

# Hadamard quantum broadcast channels

Qingle Wang<sup>\*†</sup>

Siddhartha Das<sup>†</sup>

Mark M. Wilde<sup>†‡</sup>

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## Abstract

We consider three different communication tasks for quantum broadcast channels, and we determine the capacity region of a Hadamard broadcast channel for these various tasks. We define a Hadamard broadcast channel to be such that the channel from the sender to one of the receivers is entanglement-breaking and the channel from the sender to the other receiver is complementary to this one. As such, this channel is a quantum generalization of a degraded broadcast channel, which is well known in classical information theory. The first communication task we consider is classical communication to both receivers, the second is quantum communication to the stronger receiver and classical communication to other, and the third is entanglement-assisted classical communication to the stronger receiver and unassisted classical communication to the other. The structure of a Hadamard broadcast channel plays a critical role in our analysis: the channel to the weaker receiver can be simulated by performing a measurement channel on the stronger receiver's system, followed by a preparation channel. As such, we can incorporate the classical output of the measurement channel as an auxiliary variable and solve all three of the above capacities for Hadamard broadcast channels, in this way avoiding known difficulties associated with quantum auxiliary variables.

## 1 Introduction

Broadcast channels model the communication of a single sender to multiple receivers [1]. They have been explored extensively in classical information theory [1, 2, 3, 4], with a variety of coding schemes known, including the superposition coding method [1]. The capacity of a classical broadcast channel has been solved in certain cases [4] but remains unsolved in the general case, being a well known open problem in network classical information theory.

Quantum broadcast channels were introduced in [5] and take on a particular relevance in quantum information theory, due to the no-cloning theorem [6, 7] and associated “no-go” results [8, 9, 10, 11]. A variety of information-theoretic results are now known for quantum broadcast channels. Refs. [5, 12] established a quantum generalization of the superposition coding method for sending classical information over a broadcast channel. Ref. [5] established a method for sending classical information to one receiver while sending quantum information to the other. Other

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<sup>\*</sup>State Key Laboratory of Networking and Switching Technology, Beijing University of Posts and Telecommunications, Beijing 100876, China

<sup>†</sup>Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803, USA

<sup>‡</sup>Center for Computation and Technology Louisiana State University, Baton Rouge, Louisiana 70803, USA

information-theoretic results having to do with a variety of communication tasks for quantum broadcast channels are available in [13, 14, 15, 16, 17].

In this paper, we determine the classical capacity region, the classical–quantum capacity region, and the partially entanglement-assisted classical capacity region of Hadamard quantum broadcast channels. That is, we determine the optimal rates at which a sender can transmit classical information to two receivers, the optimal rates at which a sender can communicate quantum information to one receiver and classical information to the other receiver, as well as the optimal rates at which a sender can communicate classical information to both receivers while sharing entanglement with one of them, whenever the underlying channel is a Hadamard broadcast channel. Hadamard broadcast channels are such that the sender Alice’s input is isometrically embedded in the Hilbert space of two receivers, Bob and Charlie, the channel from Alice to Charlie is entanglement-breaking [18], and the channel from Alice to Bob is complementary to the aforementioned one. The channel from Alice to Bob is known as a Hadamard channel [19], and for this reason, we call the corresponding broadcast channel a Hadamard broadcast channel. Single-sender single-receiver Hadamard channels have a complete characterization in terms of all of their capacities [20, 21, 22], and so our results here represent a further exploration of these ideas in the domain of broadcast channels. An interesting example of a Hadamard channel is a quantum-limited amplifier channel [23, 24], which has appeared in a variety of contexts in quantum information theory due to its connections with approximate cloning [25, 26, 27].

The rest of the paper proceeds as follows. In the next section, we provide a definition of a Hadamard quantum broadcast channel. Section 3 details our first main result about the classical capacity of a Hadamard broadcast channel. Therein we define the communication task, we review the rate region achievable when using the superposition coding method from [5, 12], and we detail a proof that the region is single-letter for Hadamard broadcast channels. Section 4 details our second main result about the classical–quantum capacity of a Hadamard broadcast channel. Therein we define the communication task, review the achievable rate region from [5], and thereafter give the converse proof for the classical–quantum capacity region. Section 5 gives our third main result about the partially entanglement-assisted classical capacity of a Hadamard broadcast channel. In Section 6, we conclude with a brief summary and some open questions for future research. We point the reader to [22] for basics of quantum information theory and for background on the standard notation and concepts being used in our paper.

## 2 Hadamard quantum broadcast channels

We define a quantum broadcast channel  $\mathcal{N}_{A \rightarrow BC}^H$  to be Hadamard if it has the following action on an input state  $\sigma_A$ :

$$\mathcal{N}_{A \rightarrow BC}^H(\sigma_A) \equiv \sum_{x,y} \langle \phi^x |_A \sigma_A | \phi^y \rangle_A |x\rangle \langle y|_B \otimes |\psi^x\rangle \langle \psi^y|_C, \quad (1)$$

where the vectors  $\{|\phi^x\rangle_A\}_x$  are such that they form a positive operator-valued measure (POVM)  $\sum_x |\phi^x\rangle \langle \phi^x|_A = I_A$ ,  $\{|x\rangle_B\}_x$  is an orthonormal basis, and  $\{|\psi^x\rangle_C\}_x$  is a set of states. Note that the channel in (1) is an isometric channel [22, Section 4.6.3], meaning that its can be reversed. This is a key fact that we exploit in our paper. The reduced channel to Bob is a Hadamard channel [19]

of the following form:

$$\mathcal{N}_{A \rightarrow B}^H(\sigma_A) \equiv (\text{Tr}_C \circ \mathcal{N}_{A \rightarrow BC}^H)(\sigma_A) \quad (2)$$

$$= \sum_{x,y} \langle \phi^x |_A \sigma_A | \phi^y \rangle_A \langle \psi^y | \psi^x \rangle_C |x\rangle \langle y|_B. \quad (3)$$

The reduced channel to Charlie is an entanglement-breaking channel [18] of the following form:

$$\mathcal{N}_{A \rightarrow C}^H(\sigma_A) \equiv (\text{Tr}_B \circ \mathcal{N}_{A \rightarrow BC}^H)(\sigma_A) \quad (4)$$

$$= \sum_x \langle \phi^x |_A \sigma_A | \phi^x \rangle_A | \psi^x \rangle \langle \psi^x |_C, \quad (5)$$

meaning that the action of the channel is to measure the input with respect to the POVM  $\{|\phi^x\rangle\langle\phi^x|_A\}_x$  and then prepare the state  $|\psi^x\rangle_C$  at the output if the measurement outcome is  $x$ . Such channels have the property that Bob can apply the following isometry to his output received from the channel:

$$V_{B \rightarrow BYC'} \equiv \sum_x |x\rangle_B \langle x|_B \otimes |x\rangle_Y \otimes |\psi^x\rangle_{C'}, \quad (6)$$

where system  $C'$  is isomorphic to system  $C$ , and if he then discards the  $B$  and  $Y$  systems, the effect is to simulate the channel to Charlie. That is,

$$\text{Tr}_{BY} \circ \mathcal{V}_{B \rightarrow BYC'} \circ \mathcal{N}_{A \rightarrow B}^H = \mathcal{N}_{A \rightarrow C}^H. \quad (7)$$

The channel  $\mathcal{D}_{B \rightarrow C'} \equiv \text{Tr}_{BY} \circ \mathcal{V}_{B \rightarrow BYC'}$  is known as the degrading channel [28]. We can also consider the degrading channel as arising in two steps: a measurement channel  $\mathcal{M}_{B \rightarrow Y}(\cdot) = \sum_x |x\rangle \langle x|_Y (\cdot) |x\rangle \langle x|_Y$  followed by a preparation channel  $\mathcal{P}_{Y \rightarrow C'}(\cdot) = \sum_x \langle x|(\cdot)|x\rangle_Y |\psi^x\rangle \langle \psi^x|_{C'}$ , so that

$$\mathcal{D}_{B \rightarrow C'} = \mathcal{P}_{Y \rightarrow C'} \circ \mathcal{M}_{B \rightarrow Y}. \quad (8)$$

