

# Asymmetric Quantum Dialogue in Noisy Environment

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## Abstract

A notion of asymmetric quantum dialogue (AQD) is introduced. Conventional protocols of quantum dialogue are essentially symmetric as both the users (Alice and Bob) can encode the same amount of classical information. In contrast, the scheme for AQD introduced here provides different amount of communication powers to Alice and Bob. The proposed scheme, offers an architecture, where the entangled state and the encoding scheme to be shared between Alice and Bob depends on the amount of classical information they want to exchange with each other. The general structure for the AQD scheme has been obtained using a group theoretic structure of the operators introduced in (Shukla et al., Phys. Lett. A, **377** (2013) 518). The effect of different types of noises (e.g., amplitude damping and phase damping noise) on the proposed scheme is investigated, and it is shown that the proposed AQD is robust and uses optimized amount of quantum resources.

**Keywords:** Asymmetric quantum dialogue, noise models, secure quantum communication.

## 1 Introduction

The birth of quantum cryptography in 1984 [1], followed by its variants steadily paved way for various quantum communication schemes, thereby, establishing its manifold applications (see [2] and references therein). Initial protocols [1, 3, 4, 5] of secure quantum communication were restricted to the designing of protocols for quantum key distribution (QKD), which enables a sender (Alice) to share (distribute) a key with a receiver (Bob). It was later realized that a pre-shared (pre-distributed) key is not essential for secure quantum communication, and several schemes for secure direct quantum communication were proposed [6, 7, 8, 9, 10, 11]. These schemes of secure direct quantum communication were mainly of two types: quantum secure direct communication (QSDC) and deterministic secure quantum communication (DSQC). In a QSDC scheme, the receiver can decode an information encoded by the sender without any additional information (classical communication) [6, 7, 8]. In contrast, in a scheme for DSQC, the receiver requires at least one bit of classical information to decode each bit of the message encoded by the sender [9, 10]. Different variants of DSQC and QSDC protocols have been studied in the recent past. Specifically, controlled DSQC has been studied in much detail in the recent past ([11] and references therein). Interestingly, the schemes of DSQC and QSDC are one way schemes in the sense that these schemes only allow Alice to communicate a message to Bob, but does not allow Bob to do the same. In contrast, in daily life, one often need bidirectional communication, where two parties can simultaneously communicate messages to each other, an example been classical communication via telephone. Keeping this in mind, a scheme for simultaneous bidirectional quantum communication was introduced by Ba An [12] in 2004, and was referred to as “quantum dialogue (QD)”. However, it was found that the Ba An scheme was insecure under intercept-resend attack [9]. In Ref. [9], an effort was made to modify the original Ba An scheme to provide a secure

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scheme for QD. However the effort was not successful and later a secure version of the original scheme was proposed by Ba An himself [13] using Bell states.

It is important to note that a scheme for QD is different from a system where conversation between Alice and Bob happens using two QSDCs or two DSQCs (say, one QSDC/DSQC is used for Alice to Bob communication and the other one is used for Bob to Alice communication). In QD, messages of Alice and Bob are required to be simultaneously encoded in the same channel. Further, we would like to note that this type of scheme for bidirectional direct quantum communication has been referred by various names, but schemes proposed under these alternate names are equivalent and may be referred to as quantum dialogue. Specifically, equivalent bidirectional schemes for secure direct quantum communication have been referred to as bidirectional quantum communication in [14], quantum telephone in [15, 16], quantum dialogue in [12, 17], quantum conversation in [18], etc.

The practical importance of the QD drew considerable attention of the quantum cryptography community and several schemes for QD have been proposed. For example, Xia et al. proposed a QD scheme using *GHZ* states [19] and Dong *et al.* proposed the same using tripartite *W* states [20]. Further, efforts have been made to realize QD using various approaches. For instance, protocols for QD have been proposed using (i) dense coding [9, 15, 19], (ii) entanglement swapping [21, 22], (iii) single photon [17], (iv) auxiliary particles [14], (v) qutrit states [22, 23, 24], (vi) continuous variable states [25], (vii) prior shared key for authentication [17, 26], etc. Apart from all these variants of QD, a modified QD scheme with the name quantum secure direct dialogue [27, 28] has been recently proposed, in which, a few of the Bell states are used for Alice to Bob communication, while the remaining for sending Bob's message to Alice. This is not really a scheme for QD as the information encoded by Alice and Bob does not exist simultaneously in the same channel. Thus, it is to be viewed as two QSDCs. In the scheme proposed in [27, 28], Alice and Bob were capable of sending messages of different length to each other, and this was claimed to be the advantage of quantum secure direct dialogue protocol [27, 28] over conventional protocols of QD. However, to achieve this feature they had to go beyond the domain of QD. In what follows, we will show that we can achieve this feature, strictly remaining within the domain of quantum dialogue. Further, we would like to note that almost all the Ba-An-type QD schemes proposed so far have a generalized mathematical structure [29]. Specifically, if a maximally densecodable quantum state is used as quantum channel, the encoding operations form a group under multiplication. This point will be further discussed in detail in the forthcoming sections.

