

# Polygamy relation for the Rényi- $\alpha$ entanglement of assistance in multi-qubit systems

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We prove a new polygamy relation of multi-party quantum entanglement in terms of Rényi- $\alpha$  entanglement of assistance for  $(\sqrt{7}-1)/2 \leq \alpha \leq (\sqrt{13}-1)/2$ . This class of polygamy inequality reduces to the polygamy inequality based on entanglement of assistance since Rényi- $\alpha$  entanglement is a generalization of entanglement of formation. We further show that the polygamy inequality also holds for the  $\mu$ th power of Rényi- $\alpha$  entanglement of assistance.

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One fundamental property of quantum entanglement is in its limited shareability in multi-party quantum systems[1]. For example, if the two subsystems are more entangled with each other, then they will share a less amount of entanglement with the other subsystems with specific entanglement measures. This restricted shareability of entanglement is named as the monogamy of entanglement (MoE). The concept of monogamy is an essential feature allowing for security in quantum key distribution[3]. It also plays an important role in many field of physics such as foundations of quantum mechanics[4–6], condensed matter physics[7, 8], statistical mechanics[4], and even black-hole physics[9, 10]. Monogamy inequality was first built for three-qubit systems using tangle as the bipartite entanglement measure[2], and generalized into multi-qubit systems in terms of various entanglement measures[11–42].

On the other hand, the assisted entanglement, which is a dual concept to bipartite entanglement measures, is known to have a dually monogamous or polygamous property in multi-party quantum systems. The polygamous property can be regarded as another kind of entanglement constraints in multi-qubit systems, and Gour *et al*[43] established the first dual monogamy inequality or polygamy inequality for multi-qubit systems using concurrence of assistance (CoA). For a three-qubit pure state  $|\psi\rangle_{A_1A_2A_3}$ , a polygamy inequality was introduced as:

$$C^2\left(|\psi\rangle_{A_1|A_2A_3}\right) \leq [C^a(\rho_{A_1A_2})]^2 + [C^a(\rho_{A_1A_3})]^2, \quad (1)$$

where CoA for a bipartite state  $\rho_{AB}$  is defined as:  $C^a(\rho_{AB}) = \max \sum_i p_i C(|\psi_i\rangle_{AB})$ , with the maximum is taken over all possible pure state decompositions of

$\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$  and  $C(|\psi_i\rangle_{AB})$  denotes the concurrence[44] of  $|\psi_i\rangle_{AB}$ . Furthermore, it is shown that for any pure state  $|\psi\rangle_{A_1A_2\dots A_n}$  in a  $n$ -qubit system[45], we have

$$C^2\left(|\psi\rangle_{A_1|A_2\dots A_n}\right) \leq [C^a(\rho_{A_1A_2})]^2 + \dots + [C^a(\rho_{A_1A_n})]^2. \quad (2)$$

Later, polygamy inequalities was generalized in terms of Tsallis entanglement of assistance(TEoA)[46] or unified entanglement of assistance[47], and polygamy inequalities in higher-dimensional systems were also shown using the entanglement of assistance(EoA)[48] or TEoA[49]. In this paper, we establish a new polygamy relation of multi-party quantum entanglement in terms of Rényi- $\alpha$  entropy (ER $\alpha$ E)[31]. As an important generalization of entanglement of formation(EoF), ER $\alpha$ E is a well-defined entanglement measure which has a continuous spectrum parametrized by the non-negative real parameter  $\alpha$ . It reduces to the standard EoF when  $\alpha$  tends to 1. Thus our polygamy inequalities including previous polygamy relation of EoF as a special case[48]. Furthermore, we generalize the polygamy inequalities in terms of the  $\mu$ th power of Rényi- $\alpha$  entanglement of assistance.