### 3 Classical capacity region of a Hadamard broadcast channel

#### 3.1 Definition of the classical capacity region of a broadcast channel

We begin by defining the classical capacity region of a quantum broadcast channel [5]. Let  $\mathcal{N}_{A \rightarrow BC}$  denote a quantum broadcast channel from a sender Alice to receivers Bob and Charlie. Let  $n \in \mathbb{N}$ ,  $M_B, M_C \in \mathbb{N}$ , and  $\varepsilon \in [0, 1]$ . An  $(n, M_B, M_C, \varepsilon)$  code for classical communication over the broadcast channel  $\mathcal{N}_{A \rightarrow BC}$  consists of quantum codewords  $\{\rho_{A^n}^{m_1, m_2}\}_{m_1, m_2}$  and POVMs  $\{\Lambda_{B^n}^{m_1}\}_{m_1}$  and  $\{\Gamma_{C^n}^{m_2}\}_{m_2}$  such that the message pair  $(m_1, m_2)$  is communicated with average success probability not smaller than  $1 - \varepsilon$ :

$$\frac{1}{M_B M_C} \sum_{m_1, m_2} \text{Tr}\{(\Lambda_{B^n}^{m_1} \otimes \Gamma_{C^n}^{m_2}) \mathcal{N}_{A \rightarrow BC}^{\otimes n}(\rho_{A^n}^{m_1, m_2})\} \geq 1 - \varepsilon. \quad (9)$$

Note that, in the above,  $m_1 \in \{1, \dots, M_B\}$  and  $m_2 \in \{1, \dots, M_C\}$ .

A rate pair  $(R_B, R_C)$  is achievable for classical communication on  $\mathcal{N}_{A \rightarrow BC}$  if for all  $\varepsilon \in (0, 1)$ ,  $\delta > 0$ , and sufficiently large  $n$ , there exists an  $(n, 2^{n[R_B - \delta]}, 2^{n[R_C - \delta]}, \varepsilon)$  code of the above form. The classical capacity region of  $\mathcal{N}_{A \rightarrow BC}$  is equal to the closure of all achievable rate pairs.

### 3.2 Achievable rate region for an arbitrary quantum broadcast channel

From the superposition coding result in [5, 12], we know the following achievability statement:

**Theorem 1** ([5, 12]) *Given a quantum broadcast channel  $\mathcal{N}_{A \rightarrow BC}$ , a rate pair  $(R_B, R_C)$  is achievable for classical communication on  $\mathcal{N}_{A \rightarrow BC}$  if*

$$R_B \leq I(Z; B|W)_\theta, \quad (10)$$

$$R_C \leq I(W; C)_\theta, \quad (11)$$

$$R_B + R_C \leq I(Z; B)_\theta, \quad (12)$$

where the information quantities are evaluated with respect to a state  $\theta_{WZBC}$  of the following form:

$$\sum_{w,z} p_{WZ}(w, z) |w\rangle\langle w|_W \otimes |z\rangle\langle z|_Z \otimes \mathcal{N}_{A \rightarrow BC}(\sigma_A^z), \quad (13)$$

with  $p_{WZ}$  a probability distribution and  $\{\sigma_A^z\}_z$  a set of states.

For a Hadamard quantum broadcast channel, the region above simplifies due to the structure of the channel. That is, the bound in (12) is unnecessary (redundant) for a Hadamard quantum broadcast channel. To see this, consider that the sum of (10)–(11) leads to

$$R_B + R_C \leq I(Z; B|W)_\theta + I(W; C)_\theta \quad (14)$$

$$\leq I(Z; B|W)_\theta + I(W; B)_\theta \quad (15)$$

$$= I(WZ; B)_\theta \quad (16)$$

$$= I(Z; B)_\theta + I(W; B|Z)_\theta \quad (17)$$

$$= I(Z; B)_\theta. \quad (18)$$

The second inequality follows from the data-processing inequality for mutual information and from the fact that there is a degrading channel taking system  $B$  to system  $C$  for a Hadamard broadcast channel. The next two equalities follow from the chain rule for mutual information. The last equality follows because the state of systems  $W$  and  $B$  are product when conditioned on the value in system  $Z$ . So then the achievable region for a Hadamard broadcast channel consists of (10)–(11).

### 3.3 Classical capacity of a Hadamard broadcast channel

We now show that the region specified by (10)–(11) is in fact the classical capacity region of a Hadamard broadcast channel (i.e., one can never achieve a rate outside of this region). That is, we establish the following capacity theorem for a Hadamard broadcast channel:

**Theorem 2** *The classical capacity region of a Hadamard broadcast channel  $\mathcal{N}_{A \rightarrow BC}^H$  is the set of rate pairs  $(R_B, R_C)$  such that*

$$R_B \leq I(Z; B|W)_\theta, \quad (19)$$

$$R_C \leq I(W; C)_\theta, \quad (20)$$

for some state

$$\theta_{WZA} = \sum_{w,z} p_{WZ}(w, z) |w\rangle\langle w|_W \otimes |z\rangle\langle z|_Z \otimes \varphi_A^z, \quad (21)$$

where  $p_{WZ}$  is a probability distribution, each  $\varphi_A^z$  is a pure state, and the information quantities are evaluated with respect to the state  $\theta_{WZBC} = \mathcal{N}_{A \rightarrow BC}^H(\theta_{WZA})$ .

**Proof.** The achievability part follows from a direct application of Theorem 1.

For the converse, consider an arbitrary  $(n, M_B, M_C, \varepsilon)$  code for the broadcast Hadamard channel  $\mathcal{N}_{A \rightarrow BC}^H$ . Let  $\omega_{M_1 M_2 B^n C^n}$  denote the following state:

$$\omega_{M_1 M_2 B^n C^n} \equiv \frac{1}{M_B M_C} \sum_{m_1, m_2} |m_1\rangle\langle m_1|_{M_1} \otimes |m_2\rangle\langle m_2|_{M_2} \otimes \mathcal{N}_{A \rightarrow BC}^{H \otimes n}(\rho_{A^n}^{m_1, m_2}), \quad (22)$$

so that this is the state before the receivers act with their measurements. The post-measurement state is as follows:

$$\omega_{M_1 M_2 M'_1 M'_2} \equiv \sum_{m_1, m'_1, m_2, m'_2} p(m_1, m'_1, m_2, m'_2) |m_1\rangle\langle m_1|_{M_1} \otimes |m'_1\rangle\langle m'_1|_{M'_1} \otimes |m_2\rangle\langle m_2|_{M_2} \otimes |m'_2\rangle\langle m'_2|_{M'_2}, \quad (23)$$

where

$$p(m_1, m'_1, m_2, m'_2) \equiv \frac{\text{Tr}\{(\Lambda_{B^n}^{m'_1} \otimes \Gamma_{C^n}^{m'_2}) \mathcal{N}_{A \rightarrow BC}^{H \otimes n}(\rho_{A^n}^{m_1, m_2})\}}{M_B M_C}. \quad (24)$$

From the condition in (9), it follows that

$$\frac{1}{2} \left\| \omega_{M_1 M_2 M'_1 M'_2} - \bar{\Phi}_{M_1 M'_1} \otimes \bar{\Phi}_{M_2 M'_2} \right\|_1 \leq \varepsilon, \quad (25)$$

where  $\bar{\Phi}_{M_i M'_i} \equiv \frac{1}{|M_i|} \sum_{m_i} |m_i\rangle\langle m_i|_{M_i} \otimes |m_i\rangle\langle m_i|_{M'_i}$  is the maximally correlated state for  $i \in \{1, 2\}$ .