Recently, considerable efforts have also been made to avoid information leakage in quantum dialogue protocols (see [16, 21, 22, 24] and reference therein). Some efforts have also been made to study the effect of noise on the schemes of QD [30, 31, 32]. For instance, in Ref. [30], the effect of a set of noise models on a single-particle-based and an entangled-state-based QD schemes has been investigated for a comparative analysis. Similarly, the effect of noise on the controlled version of a QD scheme is analyzed under amplitude and phase damping noise models in Ref. [31]. Further, Yang and Hwang [32] have proposed a QD scheme immune to collective noise using a set of logical bits (which are Bell states immune to collective noise) as reported in [33]. Specifically, some Bell states form a decoherence free subspace under collective noise and have been used to design various QD protocols since then [28, 32, 34]. However, all the schemes of QD proposed so far and investigated under noisy environment are symmetric in the sense that Alice and Bob encode equal amount of information. Interestingly, in practical circumstances, one often comes across situations, where one of the users communicates more (say in an online lecture class usually Professor speaks more). In such situations, we need schemes for asymmetric quantum dialogue (AQD), where the communication power (the amount of classical information one can send) is different for different users. In what follows, we have referred to standard schemes of symmetric QD as QD and have designed a new scheme of AQD, which is shown to be interesting by establishing that it would require lesser quantum resource and be less affected in a noisy channel (amplitude damping and phase damping channel).

The remaining part of this paper is organized as follows. In Section 2, the protocol of AQD is proposed, and subsequently, the group theoretic structure of the operators that can be used to implement the protocol is discussed in Section 3. In Section 4, the feasibility of the proposed AQD scheme is analyzed under the effect of amplitude damping and phase damping noise. The leakage and efficiency in the proposed AQD scheme are discussed in Section 5 and finally, the paper is concluded in Section 6.

## 2 Protocol for asymmetric quantum dialogue

Before we proceed with the AQD protocol, it would be appropriate to note that the protocol proposed in this section is a Ba-An-type protocol, and is along the line of the generalized Ba-An-type protocol for QD proposed by some of the present authors in Ref. [29], which we refer to as Shukla-Banerjee-Pathak (SBP) protocol. In SBP protocol, an  $n$ -qubit quantum state  $|\phi_1\rangle$  from the basis set  $\{|\phi_i\rangle\}$  is used. Further, a set of  $2^n$  unitary operators  $\{U_1, U_2, \dots, U_{2^n}\}$  are needed to encode an  $n$ -bit message, where the unitary operators are essentially  $j$ -qubit ( $j \leq n$ ) operators such that  $U_i|\phi_1\rangle = |\phi_i\rangle$ . It is also necessary that the set of operators  $\{U_1, U_2, \dots, U_{2^n}\}$  forms a group under multiplication after

neglecting the global phase [29]. This is important because in Ba-An-type schemes, Alice encodes  $U_A$  on the quantum state  $|\phi'_1\rangle = U_B|\phi_1\rangle$  which was produced by Bob by applying a unitary operation  $U_B$  on the initial state  $|\phi_1\rangle$ . Thus, the final state after Alice's operation becomes  $|\phi''_1\rangle = U_A U_B |\phi_1\rangle$ , which will be in the basis set  $\{|\phi_i\rangle\}$  only if  $U_A U_B = U_j$  is an element of the set  $\{U_1, U_2, \dots, U_{2^n}\}$ . Its relevance would be clear if we note that, to decode any information sent by Alice to Bob (or equivalently from Bob to Alice) one needs knowledge of either  $U_A$  or  $U_B$ . As only Alice and Bob possess this information, they can extract the information encoded by the other user if one of the user (Bob) measures  $|\phi''_1\rangle$  using  $\{|\phi_i\rangle\}$  basis set and announces the result. These conditions led to a generalized group theoretic structure for the operators to be used to implement a protocol of QD. In what follows, we use the same conditions and notations to introduce a protocol for asymmetric quantum dialogue, where Alice and Bob wish to send messages of  $m$  and  $n$  bits ( $m \neq n$  for AQD), respectively to each other. In contrast, in a conventional symmetric QD, one always had  $m = n$ . Technically, it is possible to achieve the task accomplished by AQD using a conventional scheme of QD, but that would require utilization of more quantum resources. To be precise, if  $m < n$  then Alice would require to send  $(n - m)$  auxiliary bits. Similarly, Bob would require to send  $(m - n)$  auxiliary bits, when  $m > n$ . Without loss of generality, here we may describe our protocol for AQD for the specific case  $m < n$  and show that neither Alice needs to send any auxiliary bits, nor we need to go beyond the domain of quantum dialogue (as was done in Refs. [27, 28]). The protocol for AQD can be described as follows:

**AQD1** Bob prepares  $p$  copies of an initial state  $|\phi_1\rangle$ , which is an  $n$ -qubit entangled state, i.e., he prepares  $|\phi_1\rangle^{\otimes p}$ . Subsequently, he encodes his message by applying a  $j$ -qubit unitary operator from the group of unitary operators  $\{U_1, U_2, \dots, U_{2^n}\}$ . Note that each unitary operation encodes an  $n$ -bit classical message. Hence, Bob encodes an  $np$ -bit message using all the states. The orthogonality of the information encoded states is ensured due to the specific properties of the set of unitary operators used here, i.e., operation of an operator from the set  $\{U_1, U_2, \dots, U_{2^n}\}$  gives  $U_i|\phi_1\rangle = |\phi_i\rangle$ , which is an element of the complete basis set  $\{|\phi_i\rangle\}$ .

**AQD2** Bob prepares two strings of travel and home qubits as  $P_A = [p_1(t_1, t_2, \dots, t_l), p_2(t_1, t_2, \dots, t_l), \dots, p_N(t_1, t_2, \dots, t_l)]$  and  $P_B = [p_1(h_1, h_2, \dots, h_{n-l}), p_2(h_1, h_2, \dots, h_{n-l}), \dots, p_N(h_1, h_2, \dots, h_{n-l})]$ , respectively. The string of the travel qubits contains all the  $l$  qubits on which Alice will encode her message in **AQD4**. Another string of home qubits is composed of the qubits not encoded by Bob and the Bob's encoded qubits on which Alice is not supposed to encode her secret. Then Bob prepares  $lp$  decoy qubits and concatenate them with  $P_A$  to form a larger string  $P'_A$ . Subsequently, he applies a permutation operator  $\Pi_{2lp}$  on the string  $P'_A$  to obtain a new string  $P''_A$  and sends it to Alice.

