Quantum coherence in mutually unbiased bases

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We investigate the l_1 norm of coherence of quantum states in mutually unbiased bases. We find that the sum of squared l_1 norm of coherence of the mixed state single qubit is less than two. We derive the l_1 norm of coherence of three classes of X states in nontrivial mutually unbiased bases for 4-dimensional Hilbert space is equal. We proposed "autotensor of mutually unbiased basis(AMUB)" by the tensor of mutually unbiased bases, and depict the level surface of constant the sum of the l_1 norm of coherence of Bell-diagonal states in AMUB. We find the l_1 norm of coherence of Werner states and isotropic states in AMUB is equal respectively.

I. INTRODUCTION

Quantum coherence is a special feature of quantum mechanic like entanglement and other quantum correlations. Quantum coherence is an essential factor in quantum information processing [1–3], quantum optics [4–6], quantum metrology [7–9], low-temperature thermodynamics [10–17] and quantum biology [18–23]. Recently, a structure to quantify coherence has been proposed [25], and various quantum coherence measures, such as the l_1 norm of coherence [25], trace norm of coherence [24], relative entropy of coherence [25], Tsallis relative α entropies [26] and Relative Rényi α monotones [27], have been defined. With the help of of the coherence measures, a variety of properties of quantum coherence, such as the relations between quantum correlations and quantum coherence [28–32], the freezing phenomenon of coherence [33, 34], have been studied.

Mutually unbiased bases are used in detection of quantum entanglement [35], quantum state reconstruction [36], quantum error correction [37, 38], and the mean kings problem [39, 40]. Many features of mutually unbiased bases are reviewed in reference [41]. When d is power of a prime number, maximal sets of $d+1$ mutually unbiased bases have been built for the case. Maximal sets of MUBs are an open problem [41], when the dimensionality is another composite number. Entropic uncertainty relations for $d+1$ mutually unbiased bases in d-dimensional Hilbert space were obtained in references [42, 43]. The fine-grained uncertainty relation for mutually unbiased bases is derived in [44]. The relation between mutually unbiased bases and unextendible maximally entangled is investigated in [45].

In this article, we investigate the l_1 norm of coherence of quantum states in mutually unbiased bases. We evaluate analytically the sum of squared l_1 norm of coherence of the mixed state single qubit. We derive the relation of the l_1 norm of coherence of three classes of X states in nontrivial mutually unbiased bases for 4-dimensional Hilbert space. We propose "autotensor of mutually unbiased basis(AMUB)" by the tensor of mutually unbiased bases, and depict the level surface [46] of constant the sum of the l_1 norm of coherence of Bell-diagonal states in AMUB. We obtain the relations of the l_1 norm of coherence of Werner states and isotropic states in AMUB respectively.

II. THE l_1 NORM OF COHERENCE OF QUANTUM STATES IN 2 DIMENSION MUTUALLY UNBIASED BASES

Under fixed reference basis, the l_1 norm of coherence of state ρ is defined by

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{i,j}|, \tag{1}$$

and the relative entropy of coherence is given by

$$C_r(\rho) = S(\rho_{diag}) - S(\rho), \tag{2}$$

where $S(\rho) = -Tr\rho \log \rho$ is von Neumann entropy.

A set of orthonormal bases $\{B_k\}$ for a Hilbert space $H = C^d$ where $\{B_k\} = \{|0_k\rangle, \cdots, |d-1_k\rangle$ is called mutually unbiased (MU) iff

$$|\langle i_k | j_l \rangle|^2 = \frac{1}{d}, \forall i, j \in \{0, \cdots, d-1\}, \tag{3}$$

holds for all basis vectors $|i_k\rangle$ and $|j_l\rangle$ that belong to different bases, i.e. $\forall k \neq l$.