For a bipartite pure state  $|\psi\rangle_{AB}$ , the ER $\alpha$ E is defined as

$$E_\alpha(|\psi\rangle_{AB}) := S_\alpha(\rho_A) := \frac{1}{1-\alpha} \log(\text{tr}\rho_A^\alpha), \quad (3)$$

where  $S_\alpha(\rho_A)$  is the Rényi- $\alpha$  entropy. The Rényi- $\alpha$  entropy has found important applications in characterizing quantum phases with differing computational power [50], ground state properties in many-body systems [51], and topologically ordered states [52, 53]. The ER $\alpha$ E of a bipartite mixed state  $\rho_{AB}$  can be defined using the convex roof technique

$$E_\alpha(\rho_{AB}) = \min \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}). \quad (4)$$

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It is known that Rényi- $\alpha$  entropy converges to the von Neumann entropy when  $\alpha$  tends to 1. So the entanglement Rényi- $\alpha$  entropy reduces to the EoF when  $\alpha$  tends to 1. For any two-qubit state  $\rho_{AB}$  with  $\alpha \geq (\sqrt{7} - 1)/2$ , there exist an analytic formula of ER $\alpha$ E[31, 54]

$$E_\alpha(\rho_{AB}) = f_\alpha(C(\rho_{AB})), \quad (5)$$

where

$$f_\alpha(x) = \frac{1}{1-\alpha} \log \left[ \left( \frac{\Theta(x)}{2} \right)^\alpha + \left( \frac{\Xi(x)}{2} \right)^\alpha \right], \quad (6)$$

with  $\Theta(x) = 1 + \sqrt{1-x^2}$ ,  $\Xi(x) = 1 - \sqrt{1-x^2}$ .

As a dual concept to ER $\alpha$ E, we define the Rényi- $\alpha$  entanglement of assistance(REoA) as

$$E_\alpha^a(\rho_{AB}) := \max \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}), \quad (7)$$

where the maximum is taken over all possible pure state decompositions of  $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$ .

For  $0 < \alpha < 1$ , we can derive a upper bound of REoA. From the definition of entanglement of REoA, we have

$$\begin{aligned} E_\alpha^a(\rho_{AB}) &= \max \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}) \\ &= \max \sum_i p_i S_\alpha(\rho_{iA}) \\ &\leq S_\alpha\left(\sum_i p_i \rho_{iA}\right) \\ &= S_\alpha(\rho_A), \end{aligned} \quad (8)$$

where  $\rho_{iA}$  is the reduced density matrix of  $|\psi_i\rangle_{AB}$ , and the inequality holds due to the concave property of  $S_\alpha(\rho)$  for  $0 < \alpha < 1$ [55–57]. Similarly, we can derive  $E_\alpha^a(\rho_{AB}) \leq S_\alpha(\rho_B)$ . Thus we have  $E_\alpha^a(\rho_{AB}) \leq \min\{S_\alpha(\rho_A), S_\alpha(\rho_B)\}$

Before showing the main result of this paper, we first give two lemmas as follows.

*Lemma 1.* For any two-qubit state  $\rho_{AB}$  and  $\alpha \geq (\sqrt{7} - 1)/2$ , we have

$$E_\alpha^a(\rho_{AB}) \geq f_\alpha(C^a(\rho_{AB})), \quad (9)$$

where  $E_\alpha^a(\rho_{AB})$  and  $C^a(\rho_{AB})$  are the REoA and CoA of  $\rho_{AB}$ , respectively.

*Proof.* Suppose that the optimal decomposition for  $C^a(\rho_{AB})$  is  $\{p_i, |\psi_i\rangle_{AB}\}$ , we have

$$\begin{aligned} f_\alpha(C^a(\rho_{AB})) &= f_\alpha\left(\sum_i p_i C(|\psi_i\rangle_{AB})\right) \\ &\leq \sum_i p_i f_\alpha(C(|\psi_i\rangle_{AB})) \\ &= \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}) \leq E_\alpha^a(\rho_{AB}) \end{aligned} \quad (10)$$

where in the first inequality we have used the convex property of  $f_\alpha(x)$  as a function of  $x$  for  $\alpha \geq (\sqrt{7} - 1)/2$ , and the second inequality is due to the definition of EoA.  $\square$