A detailed discussion of the various types of decoy qubit based eavesdropping check subroutines can be found in the recent literature ([35] and references therein).

**AQD3** Bob announces the permutation operator  $\Pi_{lp}$  corresponding to the decoy qubits, i.e., the correct positions of the decoy qubits on which certain decoy qubit based eavesdropping checking technique (such as BB84 subroutine, GV subroutine, etc. [2, 10, 11, 29, 36]) depending upon the choice of decoy qubits can be applied [35]. They proceed with the protocol if the detected error rate is found to be below a tolerable limit, otherwise they start afresh.

**AQD4** Bob informs Alice the order of the remaining qubits. Then Alice obtains the actual order and performs her encoding using a set of  $l$ -qubit unitary operators  $\{U'_1, U'_2, \dots, U'_{2^m}\}$  such that  $\{U'_1 \otimes I_2^{\otimes j-l}, U'_2 \otimes I_2^{\otimes j-l}, \dots, U'_{2^m} \otimes I_2^{\otimes j-l}\}$  forms a subgroup (of order  $2^m$ ) of the group  $\{U_1, U_2, \dots, U_{2^n}\}$ , which contains the operators used by Bob for encoding his message, and is a group of order  $2^n$  with  $n > m$  for AQD. Thereafter, Alice prepares  $lp$  decoy qubits and concatenates them with the original string  $P_{AB}$  to obtain a larger sequence  $P'_{AB}$ . This is followed by a permutation operation on the enlarged sequence  $P'_{AB}$  by Alice to create  $P''_{AB}$ . Finally, she sends  $P''_{AB}$  back to Bob.

Note that the Identity operator ( $I_2$ ) mentioned above, were included in the description to illustrate the nature of the subgroup that can be formed and in our case, we can visualize it as a situation in which Alice encoded her message on the qubits received by her using an operator  $U'_i$  from the group of operators  $\{U'_i\}$  and thus,  $I_2$  operators operate on the remaining qubits on which Bob had encoded his message, but not send to Alice.

**AQD5** Bob also performs an eavesdropping checking (in collaboration with Alice) as in **AQD3**. They proceed with the protocol if and only if sufficiently low error rate is obtained, otherwise they restart from **ADQ1**.

**AQD6** Alice announces the permutation operator for Bob to reorder the remaining qubits. Bob obtains  $P_{AB}$  and recombines it with  $P_B$  to measure each  $n$ -qubit entangled state in  $\{|\phi_i\rangle\}$  basis. Subsequently, Bob publicly announces the final states he had obtained on measurement. The initial and final states are publicly known. Bob knows his encoding as well, with the help of which he can decode Alice's message. Similarly, Alice obtains Bob's

message using these publicly known information and her knowledge about the encoding operation performed by her during **AQD4**.

Specific examples of the AQD scheme will be discussed in the forthcoming sections.

### 3 Group theoretic structure of the AQD protocol

To illustrate the general structure of the possible schemes for AQD, we may use a notion of the modified Pauli group introduced in [29, 36]. In [29], an operational definition of the Pauli group was used, where global phase was ignored from the group multiplication table of the Pauli group (thus, more than one element of standard Pauli group [37] which are different only in global phase would become a single element of the modified Pauli group [29]). For the convenience of the reader, the operational definition of the modified Pauli group and the notation introduced in [29] and followed in this paper are summarized in Appendix A. In what follows, we describe the group theoretic structure of the operators used by Alice and Bob to realize the proposed scheme of AQD. As the choice of the encoding operators depends on the entangled states to be shared between Alice and Bob, we restrict our discussion to a finite set of  $n$ -qubit entangled states (where we choose  $2 \leq n \leq 5$ ). Specifically, for  $n = 2$ , only entangled state of interest is a Bell state; whereas for  $n = 3$ , all arbitrary entangled state can be classified into two sub-classes: *GHZ*-type states and *W*-type states, and keeping that in mind for  $n = 3$ , we have restricted our investigation to the search of groups of unitary operators that can be used to implement QD or AQD using representative quantum states from *GHZ* class and *W* class. Similarly, for  $n = 4$  case, we have concentrated on the representative quantum states from the 9 families of 4-qubit entangled states introduced in [38]. We have already noted that all 3-qubit entangled states can be classified as *GHZ*-type states and *W*-type states. Inspired by this observation Verstraete et al. tried to classify 4-qubit entangled states into a finite set of SLOCC nonequivalent families and introduced 9 families which were referred to as  $G_{abcd}, L_{abc2}, L_{a2b2}, L_{ab3}, L_{a4}, L_{a20_{3\oplus\bar{1}}}, L_{0_{5\oplus\bar{3}}}, L_{0_{7\oplus\bar{1}}}, L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$  (for exact definitions of these families see Theorem 2 of Ref. [38]). Later on, Chterental et al. [39] and Borsten et al. [40] also obtained these nine families through different approaches. However, Gour and Wallach had later shown that actually there exist an infinite number of SLOCC non-equivalent classes for four qubit entangled states [41]. It's not our purpose to discuss these SLOCC nonequivalent classes in detail. Rather, we are interested in illustrating the group theoretic structure of the AQD scheme with some quantum states as an example, and for this purpose we have chosen representative states from different families of 4-qubit entangled states as classified in [38]. Finally, we also report the group theoretic structure of the operators that may used to implement QD/AQD using specific type of 5-quabit entangled states (namely 5-qubit Brown and cluster states).