In dimension $d = 2$, a set of three mutually unbiased bases is readily obtained from the eigenvectors of the three Pauli matrices σ_z, σ_x and σ_y:

$$\begin{aligned}
\alpha_1 &= \{\alpha_{11}, \alpha_{12}\} = \{|0\rangle, |1\rangle\}, \\
\alpha_2 &= \{\alpha_{21}, \alpha_{22}\} = \{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}, \\
\alpha_3 &= \{\alpha_{31}, \alpha_{32}\} = \{\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\}.
\end{aligned}$$

In dimension $d = 3$, there are four mutually unbiased bases as fowllow:

$$\begin{aligned}
\beta_1 &= \{\beta_{11}, \beta_{12}, \beta_{13}\} = \{|0\rangle, |1\rangle, |2\rangle\}, \\
\beta_2 &= \{\beta_{21}, \beta_{22}, \beta_{23}\} = \{\frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \omega^2|2\rangle), \frac{1}{\sqrt{3}}(|0\rangle + \omega^2|1\rangle + \omega|2\rangle)\}, \\
\beta_3 &= \{\alpha_{31}, \alpha_{32}, \alpha_{33}\} = \{\frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + \omega^2|2\rangle), \frac{1}{\sqrt{3}}(|0\rangle + \omega^2|1\rangle + |2\rangle), \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + \omega|2\rangle)\}, \\
\beta_4 &= \{\alpha_{41}, \alpha_{42}, \alpha_{43}\} = \{\frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + \omega|2\rangle), \frac{1}{\sqrt{3}}(|0\rangle + \omega|1\rangle + |2\rangle), \frac{1}{\sqrt{3}}(|0\rangle + \omega^2|1\rangle + \omega^2|2\rangle)\},
\end{aligned}$$

where $\omega = e^{i\frac{2\pi}{3}}$.

An arbitrary density matrix for a mixed state single qubit may be written as

$$\rho_s = \frac{I + \overrightarrow{r} \cdot \overrightarrow{\sigma}}{2}$$

where $\overrightarrow{r} = (x, y, z)$ is a real three-dimensional vector such that $x^2 + y^2 + z^2 \leq 1$, and $\overrightarrow{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. In particular, ρ is pure if and only if $x^2 + y^2 + z^2 = 1$.

Next, we will consider the relation of the l_1 norm of coherence among ρ_s in three mutually unbiased bases $\alpha_1, \alpha_2, \alpha_3$.

The density matrix of mixed state single qubit ρ_s in base $\alpha_1 = \{\alpha_{11}, \alpha_{12}\} = \{|0\rangle, |1\rangle\}$ is

$$\begin{aligned}\rho_s &= \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \\ &= \frac{1}{2}(1+z)|0\rangle\langle 0| + \frac{1}{2}(x-iy)|0\rangle\langle 1| + \frac{1}{2}(x+iy)|1\rangle\langle 0| + \frac{1}{2}(1-z)|1\rangle\langle 1|,\end{aligned} \tag{4}$$

Using Eq. (1) directly, the l_1 norm of coherence of state ρ_s in base α_1 is

$$C_{l_1}(\rho_s)_{\alpha_1} = |\frac{1}{2}(x-iy)| + |\frac{1}{2}(x+iy)| = \sqrt{x^2+y^2}. \tag{5}$$

The density matrix of ρ_s in base $\alpha_2 = \{\alpha_{21}, \alpha_{22}\}$ is

$$\begin{aligned}\rho_s &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= a_{11}\alpha_{21}\alpha_{21}^\dagger + a_{12}\alpha_{21}\alpha_{22}^\dagger + a_{21}\alpha_{22}\alpha_{21}^\dagger + a_{22}\alpha_{22}\alpha_{22}^\dagger \\ &= \frac{1}{2}(a_{11}+a_{12}+a_{21}+a_{22})|0\rangle\langle 0| + \frac{1}{2}(a_{11}-a_{12}+a_{21}-a_{22})|0\rangle\langle 1| \\ &\quad + \frac{1}{2}(a_{11}+a_{12}-a_{21}-a_{22})|1\rangle\langle 0| + \frac{1}{2}(a_{11}-a_{12}-a_{21}+a_{22})|1\rangle\langle 1|.\end{aligned} \tag{6}$$

