Triangle-like inequalities related to coherence and entanglement negativity

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Abstract

Quantum coherence and entanglement are two key features in quantum mechanics and play important roles in quantum information processing and quantum computation. We provide a general triangle-like inequality satisfied by the l_1-norm measure of coherence for convex combination of arbitrary n pure states of a quantum state ρ. Furthermore, we present triangle-like inequality for the convex-roof extended negativity for any states of rank 2, which gives a positive answer to a conjecture raised in [Phys. Rev. A 96, 062308 (2017)]. Detailed examples are given to illustrate the relations characterized by the triangle-like inequalities.

INTRODUCTION

Quantum coherence and entanglement are the two key features of quantum world. While quantum coherence is defined for single systems, quantum entanglement is adopted to describe the correlations in bipartite or multipartite systems. Recent developments in understanding of quantum coherence have come from the burgeoning field of quantum information science. Just like entanglement, coherence can be treated as a physical resource and has been widely studied. One important pillar of the field is the study on quantification of coherence. Since the seminal work [1] on defining a good coherence measure in terms of the resource theory, quantum coherence has been widely studied and applied to many quantum information processing [1–20].

The relative entropy and l_1-norm coherence measures are two well-known measures of coherence, especially concerning the strong monotonicity property and the closed expressions. In fact, different quantifications of coherence can greatly enrich our understanding of coherence. In particular, the distillable coherence [8, 21], the coherence of formation [8, 21], the robustness of coherence [13], the coherence measures based on entanglement [2], max-relative entropy of coherence [20], and the coherence concurrence [22, 23] have been proposed and investigated. For instance, the relative entropy coherence can be understood as the optimal rate for distilling a maximally coherent state from given states [8]. The max-relative entropy of coherence can exactly characterize the subchannel discrimination problems such that the coherent state allows for a higher probability of successfully discriminating subchannels than that of all incoherent states [20]. The robustness of coherence quantifies the advantage enabled by a quantum state in a phase discrimination task [13]. In addition, the relations between coherence and path information [24–26], the distribution of quantum coherence in multipartite systems [19], the complementarity between coherence and mixedness [27, 28] have also been studied.

In [29], the authors show that if a rank-2 state ρ can be expressed as a convex combination of two pure states, i.e., $\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$, a triangle inequality can be established, $|E(\sqrt{p_1}|\psi_1\rangle) - E(\sqrt{p_2}|\psi_2\rangle)| \leq E(\rho) \leq E(\sqrt{p_1}|\psi_1\rangle) + E(\sqrt{p_2}|\psi_2\rangle)$, where E can be either the measures of coherence or the entanglement concurrence. In this paper, we give a revise to the inequalities to make it a suitable measure. We provide detailed proofs of triangle-like inequalities in coherence measures and entanglement negativity. A general triangle-like

inequality for any convex combination of n states ρ based on the l_1 norm coherence measure is given. Furthermore, we provide entanglement negativity satisfied the triangle-like inequality for any states of rank 2, which gives a positive answer to a conjecture of the convex-roof extended negativity satisfied the triangle-like inequality [29]. At last, we give an example of entanglement negativity in a two-qubit system.

TRIANGLE-LIKE INEQUALITIES FOR MEASURES OF COHERENCE

A widely used measure of coherence is the distance-based measure [1], defined by the minimal distance between a given state and the set of incoherent quantum states $\mathcal{I}$. The incoherent states are diagonal ones in the reference basis $\{|i\rangle\}$ of a d-dimensional Hilbert space, $\delta = \sum_{i=1}^{d} \delta_i |i\rangle\langle i|$. A measure of coherence for a state ρ can be defined by

$$C_D(\rho) = \min_{\delta \in \mathcal{I}} D(\rho, \delta), \tag{1}$$

where $D(\rho, \delta)$ denotes certain distance measures of quantum states.

Typical distance-based measures of coherence are the relative entropy, the l_1-norm and the trace norm [1]. The l_1-norm measure of coherence for a state ρ is given by

$$C_{l_1}(\rho) = \min_{\delta \in \mathcal{I}} ||\rho - \delta||_{l_1} = \sum_{i \neq j} |\rho_{ij}|, \tag{2}$$

which is equal to sum of the absolute values of all off-diagonal elements of ρ.