To begin with, we note that a large number of alternate possibilities of implementing QD using  $n$ -qubit ( $2 \leq n \leq 5$ ) entangled states and various groups of unitary operators were already listed in Table 4 of our earlier work [29]. The present investigation has revealed a number of new possibilities, and they are summarized in Table 1. Specifically, the last column of Table 1 reports a new set (i.e., not reported earlier) of group of operators that may be used to implement QD or AQD using a particular type of entangled state mentioned in the first column of the table. Table 1 clearly illustrates that a scheme for QD can be implemented using a state mentioned in the 1st column of the  $i$ th row of the table, if the users use one of the groups listed in Column 3 or 4 of the same row. However, to implement an AQD, using a state mentioned in the 1st column of  $i$ th row of this table, one of the user (say, Bob, who is the first user in the sense that he prepares the quantum channel) has to use a group (say,  $G_B$ ) of order  $2^n$  listed in Column 3 or 4 of the same row, whereas the other user (Alice) would require to use another group of operators (say,  $G_A$ ) of order  $2^m$ , such that  $G_A \otimes \{I_2^{j-l}\}$  (cf. **AQD4**) forms a subgroup of  $G_B$ . In what follows, we discuss a few specific examples to extend this point, and provided a large list of allowed combination of states and such groups of operators in Table 2.

Let us consider a particular example in which a 4-qubit cluster state is used for the implementation of a scheme for QD/AQD between Alice and Bob. To begin with, let us consider the symmetric case in which both Alice and Bob wish to communicate 4 bits of classical information to each other. In this particular case, 2 travel qubits will be required, and both Alice and Bob have to encode their secrets using the operators from the group  $G_2$ . In contrast, they may also decide to go for an AQD scheme using the same quantum state and allowing Bob to encode 4 bits of classical information and Alice to encode half of that. In this particular case, Bob would still encode using the operators from  $G_2$  on two qubits (as he did in symmetric case), but would send only one qubit to Alice, who will be able to encode her 2 bits of classical information using unitary operators from  $G_1$  group on the travel qubit and send it back to Bob. Note that  $G_1 \otimes \{I_2\} = \{I_2 \otimes I_2, X \otimes I_2, iY \otimes I_2, Z \otimes I_2\} < G_2$  (i.e,  $G_1 \otimes \{I_2\}$  is a subgroup of  $G_2$ ). Subsequently, Bob will measure the final state in the suitable basis and broadcast the measurement outcome. One may argue that we can implement the AQD scheme using conventional QD, too. In that case, we have to send two travel qubits through the channel. That would increase the possibilities of being affected by the channel noise and also would involve higher quantum cost as far as the communication via the quantum channel is concerned. It is evident that if Alice needs to encode less classical information then AQD with lesser number of travel qubits is preferable and sufficient. In Table

Quantum state	SLOCC nonequiv- alent family	Group of unitary operations that can be used for QD and described in Ref. [29]	New group of unitary operations that can also be used for QD
2-qubit Bell state	Bell	$G_1$	
3-qubit $GHZ$	$GHZ$	$G_2^1(8), G_2^2(8), G_2^4(8), G_2^5(8)$	
3-qubit $GHZ$ -like	$GHZ$	$G_2^2(8), G_2^3(8), G_2^5(8), G_2^6(8), G_2^8(8), G_2^9(8)$	
4-qubit cat state	$G_{abcd}$		$G_2^1(8), G_2^2(8), G_2^4(8), G_2^5(8)$
4-qubit $W$	$L_{ab3}$	$G_2^8(8), G_2^9(8)$	
4-qubit $Q_5$	$L_{0_7 \oplus \bar{1}}$	$G_2^4(8), G_2^5(8)$	
4-qubit cluster state	$G_{abcd}$	$G_2$	$G_2^1(8), G_2^2(8), G_2^4(8), G_2^5(8)$
4-qubit $\Omega$ state	$L_{0_3 \oplus \bar{1} 0_3 \oplus \bar{1}}$	$G_2$	$G_2^i(8) : i \in \{1, \dots, 11\}$
4-qubit $Q_4$	$L_{0_5 \oplus \bar{3}}$	$G_2^6(8), G_2^7(8)$	$G_2^5(8)$
$\frac{ 0001\rangle +  0010\rangle +  0111\rangle +  1011\rangle}{2}$	$L_{ab3}$		$G_2^8(8), G_2^9(8)$
$\frac{ 0000\rangle +  0111\rangle}{\sqrt{2}}$	$L_{0_3 \oplus \bar{1} 0_3 \oplus \bar{1}}$		$G_2^4(8), G_2^5(8), G_2^8(8), G_2^9(8), G_2^{10}(8), G_2^{11}(8)$
5-qubit Brown state	-	$G_3^1(32), G_3^2(32), G_3^4(32), G_3^5(32), G_3^7(32), G_3^8(32)$	
5-qubit cluster state	-	$G_3^4(32), G_3^5(32), G_3^7(32), G_3^8(32)$	

Table 1: List of useful quantum states and corresponding operators that may be used to implement protocols for AQD and QD. Specifically, to implement a protocol of QD using a state mentioned in the 1st column of  $i$ th row of this table, both the users should use one of the groups listed in Column 3 or 4 of the same row. However, to implement an AQD, using a state mentioned in the 1st column of  $i$ th row of this table, one of the user (say Bob) would use a group ( $G_B$ ) listed in Column 3 or 4 of the same row, as was done in QD, and Alice (who is expected to communicate less) would use a smaller group  $G_A : G_A \otimes I_2^{j-l} < G_B$ , where Bob and Alice encode their message by using  $j$  and  $l$  qubit unitary operations, respectively.