As ρ_s in Eq. (4) and Eq. (6) is the same, using the method of undetermined coefficients, we obtain

$$\begin{cases} \frac{1}{2}(a_{11}+a_{12}+a_{21}+a_{22}) = \frac{1}{2}(1+z) \\ \frac{1}{2}(a_{11}-a_{12}+a_{21}-a_{22}) = \frac{1}{2}(x-iy) \\ \frac{1}{2}(a_{11}+a_{12}-a_{21}-a_{22}) = \frac{1}{2}(x+iy) \\ \frac{1}{2}(a_{11}-a_{12}-a_{21}+a_{22}) = \frac{1}{2}(1-z) \end{cases}.$$

The solution of the equation is

$$\begin{cases} a_{11} = \frac{1+x}{2} \\ a_{12} = \frac{z+iy}{2} \\ a_{21} = \frac{z-iy}{2} \\ a_{22} = \frac{1-x}{2} \end{cases}. \tag{7}$$

The l_1 norm of coherence of state ρ_s in base α_2 is

$$C_{l_1}(\rho_s)_{\alpha_2} = |\frac{1}{2}(z+iy)| + |\frac{1}{2}(z-iy)| = \sqrt{z^2+y^2}. \tag{8}$$

The density matrix of ρ_s in base $\alpha_3 = \{\alpha_{31}, \alpha_{32}\}$ by the above method is

$$\rho_s = \frac{1}{2}\begin{pmatrix} 1+y & z-ix \\ z+ix & 1-y \end{pmatrix} \tag{9}$$

The l_1 norm of coherence of state ρ_s in base α_3 is

$$C_{l_1}(\rho_s)_{\alpha_3} = |\frac{1}{2}(z-ix)| + |\frac{1}{2}(z+ix)| = \sqrt{z^2+x^2}. \tag{10}$$

As $x^2+y^2+z^2 \leq 1$, $[C_{l_1}(\rho_s)_{\alpha_1}]^2 + [C_{l_1}(\rho_s)_{\alpha_2}]^2 + [C_{l_1}(\rho_s)_{\alpha_3}]^2 \leq 2$.

III. THE l_1 NORM OF COHERENCE OF X STATES IN THE TENSOR OF 3 DIMENSION MUTUALLY UNBIASED BASES

For the three classes of X states in base $\beta_1 = \{\beta_{11}, \beta_{12}, \beta_{13}\} = \{|0\rangle, |1\rangle, |2\rangle\}$

$$\rho_X = \begin{pmatrix} x & 0 & z \\ 0 & 1-x-y & 0 \\ z & o & y \end{pmatrix}, \tag{11}$$

where x, y, z are all real number, we will consider the l_1 norm of coherence of ρ_X in the 3 dimension mutually unbiased bases $\beta_2, \beta_3, \beta_4$.

Let the density matrix of ρ_X in base $\beta_2 = \{\beta_{21}, \beta_{22}, \beta_{23}\}$ be

$$\rho_X = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \tag{12}$$

and $\rho_X = b_{11}\beta_{21}\beta_{21}^\dagger + b_{12}\beta_{21}\beta_{22}^\dagger + b_{13}\beta_{21}\beta_{23}^\dagger + b_{21}\beta_{22}\beta_{21}^\dagger + b_{22}\beta_{22}\beta_{22}^\dagger + b_{23}\beta_{22}\beta_{23}^\dagger + b_{31}\beta_{23}\beta_{21}^\dagger + b_{32}\beta_{23}\beta_{22}^\dagger + b_{33}\beta_{23}\beta_{23}^\dagger$. As ρ_X in Eq. (11) and Eq. (12) is the same, using the method of undetermined coefficients, we obtain