The reduced state for Charlie can be simulated by acting with the measurement channel  $\mathcal{M}_{B \rightarrow Y}^{\otimes n}$  followed by the preparation channel  $\mathcal{P}_{Y \rightarrow C}^{\otimes n}$ , given our assumption of a Hadamard broadcast channel. That is, we have that

$$\omega_{M_1 M_2 C^n} = \mathcal{P}_{Y \rightarrow C}^{\otimes n}(\xi_{M_1 M_2 Y^n}), \quad (26)$$

where

$$\xi_{M_1 M_2 Y^n} \equiv \mathcal{M}_{B \rightarrow Y}^{\otimes n}(\omega_{M_1 M_2 B^n}). \quad (27)$$

Let a spectral decomposition for the state  $\rho_{A^n}^{m_1, m_2}$  be as follows:

$$\rho_{A^n}^{m_1, m_2} = \sum_t p(t|m_1, m_2) \varsigma_{A^n}^{m_1, m_2, t}, \quad (28)$$

with each  $\varsigma_{A^n}^{m_1, m_2, t}$  pure, so that the following state  $\omega_{M_1 M_2 B^n C^n T}$  is an extension of  $\omega_{M_1 M_2 B^n C^n}$ :

$$\omega_{M_1 M_2 B^n C^n T} \equiv \frac{1}{M_B M_C} \sum_{m_1, m_2, t} |m_1\rangle\langle m_1|_{M_1} \otimes |m_2\rangle\langle m_2|_{M_2} \otimes \mathcal{N}_{A \rightarrow BC}^{H \otimes n}(\varsigma_{A^n}^{m_1, m_2, t}) \otimes p(t|m_1, m_2) |t\rangle\langle t|_T \quad (29)$$

For  $i \in \{1, \dots, n\}$ , let

$$\zeta_{M_1 M_2 B_i C_i Y^{i-1} T}^i \equiv (\text{id}_{M_1 M_2 B_i C_i T} \otimes \mathcal{M}_{B \rightarrow Y}^{\otimes(i-1)} \otimes \text{Tr}_{C^{i-1}})(\omega_{M_1 M_2 B^i C^i T}) \quad (30)$$

$$\begin{aligned} &= \frac{1}{M_B M_C} \sum_{m_1, m_2, t} |m_1\rangle\langle m_1|_{M_1} \otimes |m_2\rangle\langle m_2|_{M_2} \otimes \\ &\quad (\mathcal{N}_{A_i \rightarrow B_i C_i}^H \otimes [(\mathcal{M}_{B \rightarrow Y}^{\otimes(i-1)} \otimes \text{Tr}_{C^{i-1}}) \circ \mathcal{N}_{A \rightarrow BC}^{H \otimes(i-1)}]) (\zeta_{A_i}^{m_1, m_2, t}) \\ &\quad \otimes p(t|m_1, m_2)|t\rangle\langle t|_T \end{aligned} \quad (31)$$

$$\begin{aligned} &= \frac{1}{M_B M_C} \sum_{m_1, m_2, y^{i-1}, t} |m_1\rangle\langle m_1|_{M_1} \otimes |m_2\rangle\langle m_2|_{M_2} \otimes \mathcal{N}_{A_i \rightarrow B_i C_i}^H (\tau_{A_i}^{m_1, m_2, t, y^{i-1}}) \\ &\quad \otimes p(y^{i-1}|m_1, m_2, t)|y^{i-1}\rangle\langle y^{i-1}|_{Y^{i-1}} \otimes p(t|m_1, m_2)|t\rangle\langle t|_T, \end{aligned} \quad (32)$$

where

$$\left[ (\mathcal{M}_{B \rightarrow Y}^{\otimes(i-1)} \otimes \text{Tr}_{C^{i-1}}) \circ \mathcal{N}_{A \rightarrow BC}^{H \otimes(i-1)} \right] (\zeta_{A_i}^{m_1, m_2, t}) = \sum_{y^{i-1}} \tau_{A_i}^{m_1, m_2, t, y^{i-1}} \otimes p(y^{i-1}|m_1, m_2, t)|y^{i-1}\rangle\langle y^{i-1}|_{Y^{i-1}} \quad (33)$$

Taking a spectral decomposition of  $\tau_{A_i}^{m_1, m_2, t, y^{i-1}}$  as

$$\tau_{A_i}^{m_1, m_2, t, y^{i-1}} = \sum_s p(s|m_1, m_2, t, y^{i-1}) \varphi_{A_i}^{m_1, m_2, t, y^{i-1}, s}, \quad (34)$$

with each  $\varphi_{A_i}^{m_1, m_2, t, y^{i-1}, s}$  pure, we find that an extension of  $\zeta_{M_1 M_2 B_i C_i Y^{i-1} T}^i$  is

$$\begin{aligned} \zeta_{M_1 M_2 B_i C_i Y^{i-1} T S}^i &\equiv \frac{1}{M_B M_C} \sum_{m_1, m_2, t, y^{i-1}, s} |m_1\rangle\langle m_1|_{M_1} \otimes |m_2\rangle\langle m_2|_{M_2} \otimes \mathcal{N}_{A_i \rightarrow B_i C_i}^H (\varphi_{A_i}^{m_1, m_2, t, y^{i-1}, s}) \\ &\quad \otimes p(y^{i-1}|m_1, m_2, t)|y^{i-1}\rangle\langle y^{i-1}|_{Y^{i-1}} \otimes p(t|m_1, m_2)|t\rangle\langle t|_T \otimes p(s|m_1, m_2, t, y^{i-1})|s\rangle\langle s|_S. \end{aligned} \quad (35)$$

Let  $\zeta_{Q M_1 M_2 B C \bar{Y} T S}$  denote the following state:

$$\zeta_{Q M_1 M_2 B C \bar{Y} T S} \equiv \frac{1}{n} \sum_{i=1}^n |i\rangle\langle i|_Q \otimes \zeta_{M_1 M_2 B_i C_i Y^{i-1} T S}^i, \quad (36)$$

where  $\bar{Y}$  is large enough to hold the values in each  $Y^{i-1}$  (and zero-padded if need be). Let  $\zeta_{Q \bar{Q} M_1 M_2 \bar{M}_2 B C \bar{Y} T S}$  denote an extension of  $\zeta_{Q M_1 M_2 B C \bar{Y} T S}$ , such that systems  $\bar{Q}$ ,  $\bar{M}_2$ , and  $\bar{Y}$  contain a classical copy of the value in  $Q$ ,  $M_2$ , and  $\bar{Y}$ , respectively.