For mixed states, the convex-roof l_1-norm is adopted as a different measure of coherence [22]. The convex-roof l_1-norm of a mixed state ρ is given by

$$\tilde{C}_{l_1}(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_{l_1}(|\psi_i\rangle), \tag{3}$$

where the minimization is taken over all pure state decompositions of $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $\sum_i p_i = 1$, $C_{l_1}(|\psi_i\rangle)$ is the l_1-norm of the state $|\psi_i\rangle\langle\psi_i|$.

We begin with a general triangle-like inequality based on the l_1 norm measure of coherence.

[**Theorem 1**]. If a state ρ can be expressed as a convex combination of n $(n \geq 2)$ states $\rho = \sum_{i=1}^{n} p_i \rho_i$, $C_{l_1}(\rho)$ satisfies the following triangle-like inequality,

$$\frac{1}{n} \sum_{k=1}^{n} \left| G_k^{(n-1)} - p_k C_{l_1}(\rho_k) \right| \leq C_{l_1}(\rho) \leq \sum_i p_i C_{l_1}(\rho_i), \tag{4}$$

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where $G_k^{(n-1)}$, $k = 1, 2, \cdots, n$, are the low bounds of $\sum_{i \neq k}^{n} p_i C_{l_1} \left(\frac{\sum_{j \neq k}^{n} p_j \rho_j}{\sum_{t \neq k}^{n} p_t} \right)$.

[Proof]. First we consider the case of $n = 2$, i.e. $\rho = p_1 \rho_1 + p_2 \rho_2$. Then we have

$$
\begin{aligned}
C_{l_1}(\rho) &= C_{l_1}(p_1 \rho_1 + p_2 \rho_2) \\
&= \sum_{i \neq j} |p_1 \rho_{1ij} + p_2 \rho_{2ij}| \\
&\geq \left| \sum_{i \neq j} p_1 |\rho_{1ij}| - \sum_{i \neq j} p_2 |\rho_{2ij}| \right| \\
&= \left| p_1 C_{l_1}(\rho_1) - p_2 C_{l_1}(\rho_2) \right|,
\end{aligned}
\tag{5}
$$

where $G_1^{(1)} = p_2 C_{l_1}(\rho_2)$, $G_2^{(1)} = p_1 C_{l_1}(\rho_1)$. Hence (5) is a special case of the left hand of (4).

Next, we consider the case of $n = 3$, $\rho = \sum_{i=1}^{3} p_i \rho_i$. From (5), we get

$$
\begin{aligned}
C_{l_1}(\rho) &\geq \left| \sum_{j \neq i}^{3} p_j C_{l_1} \left(\frac{\sum_{t \neq i}^{3} p_t \rho_t}{\sum_{k \neq i}^{3} p_k} \right) - p_i C_{l_1}(\rho_i) \right| \\
&\geq \left| \left| p_j C_{l_1}(\rho_j) - p_k C_{l_1}(\rho_k) \right| - p_i C_{l_1}(\rho_i) \right|.
\end{aligned}
\tag{6}
$$

Summing over all the i, j, k in (6), we have

$$
C_{l_1}(\rho) \geq \frac{1}{3} \sum_{i \neq \{j,k\}, j < k} \left| \left| p_j C_{l_1}(\rho_j) - p_k C_{l_1}(\rho_k) \right| - p_i C_{l_1}(\rho_i) \right|,
\tag{7}
$$

where $G_k^{(2)} = \left| p_i C_{l_1}(\rho_i) - p_j C_{l_1}(\rho_j) \right|$, $i < j$, $k \neq \{i, j\}$, $i, j, k \in \{1, 2, 3\}$. Thus (7) can be rewritten as $C_{l_1}(\rho) \geq \frac{1}{3} \sum_{k=1}^{3} \left| G_k^{(2)} - p_k C_{l_1}(\rho_k) \right|$.