$$\begin{cases} \frac{1}{3}(b_{11}+b_{12}+b_{13}+b_{21}+b_{22}+b_{23}+b_{31}+b_{32}+b_{33}) = x \\ \frac{1}{3}(b_{11}+\omega^2 b_{12}+\omega b_{13}+b_{21}+\omega^2 b_{22}+\omega b_{23}+b_{31}+\omega^2 b_{32}+\omega b_{33}) = 0 \\ \frac{1}{3}(b_{11}+\omega b_{12}+\omega^2 b_{13}+b_{21}+\omega b_{22}+\omega^2 b_{23}+b_{31}+\omega b_{32}+\omega^2 b_{33}) = z \\ \frac{1}{3}(b_{11}+b_{12}+b_{13}+\omega b_{21}+\omega b_{22}+\omega b_{23}+\omega^2 b_{31}+\omega^2 b_{32}+\omega^2 b_{33}) = 0 \\ \frac{1}{3}(b_{11}+\omega^2 b_{12}+\omega b_{13}+\omega b_{21}+b_{22}+\omega^2 b_{23}+\omega^2 b_{31}+\omega b_{32}+b_{33}) = 1-x-y \\ \frac{1}{3}(b_{11}+\omega b_{12}+\omega^2 b_{13}+\omega b_{21}+\omega^2 b_{22}+b_{23}+\omega^2 b_{31}+b_{32}+\omega b_{33}) = 0 \\ \frac{1}{3}(b_{11}+b_{12}+b_{13}+\omega^2 b_{21}+\omega^2 b_{22}+\omega^2 b_{23}+\omega b_{31}+\omega b_{32}+\omega b_{33}) = z \\ \frac{1}{3}(b_{11}+\omega^2 b_{12}+\omega b_{13}+\omega^2 b_{21}+\omega b_{22}+b_{23}+\omega b_{31}+b_{32}+\omega^2 b_{33}) = 0 \\ \frac{1}{3}(b_{11}+\omega b_{12}+\omega^2 b_{13}+\omega^2 b_{21}+b_{22}+\omega b_{23}+\omega b_{31}+\omega^2 b_{32}+b_{33}) = y \end{cases}. \tag{13}$$

The solution of the equation is

$$\begin{cases} b_{11} = \frac{1+2z}{3}, b_{12} = \frac{(3x+z-1)-\sqrt{3}(x+2y+z-1)i}{6}, b_{13} = \frac{(3x+z-1)+\sqrt{3}(x+2y+z-1)i}{6}, \\ b_{21} = \overline{b_{12}}, b_{22} = \frac{1-z}{3}, b_{23} = \frac{(3x-2z-1)-\sqrt{3}(x+2y-2z-1)i}{6}, \\ b_{31} = \overline{b_{13}}, b_{32} = \overline{b_{23}}, b_{33} = \frac{1-3z}{3}. \end{cases} \tag{14}$$

The l_1 norm of coherence of state ρ_X in base β_2 is

$$C_{l_1}(\rho_X)_{\beta_2} = 2(|b_{12}| + |b_{13}| + |b_{23}|). \tag{15}$$

Similarly, the density matrix of ρ_X in base β_3 is

$$\rho_X = \begin{pmatrix} b_{22} & \overline{b_{12}} & b_{23} \\ b_{12} & b_{11} & b_{13} \\ \overline{b_{23}} & \overline{b_{13}} & b_{33} \end{pmatrix}, \tag{16}$$

The l_1 norm of coherence of state ρ_X in base β_3 is

$$C_{l_1}(\rho_X)_{\beta_3} = 2(|b_{12}| + |b_{13}| + |b_{23}|). \tag{17}$$

The density matrix of ρ_X in base β_4 is

$$\rho_X = \begin{pmatrix} b_{22} & b_{12} & \overline{b_{23}} \\ \overline{b_{12}} & b_{11} & \overline{b_{13}} \\ b_{23} & b_{13} & b_{33} \end{pmatrix}, \tag{18}$$

The l_1 norm of coherence of state ρ_X in base β_4 is

$$C_{l_1}(\rho_X)_{\beta_4} = 2(|b_{12}| + |b_{13}| + |b_{23}|). \tag{19}$$

At last, we find that the l_1 norm of coherence of state ρ_X in base $\beta_2, \beta_3, \beta_4$ is equal, i.e

$$C_{l_1}(\rho_X)_{\beta_2} = C_{l_1}(\rho_X)_{\beta_3} = C_{l_1}(\rho_X)_{\beta_4}. \tag{20}$$

Furthermore, let

$$\rho_\triangle = \begin{pmatrix} 1-x-y & 0 & 0 \\ 0 & x & z \\ 0 & z & y \end{pmatrix}, \tag{21}$$

and

$$\rho_\nabla = \begin{pmatrix} x & z & 0 \\ z & y & 0 \\ 0 & 0 & 1-x-y \end{pmatrix}, \tag{22}$$

where x, y, z are all real number, using above method, we can find that the l_1 norm of coherence of state $\rho_\triangle$ and ρ_∇ in base $\beta_2, \beta_3, \beta_4$ is also equal respectively.