We begin our analysis using information inequalities. Consider that

$$\log M_C = I(M_2; M_2')_{\bar{\Phi}} \quad (37)$$

$$\leq I(M_2; M_2')_{\omega} + \varepsilon \log M_C + h_2(\varepsilon), \quad (38)$$

where the inequality follows from a uniform bound for continuity of entropy [29, 30] (see also [22])

and  $h_2(\varepsilon)$  denotes the binary entropy. Continuing, we find that

$$I(M_2; M'_2)_\omega \leq I(M_2; C^n)_\omega \quad (39)$$

$$= H(C^n)_\omega - H(C^n|M_2)_\omega \quad (40)$$

$$= \sum_{i=1}^n H(C_i|C^{i-1})_\omega - H(C_i|C^{i-1}M_2)_\omega \quad (41)$$

$$\leq \sum_{i=1}^n H(C_i)_\omega - H(C_i|C^{i-1}M_2)_\omega \quad (42)$$

The first inequality follows from quantum data processing. The first equality is an expansion of the mutual information, and the second equality is an application of the chain rule for conditional entropy. The last inequality follows from the fact that conditioning does not increase entropy. Continuing,

$$(42) \leq \sum_{i=1}^n H(C_i)_{\zeta^i} - H(C_i|Y^{i-1}M_2)_{\zeta^i} \quad (43)$$

$$= \sum_{i=1}^n I(Y^{i-1}M_2; C_i)_{\zeta^i} \quad (44)$$

$$= nI(\bar{Y}M_2; C|Q)_\zeta \quad (45)$$

$$\leq nI(\bar{Y}M_2Q; C)_\zeta. \quad (46)$$

The first inequality follows due to the structure of the Hadamard broadcast channel: the systems  $C^{i-1}$  can be simulated from classical systems  $Y^{i-1}$ , which in turn can be simulated from the systems  $B^{i-1}$ . The first equality follows from the definition of mutual information. The second equality follows by using the definition of the state in (36) and the fact that conditioning on a classical system leads to a convex combination of mutual informations. The last inequality follows because  $I(\bar{Y}M_2; C|Q)_\zeta = I(\bar{Y}M_2Q; C)_\zeta - I(Q; C)_\zeta \leq I(\bar{Y}M_2Q; C)_\zeta$ .

We now handle the other rate bound. Consider that

$$\log M_B = I(M_1; M'_1)_{\bar{\Phi}} \quad (47)$$

$$\leq I(M_1; M'_1)_\omega + \varepsilon \log M_B + h_2(\varepsilon), \quad (48)$$

where the inequality follows from a uniform bound for continuity of entropy [29, 30] (see also [22]). Continuing, we find that

$$I(M_1; M'_1)_\omega \leq I(M_1; B^n M_2)_\omega \quad (49)$$

$$= I(M_1; B^n|M_2)_\omega \quad (50)$$

$$\leq I(M_1 T; B^n|M_2)_\omega \quad (51)$$

$$= H(B^n|M_2)_\omega - H(B^n|M_2 M_1 T)_\omega \quad (52)$$

$$= H(B^n|M_2)_\omega - H(C^n|M_2 M_1 T)_\omega \quad (53)$$

$$= \sum_{i=1}^n H(B_i|B^{i-1}M_2)_\omega - H(C_i|C^{i-1}M_2 M_1 T)_\omega \quad (54)$$

The first inequality follows from quantum data processing. The first equality follows from the chain rule for mutual information and the fact that  $I(M_1; M_2)_\omega = 0$ . The second inequality follows from quantum data processing for the conditional mutual information. The second equality is an expansion of the conditional mutual information. The third equality follows because the state of systems  $B^n C^n$  is pure when conditioned on classical systems  $M_1$ ,  $M_2$ , and  $T$ . The last equality applies the chain rule for conditional entropy. Continuing,

$$(54) \leq \sum_{i=1}^n H(B_i | Y^{i-1} M_2)_{\zeta^i} - H(C_i | Y^{i-1} M_2 M_1 T)_{\zeta^i} \quad (55)$$

$$\leq \sum_{i=1}^n H(B_i | Y^{i-1} M_2)_{\zeta^i} - H(C_i | Y^{i-1} M_2 M_1 T S)_{\zeta^i} \quad (56)$$

$$= \sum_{i=1}^n H(B_i | Y^{i-1} M_2)_{\zeta^i} - H(B_i | Y^{i-1} M_2 M_1 T S)_{\zeta^i} \quad (57)$$

$$= \sum_{i=1}^n I(M_1 T S; B_i | Y^{i-1} M_2)_{\zeta^i} \quad (58)$$

$$= nI(M_1 T S; B | \bar{Y} M_2 Q)_\zeta \quad (59)$$

$$\leq nI(M_1 T S \overline{Q M_2 \bar{Y}}; B | \bar{Y} M_2 Q)_\zeta. \quad (60)$$

The first inequality applies the data processing inequality for conditional entropy: the  $Y^{i-1}$  systems result from measurements of the  $B^{i-1}$  systems, and the  $C^{i-1}$  systems can be simulated by preparation channels acting on the  $Y^{i-1}$  systems. The second inequality again follows from the data processing inequality for conditional entropy. The first equality follows because the state of the  $B_i C_i$  systems is pure when conditioned on systems  $Y^{i-1}$ ,  $M_2$ ,  $M_1$ ,  $T$ , and  $S$ . The second equality follows from the definition of conditional mutual information. The third equality follows by introducing the  $Q$  system and evaluating the conditional mutual information of the state  $\zeta_{Q M_1 M_2 B C \bar{Y} T S}$ . The final inequality follows from data processing for the conditional mutual information.

Putting everything together, we find that the following inequalities hold

$$\frac{1-\varepsilon}{n} \log M_B \leq I(M_1 T S \overline{Q M_2 \bar{Y}}; B | \bar{Y} M_2 Q)_\zeta + \frac{1}{n} h_2(\varepsilon), \quad (61)$$

$$\frac{1-\varepsilon}{n} \log M_C \leq I(\bar{Y} M_2 Q; C)_\zeta + \frac{1}{n} h_2(\varepsilon). \quad (62)$$

Now identifying the systems  $M_1 T S \overline{Q M_2 \bar{Y}}$  with system  $Z$  in (21), systems  $\bar{Y} M_2 Q$  with system  $W$  in (21), and the state  $\varphi_{A_i}^{m_1, m_2, t, y^{i-1}, s}$  with  $\varphi_A^z$  in (21), we can rewrite the above inequalities as follows:

$$\frac{1-\varepsilon}{n} \log M_B \leq I(Z; B | W)_\zeta + \frac{1}{n} h_2(\varepsilon), \quad (63)$$

$$\frac{1-\varepsilon}{n} \log M_C \leq I(W; C)_\zeta + \frac{1}{n} h_2(\varepsilon). \quad (64)$$

Now that we have established that these inequalities hold for an arbitrary  $(n, M_B, M_C, \varepsilon)$  code, considering a sequence  $\{(n, M_B, M_C, \varepsilon_n)\}_n$  of them with  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , we find that the rate region is characterized by (19)–(20). ■

## 4 Classical–quantum capacity of a Hadamard broadcast channel

### 4.1 Definition of the classical–quantum capacity region of a quantum broadcast channel

We now recall the definition of the classical–quantum capacity region of a quantum broadcast channel [5]. Let  $\mathcal{N}_{A \rightarrow BC}$  denote a quantum broadcast channel from a sender Alice to receivers Bob and Charlie. Let  $n \in \mathbb{N}$ ,  $M_B, M_C \in \mathbb{N}$ , and  $\varepsilon \in [0, 1]$ . An  $(n, M_B, M_C, \varepsilon)$  code for classical–quantum communication over the broadcast channel  $\mathcal{N}_{A \rightarrow BC}$  consists of quantum codewords  $\{\rho_{RA^n}^m\}_m$ , such that  $\dim(\mathcal{H}_R) = M_B$ , a decoding channel  $\mathcal{D}_{B^n \rightarrow \hat{R}}$ , and a decoding POVM  $\{\Gamma_{C^n}^m\}_m$ . Let the state after the channel acts be as follows:

$$\omega_{MRB^n C^n} \equiv \frac{1}{M_B} \sum_m |m\rangle\langle m|_M \otimes \mathcal{N}_{A \rightarrow BC}^{\otimes n}(\rho_{RA^n}^m), \quad (65)$$

and let the state after the decoders act be as follows:

$$\omega_{MM'R\hat{R}} \equiv \sum_{m'} |m'\rangle\langle m'|_{M'} \otimes \text{Tr}_{C^n} \{ \Gamma_{C^n}^{m'} [ \mathcal{D}_{B^n \rightarrow \hat{R}}(\omega_{MRB^n C^n}) ] \}. \quad (66)$$

For an  $(n, M_B, M_C, \varepsilon)$  code, the following condition holds

$$\frac{1}{2} \|\bar{\Phi}_{MM'} \otimes \Phi_{R\hat{R}} - \omega_{MM'R\hat{R}}\|_1 \leq \varepsilon, \quad (67)$$

where  $\Phi_{R\hat{R}}$  denotes a maximally entangled state.