2, a list of groups of operators capable of implementing the proposed scheme of QD and AQD for different quantum states is presented. It is shown that there exists a large number of alternative ways (combination of groups of operators and quantum states) that may be used for the implementation of the proposed scheme for AQD.

## 4 Robustness: Effect of noise on the AQD scheme

To implement the AQD protocol described above, an entangled state is to be used, part of which (travel qubits) will travel through the channel and thus will get exposed to the environment, whereas the other qubits (home qubits) would remain with one of the users. In what follows, it will be assumed that home qubits will not be affected by noise [31]. Keeping this in mind, in this section, we aim to study the effect of widely used noise models: Amplitude damping (AD) and Phase damping (PD) on the travel qubits in the AQD protocol. We also compare the fidelity of the quantum state at the end of Alice's and Bob's encoding in noisy and ideal scenarios obtained for AQD in a particular type of noise channel, with that of the fidelity obtained for QD implemented with the same conditions.

Before we discuss the effect of noise on the proposed AQD scheme, we will briefly introduce the strategy adopted here to quantify this. If Bob prepares an  $n$ -qubit pure entangled state  $\rho_{\text{initial}} = [|\psi\rangle\langle\psi|]_{h+t}$  and keeps the home qubits ( $h$ ) with himself after sending travel qubits ( $t$ ) to Alice, where  $t = n - h$ , thus the noise acts on the travel qubits only. The transformed quantum state in the presence of noise can be written as

$$\rho_k = \sum_{i_j} I_2^{\otimes h} \otimes E_{i_1}^k \otimes \dots \otimes E_{i_j}^k \dots \otimes E_{i_t}^k \rho_{\text{initial}} \left( I_2^{\otimes h} \otimes E_{i_1}^k \otimes \dots \otimes E_{i_j}^k \dots \otimes E_{i_t}^k \right)^\dagger, \quad (1)$$

where  $E_{i_j}^k$  are the suitable Kraus operators for AD or PD channels, which will be discussed in detail in the following subsections.

The effect of noise can be quantitatively obtained using a distance based measure fidelity between the quantum state  $\rho_k$  obtained in the presence of noisy channels with the one in ideal condition. Suppose the quantum state in the noisy channel after the encoding of Alice and Bob is  $\rho'_k$ , while it was expected to be  $|\psi'\rangle$  in the absence of noise. In this case, the fidelity is

$$F_k = \langle \psi' | \rho'_k | \psi' \rangle, \quad (2)$$



Quantum state	AQD				QD		
	$N_T$	Operation of B	Operation of A	c-bits (B:A)	$N_T$	Operation of B or A	c-bits (B:A)
2-qubit Bell state	1	$g_i : i \in \{1, 2, 3\}$	$G_1$	1:2	1	$G_1$	2:2
	1	$G_1$	$g_i : i \in \{1, 2, 3\}$	2:1			
3-qubit GHZ	1	$G_2^i(8) : i \in \{4, 5\}$	$g_i : i \in \{1, 2, 3\}$	3:1	2	$G_2^i(8) : i \in \{4, 5\}$	3:3
	1	$G_2^i(8) : i \in \{4, 5\}$	$G_1$	3:2			
4-qubit cluster state and $\Omega$ state	1	$G_2$	$g_i : i \in \{1, 2, 3\}$	4:1	2	$G_2$	4:4
	1	$G_2$	$G_1$	4:2			
	2	$G_2$	$G_2^i(8) : i \in \{1, \dots, 6\}$	4:3			
5-qubit Brown state	1	$G_2^i(8) : i \in \{1, 2, 4, 5, 7, 8\}$	$g_i : i \in \{1, 2, 3\}$	5:1	3	$G_2^i(8) : i \in \{1, 2, 4, 5, 7, 8\}$	5:5
	1	$G_2^i(8) : i \in \{1, 2, 4, 5, 7, 8\}$	$G_1$	5:2			
	2	$G_2^i(8) : i \in \{1, 2, 4, 5, 7, 8\}$	$G_2^i(8) : i \in \{1, \dots, 6\}$	5:3			
	2	$G_2^i(8) : i \in \{1, 2, 4, 5, 7, 8\}$	$G_2$	5:4			

Table 2: Asymmetric quantum dialogue (AQD) and quantum dialogue (QD) between Alice (A) and Bob (B).  $N_T$  is the number of travel qubits and  $g_1 = \{I_2, X\}$ ,  $g_2 = \{I_2, iY\}$ , and  $g_3 = \{I_2, Z\}$ . The notations used are elaborated in Appendix A and are consistent with the previous table.

which is the square of the conventional fidelity expression and has also been used in the past.

It would be worth mentioning here that we have obtained average fidelity considering all possible encoding of Alice and Bob. Further, we have considered here a special case with 4-qubit cluster state as initial state, which will be utilized as quantum channel. AQD scheme gives a certain advantage in the implementation of the scheme as it will be less affected due to noise than the QD scheme. This is discussed in following subsections.

