

# Uncertainty under Quantum Measures and Quantum Memory

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The uncertainty principle restricts potential information one gains about physical properties of the measured particle. However, if the particle is prepared in entanglement with a quantum memory, the corresponding entropic uncertainty relation will vary. Based on the knowledge of correlations between the measured particle and quantum memory, we have investigated the entropic uncertainty relations for two and multiple measurements, and generalized the lower bounds on the sum of Shannon entropies without quantum side information to those that allow quantum memory. In particular, we have obtained generalization of Kaniewski-Tomamichel-Wehner's bound for effective measures and majorization bounds for noneffective measures to allow quantum side information. Furthermore, we have derived several strong bounds for the entropic uncertainty relations in the presence of quantum memory for two and multiple measurements. Finally, potential applications of our results to entanglement witnesses are discussed via the entropic uncertainty relation in the absence of quantum memory.

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## I. INTRODUCTION

Heisenberg's uncertainty principle [1] bounds the limit of measurement outcomes of two incompatible observables, which reveals a fundamental difference between the classical and quantum mechanics. After intensive studies of the principle in terms of standard deviations of the measurements, entropies have stood out to be a natural and important alternative formulation of the uncertainty principle [2]. The importance of entropic uncertainty relations is solidified by a variety of applications, ranging from entanglement witnessing to quantum cryptography.

The first entropic uncertainty relation of observables with finite spectrum was given by Deutsch [3] and then improved by Maassen and Uffink [4], who gave the celebrated MU bound: if two incompatible measurements  $M_1 = \{|u_{i_1}^1\rangle\}$  and  $M_2 = \{|u_{i_2}^2\rangle\}$  are chosen on the particle  $A$ , then the uncertainty is bounded below by

$$H(M_1) + H(M_2) \geq \log_2 \frac{1}{c_1}, \quad (1)$$

where  $H(M_i)$  is the *Shannon entropy* of the probability distribution induced by measurement  $M_i$  and  $c_1 = \max_{i_1, i_2} |\langle u_{i_1}^1 | u_{i_2}^2 \rangle|^2$  denotes the largest overlap between the observables. On the other hand, a mixed state is expected to have more uncertainty, as (1) can be reinforced by adding the complementary term of the *von Neumann entropy*  $H(A) = S(\rho_A)$ :

$$H(M_1) + H(M_2) \geq \log_2 \frac{1}{c_1} + H(A). \quad (2)$$

The entropy  $H(A)$  measures the amount of uncertainty induced by the mixing status of the state  $\rho_A$ : if the state is pure, then  $H(A) = 0$ , and if the state is a mixed state, then  $H(A) > 0$ . Therefore the corresponding bound (2) is stronger than (1) even though there is no auxiliary quantum system such as a quantum memory. We refer to  $\log_2 \frac{1}{c_1}$  as the classical part  $B_{MU}$  and call  $H(A)$  the mixing part of the bound for the entropic uncertainty relation since it measures the mixing status of the particle.

Most of the bounds for entropic uncertainty relations in the absence of quantum memory contain two parts: (i) the classical part  $B_C$ , for instance, Maassen and Uffink's bound [4], Coles and Piani's bound [5], or our recent bound [6]; (ii) the mixing part  $H(A)$ , which describes the information pertaining to the mixing status of the particle  $\rho_A$ . We note that both the Kaniewski-Tomamichel-Wehner bound [7] based on effective anti-commutator and the direct-sum majorization bound [8] only involve with the classical part and have no mixing parts. For more details, see Sec. II.

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Obviously, not all the bounds  $B_C$  can be generalized to the case with quantum memory by simply adding an extra term  $H(A|B)$ . Therefore it is an interesting problem to extend the entropic uncertainty relations in the absence of quantum memory to those with quantum memory.

In this paper, we will solve the extension problem by answering three questions: (i) Can the uncertainty relation in the absence of quantum memory be generalized to the case with quantum side information? (ii) Are there other indices besides  $H(A|B)$  to quantify the amount of entanglement between the measured particle and quantum memory? (iii) Can two pairs of observables sharing the same overlaps between bases have different entropic uncertainty relations? Besides answering these questions in detail we will give a couple of strong entropic uncertainty relations in the presence of quantum memory.

## II. GENERALIZED ENTROPIC UNCERTAINTY RELATIONS

Strengthening the bound for the entropic uncertainty relation is an interesting problem arising from quantum theory. One of the main issues in this direction is how to extend the entropic uncertainty relation to allow for quantum side information. Several approaches have been devoted to seek for stronger bounds for the entropic uncertainty relations (e.g. majorization-based uncertainty relations, direct-sum majorization relations, uncertainty relations based on effective anti-commutators and so on). However it is still unclear how to implement these methods to allow for quantum side information. In this section we will show that it is possible to generalize all uncertainty relations for the sum of Shannon entropies to allow for quantum side information by using the *Holevo inequality*.

Before analyzing our main techniques and results, let us first discuss the modern formulation of the uncertainty principle, the so-called *guessing game* (also known as the *uncertainty game*), which highlights its relevance with quantum cryptography. We can imagine there are two observers, Alice and Bob. Before the game initiates, they agree on two measurements  $M_1$  and  $M_2$ . The guessing game proceeds as follows: Bob, can prepare an arbitrary state  $\rho_A$  which he will send to Alice. Alice then randomly chooses to perform one of measurements and records the outcome. After telling Bob the choices of her measurements, Bob can win the game if he correctly guesses Alice's outcome. Nevertheless, the uncertainty principle tells us that Bob cannot win the game under the condition of incompatible measurements.

What if Bob prepares a bipartite quantum state  $\rho_{AB}$  and sends only the particle  $A$  to Alice? Equivalently, what if Bob has nontrivial *quantum side information* about Alice's system? Or, what if all information Bob has on the particle  $\rho_A$  is beyond the classical description, for example, information on its density matrix? Berta *et al.* [9] answered these questions and generalized the uncertainty relation (1) to the case with an auxiliary quantum system  $B$  known as quantum memory.