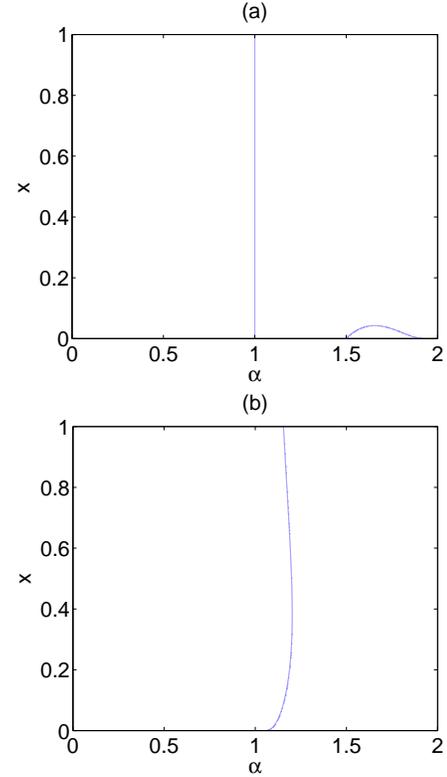


FIG. 1: (color online) The plots of the equations(a) $\partial h_\alpha/\partial x = 0$ ; (b) $\partial h_\alpha/\partial \alpha = 0$  for  $0 \leq x \leq 1, 0 \leq \alpha \leq 2$ .

*Lemma 2.* For any  $(\sqrt{7} - 1)/2 \leq \alpha \leq (\sqrt{13} - 1)/2$  and the function  $f_\alpha(x)$  defined on the domain  $\mathcal{D} = \{(x, y) | 0 \leq x, y \leq 1, 0 \leq x^2 + y^2 \leq 1\}$ , we have

$$f_\alpha(\sqrt{x^2 + y^2}) \leq f_\alpha(x) + f_\alpha(y). \quad (11)$$

*Proof.* We define a two-variable function

$$g_\alpha(x, y) = f_\alpha(\sqrt{x^2 + y^2}) - f_\alpha(x) - f_\alpha(y), \quad (12)$$

on the domain  $\mathcal{D}$ . Then it is sufficient to show that  $g_\alpha(x, y)$  is a non-negative function on  $\mathcal{D}$ . Since  $g_\alpha(x, y)$  is analytic in the interior of  $\mathcal{D}$ , and continuous on  $\mathcal{D}$ , its maximum or minimum value arises only on the critical points or on the boundary of  $\mathcal{D}$ . The critical points of  $g_\alpha(x, y)$  satisfy the condition

$$\nabla g_\alpha(x, y) = \left( \frac{\partial g_\alpha(x, y)}{\partial x}, \frac{\partial g_\alpha(x, y)}{\partial y} \right) = (0, 0), \quad (13)$$

where

$$\begin{aligned} &\frac{\partial g_\alpha(x, y)}{\partial x} \\ &= \frac{Cx \left[ \left( \Theta(\sqrt{x^2 + y^2}) \right)^{\alpha-1} - \left( \Xi(\sqrt{x^2 + y^2}) \right)^{\alpha-1} \right]}{\sqrt{1-x^2-y^2} \left[ \left( \Xi(\sqrt{x^2 + y^2}) \right)^\alpha + \left( \Theta(\sqrt{x^2 + y^2}) \right)^\alpha \right]} \\ &= \frac{Cx \left[ \left( \Theta(x) \right)^{\alpha-1} - \left( \Xi(x) \right)^{\alpha-1} \right]}{\sqrt{1-x^2} \left[ \left( \Xi(x) \right)^\alpha + \left( \Theta(x) \right)^\alpha \right]}, \end{aligned} \quad (14)$$

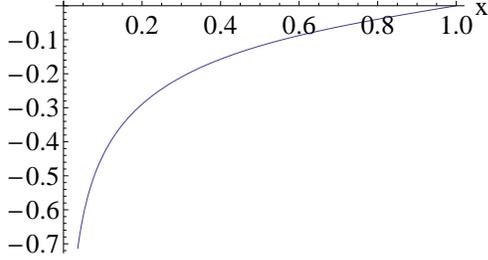


FIG. 2: (color online). The plot of  $h_\alpha(x)|_{\alpha=(\sqrt{13}-1)/2}$  as a function of  $x$  for  $0 \leq x \leq 1$ .