(4) reduces to (7) for $n = 3$. Now suppose That Theorem 1 holds for $n = m$. Consider the case of $n = m + 1$, $\rho = \sum_{i=1}^{m+1} p_i \rho_i$, we have

$$
\begin{aligned}
C_{l_1}(\rho) &= C_{l_1} \left(\sum_{i \neq k}^{m+1} p_i \rho_i + p_k \rho_k \right) \\
&\geq \left| \sum_{i \neq k}^{m+1} p_i C_{l_1} \left(\frac{\sum_{j \neq k}^{m+1} p_j \rho_j}{\sum_{t \neq k}^{m+1} p_t} \right) - p_k C_{l_1}(\rho_k) \right| \\
&\geq \left| G_k^{(m)} - p_k C_{l_1}(\rho_k) \right|.
\end{aligned}
\tag{8}
$$

Summing over all the k in (8), we have

$$
C_{l_1}(\rho) \geq \frac{1}{m+1} \sum_{k=1}^{m+1} \left| G_k^{(m)} - p_k C_{l_1}(\rho_k) \right|.
\tag{9}
$$

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Therefore, the left inequality of (4) is proved. The right inequality in (4) is obvious due to the convex property of the l_1-norm coherence. $\square$

Note that here the decomposed states $\rho_1, \cdots, \rho_n$ can be either pure or mixed states. For pure state decompositions, we have

[**Theorem 2**]. If ρ has a linearly independent pure state decomposition $|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_n\rangle$, $\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|$, then the convex-roof l_1-norm coherence of multiqubit state ρ satisfies the following triangle-like inequality:

$$\frac{1}{n} \sum_{k=1}^{n} \left| \tilde{G}_k^{(n-1)} - p_k \tilde{C}_{l_1}(|\psi_k\rangle) \right| \leq \tilde{C}_{l_1}(\rho) \leq \sum_i p_i \tilde{C}_{l_1}(|\psi_i\rangle), \tag{10}$$

where $\tilde{C}_{l_1}(|\psi_i\rangle) = C_{l_1}(|\psi_i\rangle)$ is the convex-roof l_1-norm coherence of pure states $|\psi_i\rangle$ $i = 1, 2, \cdots, n$, $\tilde{G}_k^{(n-1)}$, $k = 1, 2, \cdots, n$, are the low bounds of $\sum_{i \neq k}^{n} p_i \tilde{C}_{l_1} \left(\frac{\sum_{j \neq k}^{n} p_j \rho_j}{\sum_{t \neq k}^{n} p_t} \right)$.

[Proof]. The second inequality of (10) is due to that the convex-roof l_1-norm coherence $\tilde{C}_{l_1}(\rho)$ is a sum of the minimal decomposition of ρ, while $\sum_i p_i C_{l_1}(|\psi_i\rangle)$ can be regarded as a sum of a general decomposition of ρ.

Assume that $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$ is the optimal decomposition of $\tilde{C}_{l_1}(\rho)$. We have

$$\begin{aligned} \tilde{C}_{l_1}(\rho) &= \sum_k p_k C_{l_1}(|\psi_k\rangle) \\ &= \sum_k p_k \sum_{i \neq j} \left| \langle i|\psi_k\rangle\langle\psi_k|j\rangle \right| \\ &\geq \sum_{i \neq j} \left| \langle i| \sum_k p_k |\psi_k\rangle\langle\psi_k|j\rangle \right| \\ &= \sum_{i \neq j} |\rho_{ij}| = C_{l_1}(\rho). \end{aligned} \tag{11}$$

From Theorem 1, we obtain (10), which completes the proof. $\square$

Remark 1. Theorem 1 is the main result in [29], where $n = 2$. There is a potential question of whether other quantum measures also satisfy triangle inequalities. One may also consider other measures of coherence, and entanglement measures such as relative entropy and the entanglement of formation. However, these measures are usually difficult to calculate for given states in general. Hence, the triangle-like inequalities associated with these measures are still open problems.

TRIANGLE-LIKE INEQUALITIES IN ENTANGLEMENT NEGATIVITY

Concurrence is a well-known measure of entanglement [30–34]. For a general bipartite pure state $|\psi\rangle_{AB}$ in $H_A \otimes H_B$, the concurrence is defined by [33, 35, 36], $C(|\psi\rangle_{AB}) = \sqrt{2[1 - \mathrm{Tr}(\rho_A^2)]}$, with ρ_A the reduced density matrix by tracing over the subsystem B, $\rho_A = \mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$. The concurrence for a bipartite mixed state ρ_{AB} is defined by the convex-roof extension $C(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$, where the minimum is taken over all possible decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $p_i \geq 0$ and $\sum_i p_i = 1$ and $|\psi_i\rangle \in H_A \otimes H_B$.