It is now possible for Bob to experience no uncertainty at all when equipped himself with quantum memory, and Bob's uncertainty about the result of measurements on Alice's system is bounded by

$$H(M_1|B) + H(M_2|B) \geq \log_2 \frac{1}{c_1} + H(A|B), \quad (3)$$

where  $H(M_1|B) = H(\rho_{M_1B}) - H(\rho_B)$  is the conditional entropy with  $\rho_{M_1B} = \sum_j (|u_j\rangle\langle u_j| \otimes I)(\rho_{AB})(|u_j\rangle\langle u_j| \otimes I)$  (similarly for  $H(M_2|B)$ ), and the term  $H(A|B) = H(\rho_{AB}) - H(\rho_B)$  is related to the entanglement between the measured particle  $A$  and the quantum memory  $B$ .

On the other hand, entropic uncertainty relation without quantum memory can be roughly divided into two categories. If the measure of incompatibility is effective (state-dependent), one can follow Kaniewski, Tomamichel and Wehner's approach to obtain bounds (e.g.  $B_{ac}$  [7]) based on effective anticommutators. Otherwise one can derive strong bounds (e.g.  $B_{Maj1}$ ,  $B_{Maj2}$ ,  $B_{RPZ1}$ ,  $B_{RPZ2}$ ,  $B_{RPZ3}$  [8]) based on majorization, or bounds (e.g.  $B_{CP}$  [5]) constructed by the monotonicity of relative entropy under quantum channels. Note that Maassen and Uffink's bound  $B_{MU}$  [4], Coles and Piani's bound  $B_{CP}$  [5] are still valid in the presence of quantum memory by adding an extra term  $H(A|B)$ . All these bounds can be generalized to allow for quantum side information.

Suppose we are given a quantum state  $\rho_{AB}$  and a pair of observables,  $M_m$  ( $m = 1, 2$ ). Define the *classical correlation* of state  $\rho_{AB}$  with respect to the measurement  $M_m$  by

$$H(\rho_B) - S_m \quad (4)$$

with

$$S_m = \sum_{i_m} p_{i_m}^m H(\rho_{B_{i_m}}^m),$$

where  $\rho_{B_{i_m}}^m = \text{Tr}_A(|u_{i_m}^m\rangle\langle u_{i_m}^m| \rho_{AB})/p_{i_m}^m$  and  $(p_{i_m}^m)_{i_m}$  is the probability vector according to the measurement  $M_m$ .

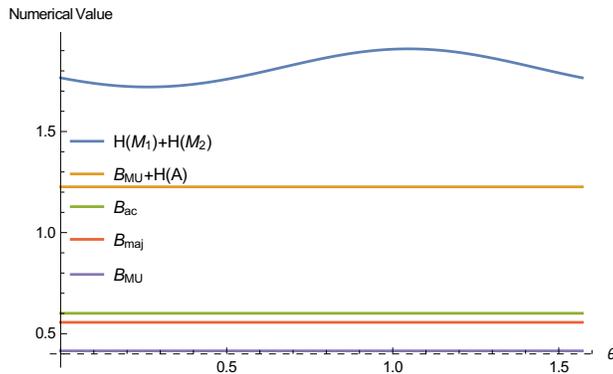


FIG. 1: Comparison of bounds for the quantum state  $\rho_A$  in (9). The blue, orange, green, red and purple curves are respectively the entropic sum  $H(M_1) + H(M_2)$ , the entropic bound  $B_{MU} + H(A)$ , the entropic bound  $B_{ac}$ , the entropic bound  $B_{maj}$ , and Maassen and Uffink's bound  $B_{MU}$ . The dashed line is  $\theta$ -axis.

It follows from definition and *Holevo's inequality* [10] that the entropic uncertainty relation in the presence of quantum memory can be written as

$$H(M_1|B) + H(M_2|B) = H(M_1) + H(M_2) - 2H(\rho_B) + S_1 + S_2, \quad (5)$$

where  $H(M_1), H(M_2)$  are the Shannon entropies of the state  $\rho_A$ . Suppose  $B_C$  is a lower bound of the entropic sum  $H(M_1) + H(M_2)$ , then

$$H(M_1|B) + H(M_2|B) \geq B_C - 2H(B) + S_1 + S_2. \quad (6)$$

We analyze the lower bound according to various types of  $B_C$  as follows. In Table 1, we list the various bounds such as  $B_{MU}$ ,  $B_{CP}$ , etc. and their references.

(i) Bounds [4, 5, 8] that contain a nonnegative state-dependent term  $H(A) = S(\rho_A)$ , the *von Neumann* entropy (*mixing part*):

$$\begin{aligned} H(M_1) + H(M_2) &\geq B_{MU} + H(A); \\ H(M_1) + H(M_2) &\geq B_{CP} + H(A); \\ H(M_1) + H(M_2) &\geq B_{RPZm} + H(A). \quad (m = 1, 2, 3) \end{aligned} \quad (7)$$

(ii) Bounds [7, 8, 11, 12] without the mixing term  $H(A)$ :

$$\begin{aligned} H(M_1) + H(M_2) &\geq B_{ac}, \\ H(M_1) + H(M_2) &\geq B_{Majm}. \quad (m = 1, 2) \end{aligned} \quad (8)$$

Although both effective anticommutators and majorization approach play an important role in improving the bound for entropic uncertainty relations, even the strengthened Maassen and Uffink's bound  $B_{MU} + H(A)$  can be tighter than the majorization bound  $B_{Maj1}$  [8] and Kaniewski-Tomamichel-Wehner's bound  $B_{ac}$  [7] if the mixing part is absent. To see this, we consider a family of quantum states

$$\rho_A = \frac{1}{2} \begin{pmatrix} \cos^2 \theta + \frac{1}{2} & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta + \frac{1}{2} \end{pmatrix}, \quad (9)$$

where  $0 \leq \theta \leq \pi/2$  with the measurements  $M_1 = \{(1, 0), (0, 1)\}$  and  $M_2 = \{(1/2, -\sqrt{3}/2), (\sqrt{3}/2, 1/2)\}$ . The relations among  $H(M_1) + H(M_2)$ ,  $B_{Maj}$  [8],  $B_{ac}$  [7],  $B_{MU}$  [4] and  $B_{MU} + H(A)$  are shown in FIG. 1. The maximum overlap is  $c_1 = 3/4$ , and it is known [7] that the bound  $B_{ac}$  outperforms  $B_{Maj}$ . Moreover, the picture shows that the quantity  $B_{MU} + H(A)$  is tighter than either  $B_{Maj}$  or  $B_{ac}$ .