and

$$\begin{aligned} & \frac{\partial g_\alpha(x, y)}{\partial y} \\ &= \frac{Cy \left[ \left( \Theta(\sqrt{x^2 + y^2}) \right)^{\alpha-1} - \left( \Xi(\sqrt{x^2 + y^2}) \right)^{\alpha-1} \right]}{\sqrt{1-x^2-y^2} \left[ \left( \Xi(\sqrt{x^2 + y^2}) \right)^\alpha + \left( \Theta(\sqrt{x^2 + y^2}) \right)^\alpha \right]} \\ &- \frac{Cy \left[ \left( \Theta(y) \right)^{\alpha-1} - \left( \Xi(y) \right)^{\alpha-1} \right]}{\sqrt{1-y^2} \left[ \left( \Xi(y) \right)^\alpha + \left( \Theta(y) \right)^\alpha \right]}. \end{aligned} \quad (15)$$

Suppose that there exists  $(x_0, y_0)$  in the interior of  $\mathcal{D}$  such that  $\nabla g_\alpha(x_0, y_0) = (0, 0)$ . From Eq.(14) and Eq.(15), we have

$$l_\alpha(x_0) = l_\alpha(y_0), \quad (16)$$

where  $l_\alpha(x)$  is defined as

$$l_\alpha(x) := \frac{\left[ \left( \Theta(x) \right)^{\alpha-1} - \left( \Xi(x) \right)^{\alpha-1} \right]}{\sqrt{1-x^2} \left[ \left( \Xi(x) \right)^\alpha + \left( \Theta(x) \right)^\alpha \right]}, \quad (17)$$

for  $0 < x < 1$ . We divide the proof into two cases. We first show that  $l_\alpha(x)$  is a strictly monotone-decreasing function for  $0 < x < 1, 1 < \alpha < (\sqrt{13}-1)/2$ , then it is sufficient to consider the first-order derivative of  $l_\alpha(x)$ . After a direct calculation, we have

$$\begin{aligned} \frac{dl_\alpha(x)}{dx} &= \frac{\alpha x \left[ \left( \Theta(x) \right)^{\alpha-1} - \left( \Xi(x) \right)^{\alpha-1} \right]^2}{(1-x^2) \left[ \left( \Xi(x) \right)^\alpha + \left( \Theta(x) \right)^\alpha \right]} \\ &- \frac{(\alpha-1)x \left[ \left( \Theta(x) \right)^{\alpha-2} + \left( \Xi(x) \right)^{\alpha-2} \right]}{(1-x^2) \left[ \left( \Xi(x) \right)^\alpha + \left( \Theta(x) \right)^\alpha \right]} \\ &+ \frac{x \left[ \left( \Theta(x) \right)^{\alpha-1} - \left( \Xi(x) \right)^{\alpha-1} \right]}{\sqrt{(1-x^2)^3} \left[ \left( \Xi(x) \right)^\alpha + \left( \Theta(x) \right)^\alpha \right]}. \end{aligned} \quad (18)$$

In order to show the negativity of the first-order derivative of  $l_\alpha(x)$ , let us consider the value of the two-variable function  $h_\alpha(x) := dl_\alpha(x)/dx$  on the domain

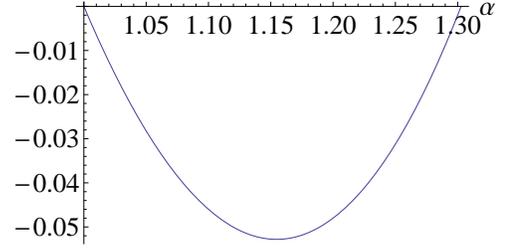


FIG. 3: (color online). The plot of  $h_\alpha(x)|_{x \rightarrow 1}$  as a function of  $\alpha$  for  $1 \leq \alpha \leq (\sqrt{13}-1)/2$ .

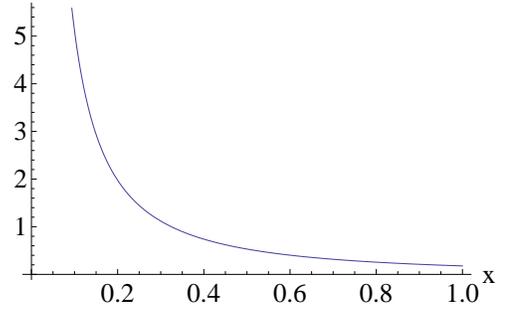


FIG. 4: (color online). The plot of  $h_\alpha(x)|_{\alpha=(\sqrt{7}-1)/2}$  as a function of  $x$  for  $0 \leq x \leq 1$ .