Another well-known measure of bipartite entanglement is the negativity. Given a bipartite state ρ_{AB} in $H_A \otimes H_B$, the negativity is defined by [37], $N(\rho_{AB}) = (||\rho_{AB}^{T_A}|| - 1)/2$, where $\rho_{AB}^{T_A}$ is the partial transpose of ρ_{AB} with respect to the subsystem A, $||X||$ denotes the trace norm of X, $||X|| = \mathrm{Tr}\sqrt{XX^\dagger}$. Negativity is a computable measure of entanglement, and is a convex function of ρ_{AB}. It vanishes if and only if ρ_{AB} is separable for the $2 \otimes 2$ and $2 \otimes 3$ systems [38]. For simplicity, we use the following definition of negativity, $N(\rho_{AB}) = ||\rho_{AB}^{T_A}|| - 1$. For a mixed state ρ_{AB}, the convex-roof extended negativity (CREN) is defined by

$$N_c(\rho_{AB}) = \min \sum_i p_i N(|\psi_i\rangle_{AB}),$$

where the minimum is taken over all possible pure state decompositions $\{p_i,\ |\psi_i\rangle_{AB}\}$ of ρ_{AB}. CREN gives a perfect discrimination of positive partial transposed bound entangled states and separable states in any bipartite quantum systems [39, 40].

For any bipartite pure state $|\psi\rangle_{AB}$ in a $d \otimes d$ quantum system with Schmidt rank 2, $|\psi\rangle_{AB} = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle$, one has $N(|\psi\rangle_{AB}) = || |\psi\rangle\langle\psi|^{T_B} || - 1 = 2\sqrt{\lambda_0\lambda_1} = \sqrt{2(1 - \mathrm{Tr}\rho_A^2)} = C(|\psi\rangle_{AB})$. Namely, the negativity is equivalent to the concurrence for any pure state with Schmidt rank 2. Consequently it follows that for any two-qubit mixed state $\rho_{AB} = \sum p_i |\psi_i\rangle_{AB}\langle\psi_i|$,

$$N_c(\rho_{AB}) = \min \sum_i p_i N(|\psi_i\rangle_{AB}) = \min \sum_i p_i C(|\psi_i\rangle_{AB}) = C(\rho_{AB}), \tag{12}$$

For a general bipartite pure state $\rho_{AB} = |\psi\rangle\langle\psi|$, the concurrence can be written as [41], $C(|\psi\rangle) = 2\sqrt{\sum_{i<j}^{d_1(d_1-1)/2} \sum_{k<l}^{d_1(d_1-1)/2} |a_{ik}a_{jl} - a_{il}a_{jk}|^2}$, with $|\psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} a_{ij}|ij\rangle$. It is easy to obtain $p_i C(|\psi_i\rangle) = C(\sqrt{p_i}|\psi_i\rangle)$, $i = 1, 2$. From the result of Theorem 3 in [29], one has,

$$\left| p_1 C(|\psi_1\rangle) - p_2 C(|\psi_2\rangle) \right| \leq C(\rho) \leq p_1 C(|\psi_1\rangle) + p_2 C(|\psi_2\rangle), \tag{13}$$

where $\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$ is a two-qubit mixed state.

Combining (12) and (13), we have

$$\left| p_1 N_c(|\psi_1\rangle) - p_2 N_c(|\psi_2\rangle) \right| \leq N_c(\rho) \leq p_1 N_c(|\psi_1\rangle) + p_2 N_c(|\psi_2\rangle), \tag{14}$$

and with a similiar method of Theorem 1, we generalize (14) to multiqubit states.