In the above discussion the value  $H(A)$  is a constant, so all the bounds appeared in FIG. 1 are straight lines. Now let's turn to the quantum states given by

$$\rho_A = \frac{1}{2} \begin{pmatrix} \cos^2 \theta & 0 \\ 0 & \sin^2 \theta \end{pmatrix}, \quad (10)$$

where  $0 \leq \theta \leq \pi/2$  with the same measurements as above. The relations among  $H(M_1) + H(M_2)$ ,  $B_{Maj}$ ,  $B_{ac}$ ,  $B_{MU}$  and  $B_{MU} + H(A)$  are depicted in FIG. 2, once again the strengthened Maassen-Uffink's bound  $B_{MU} + H(A)$  outperforms both  $B_{Maj}$  and  $B_{ac}$ . In the neighborhood of  $\theta = \pi/4$ , the bound  $B_{MU} + H(A)$  gives the best estimate.

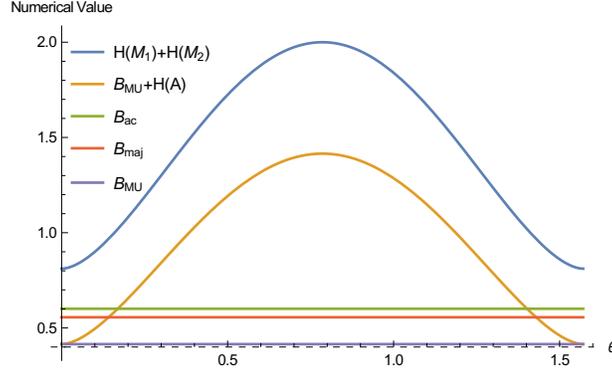


FIG. 2: Comparison of bounds for quantum state  $\rho_A$  from (10). The blue, orange, green, red, and purple curves are respectively the entropic bounds  $H(M_1) + H(M_2)$ ,  $B_{MU} + H(A)$ ,  $B_{ac}$ ,  $B_{maj}$ , and Maassen and Uffink's bound  $B_{MU}$ . The dashed line is  $\theta$ -axis.

### III. QUANTUM MEASURES

The existence of quantum memory translates into additional information on the uncertainty relation. We introduce the notion of *quantum measure* to describe the relationship between measured particle and quantum memory. There are two types of quantum measures.

The first type of quantum measure on entropic uncertainty relations is the mutual information between measured particle  $A$  and quantum memory  $B$ , which comes from the conditional von Neumann entropy [9]

$$H(A|B) = H(A) - I(A : B) \quad (11)$$

with  $I(A : B) = H(A) + H(B) - H(A, B)$  and  $H(A, B) = H(\rho_{AB})$ . Let  $Q_1 = -I(A : B)$  be the *first quantum measure*, as  $H(A)$  counts for the mixing level for measured particle  $A$ . Then the bounds for the entropic uncertainty relation in the presence of quantum memory consist of three parts: the bound  $B_C$  for the sum of Shannon entropies, the *mixing part*  $H(A)$  and the first quantum measure  $Q_1$

$$\begin{aligned} H(M_1|B) + H(M_2|B) &\geq B_{MU} + H(A) + Q_1, \\ H(M_1|B) + H(M_2|B) &\geq B_{CP} + H(A) + Q_1, \end{aligned} \quad (12)$$

where  $B_{MU} = -\log c_1$ ,  $B_{CP} = -\log c_1 + \frac{1-\sqrt{c_1}}{2} \log \frac{c_1}{c_2}$ , and  $c_2$  is the second largest entry of the matrix  $(| \langle u_{i_1}^1 | u_{i_2}^2 \rangle |^2)_{i_1 i_2}$ .

A more natural and less restrictive *quantum measure* is  $-2H(B) + S_1 + S_2$  discussed in Sec. II. Let  $Q_2 = -2H(B) + S_1 + S_2$  be the *second quantum measure*, then we can generalize all the bounds for the sum of Shannon entropies to allow for quantum side information. Namely we have

$$\begin{aligned} H(M_1|B) + H(M_2|B) &\geq B_{MU} + H(A) + Q_2, \\ H(M_1|B) + H(M_2|B) &\geq B_{CP} + H(A) + Q_2, \end{aligned} \quad (13)$$

Clearly, both Maassen and Uffink's bound  $B_{MU}$  and Coles and Piani's bound  $B_{CP}$  are valid with or without quantum side information, with the mixing part  $H(A)$  in the former case or the conditional entropy  $H(A|B)$  in the latter. Mathematically, the relation says that

$$\begin{aligned} H(M_1) + H(M_2) &\geq B_{CC} + H(A), \\ H(M_1|B) + H(M_2|B) &\geq B_{CC} + H(A|B), \end{aligned} \quad (14)$$

where  $B_{CC} = B_{MU}$  or  $B_{CP}$ . The term  $B_{CC}$  will be referred as the *consistent classical part* of the bound for the entropic uncertainty relation. In place of  $B_{MU}$  and  $B_{CP}$  in (14), we have recently given a new consistent classical part  $B$ , which is a tighter bound depending on all overlaps between incompatible observables [6]:

$$B = \log_2 \frac{1}{c_1} + \frac{1-\sqrt{c_1}}{2} \log_2 \frac{c_1}{c_2} + \frac{2-\Omega_4}{2} \log_2 \frac{c_2}{c_3} + \cdots + \frac{2-\Omega_{2(d-1)}}{2} \log_2 \frac{c_{d-1}}{c_d}, \quad (15)$$