IV. THE l_1 NORM OF COHERENCE OF BELL-DIAGONAL STATES IN THE TENSOR OF 2 DIMENSION MUTUALLY UNBIASED BASES

In this section, we extend the concept of mutually unbiased basis by the tensor.

Definition. For the set of mutually unbiased bases $\{B_k\}$ for a Hilbert space $H = C^d$ where $\{B_k\} = \{|0_k\rangle, \cdots, |d-1_k\rangle\}$, we call the set $\{\gamma_k\} = \{|i\rangle_k \otimes |i\rangle_k | \forall i, j \in \{0, \cdots, d-1\}\}$ *autotensor of mutually unbiased basis*(AMUB) if

$$|(\langle i|_k \otimes \langle j|_k)(|m\rangle_l \otimes |n\rangle_l)| = \frac{1}{d}, \tag{23}$$

where $k \neq l$. Furthermore, we can construct a set of AMUB by $d = 2$ dimension mutually unbiased bases. For example, let

$$\begin{aligned} \gamma_1 &= \{\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}\} = \{\alpha_{11} \otimes \alpha_{11}, \alpha_{11} \otimes \alpha_{12}, \alpha_{12} \otimes \alpha_{11}, \alpha_{12} \otimes \alpha_{12}\}, \\ \gamma_2 &= \{\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}\} = \{\alpha_{21} \otimes \alpha_{21}, \alpha_{21} \otimes \alpha_{22}, \alpha_{22} \otimes \alpha_{21}, \alpha_{22} \otimes \alpha_{22}\}, \\ \gamma_3 &= \{\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}\} = \{\alpha_{31} \otimes \alpha_{31}, \alpha_{31} \otimes \alpha_{32}, \alpha_{32} \otimes \alpha_{31}, \alpha_{32} \otimes \alpha_{32}\}. \end{aligned}$$

Next, we will consider the relation of the coherence of quantum states in above AMUB.

A two-qubit Bell-diagonal states can be written as

$$\rho_B = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i), \tag{24}$$

where $\{\sigma_i\}_{i=1}^3$ are the Pauli matrices, and $c_1, c_2, c_3 \in [-1, 1]$. The density matrix of ρ_B in base $\gamma_1 = \{\gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}\} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is:

$$\rho_B = \frac{1}{4} \begin{pmatrix} 1+c_3 & 0 & 0 & c_1-c_2 \\ 0 & 1-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-c_3 & 0 \\ c_1-c_2 & 0 & 0 & 1+c_3 \end{pmatrix} \tag{25}$$

The l_1 norm of coherence of state ρ_B in base γ_1 is

$$C_{l_1}(\rho_B)_{\gamma_1} = 2(|\frac{1}{4}(c_1 - c_2)| + |\frac{1}{4}(c_1 + c_2)|) = \frac{1}{2}(|(c_1 - c_2)| + |(c_1 + c_2)|). \tag{26}$$

Let the density matrix of ρ_B in base $\gamma_2 = \{\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}\}$ is

$$\rho_B = \begin{pmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}, \tag{27}$$

and $\rho_B = d_{11}\gamma_{21}\gamma_{21}^\dagger + d_{12}\gamma_{21}\gamma_{22}^\dagger + d_{13}\gamma_{21}\gamma_{23}^\dagger + d_{14}\gamma_{21}\gamma_{24}^\dagger + d_{21}\gamma_{22}\gamma_{21}^\dagger + d_{22}\gamma_{22}\gamma_{22}^\dagger + d_{23}\gamma_{22}\gamma_{23}^\dagger + d_{24}\gamma_{22}\gamma_{24}^\dagger + d_{31}\gamma_{23}\gamma_{21}^\dagger + d_{32}\gamma_{23}\gamma_{22}^\dagger + d_{33}\gamma_{23}\gamma_{23}^\dagger + d_{34}\gamma_{23}\gamma_{24}^\dagger + d_{41}\gamma_{24}\gamma_{21}^\dagger + d_{42}\gamma_{24}\gamma_{22}^\dagger + d_{43}\gamma_{24}\gamma_{23}^\dagger + d_{44}\gamma_{24}\gamma_{24}^\dagger$. As ρ_B in Eq. (25) and Eq. (27) is the same, using the method of undeterminated coefficients, we obtain