[**Theorem 3**]. If a multiqubit state ρ can be expressed as a convex combination of n ($n \geq 2$) states $\rho = \sum_{i=1}^{n} p_i \rho_i$, with $\rho_i = |\psi_i\rangle\langle\psi_i|$, $i = 1, 2, \cdots, n$, then the negativity $N_c(\rho)$ satisfies the triangle-like inequality:

$$\frac{1}{n}\sum_{k=1}^{n}\left| H_k^{(n-1)} - p_k N_c(\rho_k) \right| \leq N_c(\rho) \leq \sum_i p_i N_c(\rho_i), \tag{15}$$

where $H_k^{(n-1)}$, $k = 1, 2, \cdots, n$, are the low bounds of $\sum_{i\neq k}^{n} p_i N_c \left(\frac{\sum_{j\neq k}^{n} p_j \rho_j}{\sum_{t\neq k}^{n} p_t} \right)$, $H_1^{(1)} = p_2 N_c(\rho_2)$, $H_2^{(1)} = p_1 N_c(\rho_1)$.

[Proof]. First, we consider the case of $n = 3$, $\rho = \sum_{i=1}^{3} p_i \rho_i$. From Eq. (14), we get

$$\begin{aligned} N_c(\rho) &\geq \left| \sum_{j\neq i}^{3} p_j N_c \left(\frac{\sum_{t\neq i}^{3} p_t \rho_t}{\sum_{k\neq i}^{3} p_k} \right) - p_i N_c(\rho_i) \right| \\ &\geq \left| \left| p_j N_c(\rho_j) - p_k N_c(\rho_k) \right| - p_i N_c(\rho_i) \right|. \end{aligned} \tag{16}$$

Summing over all the i, j, k in (16), we have

$$N_c(\rho) \geq \frac{1}{3} \sum_{i\neq\{j,k\}, j<k} \left| \left| p_j N_c(\rho_j) - p_k N_c(\rho_k) \right| - p_i N_c(\rho_i) \right|, \tag{17}$$

where $H_k^{(2)} = \left| p_i N_c(\rho_i) - p_j N_c(\rho_j) \right|$, $i \leq j$, $k \neq \{i,j\}$, $i, j, k \in \{1, 2, 3\}$. Then (17) can be rewritten as $N_c(\rho) \geq \frac{1}{3}\sum_{k=1}^{3} \left| H_k^{(2)} - p_k N_c(\rho_k) \right|$.

(15) reduces to (17) for $n = 3$. Suppose that Theorem 3 holds for $n = m$. Consider the state $\rho = \sum_{i=1}^{m+1} p_i \rho_i$, we have

$$\begin{aligned} N_c(\rho) &= N_c \left(\sum_{i\neq k}^{m+1} p_i \rho_i + p_k \rho_k \right) \\ &\geq \left| \sum_{i\neq k}^{m+1} p_i N_c \left(\frac{\sum_{j\neq k}^{m+1} p_j \rho_j}{\sum_{t\neq k}^{m+1} p_t} \right) - p_k N_c(\rho_k) \right| \\ &\geq \left| H_k^{(m)} - p_k N_c(\rho_k) \right|. \end{aligned} \tag{18}$$
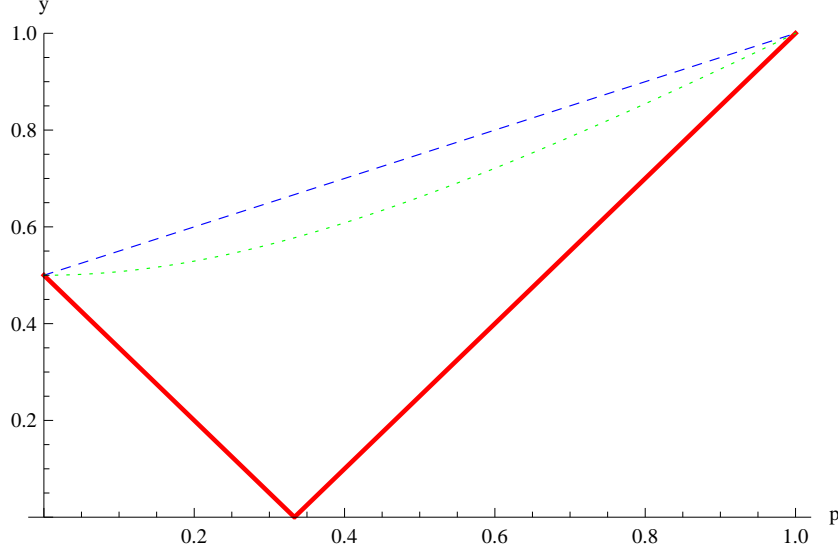
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FIG. 1: Solid (red) line denotes the value of $\left| pN_c(|\psi_1\rangle) - (1-p)N_c(|\psi_2\rangle) \right|$ for given p, dashed (blue) line for $pN_c(|\psi_1\rangle) + (1-p)N_c(|\psi_2\rangle)$, and dotted (green) line for $N_c(\rho)$.