TABLE I: Comparison among bounds for entropic uncertainty relations with and without quantum memory

Reference	Lower bound for $H(M_1) + H(M_2)$	Lower bound for $H(M_1 B) + H(M_2 B)$
[4]	$B_{MU} + H(A)$	$B_{MU} + H(A) + Q_1$ (or $Q_2$ )
[5]	$B_{CP} + H(A)$	$B_{CP} + H(A) + Q_1$ (or $Q_2$ )
[6]	$B + H(A)$	$B + H(A) + Q_1$ (or $Q_2$ )
[7]	$B_{ac}$	$B_{ac} + Q_2$
[8]	$B_{Maj1}$	$B_{Maj1} + Q_2$
[8]	$B_{Maj2}$	$B_{Maj2} + Q_2$
[8]	$B_{RPZ1} + H(A)$	$B_{RPZ1} + H(A) + Q_2$
[8]	$B_{RPZ2} + H(A)$	$B_{RPZ2} + H(A) + Q_2$
[8]	$B_{RPZ3} + H(A)$	$B_{RPZ3} + H(A) + Q_2$

where  $c_i$  is the  $i$ -th largest overlap among  $c_{jk}$ :  $c_1 \geq c_2 \geq c_3 \geq \dots \geq c_{d^2}$ , and  $\Omega_k$  is the  $k$ -th element of majorization bound for measurements  $M_1$  and  $M_2$  [6]. In general the bound  $B$  is always tighter than  $B_{CP}$ , except possibly when two orthonormal bases are mutually unbiased.

We continue discussing the quantum measure of the entropic uncertainty relation with a consistent classical part. When quantum memory is present, there are infinitely many quantum measures. For any  $\lambda \in [0, 1]$  one has that

$$H(M_1|B) + H(M_2|B) \geq B_{CC} + H(A) + Q(\lambda), \quad (16)$$

where

$$Q(\lambda) := -\lambda I(A : B) + (1 - \lambda)(-2H(B) + S_1 + S_2) \quad (17)$$

is a new quantum measure for the entropic uncertainty relation with a consistent part. Here we have used a weighted sum of quantum measures similar to [13]. Note that the weight is applied on the quantum measures instead of the uncertainty relations. Through this simple process, we can always get a better lower bound without worrying which quantum measure is tighter than the other. Aside from its own significance, the new quantum measure  $Q(\lambda)$  is expected to be useful for future quantum technologies such as entanglement witnessing.

The quantum measure  $Q_2$  has two desirable features. First, with the help of the *second quantum measure* we can extend all previous bounds of the entropic sum (Shannon entropy) to allow for the quantum side information without restrictive constraints. The comparison of some of the existing results is given together with their extensions in the presence of quantum side information in TABLE. 1. Second,  $Q_2$  can sometimes outperform  $Q_1$  to give tighter bounds for the entropic uncertainty relation in the presence of quantum memory. For more details, see Sec. IV.

Third, by taking the maximum over  $Q_2 - Q_1$  and zero, we derive that

$$\max\{0, Q_2 - Q_1\}, \quad (18)$$

is another bound, i.e.  $H(M_1|B) + H(M_2|B) \geq B_C + H(A|B) + \max\{0, Q_2 - Q_1\}$ , which coincides with the main quantity used in the recent paper [14, Eq (12)] for a strong uncertainty relation in the presence of quantum memory. We point it out that our result is more general than simply using  $\max\{0, Q_2 - Q_1\}$ . In fact,  $B + H(A) + \max\{Q_1, Q_2\}$  is tighter than the outcomes from [14]. In [6] we have given a detailed and rigorous proof on the lower bound.

#### IV. INFLUENCE OF INCOMPATIBLE OBSERVABLES

Let us consider two pairs of incompatible observables  $M_1, M_2$  and  $M_3, M_4$  with the same overlaps  $c_{jk}$ . Then the bounds for the Shannon entropic sum  $H(M_1) + H(M_2)$  on measured particle  $A$  will coincide with that of  $H(M_3) + H(M_4)$ , since their bounds only depend on the overlaps  $c_{jk}$ . If there is quantum memory  $B$  present, the same relation holds for the bounds with the first quantum measure  $Q_1$ , since their bounds also depend only on  $c_{jk}$  and  $H(A|B)$ .

However, the situation is quite different by utilizing the *second quantum measure*. Even when two pairs of incompatible observables  $M_1, M_2$  and  $M_3, M_4$  share the same overlaps, the corresponding bounds may differ. This interesting phenomenon may be useful in physical experiments: the total uncertainty can be decreased by choosing suitable incompatible observables.

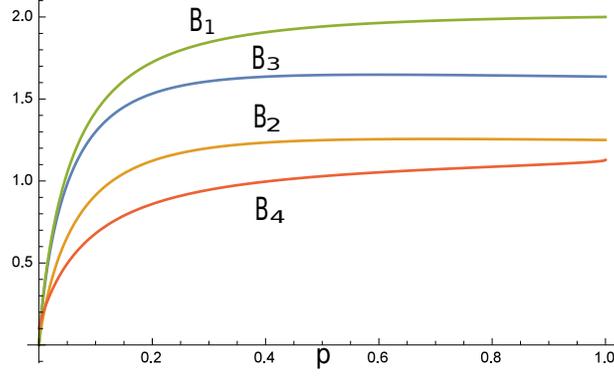


FIG. 3: Comparison of bounds for entangled quantum state  $\rho_{AB}$ . The green curve is the entropic bound  $\mathcal{B}_1$ , the blue curve is the entropic bound  $\mathcal{B}_3$ , the orange curve is the entropic bound  $\mathcal{B}_2$  and the red curve is the entropic bound  $\mathcal{B}_4$ .