A rate pair  $(Q_B, R_C)$  is achievable for classical–quantum communication on  $\mathcal{N}_{A \rightarrow BC}$  if for all  $\varepsilon \in (0, 1)$ ,  $\delta > 0$ , and sufficiently large  $n$ , there exists an  $(n, 2^{n[Q_B - \delta]}, 2^{n[R_C - \delta]}, \varepsilon)$  code of the above form. The classical–quantum capacity region of  $\mathcal{N}_{A \rightarrow BC}$  is equal to the closure of all achievable rate pairs.

### 4.2 Achievable rate region for an arbitrary quantum broadcast channel

From [5], we know the following achievability statement:

**Theorem 3 ([5])** *Given a quantum broadcast channel  $\mathcal{N}_{A \rightarrow BC}$ , a rate pair  $(Q_B, R_C)$  is achievable for classical–quantum communication on  $\mathcal{N}_{A \rightarrow BC}$  if*

$$Q_B \leq I(R)BW_\theta, \quad (68)$$

$$R_C \leq \min\{I(W; B)_\theta, I(W; C)_\theta\}, \quad (69)$$

where the information quantities are evaluated with respect to a state  $\theta_{WRBC}$  of the following form:

$$\theta_{WRBC} \equiv \sum_w p_W(w) |w\rangle\langle w|_W \otimes \mathcal{N}_{A \rightarrow BC}(\varphi_{RA}^w), \quad (70)$$

with  $p_W$  a probability distribution and  $\{\varphi_{RA}^w\}_w$  a set of pure states.

### 4.3 Classical–quantum capacity region for Hadamard broadcast channels

**Theorem 4** *The classical–quantum capacity region of a Hadamard broadcast channel  $\mathcal{N}_{A \rightarrow BC}^H$  is the set of rate pairs  $(Q_B, R_C)$  such that*

$$Q_B \leq I(R)BW_\theta, \quad (71)$$

$$R_C \leq I(W; C)_\theta, \quad (72)$$

for some state

$$\theta_{WRA} \equiv \sum_w p_W(w) |w\rangle\langle w|_W \otimes \varphi_{RA}^w, \quad (73)$$

where  $p_W$  is a probability distribution, each  $\varphi_{RA}^w$  is a pure state, and the information quantities are evaluated with respect to the state  $\theta_{WRBC} = \mathcal{N}_{A \rightarrow BC}^H(\theta_{WRA})$ .

**Proof.** The achievability part follows as a direct consequence of Theorem 3, by combining data processing with the fact that a Hadamard broadcast channel is degradable.

To begin the proof of the converse part, let a spectral decomposition for the state  $\rho_{RA^n}^m$  be as follows:

$$\rho_{RA^n}^m = \sum_t p(t|m) \zeta_{RA^n}^{m,t}, \quad (74)$$

with each  $\zeta_{RA^n}^{m,t}$  pure, so that an extension of the state  $\omega_{MRB^n C^n}$  in (65) is as follows:

$$\omega_{MRB^n C^n T} = \frac{1}{M_B} \sum_{m,t} |m\rangle\langle m|_M \otimes \mathcal{N}_{A \rightarrow BC}^{H \otimes n}(\zeta_{RA^n}^{m,t}) \otimes p(t|m) |t\rangle\langle t|_T. \quad (75)$$

For  $i \in \{1, \dots, n\}$ , let

$$\zeta_{MRB_i C_i Y^{i-1} T}^i \equiv (\text{id}_{MRB_i C_i T} \otimes \mathcal{M}_{B \rightarrow Y}^{\otimes(i-1)} \otimes \text{Tr}_{C^{i-1}})(\omega_{MRB^i C^i T}) \quad (76)$$

$$\begin{aligned} &= \frac{1}{M_B} \sum_{m,t} |m\rangle\langle m|_M \otimes (\mathcal{N}_{A_i \rightarrow B_i C_i}^H \otimes [(\mathcal{M}_{B \rightarrow Y}^{\otimes(i-1)} \otimes \text{Tr}_{C^{i-1}}) \circ \mathcal{N}_{A \rightarrow BC}^{H \otimes(i-1)}]) (\zeta_{RA^i}^{m,t}) \\ &\quad \otimes p(t|m) |t\rangle\langle t|_T \end{aligned} \quad (77)$$

$$\begin{aligned} &= \frac{1}{M_B} \sum_{m, y^{i-1}, t} |m\rangle\langle m|_M \otimes \mathcal{N}_{A_i \rightarrow B_i C_i}^H(\tau_{RA_i}^{m,t, y^{i-1}}) \otimes p(y^{i-1}|m, t) |y^{i-1}\rangle\langle y^{i-1}|_{Y^{i-1}} \\ &\quad \otimes p(t|m) |t\rangle\langle t|_T, \end{aligned} \quad (78)$$

$$\otimes p(t|m) |t\rangle\langle t|_T, \quad (79)$$

where

$$\left[ (\mathcal{M}_{B \rightarrow Y}^{\otimes(i-1)} \otimes \text{Tr}_{C^{i-1}}) \circ \mathcal{N}_{A \rightarrow BC}^{H \otimes(i-1)} \right] (\zeta_{RA^i}^{m,t}) = \sum_{y^{i-1}} \tau_{RA_i}^{m,t, y^{i-1}} \otimes p(y^{i-1}|m, t) |y^{i-1}\rangle\langle y^{i-1}|_{Y^{i-1}}. \quad (80)$$

Letting  $\varphi_{SRA_i}^{m,t, y^{i-1}}$  denote a purification of  $\tau_{RA_i}^{m,t, y^{i-1}}$ , we find that an extension of  $\zeta_{MRB_i C_i Y^{i-1} T}^i$  is

$$\begin{aligned} \zeta_{MSRB_i C_i Y^{i-1} T}^i &\equiv \frac{1}{M_B} \sum_{m, t, y^{i-1}, s} |m\rangle\langle m|_M \otimes \mathcal{N}_{A_i \rightarrow B_i C_i}^H(\varphi_{SRA_i}^{m,t, y^{i-1}}) \\ &\quad \otimes p(y^{i-1}|m, t) |y^{i-1}\rangle\langle y^{i-1}|_{Y^{i-1}} \otimes p(t|m) |t\rangle\langle t|_T. \end{aligned} \quad (81)$$

Let  $\zeta_{QMSRBC\bar{Y}T}$  denote the following state:

$$\zeta_{QMSRBC\bar{Y}T} \equiv \frac{1}{n} \sum_{i=1}^n |i\rangle\langle i|_Q \otimes \zeta_{MSRB_i C_i Y^{i-1} T}, \quad (82)$$

where  $\bar{Y}$  is large enough to hold the values in each  $Y^{i-1}$  (and zero-padded if need be).

The following information bound is a consequence of reasoning identical to that in (37)–(46):

$$\frac{1-\varepsilon}{n} \log M_C \leq I(W; C)_\zeta + \frac{1}{n} h_2(\varepsilon), \quad (83)$$

identifying  $W$  as  $\bar{Y}MTQ$ . (To get this bound, we require a final step of data processing to have system  $T$  be included in  $W$ .)