#### 4.1 Effect of amplitude damping (AD) noise

The effect of AD noise in the quantum state  $\rho_{\text{initial}}$  is characterized by the following Kraus operators [37]:

$$E_0^A = |0\rangle\langle 0| + \sqrt{1 - \eta_A}|1\rangle\langle 1|, \quad E_1^A = \sqrt{\eta_A}|0\rangle\langle 1|, \quad (3)$$

where  $\eta_A$  ( $0 \leq \eta_A \leq 1$ ) is the decoherence rate and describes the probability of error due to AD channel.

The fidelity expressions are obtained for QD/AQD (with 2 travel qubits) and AQD (with 1 travel qubit) as

$$F_{AD}^{\text{QD/AQD}} = \frac{1}{8} (\eta^4 - 4\eta^3 + 12\eta^2 - 16\eta + 8) \quad (4)$$

and

$$F_{AD}^{\text{AQD}} = \frac{1}{4} (\eta - 2)^2, \quad (5)$$

respectively. We have observed that fidelity expression depends on the number of travel qubits (which depends on the amount of classical information Alice wants to encode) and it does not depend on Bob's encoding. Specifically, it does not depend on how much classical information has been encoded by Bob, and how many qubits have been used for Bob's encoding. This is not surprising as we have considered that the home qubits don't get affected by the channel noise. Thus, as long as we are considering that only travel qubits get affected by the channel noise, our strategy for designing new schemes should be such that a proposed scheme would utilize a minimum number of travel qubits. More precisely, the number of travel qubits should be equal to the minimum number of qubits that Alice requires to encode her message. For example, one travel qubit would be sufficient as long as Alice's message is up to 2 bits. Further, the presence of quadratic terms in the fidelity expression of the encoded quantum state in AQD (with 1 travel qubit) are the signature of to and fro travel of the single qubit between Bob and Alice. Similarly, the quartic terms in the fidelity expression for QD/AQD with 2 travel qubits is the signature of to and fro travel of qubits between Bob and Alice.

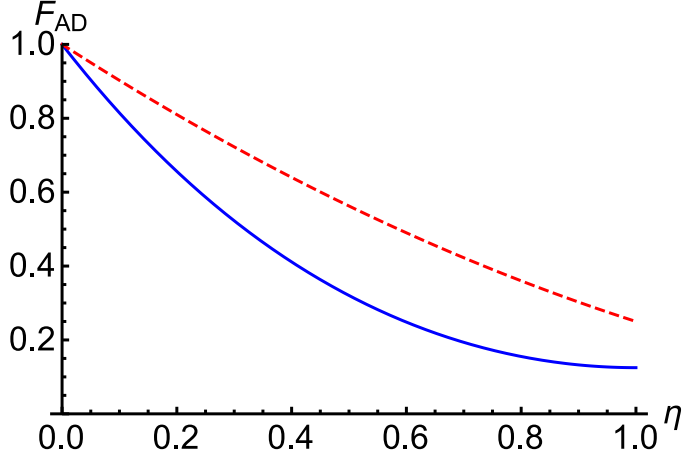


Figure 1: (Color online) The effect of AD noise on the asymmetric QD depends on the number of travel qubits. Here, 4-qubit cluster state is used as quantum channel. The smooth (blue) and dashed (red) lines correspond to 2 and 1 travel qubits, respectively.

To study the effect of AD channels, we have shown the variation of both the fidelity expressions in Fig. 1, which illustrates that the QD scheme is more affected due to noise. Quantitatively, it can be seen that the obtained fidelity of the quantum state carrying information in the AQD scheme is always greater than that of the symmetric one.

## 4.2 Effect of phase damping (PD) noise

The effect of phase damping on the travel qubit is characterized by the following Kraus operators [37]

$$\begin{aligned} E_0^P &= \sqrt{1-\eta_P} \otimes I, \\ E_1^P &= \sqrt{\eta_P} |0\rangle\langle 0|, \\ E_2^P &= \sqrt{\eta_P} |1\rangle\langle 1|, \end{aligned} \quad (6)$$

where  $\eta_P$  ( $0 \leq \eta_P \leq 1$ ) is the decoherence rate for the phase damping.

Similar to AD noise case, the fidelity expression for the symmetric or asymmetric QD schemes with 2 travel qubits is

$$F_{PD}^{\text{sy}} = \frac{1}{2} ((\eta - 1)^4 + 1), \quad (7)$$

whereas

$$F_{PD}^{\text{asy}} = \frac{1}{2} (\eta^2 - 2\eta + 2) \quad (8)$$

is obtained with 1 travel qubit, when the travel qubits are passing through a PD channel. Therefore, the presence of quadratic (quartic) terms are signature of forward and backward communication of 1 (2) travel qubit(s).

The variation of both the fidelity expressions in Fig. 2 further establishes that the AQD scheme is less affected and is preferable over the symmetric one. Though, for very large values of decoherence rate, the obtained fidelity of the information encoded quantum state in both the cases become the same, otherwise the fidelity for the AQD scheme is always greater than that of the symmetric one.