As an example, consider the following  $2 \times 4$  bipartite state,

$$\rho_{AB} = \frac{1}{1+7p} \begin{pmatrix} p & 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & p & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+p}{2} & 0 & 0 & \frac{\sqrt{1-p^2}}{2} \\ p & 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 & p & 0 \\ 0 & 0 & p & 0 & \frac{\sqrt{1-p^2}}{2} & 0 & 0 & \frac{1+p}{2} \end{pmatrix}, \quad (19)$$

which is known to be entangled for  $0 < p < 1$  [15]. We take system  $A$  as the quantum memory and measurements are performed on system  $B$ . Choose the incompatible observables  $M_1 = \{|u_i^1\rangle\}$  and  $M_2 = \{|u_i^2\rangle\}$  as the first pair of measurements

$$\begin{aligned} |u_1^1\rangle &= \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, 0\right)^\dagger, |u_2^1\rangle = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)^\dagger, \\ |u_3^1\rangle &= \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^\dagger, |u_4^1\rangle = \left(0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^\dagger; \\ |u_1^2\rangle &= \frac{1}{\sqrt{6}}(\sqrt{2}, \sqrt{2}, \sqrt{2}, 0)^\dagger, |u_2^2\rangle = \frac{1}{\sqrt{6}}(\sqrt{3}, 0, -\sqrt{3}, 0)^\dagger, \\ |u_3^2\rangle &= \frac{1}{\sqrt{6}}(1, -2, 1, 0)^\dagger, |u_4^2\rangle = (0, 0, 0, 1)^\dagger, \end{aligned} \quad (20)$$

then take  $M_3 = M_2$  and  $M_4 = \{|u_i^3\rangle\}$  such that

$$\begin{aligned} |u_j^1\rangle &\neq |u_j^3\rangle, \\ |\langle u_j^2 | u_k^3 \rangle|^2 &= |\langle u_j^1 | u_k^2 \rangle|^2. \end{aligned} \quad (21)$$

Therefore, the basis  $M_4$  is obtained as

$$\begin{aligned} (|u_1^2\rangle, |u_2^2\rangle, |u_3^2\rangle, |u_4^2\rangle) &= U(|u_1^1\rangle, |u_2^1\rangle, |u_3^1\rangle, |u_4^1\rangle), \\ (|u_1^3\rangle, |u_2^3\rangle, |u_3^3\rangle, |u_4^3\rangle) &= U(|u_1^2\rangle, |u_2^2\rangle, |u_3^2\rangle, |u_4^2\rangle), \end{aligned} \quad (22)$$

where the matrix  $U$  is easily fixed from (20).

Set  $\mathcal{B}_1 = H(B)$ ,  $\mathcal{B}_2 = H(B|A)$ ,  $\mathcal{B}_3 = H(B) - 2H(A) + S_1 + S_2$ ,  $\mathcal{B}_4 = H(B) - 2H(A) + S_2 + S_3$  and  $\mathcal{B}_c := B$  (cf. (15)). If there is no quantum memory, the entropic uncertainty relations are obtained as

$$\begin{aligned} H(M_1) + H(M_2) &\geq \mathcal{B}_c + \mathcal{B}_1, \\ H(M_3) + H(M_4) &\geq \mathcal{B}_c + \mathcal{B}_1, \end{aligned} \quad (23)$$

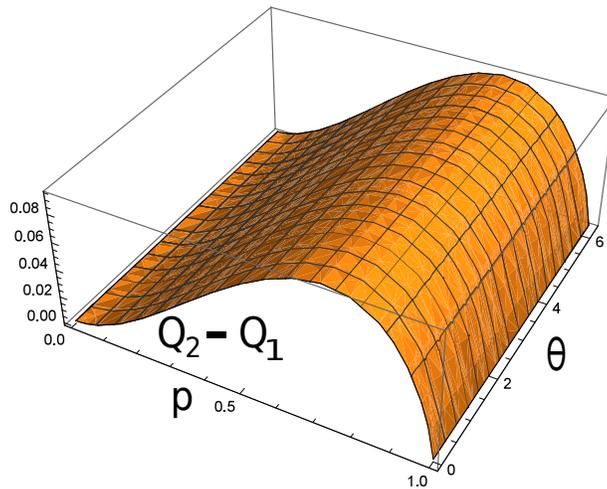


FIG. 4: The difference between the bound of entropic uncertainty relations in the presence of quantum memory with the *second quantum measure*  $Q_2$  and the bound of entropic uncertainty relations in the presence of quantum memory with the *first quantum measure*  $Q_1$ .

where the bounds are the same due to identical overlaps between the bases. In the presence of quantum memory, using the *first quantum measure*  $Q_1$  as the extra term to describe the amount of correlations between measured particle and quantum memory, we have that

$$\begin{aligned} H(M_1|A) + H(M_2|A) &\geq \mathcal{B}_c + \mathcal{B}_2, \\ H(M_3|A) + H(M_4|A) &\geq \mathcal{B}_c + \mathcal{B}_2, \end{aligned} \quad (24)$$

so their bounds coincide again. Finally, choosing the *second quantum measure*  $Q_2$  for the correlations between measured particle and quantum memory, we derive that

$$\begin{aligned} H(M_1|A) + H(M_2|A) &\geq \mathcal{B}_c + \mathcal{B}_3, \\ H(M_3|A) + H(M_4|A) &\geq \mathcal{B}_c + \mathcal{B}_4, \end{aligned} \quad (25)$$

and this time their bounds are different from each other. Therefore when the measured particle and quantum memory are entangled, the uncertainty is decreased through suitable incompatible observables. Since all the bounds contain  $\mathcal{B}_c$ , we only need to compare  $\mathcal{B}_1 = H(B)$ ,  $\mathcal{B}_2 = H(B|A)$ ,  $\mathcal{B}_3 = H(B) - 2H(A) + S_1 + S_2$  and  $\mathcal{B}_4 = H(B) - 2H(A) + S_2 + S_3$  for two pairs of measurements.

