangle_{A_{1} \mid A_{2} \ldots A_{n}}\right) \leq \sum_{i=2}^{n} E\left(\rho_{A_{1} A_{i}}\right) \tag{16}
\end{equation*}
$$

where $\rho_{A_{1} A_{i}}, 2 \leq i \leq n$, is the 2-qubit reduced density matrix of $|\psi\rangle$.
[Proof] For the $n$-qubit generalized W -class states $|\psi\rangle$, we have

$$
\begin{aligned}
E\left(|\psi\rangle_{A_{1} \mid A_{2} \ldots A_{n}}\right) & =f\left(C^{2}\left(|\psi\rangle_{A_{1} \mid A_{2} \ldots A_{n}}\right)\right) \\
& =f\left(\sum_{i=2}^{n} C^{2}\left(\rho_{A_{1} A_{i}}\right)\right) \\
& \leq \sum_{i=2}^{n} f\left(C^{2}\left(\rho_{A_{1} A_{i}}\right)\right) \\
& =\sum_{i=2}^{n} E\left(\rho_{A_{1} A_{i}}\right)
\end{aligned}
$$

where for simplify, we have denoted $f(x)=h\left(\frac{1+\sqrt{1-x}}{2}\right)$ with $h(x)=-x \log _{2}(x)-(1-x) \log _{2}(1-x)$. We have used in the first and last equalities that the entanglement of formation obeys the relation $E(\rho)=f\left(C^{2}(\rho)\right)$ for a bipartite $2 \otimes D, D \geq 2$, quantum state $\rho[19]$. The second equality is due to the fact that $C^{2}\left(|\psi\rangle_{A_{1} \ldots A_{n}}\right)=\sum_{i=2}^{n} C^{2}\left(\rho_{A_{1} A_{i}}\right)$. The inequality is due to the fact $f(x+y) \leq f(x)+f(y)$.

As for the entanglement of assistance, we have the following conclusion.
Theorem 4 For the $n$-qubit generalized $W$-class states $|\psi\rangle \in H_{A_{1}} \otimes H_{A_{2}} \otimes \ldots \otimes H_{A_{n}}$, we have

$$
\begin{equation*}
E\left(\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}\right) \leq \sum_{i=1}^{m-1} E_{a}\left(\rho_{A_{1} A_{j_{i}}}\right) \tag{17}
\end{equation*}
$$

where $\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}$ is the m-qubit reduced density matrix of $|\psi\rangle, 2 \leq m \leq n$.
[Proof] From the lemma 2 of Ref. [7], one has $\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}$ of $|\psi\rangle$ is a mixture of a generalized $W$ class state and vacuum. Then, we have

$$
\begin{aligned}
E\left(\rho_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}\right) & \leq \sum_{h} p_{h} E\left(|\psi\rangle_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}^{h}\right) \\
& \leq \sum_{h} p_{h} \sum_{i=1}^{m-1} E\left(\rho_{A_{1} A_{j_{i}}}^{h}\right) \\
& =\sum_{i=1}^{m-1}\left[\sum_{h} p_{h} E\left(\rho_{A_{1} A_{j_{i}}}^{h}\right)\right] \\
& \leq \sum_{i=1}^{m-1}\left[\sum_{h} p_{h}\left(\sum_{j} q_{j} E\left(\left|\psi_{j}\right\rangle_{A_{1} A_{j_{i}}}^{h}\left\langle\psi_{j}\right|\right)\right)\right] \\
& =\sum_{i=1}^{m-1} \sum_{h j} p_{h} q_{j} E\left(\left|\psi_{j}\right\rangle_{A_{1} A_{j_{i}}}^{h}\left\langle\psi_{j}\right|\right) .
\end{aligned}
$$

We obtain the first inequality by noting that $|\psi\rangle_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}^{h}$ is a generalized $W$ class state or vacuum [7]. When $|\psi\rangle_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}^{h}$ is a generalized $W$ class state, then we have $E\left(|\psi\rangle_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}^{h}\right) \leq \sum_{i=1}^{m-1} E\left(\rho_{A_{1} A_{j_{i}}}^{h}\right)$; When $|\psi\rangle_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}^{h}$ is a vacuum, then we have $E\left(|\psi\rangle_{A_{1} \mid A_{j_{1}} \ldots A_{j_{m-1}}}^{h}\right)=0 \leq \sum_{i=1}^{m-1} E\left(\rho_{A_{1} A_{j_{i}}}^{h}\right)$. The second inequality is due to the definition of the entanglement of formation (12) for mixed quantum states. Since $\sum_{h j} p_{h} q_{j}=1$ and $\sum_{h j} p_{h} q_{j}\left|\psi_{j}\right\rangle_{A_{1} A_{j_{i}}}^{h}\left\langle\psi_{j}\right|$ is a pure decomposition of $\rho_{A_{1} A_{j_{i}}}$, we have (17).

## IV. CONCLUSIONS AND REMARKS

Entanglement monogamy is a fundamental property of multipartite entangled states. We have shown the monogamy for the $x$-power of concurrence of assistance $C_{a}\left(\rho_{A_{1} \mid A_{j_{i}} \ldots A_{j_{m-1}}}\right)$ of the $m$-qubit reduced density matrices, $2 \leq m \leq$
$n$, for the $n$-qubit generalized $W$-class states. The monogamy relations for the entanglement of formation and the entanglement of assistance the monogamy relation for the $n$-qubit generalized W-class states have been also investigated. These relations give rise to the restrictions of entanglement distribution among the qubits in generalized $W$-class states.

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