## 5 Leakage and efficiency of the proposed scheme

Leakage is inherent in the Ba-An-type QD (for a discussion on this see [2, 29]), and the same limitation is applicable to the proposed AQD protocol, too. In brief, leakage can be thought of as the difference between the total information sent by both the legitimate users and the minimum information required by Eve to extract that information. Specifically, Eve requires the encoding information of at least one of the users to extract the information of the other user. For instance, in Ba An QD scheme, the total amount of encoded classical information is 4 bits and minimum encoding of a party is 2 bits (i.e., Eve requires 2 bits of minimum information) resulting in leakage of 2 bits. In case of AQD, the minimum requirement for Eve becomes less than QD hence resulting in increase of leakage. However, the leakage can be avoided if the initial state is unknown to the users. In Ba An's original scheme [12], it is explicitly mentioned (in connection with the choice of initial state) that "The choice may be random or in some secret fashion unknown to

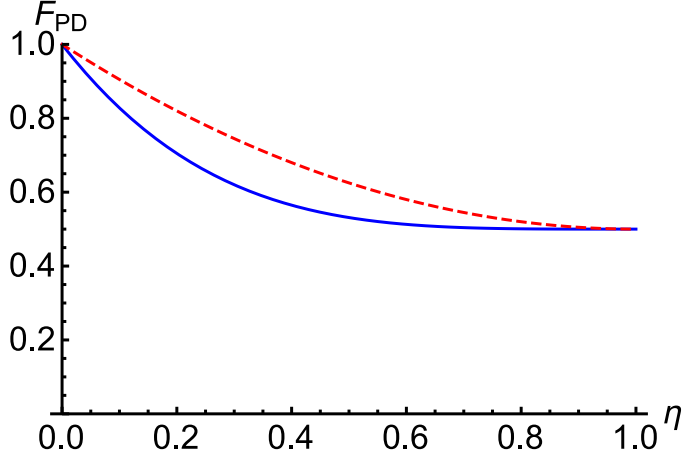


Figure 2: (Color online) The dependence of the fidelity of the information encoded quantum state of the AQD schemes on the number of travel qubits is shown, when subjected to PD noise using 4-qubit cluster state as quantum channel. The smooth (blue) and dashed (red) lines correspond to 2 and 1 travel qubits, respectively.

Eve". The choice signifies which particular Bell state is selected among the four Bell states. Here, we have taken this approach for AQD and we conjecture that if we can send the initial state to the users by QSDC (secret fashion) then the leakage can be nullified.

We may now look at the qubit efficiency of the scheme with and without QSDC. To do so, we use following measure of qubit efficiency [42]

$$\eta = \frac{c}{q + b}, \quad (9)$$

where  $c$  denotes the total number of transmitted classical bits (message bits),  $q = Q + 2t$  denotes the total number of qubits used with  $Q$ -qubit entangled state as quantum channel and  $t$  travel qubits (corresponding to decoy qubits),  $b$  is the number of classical bits exchanged for decoding of the message (classical communication used for checking of eavesdropping is not counted). If we consider the example of Bell-state-based Ba-An-type QD protocol, then  $c = 4$ ,  $q = 2 + 2 = 4$  and  $b = 2$ , and thus efficiency is calculated to be 67%. Now, if we use a Bell state for QSDC of the information regarding initial state, then total  $q = 8$  and thus efficiency is less. However, we know that we need to send the initial state, just once. Thus, even if we have to send  $n$  c-bits via the main part of the QD scheme, we still need to communicate 2 bits of classical information regarding the initial state through a QSDC scheme. This would require 4 extra qubits (2 qubits for channel and 2 qubits for eavesdropping checking) and thus in the  $n \rightarrow \infty$  limit the efficiency would become the same as that in the QD scheme without QSDC. This is so because in this particular case,  $\eta = \lim_{n \rightarrow \infty} \frac{4n}{(2n+2n)+4+2n} = \lim_{n \rightarrow \infty} \frac{4n}{6(n+\frac{4}{3})} \approx 67\%$ .

As we have discussed the specific case of 4-qubit cluster state as a quantum channel for the discussion of the effect of noise in the previous section, it would be relevant to calculate and discuss the efficiency for the same. Specifically, in the QD protocol with 4-qubit cluster state, the efficiency would be  $\eta = \frac{8}{(4+4)+4} = 67\%$ . However, in AQD protocol, we decrease the value of  $c$  and  $t$  by 2 and 1 respectively to obtain qubit efficiency  $\eta = 60\%$ . It may appear that the efficiency will always decrease with AQD protocols when compared with their QD counterparts. A contrary example would be of GHZ state, where the efficiency can be calculated to be 60% and 62.5% for QD and AQD protocols, respectively. It proves that if we have a multipartite state with odd number of qubits then efficiency can be increased in AQD protocols. Further, it has been already established that using QSDC for sending the initial state information, the efficiency equal to that of the protocols without using QSDC can be obtained for a large number of copies of the quantum channel.

## 6 Conclusion

We have designed a protocol for AQD which is different from the standard protocols of QD in the sense that the communication powers of the users are different. Interestingly, the task is performed without violating the requirements of QD. In the process of design and analysis of this protocol, we have obtained several interesting results. For example, we obtained a large number of new alternatives (in terms of groups of operators and corresponding quantum states), which can be used to implement QD (cf. last column of 1). Secondly, we obtained a group theoretic structure of the



operators that may be used to realize the proposed scheme for AQD. The analysis performed on the proposed scheme has also revealed that the proposed AQD involves more leakage in comparison to its QD counterpart. However, the leakage can be completely circumvented by including a QSDC scheme for sharing the information about the initial state prepared by Bob. Further, as the number of travel qubits is reduced in AQD (in comparison to an equivalent QD scheme), the effect of various noise models is also reduced. Further, using GHZ state as an example, it has been shown that the qubit efficiency of the proposed AQD is higher than the corresponding scheme of QD. The present AQD scheme can be easily extended to design a controlled AQD scheme, where a controller (Charlie) prepares the quantum state, which is followed by his announcement of initial state after the HJRSP scheme (except the unitary operations of Bob, which would depend on the initial state prepared by Charlie) has been faithfully implemented by Alice and Bob. Charlie can also send the information regarding initial state using QSDC scheme to both the legitimate users, Alice and Bob. Thus, in brief, present work introduces the concept of AQD, which is shown to be much beneficial (in both noiseless and noisy environments) than a QD in a situation where the users are not required to communicate equal amount of information, and the proposed scheme can also be extended to develop a relevant scheme for controlled-quantum communication.