In FIG. 3, the comparison is done for  $\mathcal{B}_1$ ,  $\mathcal{B}_2$ ,  $\mathcal{B}_3$  and  $\mathcal{B}_4$ , which shows how the *second quantum measure* works for selected pairs of incompatible observables. The bound  $\mathcal{B}_3$  (with the *second quantum measure*) provides the best estimation for the entropic sum in the presence of quantum memory, while the bound  $\mathcal{B}_2$  (with the *first quantum measure*) gives a weaker approximation. The *second quantum measure* does not always outperform the *first quantum measure*, since  $\mathcal{B}_4$  is typically worse than  $\mathcal{B}_2$ . However, comparing the bound  $\mathcal{B}_3$  with  $\mathcal{B}_4$ , we find that the uncertainty from measurements can be weakened by selecting appropriate measurements even if each pair of incompatible observables shares the same overlaps.

To illustrate improvement of the bound in the presence of quantum memory, we compare the bound based on the *second quantum measure* with that based on the *first quantum measure*. As a first step, choose the initial state as *Werner State*  $\rho_{AB} = \frac{1}{4}(1-p)I + p|B_1\rangle\langle B_1|$  with  $0 < p < 1$ , and  $|B_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  the *Bell State*. Take  $|u_1^1\rangle = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ ,  $|u_2^1\rangle = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ ;  $|u_1^2\rangle = (\cos\theta, -\sin\theta)$ ,  $|u_2^2\rangle = (\sin\theta, \cos\theta)$  with  $0 < \theta < 2\pi$ , then the difference between the bound with *second quantum measure* and the bound with the *first quantum measure* is illustrated in FIG. 4. The nonnegativity of the surface shows that our newly constructed bound with the *second quantum measure* can outperform the bound with the *first quantum measure* everywhere in this case.

Using quantum measures we have shown that it is possible to reduce the total uncertainties coming from incompatibility of the observables by an appropriate choice. However, when the measured particle and quantum memory are maximally entangled, both the first and second *quantum measure* equal to  $-\log_2 d$ . We sketch a proof of this statement in Appendix A.

## V. STRONG ENTROPIC UNCERTAINTY RELATIONS IN THE PRESENCE OF QUANTUM MEMORY

In this section, we derive several strong entropic uncertainty relations in the presence of quantum memory by utilizing both the relevant bounds for the sum of Shannon entropies and optimal selection of quantum measures. Recall that the bounds of entropic uncertainty relations in the presence of quantum memory contain three ingredients: the classical part  $B_C$ , the mixing part  $H(A)$  (which is not necessarily existent, e.g., the majorization bounds [8, 11, 12] and  $B_{ac}$  [7]), and the quantum measures  $Q_i$  ( $i = 1, 2$ ).

Let  $\rho_{AB}$  be a bipartite quantum state, and  $M_i$  ( $i = 1, 2$ ) two nondegenerate incompatible observables on the system  $A$ . We take system  $B$  as the quantum memory. A simple lower bound for the entropic sum in the presence of quantum memory can be obtained as follows. Note that the *consistent classical part*  $B_{CC}$  is valid with both quantum measures  $Q_i$ , therefore for  $i = 1, 2$

$$H(M_1|B) + H(M_2|B) \geq B_{CC} + H(A) + Q_i. \quad (26)$$

As the bound  $B$  in (15) is the tightest, so the strongest lower bound for the entropic sum in the presence of quantum memory with *consistent classical part* is given by

$$\mathfrak{B}_{CC} := B + H(A) + \max\{Q_1, Q_2\}. \quad (27)$$

Without the help of the consistent classical part, all other classical parts  $B_C$  can be estimated in the same way.

$$H(M_1|B) + H(M_2|B) \geq \mathfrak{B}_C + H(A) + Q_2. \quad (28)$$

Note that for  $B_C = B_{ac}$  or  $B_{Maj}$ , there is no mixing part  $H(A)$  on the right-hand side of (28). Taking the maximum over all possible  $B_C$ 's we obtain a lower bound

$$\mathfrak{B}_C := \max\{B_{ac}, B_{Maj1}, B_{Maj2}, B_{RPZ1} + H(A), B_{RPZ2} + H(A), B_{RPZ3} + H(A)\} + Q_2, \quad (29)$$

Clearly both the lower bounds  $\mathfrak{B}_C$  and  $\mathfrak{B}_{CC}$  can be combined into a hybrid bound for the uncertainty relation in the presence of quantum memory:

$$H(M_1|B) + H(M_2|B) \geq \max\{\mathfrak{B}_C, \mathfrak{B}_{CC}\}, \quad (30)$$

where  $\mathfrak{B}_C$  and  $\mathfrak{B}_{CC}$  are given by (27) and (28) respectively.

We now extend our results to the general case of  $L$ -partite particles ( $L \geq 3$ ) with  $N$  incompatible observables ( $N \geq 3$ ). Assume the measured system is the  $l_1$ -partite subsystem and the quantum memory is the remaining  $l_2$ -partite subsystem, where  $l_2 = L - l_1$  and  $l_1 \geq 2$ .

Suppose that the  $N$  measurements  $M_1, M_2, \dots, M_N$  are given by the bases  $M_m = \{|u_{i_m}^m\rangle\}$ . Let system  $A$  be the measured particle ( $l_1$ -partite) and  $B$  the quantum memory ( $l_2$ -partite). The probability distributions

$$p_{i_m}^m = \langle u_{i_m}^m | \rho_A | u_{i_m}^m \rangle,$$

have a majorization bound [16]:

$$(p_{i_m}^m) \prec \omega = \sup_{M_m} (p_{i_m}^m), \quad (31)$$

which is state-independent. For different correlations between particles, there may exist different kind of state-independent  $\omega$  called the *uniform entanglement frames* [17]. In fact, if the majorization bound is written as  $\omega = (\Omega_1, \Omega_2 - \Omega_1, \dots, 1 - \Omega_{d-1})$ , then we have

$$\sum_{m=1}^N H(M_m|B) \geq (N-1)H(A|B) - \log_2 b_1 + (1 - \Omega_1) \log_2 \frac{b_1}{b_2} + \dots + (1 - \Omega_{d-1}) \log_2 \frac{b_{d-1}}{b_d}, \quad (32)$$

where  $b_i$  is the  $i$ -th largest element among all

$$\left\{ \sum_{i_2 \dots i_{N-1}} \max_{i_1} [c(u_{i_1}^1, u_{i_2}^2)] \prod_{m=2}^{N-1} c(u_{i_m}^m, u_{i_{m+1}}^{m+1}) \right\}$$

over the indices  $i_N$  and  $c(u_{i_m}^m, u_{i_{m+1}}^{m+1}) = |\langle u_{i_m}^m | u_{i_{m+1}}^{m+1} \rangle|^2$ . A complete proof of the relation (32) is given in Appendix B.