$\mathcal{D}_1 = \{(\alpha, x) | 1 \leq \alpha \leq (\sqrt{13}-1)/2, 0 \leq x \leq 1\}$ . The maximum or minimum values of  $h_\alpha(x)$  can arise only at the critical points or on the boundary of  $\mathcal{D}_1$ . The critical points of  $h_\alpha(x)$  satisfy the condition  $\nabla h_\alpha(x) = (\partial h_\alpha(x)/\partial \alpha, \partial h_\alpha(x)/\partial x) = (0, 0)$ . It is shown in Fig.1(a) and (b) that there are no common solutions on the interior of domain  $\mathcal{D}_1$  which indicate that  $h_\alpha(x)$  has no critical points on the interior of  $\mathcal{D}_1$ . Then we consider the function value of  $h_\alpha(x)$  on the boundary of  $\mathcal{D}_1$ . If  $\alpha = 1$ , we have  $h_\alpha(x)|_{\alpha=1} = 0$ . If  $\alpha = (\sqrt{13}-1)/2$ , we plot  $h_\alpha(x)|_{\alpha=(\sqrt{13}-1)/2}$  as a function of  $x$  in Fig.2, which illustrates that  $h_\alpha(x)|_{\alpha=(\sqrt{13}-1)/2}$  is a monotone-increasing function for  $0 \leq x \leq 1$  and obtain its maximum value 0 on  $x = 1$ . When  $x \rightarrow 1$ , we have  $h_\alpha(x)|_{x \rightarrow 1} = 2(q^3 - 4q + 3)/3$  which is always negative for  $1 < \alpha < (\sqrt{13}-1)/2$  as shown in Fig.3. Thus we have shown that  $h_\alpha(x)$  is always negative on the interior of domain  $\mathcal{D}_1$  which indicate that  $l_\alpha(x)$  is a strictly monotone-decreasing function for  $0 < x < 1, 1 < \alpha < (\sqrt{13}-1)/2$ . Similarly, we can show that  $l_\alpha(x)$  is a strictly monotone-increasing function for  $(\sqrt{7}-1) < \alpha < 1$ . In this case, it is enough to prove the non-negative of the function  $h_\alpha(x) := dl_\alpha(x)/dx$  on the domain  $\mathcal{D}_2 = \{(\alpha, x) | (\sqrt{7}-1)/2 \leq \alpha \leq 1, 0 \leq x \leq 1\}$ . Because  $h_\alpha(x)$  has no critical points on the interior of  $\mathcal{D}_2$  as shown in Fig.1, we consider the function value of  $h_\alpha(x)$  on the boundary of  $\mathcal{D}_2$ . If  $x \rightarrow 1$ , we can verify that the function  $h_\alpha(x)|_{x \rightarrow 1}$  is always positive for  $(\sqrt{7}-1)/2 <$

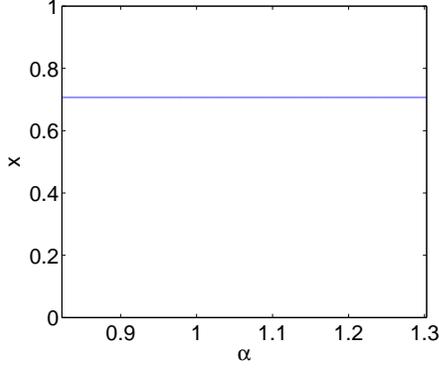


FIG. 5: (color online). The plot of  $\partial m_\alpha(x)/\partial x = 0$  for  $0 \leq x \leq 1$ ,  $(\sqrt{7} - 1)/2 \leq \alpha \leq (\sqrt{13} - 1)/2$ .

$\alpha < 1$ . If  $\alpha = (\sqrt{7} - 1)/2$ , it is shown in Fig.4 that  $h_\alpha(x)|_{\alpha=(\sqrt{7}-1)/2}$  is always positive for  $0 < x < 1$ . Therefore,  $h_\alpha(x)$  is always positive for  $(\sqrt{7} - 1)/2 < \alpha < 1$  which indicates that  $l_\alpha(x)$  is a strictly monotone-increasing function in this case. Combining Eq.(16) we can derive  $x_0 = y_0$ . However, from Eq.(13)-(15) and  $\nabla g_\alpha(x_0, x_0) = (0, 0)$  we have  $l_\alpha(\sqrt{2}x_0) = l_\alpha(x_0)$  which contradicts to the strict monotonicity of  $l_\alpha(x)$  for  $0 < x < 1$ . Therefore, we conclude that  $g_\alpha(x, y)$  has no critical points in the interior of  $\mathcal{D}$ .