Summing over all the k in (18), we have

$$N_c(\rho) \quad \geq \quad \frac{1}{m+1} \sum_{k=1}^{m+1} \left| H_k^{(m)} - p_k N_c(\rho_k) \right|. \tag{19}$$

Therefore, the left inequality of (15) is proved. The right inequality in (15) is obvious due to the convex property of the negativity. □

Remark 2. Theorem 3 can be used to detect the entanglement of a multiqubit mixed state. If the product of the probability distribution of $|\psi_i\rangle$ and its corresponding negativity is different from that of $|\psi_j\rangle$, ρ must be entangled. If all $|\psi_i\rangle$, $i = 1, 2, \cdots, n$, are partial transpose positive states, ρ must be separable. In the following example, one can find that for $p = \frac{1}{3}$, the left hand side of Theorem 3 is 0, but the state ρ is entangled.

Example. Consider a two-qubit mixed state $\rho = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2|$, where $|\psi_1\rangle = \sqrt{\frac{3}{8}}(|00\rangle + |11\rangle) + \sqrt{\frac{1}{8}}i(|01\rangle + |10\rangle)$, $|\psi_2\rangle = \sqrt{\frac{3}{8}}(|00\rangle + |11\rangle) + \sqrt{\frac{1}{8}}(|01\rangle + |10\rangle)$, $i = \sqrt{-1}$. From the definition of negativity, we have $N_c(|\psi_1\rangle) = 1$, $N_c(|\psi_2\rangle) = \frac{1}{2}$ and $N_c(\rho) = \frac{1}{2}\sqrt{1 + 3p^2}$. The relations among $\left| pN_c(|\psi_1\rangle) - (1-p)N_c(|\psi_2\rangle) \right|$, $N_c(\rho)$ and $pN_c(|\psi_1\rangle) + (1-p)N_c(|\psi_2\rangle)$ are shown in Fig. 1.

In fact, the bounds of $N_c(\rho)$ in (15) depends on the pure state decompositions of ρ. Consider a new decomposition $\rho = p|\psi_1'\rangle\langle\psi_1'| + (1-p)|\psi_2'\rangle\langle\psi_2'|$, where $(\sqrt{p}|\psi_1'\rangle, \sqrt{1-p}|\psi_2'\rangle)^T =$

$U(\sqrt{p}|\psi_1\rangle, \sqrt{1-p}|\psi_2\rangle)^T,$

$$U = \begin{pmatrix} \cos\alpha e^{i\beta} & \sin\alpha e^{i\gamma} \\ -\sin\alpha e^{-i\gamma} & \cos\alpha e^{-i\beta} \end{pmatrix}.$$

The relation still holds,

$$\left| p_1 N_c(|\psi_1'\rangle) - p_2 N_c(|\psi_2'\rangle) \right| \leq N_c(\rho) \leq p_1 N_c(|\psi_1'\rangle) + p_2 N_c(|\psi_2'\rangle).$$

In particular, under all possible unitary transformations U, the largest upper bound of $N_c(\rho)$ is just the convex-roof extended negativity of assistance [42], $N_a(\rho) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i N(|\psi_i\rangle)$.

CONCLUSION

We have provided a general triangle-like inequality for any n pure states combinations of ρ, based on the l_1-norm coherence measure. Furthermore, we have presented the triangle-like inequality satisfied by the convex-roof extended negativity for any states of rank 2, giving a positive answer to the conjecture raised in [29]. By a detailed example of a two-qubit state, the relations characterized by the triangle-like inequalities have been explicitly displayed. These results may highlight the further investigations on quantum coherence, quantum entanglement and even other quantum correlations related to steerability and non-locality.

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