$$\begin{cases}
d_{11} + d_{12} + d_{13} + d_{14} + d_{21} + d_{22} + d_{23} + d_{24} + d_{31} + d_{32} + d_{33} + d_{34} + d_{41} + d_{42} + d_{43} + d_{44} = 1 + c_3 \\
d_{11} - d_{12} + d_{13} - d_{14} + d_{21} - d_{22} + d_{23} - d_{24} + d_{31} - d_{32} + d_{33} - d_{34} + d_{41} - d_{42} + d_{43} - d_{44} = 0 \\
d_{11} + d_{12} - d_{13} - d_{14} + d_{21} + d_{22} - d_{23} - d_{24} + d_{31} + d_{32} - d_{33} - d_{34} + d_{41} + d_{42} - d_{43} - d_{44} = 0 \\
d_{11} - d_{12} - d_{13} + d_{14} + d_{21} - d_{22} - d_{23} + d_{24} + d_{31} - d_{32} - d_{33} + d_{34} + d_{41} - d_{42} - d_{43} + d_{44} = c_1 - c_2 \\
d_{11} + d_{12} + d_{13} + d_{14} - d_{21} - d_{22} - d_{23} - d_{24} + d_{31} + d_{32} + d_{33} + d_{34} - d_{41} - d_{42} - d_{43} - d_{44} = 0 \\
d_{11} - d_{12} + d_{13} - d_{14} - d_{21} + d_{22} - d_{23} + d_{24} + d_{31} - d_{32} + d_{33} - d_{34} - d_{41} + d_{42} - d_{43} + d_{44} = 1 - c_3 \\
d_{11} + d_{12} - d_{13} - d_{14} - d_{21} - d_{22} + d_{23} + d_{24} + d_{31} + d_{32} - d_{33} - d_{34} - d_{41} - d_{42} + d_{43} + d_{44} = c_1 + c_2 \\
d_{11} - d_{12} - d_{13} + d_{14} - d_{21} + d_{22} + d_{23} - d_{24} + d_{31} - d_{32} - d_{33} + d_{34} - d_{41} + d_{42} + d_{43} - d_{44} = 0 \\
d_{11} + d_{12} + d_{13} + d_{14} + d_{21} + d_{22} + d_{23} + d_{24} - d_{31} - d_{32} - d_{33} - d_{34} - d_{41} - d_{42} - d_{43} - d_{44} = 0 \\
d_{11} - d_{12} + d_{13} - d_{14} + d_{21} - d_{22} + d_{23} - d_{24} - d_{31} + d_{32} - d_{33} + d_{34} - d_{41} + d_{42} - d_{43} + d_{44} = c_1 + c_2 \\
d_{11} + d_{12} - d_{13} - d_{14} + d_{21} + d_{22} - d_{23} - d_{24} - d_{31} - d_{32} + d_{33} + d_{34} - d_{41} - d_{42} + d_{43} + d_{44} = 1 - c_3 \\
d_{11} - d_{12} - d_{13} + d_{14} + d_{21} - d_{22} - d_{23} + d_{24} - d_{31} + d_{32} + d_{33} - d_{34} - d_{41} + d_{42} + d_{43} - d_{44} = 0 \\
d_{11} + d_{12} + d_{13} + d_{14} - d_{21} - d_{22} - d_{23} - d_{24} - d_{31} - d_{32} - d_{33} - d_{34} + d_{41} + d_{42} + d_{43} + d_{44} = c_1 - c_2 \\
d_{11} - d_{12} + d_{13} - d_{14} - d_{21} + d_{22} - d_{23} + d_{24} - d_{31} + d_{32} - d_{33} + d_{34} + d_{41} - d_{42} + d_{43} - d_{44} = 0 \\
d_{11} + d_{12} - d_{13} - d_{14} - d_{21} - d_{22} + d_{23} + d_{24} - d_{31} - d_{32} + d_{33} + d_{34} + d_{41} + d_{42} - d_{43} - d_{44} = 0 \\
d_{11} - d_{12} - d_{13} + d_{14} - d_{21} + d_{22} + d_{23} - d_{24} - d_{31} + d_{32} + d_{33} - d_{34} + d_{41} - d_{42} - d_{43} + d_{44} = 1 + c_3
\end{cases}. \tag{28}$$