Next, we consider the function value of  $g_\alpha(x, y)$  on the boundary of  $\mathcal{D}$ . If  $x = 0$  or  $y = 0$ , it is direct to check that  $g_\alpha(x, y) = 0$ . When  $x^2 + y^2 = 1$ ,  $g_\alpha(x, y)$  becomes a two-variable function

$$m_\alpha(x) := 1 - \frac{1}{1-a} \log \left[ \left( \frac{\Theta(x)}{2} \right)^a + \left( \frac{\Xi(x)}{2} \right)^a \right] - \frac{1}{1-a} \log \left[ \left( \frac{\Theta(\sqrt{1-x^2})}{2} \right)^a + \left( \frac{\Xi(\sqrt{1-x^2})}{2} \right)^a \right] \quad (19)$$

As shown in Fig.5,  $\partial m_\alpha(x)/\partial x = 0$  has only one solution  $x = 1/\sqrt{2}$  on the domain  $\mathcal{D}_3 = \{(\alpha, x) | (\sqrt{7} - 1)/2 \leq \alpha \leq (\sqrt{13} - 1)/2, 0 \leq x \leq 1\}$ . On the other hand, we plot  $\partial m_\alpha(x)/\partial \alpha|_{x=1/\sqrt{2}}$  in Fig.6 and we can see that the function is always positive for  $(\sqrt{7} - 1)/2 \leq \alpha \leq (\sqrt{13} - 1)/2$ , which shows that  $m_\alpha(x)$  has no critical points on the interior of domain  $\mathcal{D}_3$ . Then we consider the value of  $m_\alpha(x)$  on the boundary of  $\mathcal{D}_3$ . If  $x = 0$  or  $1$ , we have  $m_\alpha(x) = 0$ . When  $\alpha = (\sqrt{7} - 1)/2$  or  $\alpha = (\sqrt{13} - 1)/2$ , it is direct to check that  $m_\alpha(x)$  is always a non-positive function. In Fig.7 we plot  $m_\alpha(x)$  as a function of  $x$  and  $\alpha$ , which illustrates our result.

Combining the case for  $\alpha = 1$  which has been proved in Ref.[48], we have completed the proof of Lemma 2.  $\square$

Now we can prove the main result of this paper.

*Theorem.* For  $(\sqrt{7} - 1)/2 \leq \alpha \leq (\sqrt{13} - 1)/2$ , and any  $n$ -qubit state  $\rho_{A_1 A_2 \dots A_n}$ , we have

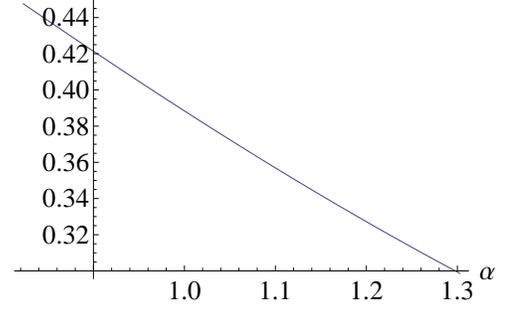


FIG. 6: (color online). The plot of  $\partial m_\alpha(x)/\partial \alpha|_{x=1/\sqrt{2}}$  as a function of  $\alpha$  for  $(\sqrt{7} - 1)/2 \leq \alpha \leq (\sqrt{13} - 1)/2$ .