Besides giving theoretical improvement of the uncertainty relation, our result has potential applications in other areas of quantum theory. For example, it can be utilized in designing new entanglement detector. To witness entanglement, one considers a source that emits a bipartite state  $\rho_A$ . One defines the probability distributions of incompatible observables  $M_m$  ( $m = 1, \dots, N$ ) as usual:

$$p_{i_m}^m = \langle u_{i_m}^m | \rho_A | u_{i_m}^m \rangle.$$

If the bipartite state  $\rho_A$  is separable, then there exists a vector  $\omega^{sep} = (\Omega_1^{sep}, \Omega_2^{sep} - \Omega_1^{sep}, \dots, 1 - \Omega_{d-1}^{sep})$  such that

$$(p_{i_m}^m) \prec \omega^{sep}. \quad (33)$$

Subsequently we have

$$\sum_{m=1}^N H(M_m) \geq (N-1)H(A) - \log_2 b_1 + (1 - \Omega_1^{sep}) \log_2 \frac{b_1}{b_2} + \dots + (1 - \Omega_{d-1}^{sep}) \log_2 \frac{b_{d-1}}{b_d}, \quad (34)$$

with other notations are the same with (32). If there exists another quantum state  $\rho'_A$  with

$$\sum_{m=1}^N H(M_m) < (N-1)H(A') - \log_2 b_1 + (1 - \Omega_1^{sep}) \log_2 \frac{b_1}{b_2} + \dots + (1 - \Omega_{d-1}^{sep}) \log_2 \frac{b_{d-1}}{b_d}, \quad (35)$$

where  $H(A') = S(\rho'_A)$ , then state  $\rho'_A$  must be entangled since it violates the majorization bound for separable states. As this method is based on *uniform entanglement frames* and the entropic uncertainty relations, the witnessed entanglement does not involve with quantum memory.

Similarly, the second quantum measure enables us to generalize the strong entropic uncertainty relations for multiple measurements [18] (i.e. *admixture bound*) to allow for quantum side information. By taking the maximum over (32) and the *admixture bound* in the presence of quantum memory, we obtain a strong entropic uncertainty relation with quantum memory for multi-measurements which will be useful in handling quantum cryptography tasks and general quantum information processings.

## VI. CONCLUSIONS

We have extended all uncertainty relations for Shannon entropies to allow for quantum side information, first in the case of two incompatible observables and then for multi-observables. Using the *second quantum measure* we have characterized the correlations between measured particle and quantum memory. Our uncertainty relations are universal and capture the intrinsic nature of the uncertainty in the presence of quantum memory. Moreover, we have observed that the uncertainties in the presence of quantum memory decrease under appropriate selection of incompatible observables. Finally, we have derived several strong bounds for the entropic uncertainty relation in the presence of quantum memory. We have also discussed applications of our result to entanglement witnesses with or without quantum memory.

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## VII. APPENDIX A: MAXIMAL ENTANGLEMENT

Let  $\rho_{AB}$  be a bipartite quantum state, and  $M_1, M_2$  a pair of incompatible observables. Suppose that the measured particle  $A$  and quantum memory  $B$  are maximally entangled. We will show that both the first and second quantum measures coincide with each other. Recall that the first quantum measure  $Q_1$  was defined in Sec. III and the combination of the quantum measure and mixing part is  $H(A) + Q_1 = H(A|B) = -\log_2 d$ .

Recall that the second quantum measure is given by  $Q_2 = -2H(B) + S_1 + S_2$ , where

$$S_1 = \sum_{i_1} p_{i_1}^1 H(\rho_{B_{i_1}}^1), \quad (36)$$

$$S_2 = \sum_{i_2} p_{i_2}^2 H(\rho_{B_{i_2}}^2). \quad (37)$$

From  $p_{i_m}^m = \langle u_{i_m}^m | \rho_A | u_{i_m}^m \rangle$  and  $[u_{i_m}^m] \equiv |u_{i_m}^m\rangle\langle u_{i_m}^m|$  ( $m = 1, 2$ ), it follows that

$$\rho_{B_{i_1}}^1 = \frac{\text{Tr}_A([u_{i_1}^1]\rho_{AB})}{p_{i_1}^1}, \quad (38)$$

$$\rho_{B_{i_2}}^2 = \frac{\text{Tr}_A([u_{i_2}^2]\rho_{AB})}{p_{i_2}^2}. \quad (39)$$

One can use the formula to compute the second quantum measure  $Q_2$  if the state is the maximally entangled quantum state  $\rho_{AB} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$ . For simplicity, we only consider the case  $d = 3$  while the high dimensional case can be similarly done. For the projective rank-1 measurements on system  $A$ , set  $|u_{i_1}^1\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$  with  $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$ , then

$$[u_{i_1}^1] = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* & \alpha\gamma^* \\ \beta\alpha^* & |\beta|^2 & \beta\gamma^* \\ \gamma\alpha^* & \gamma\beta^* & |\gamma|^2 \end{pmatrix}, \quad (40)$$

and

$$\rho_{B_{i_1}}^1 = \begin{pmatrix} |\alpha|^2 & \beta\alpha^* & \gamma\alpha^* \\ \alpha\beta^* & |\beta|^2 & \gamma\beta^* \\ \alpha\gamma^* & \beta\gamma^* & |\gamma|^2 \end{pmatrix}. \quad (41)$$

Since the density matrix  $\rho_{B_{i_1}}^1$  is rank 1, it follows that

$$H(\rho_{B_{i_1}}^1) = 0, \quad (42)$$

which implies that  $S_1 = S_2 = 0$ . Therefore

$$H(A) + Q_1 = H(A) + Q_2 = -\log_2 d,$$

where the last equality implies that the first quantum measure coincide with the second index when the measured particle and quantum memory are maximally entangled.