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## Appendix A: Modified Pauli groups and the notation used

Modified Pauli groups and the notations used to denote them in the present work was introduced earlier in [29, 36]. Here, for consistency, we briefly summarize the definition and the notation used.

It is easy to verify that the set of Pauli operators  $\{I_2, \sigma_x, i\sigma_y, \sigma_z\}$  forms a group under multiplication  $G'_1 = \{\pm I_2, \pm iI_2, \pm\sigma_x, \pm i\sigma_x, \pm\sigma_y, \pm i\sigma_y, \pm\sigma_z, \pm i\sigma_z\}$  (cf. Section 10.5.1 of [37]). The closure property of  $G'_1$  is satisfied under normal matrix multiplication because of the inclusion of  $\pm 1$  and  $\pm i$ . However, if any of the operators  $\sigma_i, -\sigma_i, i\sigma_i$  or  $-i\sigma_i$  operates on a quantum state, the effect would be the same. Keeping this in mind, if global phase is ignored from the product of matrices (which is consistent with quantum mechanics), we obtain a modified Pauli group  $G_1 = \{I_2, \sigma_x, i\sigma_y, \sigma_z\} = \{I_2, X, iY, Z\}$ . Clearly, under the above defined multiplication rule,  $G_1$  is an Abelian group of order 4 and its generators are  $\langle X, Z \rangle, \langle X, iY \rangle$  and  $\langle iY, Z \rangle$ . Similarly, we may define the modified generalized Pauli group  $G_n = G_1^{\otimes n}$  as a group of order  $2^{2^n} = 4^n$  and whose elements are all  $n$ -fold tensor products of Pauli matrices [29]. For example,

$$\begin{aligned} G_2 &= G_1 \otimes G_1 = \{I_2, X, iY, Z\} \otimes \{I_2, X, iY, Z\} \\ &= \{I_2 \otimes I_2, I_2 \otimes X, I_2 \otimes iY, I_2 \otimes Z, X \otimes I_2, X \otimes X, \\ &\quad X \otimes iY, X \otimes Z, iY \otimes I_2, iY \otimes X, iY \otimes iY, \\ &\quad iY \otimes Z, Z \otimes I_2, Z \otimes X, Z \otimes iY, Z \otimes Z\}. \end{aligned} \quad (\text{A.1})$$

In Ref. [29], it was discussed in detail, how to construct subgroups of  $G_2$ . Here, we list 11 subgroups of  $G_2$ , which are used in this paper (each is of order 8):

$$\begin{aligned} G_2^1(8) &= \{I_2 \otimes I_2, X \otimes I_2, iY \otimes I_2, Z \otimes I_2, I_2 \otimes X, X \otimes X, iY \otimes X, Z \otimes X\}, \\ G_2^2(8) &= \{I_2 \otimes I_2, X \otimes I_2, iY \otimes I_2, Z \otimes I_2, I_2 \otimes iY, X \otimes iY, iY \otimes iY, Z \otimes iY\}, \\ G_2^3(8) &= \{I_2 \otimes I_2, X \otimes I_2, iY \otimes I_2, Z \otimes I_2, I_2 \otimes Z, X \otimes Z, iY \otimes Z, Z \otimes Z\}, \\ G_2^4(8) &= \{I_2 \otimes I_2, I_2 \otimes X, I_2 \otimes iY, I_2 \otimes Z, X \otimes I_2, X \otimes X, X \otimes iY, X \otimes Z\}, \\ G_2^5(8) &= \{I_2 \otimes I_2, I_2 \otimes X, I_2 \otimes iY, I_2 \otimes Z, iY \otimes I_2, iY \otimes X, iY \otimes iY, iY \otimes Z\}, \\ G_2^6(8) &= \{I_2 \otimes I_2, I_2 \otimes X, I_2 \otimes iY, I_2 \otimes Z, Z \otimes I_2, Z \otimes X, Z \otimes iY, Z \otimes Z\}, \\ G_2^7(8) &= \{I_2 \otimes I_2, I_2 \otimes Z, Z \otimes I_2, Z \otimes Z, X \otimes X, iY \otimes X, X \otimes iY, iY \otimes iY\}, \\ G_2^8(8) &= \{I_2 \otimes I_2, Z \otimes Z, X \otimes iY, iY \otimes X, I_2 \otimes X, Z \otimes iY, iY \otimes I_2, X \otimes Z\}, \\ G_2^9(8) &= \{I_2 \otimes I_2, Z \otimes Z, X \otimes iY, iY \otimes X, X \otimes I_2, iY \otimes Z, Z \otimes X, I_2 \otimes iY\}, \\ G_2^{10}(8) &= \{I_2 \otimes I_2, X \otimes I_2, I_2 \otimes X, X \otimes X, Z \otimes Z, iY \otimes Z, Z \otimes iY, iY \otimes iY\}, \\ G_2^{11}(8) &= \{I_2 \otimes I_2, iY \otimes I_2, I_2 \otimes iY, iY \otimes iY, Z \otimes Z, Z \otimes X, X \otimes Z, X \otimes X\}, \end{aligned} \quad (\text{A.2})$$

where  $G_n^j(m)$  denotes  $j^{\text{th}}$  subgroup of order  $m < 4^n$  of the group  $G_n$  whose order is  $4^n$ .

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