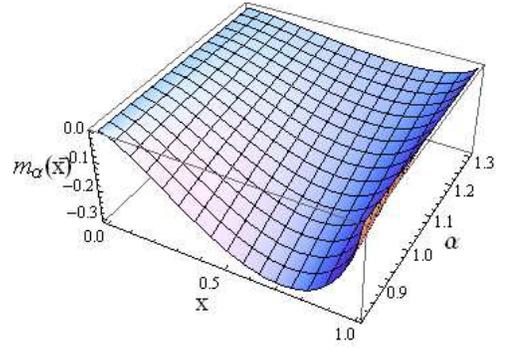


FIG. 7: (color online). The plot of  $m_\alpha(x)$  as a function of  $x$  and  $\alpha$  for  $0 \leq x \leq 1$ ,  $(\sqrt{7} - 1)/2 \leq \alpha \leq (\sqrt{13} - 1)/2$ .

$$E_\alpha^a(\rho_{A_1|A_2 \dots A_n}) \leq E_\alpha^a(\rho_{A_1 A_2}) + \dots + E_\alpha^a(\rho_{A_1 A_n}), \quad (20)$$

where  $E_\alpha^a(\rho_{A_1|A_2 \dots A_n})$  denotes the REoA in the partition  $A_1|A_2 \dots A_n$ , and  $E_\alpha^a(\rho_{A_1 A_i})$  is the REoA of the two-qubit subsystem  $A_1 A_i$  for  $i = 2, \dots, n$ .

*Proof.* We first prove the polygamy relation for the pure state  $|\psi\rangle_{A_1|A_2 \dots A_n}$ . Assuming that  $C^2(\rho_{A_1|A_2 \dots A_n}) \leq [C^a(\rho_{A_1 A_2})]^2 + \dots + [C^a(\rho_{A_1 A_n})]^2 \leq 1$  in Eq.(2), then we have

$$\begin{aligned} E_\alpha \left( |\psi\rangle_{A_1|A_2 \dots A_n} \right) &= f_\alpha \left( C(\rho_{A_1|A_2 \dots A_n}) \right) \\ &\leq f_\alpha \left( \sqrt{[C^a(\rho_{A_1 A_2})]^2 + \dots + [C^a(\rho_{A_1 A_n})]^2} \right) \\ &\leq f_\alpha \left( C^a(\rho_{A_1 A_2}) \right) \\ &+ f_\alpha \left( \sqrt{[C^a(\rho_{A_1 A_3})]^2 + \dots + [C^a(\rho_{A_1 A_n})]^2} \right) \\ &\leq f_\alpha \left( C^a(\rho_{A_1 A_2}) \right) + \dots + f_\alpha \left( C^a(\rho_{A_1 A_n}) \right) \\ &\leq E_\alpha^a(\rho_{A_1 A_2}) + \dots + E_\alpha^a(\rho_{A_1 A_n}), \end{aligned} \quad (21)$$

where in the first inequality we have used the monotonically increasing property of  $f_\alpha(x)$  for  $\alpha \geq (\sqrt{7} - 1)/2$ , the sec-

ond and third inequalities are obtained by the successive application of Lemma 2, and the last inequality is due to Lemma 1.

Then we consider the case  $C^2(\rho_{A_1|A_2\dots A_n}) \leq 1 \leq [C^a(\rho_{A_1A_2})]^2 + \dots + [C^a(\rho_{A_1A_n})]^2$ . There must exist  $k \in \{2, \dots, n-1\}$  such that  $[C^a(\rho_{A_1A_2})]^2 + \dots + [C^a(\rho_{A_1A_k})]^2 \leq 1$ ,  $[C^a(\rho_{A_1A_2})]^2 + \dots + [C^a(\rho_{A_1A_{k+1}})]^2 > 1$ . By defining  $T := [C^a(\rho_{A_1A_2})]^2 + \dots + [C^a(\rho_{A_1A_{k+1}})]^2 - 1 > 0$ , we can derive

$$\begin{aligned} E_\alpha(|\psi\rangle_{A_1|A_2\dots A_n}) &= f_\alpha(C(\rho_{A_1|A_2\dots A_n})) \leq f_\alpha(1) \\ &= f_\alpha\left(\sqrt{[C^a(\rho_{A_1A_2})]^2 + \dots + [C^a(\rho_{A_1A_{k+1}})]^2 - T}\right) \\ &\leq f_\alpha\left(\sqrt{[C^a(\rho_{A_1A_2})]^2 + \dots + [C^a(\rho_{A_1A_k})]^2}\right) \\ &+ f_\alpha\left(\sqrt{[C^a(\rho_{A_1A_{k+1}})]^2 - T}\right) \\ &\leq f_\alpha(C^a(\rho_{A_1A_2})) + \dots + f_\alpha(C^a(\rho_{A_1A_{k+1}})) \\ &\leq E_\alpha^a(\rho_{A_1A_2}) + \dots + E_\alpha^a(\rho_{A_1A_n}), \end{aligned} \quad (22)$$

where we have used the monotonically increasing property of  $f_\alpha(x)$  in the first equality, in the second inequality we have used Lemma 2, the third inequality is obtained by the successive application of Lemma 2, and the last inequality is due to Lemma 1.