We now prove the other bound. Consider that

$$\log M_B = I(R)\hat{R}_\Phi \quad (84)$$

$$\leq I(R)\hat{R}_\omega + 2\varepsilon \log M_B + g(\varepsilon), \quad (85)$$

where the inequality follows from the main result of [31] and  $g(\varepsilon) \equiv (1+\varepsilon)\log_2(1+\varepsilon) - \varepsilon\log_2(\varepsilon)$ , with the property that  $\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 0$ . Continuing, we find that

$$I(R)\hat{R}_\omega \leq I(R)B^n MT)_\omega \quad (86)$$

$$= H(B^n | MT)_\omega - H(RB^n | MT)_\omega \quad (87)$$

$$= H(B^n | MT)_\omega - H(C^n | MT)_\omega \quad (88)$$

$$= \sum_{i=1}^n H(B_i | B^{i-1} MT)_\omega - H(C_i | C^{i-1} MT)_\omega \quad (89)$$

The first inequality follows from quantum data processing. The first equality follows from definitions. The second equality follows because the state of systems  $RB^n C^n$  is pure when conditioned on systems  $MT$ . The third equality follows from the chain rule for conditional entropy. Continuing,

$$(89) \leq \sum_{i=1}^n H(B_i | Y^{i-1} MT)_{\zeta^i} - H(C_i | Y^{i-1} MT)_{\zeta^i} \quad (90)$$

$$= \sum_{i=1}^n H(B_i | Y^{i-1} MT)_{\zeta^i} - H(SRB_i | Y^{i-1} MT)_{\zeta^i} \quad (91)$$

$$= \sum_{i=1}^n I(SR)_{B_i Y^{i-1} MT)_{\zeta^i} \quad (92)$$

$$= nI(SR)_{B\bar{Y}MTQ)_\zeta. \quad (93)$$

The inequality follows from the structure of a Hadamard channel: systems  $Y^{i-1}$  can be simulated by systems  $B^{i-1}$  and systems  $C^{i-1}$  can be simulated by systems  $Y^{i-1}$ . The second equality follows because the state of systems  $SRB_i$  is pure when conditioned on systems  $Y^{i-1}MT$ . The third equality is by definition, and the last follows by evaluating the coherent information of the given

state and systems. Putting everything together leads to the following two bounds:

$$\frac{1-2\varepsilon}{n} \log M_B \leq I(R; BW)_\zeta + \frac{1}{n} g(\varepsilon), \quad (94)$$

$$\frac{1-\varepsilon}{n} \log M_C \leq I(W; C)_\zeta + \frac{1}{n} h_2(\varepsilon), \quad (95)$$

relabeling  $R$  as  $SR$  and taking  $W$  as  $\overline{Y}MTQ$ , as stated above. Now that we have established that these inequalities hold for an arbitrary  $(n, M_B, M_C, \varepsilon)$  classical-quantum code, considering a sequence  $\{(n, M_B, M_C, \varepsilon_n)\}_n$  of them with  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , we find that the rate region is characterized by (71)–(72). ■

## 5 Partially entanglement-assisted classical capacity region of a Hadamard broadcast channel

### 5.1 Definition of the partially entanglement-assisted classical capacity region of a broadcast channel

We now define the partially entanglement-assisted classical capacity region of a quantum broadcast channel [5]. Let  $\mathcal{N}_{A \rightarrow BC}$  denote a quantum broadcast channel from a sender Alice to receivers Bob and Charlie. Let  $n \in \mathbb{N}$ ,  $M_B, M_C \in \mathbb{N}$ , and  $\varepsilon \in [0, 1]$ . An  $(n, M_B, M_C, \varepsilon)$  partially entanglement-assisted code for classical communication over the broadcast channel  $\mathcal{N}_{A \rightarrow BC}$  consists of a shared (generally mixed) entangled state  $\Psi_{R_0 R'_0}$ , such that Alice has system  $R'_0$  and Bob has system  $R_0$ . Such a code also consists of a set of encoding channels  $\{\mathcal{E}_{R'_0 \rightarrow A^n}^{m_1, m_2}\}_{m_1, m_2}$ , and POVMs  $\{\Lambda_{R_0 B^n}^{m_1}\}_{m_1}$  and  $\{\Gamma_{C^n}^{m_2}\}_{m_2}$  such that the message pair  $(m_1, m_2)$  is communicated with average success probability not smaller than  $1 - \varepsilon$ :

$$\sum_{m_1, m_2} \frac{\text{Tr}\{(\Lambda_{R_0 B^n}^{m_1} \otimes \Gamma_{C^n}^{m_2}) \mathcal{N}_{A \rightarrow BC}^{\otimes n}(\mathcal{E}_{R'_0 \rightarrow A^n}^{m_1, m_2}(\Psi_{R_0 R'_0}))\}}{M_B M_C} \geq 1 - \varepsilon. \quad (96)$$

A rate pair  $(R_B, R_C)$  is achievable for partially entanglement-assisted classical communication on  $\mathcal{N}_{A \rightarrow BC}$  if for all  $\varepsilon \in (0, 1)$ ,  $\delta > 0$ , and sufficiently large  $n$ , there exists an  $(n, 2^{n[R_B - \delta]}, 2^{n[R_C - \delta]}, \varepsilon)$  code of the above form. The classical capacity region of  $\mathcal{N}_{A \rightarrow BC}$  is equal to the closure of all achievable rate pairs.

### 5.2 Achievable rate region for an arbitrary quantum broadcast channel

We now argue an achievability statement based on some prior coding schemes from [32, 28, 33]:

**Theorem 5** *Given a quantum broadcast channel  $\mathcal{N}_{A \rightarrow BC}$ , a rate pair  $(R_B, R_C)$  is achievable for partially entanglement-assisted classical communication on  $\mathcal{N}_{A \rightarrow BC}$  if*

$$R_B \leq I(R; B|W)_\theta, \quad (97)$$

$$R_C \leq \min\{I(W; B)_\theta, I(W; C)_\theta\}, \quad (98)$$

where the information quantities are evaluated with respect to a state  $\theta_{WRBC}$  of the following form:

$$\sum_w p_W(w) |w\rangle\langle w|_W \otimes \mathcal{N}_{A \rightarrow BC}(\varphi_{RA}^w), \quad (99)$$

with  $p_W$  a probability distribution and  $\{\varphi_{RA}^w\}_w$  a set of states.

**Proof.** We merely sketch a proof rather than work out the details, mainly because the ideas for it have been used in a variety of contexts. The idea is similar to that used in trade-off coding for transmitting both classical and quantum information over a single-sender single-receiver quantum channel (see in particular [22, Theorem 22.5.1]). Fix a probability distribution  $p_W(w)$  and a corresponding set of pure states  $\{\varphi_{RA}^w\}_w$ . Pick a typical type class  $T_t$ , meaning the set of all sequences with the same empirical distribution  $t(w)$  that deviates from the true distribution  $p_W(w)$  by no more than  $\delta > 0$ . All the sequences in the same type class are related to one another by a permutation, and all of them are strongly typical. Now we suppose that Alice and Bob share the state

$$\varphi_{R^n A^n}^{w^n} \equiv \varphi_{R_1 A_1}^{w_1} \otimes \cdots \otimes \varphi_{R_n A_n}^{w_n}, \quad (100)$$

where the sequence  $w^n \in T_t$ , Alice has the  $A^n$  systems, and Bob has the  $R^n$  systems. The reduced state after tracing over Bob's systems is  $\varphi_{A^n}^{w^n} = \varphi_{A_1}^{w_1} \otimes \cdots \otimes \varphi_{A_n}^{w_n}$ .