### VIII. APPENDIX B: MULTIPLE MEASUREMENTS

For an  $L$ -partite state  $\rho$ , divide the whole system into two parts: the measured subsystem  $A$  and the remaining subsystem as quantum memory  $B$ , then we can still denote the quantum state as  $\rho_{AB}$ . Given  $N$  measurements  $M_1, M_2, \dots, M_N$ , to find a lower bound for the entropic uncertainty relations in the presence of quantum memory we use basic properties of the relative entropy as follows:

$$\begin{aligned} S(\rho_{AB} \parallel \sum_{i_1} [u_{i_1}^1] \rho_{AB} [u_{i_1}^1]) &\geq S([u_{i_2}^2] \rho_{AB} [u_{i_2}^2] \parallel \sum_{i_1, i_2} c(u_{i_1}^1, u_{i_2}^2) [u_{i_2}^2] \otimes Tr_A([u_{i_1}^1] \rho_{AB})) \\ &= S(\rho_{AB} \parallel \sum_{i_1, i_2} c(u_{i_1}^1, u_{i_2}^2) [u_{i_2}^2] \otimes Tr_A([u_{i_1}^1] \rho_{AB})) + H(A|B) - H(M_2|B), \end{aligned} \quad (43)$$

where  $c(u_{i_1}^1, u_{i_2}^2) = |\langle u_{i_1}^1 | u_{i_2}^2 \rangle|^2$ ,  $[u_{i_m}^m] = |u_{i_m}^m\rangle\langle u_{i_m}^m|$ , and  $S(\rho \parallel \sigma) = \text{Tr}(\rho(\log \rho - \log \sigma))$  stands for the relative entropy. Inductively the generalized lower bound is given as follows

$$-NH(A|B) + \sum_{m=1}^N H(M_m|B) \geq S(\rho_{AB} \parallel \sum_{i_N} [u_{i_N}^N] \otimes \beta_{i_N}^N), \quad (44)$$

where  $p_{i_1}^1 \rho_{B_{i_1}}^1 = Tr_A([u_{i_1}^1] \rho_{AB})$  and

$$\beta_{i_N}^N = \sum_{i_1, \dots, i_{N-1}} p_{i_1}^1 \rho_{B_{i_1}}^1 \prod_{m=1}^{N-1} c(u_{i_m}^m, u_{i_{m+1}}^{m+1})$$

Taking maximum over indices  $i_2, \dots, i_{N-1}$  and writing

$$\sum_{i_2, \dots, i_{N-1}} \max_{i_1} [c(u_{i_1}^1, u_{i_2}^2)] \prod_{m=2}^{N-1} c(u_{i_m}^m, u_{i_{m+1}}^{m+1}) = b(i_N), \quad (45)$$

we have that

$$S(\rho_{AB} \parallel \sum_{i_N} [u_{i_N}^N] \otimes \beta_{i_N}^N) \geq -H(A|B) - \sum_{i_N} p_{i_N}^N \log_2 b(i_N), \quad (46)$$

where  $p_{i_N}^N = Tr([u_{i_N}^N] \rho_A)$ . We arrange the numerical values  $b(i_N)$  in descending order:

$$b_1 \geq b_2 \geq \dots \geq b_d, \quad (47)$$

so  $b_i$  is the  $i$ -th largest element among all  $b(i_N)$  (counting multiplicity). Denote by  $p_i^N$  the corresponding probability. Therefore

$$S(\rho_{AB} \parallel \sum_{i_N} [u_{i_N}^N] \otimes \beta_{i_N}^N) \geq -H(A|B) - \log_2 b_1 + (1 - p_1) \log_2 \frac{b_1}{b_2} + \dots + (1 - p_1 - \dots - p_{d-1}) \log_2 \frac{b_{d-1}}{b_d}. \quad (48)$$

If the measured particle is  $l_1$ -partite and the quantum memory is a  $l_2$ -partite particle such that  $l_1 + l_2 = L, l_1 \geq 2$ , then there exists a state-independent majorization bound [17]  $\omega = (\Omega_1, \Omega_2 - \Omega_1, \dots, 1 - \Omega_{d-1})$  corresponding to the structure of the measured particle. Note that

$$\begin{aligned} 1 - p_1 &\geq 1 - \Omega_1, \\ 1 - p_1 - p_2 &\geq 1 - \Omega_2, \\ &\dots\dots \\ 1 - p_1 - \dots - p_{d-1} &\geq 1 - \Omega_{d-1}, \end{aligned}$$

which imply that

$$S(\rho_{AB} \parallel \sum_{i_N} [u_{i_N}^N] \otimes \beta_{i_N}^N) \geq -H(A|B) - \log_2 b_1 + (1 - \Omega_1) \log_2 \frac{b_1}{b_2} + \dots + (1 - \Omega_{d-1}) \log_2 \frac{b_{d-1}}{b_d}. \quad (49)$$

Hence the entropic uncertainty relation is written as

$$\sum_{m=1}^N H(M_m|B) \geq (N - 1)H(A|B) - \log_2 b_1 + (1 - \Omega_1) \log_2 \frac{b_1}{b_2} + \dots + (1 - \Omega_{d-1}) \log_2 \frac{b_{d-1}}{b_d}, \quad (50)$$

which provides a substantial improvement over  $(N - 1)H(A|B) - \log_2 b_1$ , the term contained in the presence of quantum memory. Therefore, the new bound is the tightest one with *consistent classical part* till now. By taking all permutations on the index of (50) first, and computing the maximum over all possibilities, we obtain an optimal lower bound in the presence of quantum memory. One can also use uniform entanglement frames [17] to give a degenerate uncertainty inequality in the absence of quantum memory.