Using the polygamy relation for the pure state we can prove the Theorem in the mixed state. Suppose that the optimal decomposition for  $E_\alpha^a(\rho_{A_1|A_2\dots A_n})$  is  $\{p_j, |\psi_j\rangle_{A_1|A_2\dots A_n}\}$ , then we have

$$\begin{aligned} E_\alpha^a(\rho_{A_1|A_2\dots A_n}) &= \sum_i p_j E_\alpha(|\psi_j\rangle_{A_1|A_2\dots A_n}) \\ &\leq \sum_j p_j \left( E_\alpha^a(\rho_{A_1A_2}^j) + \dots + E_\alpha^a(\rho_{A_1A_n}^j) \right) \\ &\leq E_\alpha^a(\rho_{A_1A_2}) + \dots + E_\alpha^a(\rho_{A_1A_n}), \end{aligned} \quad (23)$$

where  $\rho_{A_1A_j}^j$  is the reduced density matrix of  $|\psi_j\rangle_{A_1A_2\dots A_n}$  onto the two-qubit subsystem  $A_1A_j$  for each  $i = 2, \dots, n$ , in the first inequality we have used the polygamy relation for each pure state decomposition state  $|\psi_j\rangle_{A_1|A_2\dots A_n}$

$$E_\alpha(|\psi_j\rangle_{A_1|A_2\dots A_n}) \leq E_\alpha^a(\rho_{A_1A_2}^j) + \dots + E_\alpha^a(\rho_{A_1A_n}^j), \quad (24)$$

and the last inequality is due to the definition of REoA for each  $\rho_{A_1A_i}$ . Thus we have completed the proof of Theorem.  $\square$

Furthermore, we can establish the following  $\mu$ th power polygamy inequalities for the Rényi- $\alpha$  entanglement of assistance.

*Corollary.* For  $(\sqrt{7}-1)/2 \leq \alpha \leq (\sqrt{13}-1)/2$ ,  $0 \leq \mu \leq 1$ , and any  $n$ -qubit state  $\rho_{A_1A_2\dots A_n}$ , we have

$$[E_\alpha^a(\rho_{A_1|A_2\dots A_n})]^\mu \leq [E_\alpha^a(\rho_{A_1A_2})]^\mu + \dots + [E_\alpha^a(\rho_{A_1A_n})]^\mu \quad (25)$$

This inequality holds because  $[E_\alpha^a(\rho_{A_1|A_2\dots A_n})]^\mu \leq [E_\alpha^a(\rho_{A_1A_2}) + \dots + E_\alpha^a(\rho_{A_1A_n})]^\mu \leq [E_\alpha^a(\rho_{A_1A_2})]^\mu + \dots + [E_\alpha^a(\rho_{A_1A_n})]^\mu$ , where the last inequality is due to the concave property of  $x^\mu$  for  $0 \leq \mu \leq 1$ .

By introducing the dual concept of REoA, we have established polygamy relations for the Rényi- $\alpha$  entanglement of assistance in multi-qubit systems. We have also generalized the polygamy inequalities into the  $\mu$ th power of REoA. These derived polygamy relations provide a lower bound for distribution of bipartite REoA in a multi-party system. The monogamy and polygamy relations are not only fundamental property of entanglement in multi-party systems but also provide us an efficient way of characterizing multipartite entanglement. In Ref. [39], we have proved that squared Rényi- $\alpha$  entanglement with the order  $\alpha \geq (\sqrt{7}-1)/2$  obeys a general monogamy relation in an arbitrary  $n$ -qubit mixed state. It is further shown that we can construct the multipartite entanglement indicators in terms of ER $\alpha$ E which still work well even when the indicators based on the concurrence and EoF lose their efficacy. Thus our polygamy inequalities together with previous monogamy inequalities in terms of ER $\alpha$ E might provide a useful tool to understand the property of multi-party quantum entanglement.

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