The main idea is for Alice to layer the messages on top of each other as in superposition coding. She first encodes a classical message as a permutation of the sequence  $w^n$  (using a constant-composition code as discussed in [22, Section 20.3.1]), sending the  $A^n$  systems of the state  $\varphi_{R^n A^n}^{w^n}$  through  $n$  uses of the broadcast channel. Bob and Charlie then decode, and Bob neglects his systems  $R^n$  in this first decoding step. It is possible for each of them to decode reliably as long as the rate  $R_C \leq \min\{I(W; B)_\theta, I(W; C)_\theta\}$ . At the same time, Alice can encode another message intended exclusively for Bob as an entanglement-assisted code into the states  $\varphi_{R^n A^n}^{w^n}$ . This is possible by arranging the sequence of states  $\varphi_{R^n A^n}^{w^n}$  into  $|W|$  blocks of  $\approx np_W(w)$  i.i.d. states of the form  $\varphi_{RA}^w$ . For each block, she employs the coding scheme of [33] for entanglement-assisted coding at a rate  $I(R; B)_{\mathcal{N}(\varphi^w)}$ , such that the total rate for the message intended for Bob is  $\sum_w p_W(w) I(R; B)_{\mathcal{N}(\varphi^w)} = I(R; B|W)_\theta$ . So, in a second decoding step after Bob determines which permutation of the sequence  $w^n$  Alice transmitted, he can rearrange his systems  $R^n A^n$  into the above standard form to decode the message encoded in the entanglement-assisted codes. This gives the achievable rate region above. See the discussion in the proof of [22, Theorem 22.5.1] for more details. ■

### 5.3 Partially entanglement-assisted classical capacity of a Hadamard broadcast channel

We now determine the partially entanglement-assisted classical capacity region of a Hadamard broadcast channel:

**Theorem 6** *The partially entanglement-assisted classical capacity region of a Hadamard broadcast channel  $\mathcal{N}_{A \rightarrow BC}^H$  is the set of rate pairs  $(R_B, R_C)$  such that*

$$R_B \leq I(R; B|W)_\theta, \quad (101)$$

$$R_C \leq I(W; C)_\theta, \quad (102)$$

for some state

$$\theta_{WRA} = \sum_w p_W(w) |w\rangle\langle w|_W \otimes \varphi_{RA}^w, \quad (103)$$

where  $p_W$  is a probability distribution, each  $\varphi_{RA}^w$  is a pure state, and the information quantities are evaluated with respect to the state  $\theta_{WRBC} = \mathcal{N}_{A \rightarrow BC}^H(\theta_{WRA})$ .

**Proof.** The achievability part follows from a direct application of Theorem 5, by combining data processing with the fact that a Hadamard broadcast channel is degradable.

For the converse, consider an arbitrary  $(n, M_B, M_C, \varepsilon)$  code for the broadcast Hadamard channel  $\mathcal{N}_{A \rightarrow BC}^H$ . Let  $\omega_{M_1 M_2 R_0 B^n C^n}$  denote the following state:

$$\omega_{M_1 M_2 R_0 B^n C^n} \equiv \frac{1}{M_B M_C} \sum_{m_1, m_2} |m_1\rangle\langle m_1|_{M_1} \otimes |m_2\rangle\langle m_2|_{M_2} \otimes \mathcal{N}_{A \rightarrow BC}^{H \otimes n}(\mathcal{E}_{R_0' \rightarrow A^n}^{m_1, m_2}(\Psi_{R_0 R_0'})), \quad (104)$$

so that this is the state before the receivers act with their measurements. The post-measurement state is as follows:

$$\omega_{M_1 M_2 M_1' M_2'} \equiv \sum_{m_1, m_1', m_2, m_2'} p(m_1, m_1', m_2, m_2') |m_1\rangle\langle m_1|_{M_1} \otimes |m_1'\rangle\langle m_1'|_{M_1'} \otimes |m_2\rangle\langle m_2|_{M_2} \otimes |m_2'\rangle\langle m_2'|_{M_2'}, \quad (105)$$

where

$$p(m_1, m_1', m_2, m_2') = \frac{\text{Tr}\{(\Lambda_{R_0 B^n}^{m_1'} \otimes \Gamma_{C^n}^{m_2'}) \mathcal{N}_{A \rightarrow BC}^{H \otimes n}(\mathcal{E}_{R_0' \rightarrow A^n}^{m_1, m_2}(\Psi_{R_0 R_0'}))\}}{M_B M_C}. \quad (106)$$

From the condition in (96), it follows that

$$\frac{1}{2} \left\| \omega_{M_1 M_2 M_1' M_2'} - \bar{\Phi}_{M_1 M_1'} \otimes \bar{\Phi}_{M_2 M_2'} \right\|_1 \leq \varepsilon, \quad (107)$$

where  $\bar{\Phi}_{M_i M_i'} \equiv \frac{1}{|M_i|} \sum_{m_i} |m_i\rangle\langle m_i|_{M_i} \otimes |m_i\rangle\langle m_i|_{M_i'}$  is the maximally correlated state for  $i \in \{1, 2\}$ .

For a fixed  $m_2$ , let  $\zeta_{M_1 R_0 A^n}^{m_2}$  denote the following state:

$$\zeta_{M_1 R_0 A^n}^{m_2} \equiv \frac{1}{M_B} \sum_{m_1} |m_1\rangle\langle m_1|_{M_1} \otimes \mathcal{E}_{R_0' \rightarrow A^n}^{m_1, m_2}(\Psi_{R_0 R_0'}), \quad (108)$$

and let  $\zeta_{T M_1 R_0 A^n}^{m_2}$  denote a purification of it, so that

$$\omega_{M_1 M_2 R_0 T B^n C^n} \equiv \frac{1}{M_C} \sum_{m_2} |m_2\rangle\langle m_2|_{M_2} \otimes \mathcal{N}_{A \rightarrow BC}^{H \otimes n}(\zeta_{T M_1 R_0 A^n}^{m_2}) \quad (109)$$

is an extension of  $\omega_{M_1 M_2 R_0 B^n C^n}$ . For  $i \in \{1, \dots, n\}$ , let

$$\begin{aligned} & \zeta_{M_1 M_2 R_0 B_i C_i Y^{i-1} T}^i \\ & \equiv (\text{id}_{M_1 M_2 R_0 B_i C_i T} \otimes \mathcal{M}_{B \rightarrow Y}^{\otimes(i-1)} \otimes \text{Tr}_{C^{i-1}})(\omega_{M_1 M_2 R_0 B^i C^i T}) \end{aligned} \quad (110)$$

$$= \frac{1}{M_C} \sum_{m_2} |m_2\rangle\langle m_2|_{M_2} \otimes (\mathcal{N}_{A_i \rightarrow B_i C_i}^H \otimes [(\mathcal{M}_{B \rightarrow Y}^{\otimes(i-1)} \otimes \text{Tr}_{C^{i-1}}) \circ \mathcal{N}_{A \rightarrow BC}^{H \otimes(i-1)}])(\zeta_{T M_1 R_0 A^i}^{m_2}) \quad (111)$$

$$= \frac{1}{M_C} \sum_{m_2, y^{i-1}} |m_2\rangle\langle m_2|_{M_2} \otimes \mathcal{N}_{A_i \rightarrow B_i C_i}^H(\tau_{T M_1 R_0 A_i}^{m_2, y^{i-1}}) \otimes p(y^{i-1} | m_2) |y^{i-1}\rangle\langle y^{i-1}|_{Y^{i-1}}, \quad (112)$$

where

$$\left[ (\mathcal{M}_{B \rightarrow Y}^{\otimes(i-1)} \otimes \text{Tr}_{C^{i-1}}) \circ \mathcal{N}_{A \rightarrow BC}^{H \otimes(i-1)} \right] (\zeta_{T M_1 R_0 A^i}^{m_2}) = \sum_{y^{i-1}} \tau_{T M_1 R_0 A_i}^{m_2, y^{i-1}} \otimes p(y^{i-1} | m_2) |y^{i-1}\rangle\langle y^{i-1}|_{Y^{i-1}} \quad (113)$$

Taking a purification of  $\tau_{TM_1R_0A_i}^{m_2, y^{i-1}}$  as  $\varphi_{STM_1R_0A_i}^{m_2, y^{i-1}}$ , we find that an extension of  $\zeta_{M_1M_2R_0B_iC_iY^{i-1}T}^i$  is

$$\zeta_{M_1M_2R_0B_iC_iY^{i-1}TS}^i \equiv \frac{1}{M_C} \sum_{m_2, y^{i-1}} |m_2\rangle\langle m_2|_{M_2} \otimes \mathcal{N}_{A_i \rightarrow B_iC_i}^H(\varphi_{STM_1R_0A_i}^{m_2, y^{i-1}}) \otimes p(y^{i-1}|m_2)|y^{i-1}\rangle\langle y^{i-1}|_{Y^{i-1}}. \quad (114)$$

Let  $\zeta_{QM_1M_2R_0BC\bar{Y}TS}$  denote the following state:

$$\zeta_{QM_1M_2R_0BC\bar{Y}TS} \equiv \frac{1}{n} \sum_{i=1}^n |i\rangle\langle i|_Q \otimes \zeta_{M_1M_2R_0B_iC_iY^{i-1}TS}^i, \quad (115)$$

where  $\bar{Y}$  is large enough to hold the values in each  $Y^{i-1}$  (and zero-padded if need be).

The following information bound is a consequence of reasoning identical to that in (37)–(46):

$$\frac{1-\varepsilon}{n} \log M_C \leq I(W; C)_\zeta + \frac{1}{n} h_2(\varepsilon), \quad (116)$$

identifying  $W$  as  $\bar{Y}M_2Q$ .

We now handle the other rate bound. Consider that

$$\log M_B = I(M_1; M'_1)_{\bar{\Phi}} \quad (117)$$

$$\leq I(M_1; M'_1)_\omega + \varepsilon \log M_B + h_2(\varepsilon), \quad (118)$$

where the inequality follows from a uniform bound for continuity of entropy [29, 30] (see also [22]). Continuing, we find that

$$I(M_1; M'_1)_\omega \leq I(M_1; B^n R_0 M_2)_\omega \quad (119)$$

$$= I(M_1; B^n R_0 | M_2)_\omega \quad (120)$$

$$= I(M_1 R_0; B^n | M_2)_\omega + I(M_1; R_0 | M_2)_\omega - I(B^n; R_0 | M_2)_\omega \quad (121)$$

$$\leq I(M_1 R_0; B^n | M_2)_\omega \quad (122)$$

$$\leq I(M_1 R_0 T; B^n | M_2)_\omega \quad (123)$$

$$= H(B^n | M_2)_\omega - H(B^n | M_1 R_0 T M_2)_\omega \quad (124)$$

$$= H(B^n | M_2)_\omega + H(B^n | C^n M_2)_\omega \quad (125)$$

The first inequality follows from quantum data processing. The first equality follows from the chain rule for mutual information and the fact that  $I(M_1; M_2)_\omega = 0$ . The second equality is an identity for conditional mutual information. The second inequality follows because  $I(M_1; R_0 | M_2)_\omega = 0$  (i.e., the reduced state on these systems is a product state) and  $I(B^n; R_0 | M_2)_\omega \geq 0$ . The third inequality follows from quantum data processing for the conditional mutual information. The third equality is an expansion of the conditional mutual information. The third equality follows because the state

of systems  $M_1 R_0 T B^n C^n$  is pure when conditioned on classical system  $M_2$ . Continuing,

$$(125) = \sum_{i=1}^n H(B_i | B^{i-1} M_2)_\omega + H(B_i | B^{i-1} C^n M_2)_\omega \quad (126)$$

$$\leq \sum_{i=1}^n H(B_i | Y^{i-1} M_2)_{\zeta^i} + H(B_i | Y^{i-1} C_i M_2)_{\zeta^i} \quad (127)$$

$$= \sum_{i=1}^n H(B_i | Y^{i-1} M_2)_{\zeta^i} - H(B_i | R_0 S T M_1 Y^{i-1} M_2)_{\zeta^i} \quad (128)$$

$$= \sum_{i=1}^n I(R_0 S T M_1; B_i | Y^{i-1} M_2)_{\zeta^i} \quad (129)$$

$$= I(R_0 S T M_1; B | \bar{Y} M_2 Q)_\zeta. \quad (130)$$

The first equality follows from the chain rule for conditional entropy. The first inequality applies the data processing inequality for conditional entropy: the  $Y^{i-1}$  systems result from measurements of the  $B^{i-1}$  systems and then we discard the  $C^{i-1}$  systems. The second equality follows because the state of the  $R_0 S T M_1 B_i C_i$  systems is pure when conditioned on systems  $Y^{i-1}$  and  $M_2$ . The second equality follows from the definition of conditional mutual information. The third equality follows by introducing the  $Q$  system and evaluating the conditional mutual information of the state  $\zeta_{Q M_1 M_2 R_0 B C \bar{Y} T S}$ .

Putting everything together, we find that the following inequalities hold

$$\frac{1-\varepsilon}{n} \log M_B \leq I(R_0 S T M_1; B | \bar{Y} M_2 Q)_\zeta + \frac{1}{n} h_2(\varepsilon), \quad (131)$$

$$\frac{1-\varepsilon}{n} \log M_C \leq I(\bar{Y} M_2 Q; C)_\zeta + \frac{1}{n} h_2(\varepsilon). \quad (132)$$

Now identifying the systems  $R_0 S T M_1$  with system  $R$  in (103), systems  $\bar{Y} M_2 Q$  with system  $W$  in (103), and the state  $\varphi_{S T M_1 R_0 A_i}^{m_2, y^{i-1}}$  with  $\varphi_{R A}^w$  in (103), we can rewrite the above inequalities as follows:

$$\frac{1-\varepsilon}{n} \log M_B \leq I(R; B | W)_\zeta + \frac{1}{n} h_2(\varepsilon), \quad (133)$$

$$\frac{1-\varepsilon}{n} \log M_C \leq I(W; C)_\zeta + \frac{1}{n} h_2(\varepsilon). \quad (134)$$

Now that we have established that these inequalities hold for an arbitrary  $(n, M_B, M_C, \varepsilon)$  code, considering a sequence  $\{(n, M_B, M_C, \varepsilon_n)\}_n$  of them with  $\varepsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ , we find that the rate region is characterized by (101)–(102). ■

**Remark 7** *In all of the capacity theorems established here (Theorems 2, 4, and 6), the rate  $R_C$  can be replaced by the sum of a rate  $R_{BC}$  and  $R_C$ , where  $R_{BC}$  is the rate of a common message intended for both Bob and Charlie, while  $R_C$  is the rate of a message intended for Charlie. This is because all of our converses go through with this modification, and at the same time, all of the achievability parts make use of a superposition coding technique, in which Bob first decodes the message intended for Charlie before decoding the message intended for him.*

## 6 Conclusion

This paper solves the classical capacity, classical–quantum capacity, and partially entanglement-assisted classical capacity of Hadamard broadcast channels. As such, these channels might naturally be viewed as a quantum extension of the notion of a degraded broadcast channel. Essential in all of our analyses is the structure of a Hadamard broadcast channel in which the  $C$  system can be simulated in two steps: first a measurement channel taking the  $B$  system to a classical  $Y$  system and then a preparation channel taking the classical  $Y$  system to the  $C$  system. This structure allows for a classical auxiliary variable to include the classical  $Y$  system in each of the problems we considered, as is common in network classical information theory [4].

Much remains to be understood about including fully quantum systems in auxiliary variables, but there has been some progress on this front [34] and various information-theoretic tasks have been characterized using auxiliary quantum variables [35, 36, 37, 38]. However, in many of these cases, it is not known whether a bound can be placed on the dimension of an auxiliary quantum system and so quantities involving quantum auxiliary variables are not known to be tractable. At the least, the structure of a Hadamard broadcast channel allows for circumventing this problem and yields a complete characterization of some of its capacities.

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