The solution of the equation is

$$\begin{cases}
d_{11} = \frac{1+c_1}{4}, d_{12} = 0, d_{13} = 0, d_{14} = \frac{c_3-c_2}{4}, \\
d_{21} = 0, d_{22} = \frac{1-c_1}{4}, d_{23} = \frac{c_3+c_2}{4}, d_{24} = 0, \\
d_{31} = 0, d_{32} = \frac{c_3+c_2}{4}, d_{23} = \frac{1-c_1}{4}, d_{24} = 0, \\
d_{41} = \frac{c_3-c_2}{4}, d_{42} = 0, d_{43} = 0, d_{44} = \frac{1+c_1}{4}.
\end{cases} \tag{29}$$

So, the density matrix of ρ_B in base $\gamma_2 = \{\gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{24}\}$ is

$$\rho_B = \frac{1}{4}\begin{pmatrix} 1 + c_1 & 0 & 0 & c_3 - c_2 \\ 0 & 1 - c_1 & c_3 + c_2 & 0 \\ 0 & c_3 + c_2 & 1 - c_1 & 0 \\ c_3 - c_2 & 0 & 0 & 1 + c_1 \end{pmatrix}. \tag{30}$$

The l_1 norm of coherence of state ρ_B in base γ_2 is

$$C_{l_1}(\rho_B)_{\gamma_2} = 2(|\frac{1}{4}(c_3 - c_2)| + |\frac{1}{4}(c_3 + c_2)|) = \frac{1}{2}(|(c_3 - c_2)| + |(c_3 + c_2)|). \tag{31}$$

Similarly, the density matrix of ρ_B in base $\gamma_3 = \{\gamma_{31}, \gamma_{32}, \gamma_{33}, \gamma_{34}\}$ is

$$\rho_B = \frac{1}{4}\begin{pmatrix} 1 + c_2 & 0 & 0 & c_3 - c_1 \\ 0 & 1 - c_2 & c_3 + c_1 & 0 \\ 0 & c_3 + c_1 & 1 - c_2 & 0 \\ c_3 - c_1 & 0 & 0 & 1 + c_2 \end{pmatrix}. \tag{32}$$

The l_1 norm of coherence of state ρ_B in base γ_3 is

$$C_{l_1}(\rho_B)_{\gamma_3} = 2(|\frac{1}{4}(c_3 - c_1)| + |\frac{1}{4}(c_3 + c_1)|) = \frac{1}{2}(|(c_3 - c_1)| + |(c_3 + c_1)|). \tag{33}$$

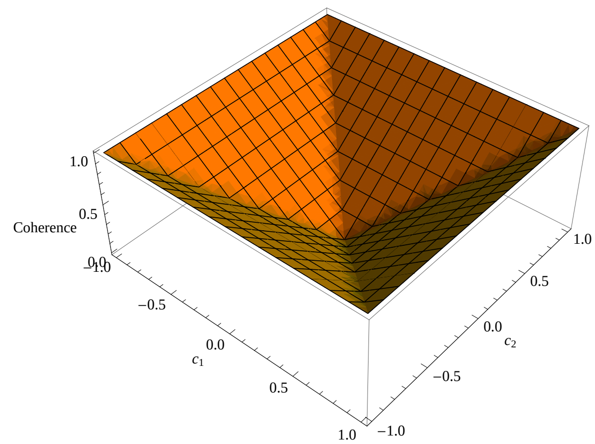

FIG. 1: (Color online) The l_1 norm of coherence of Bell-diagonal states ρ_B in base γ_1 as a function of c_1 and c_2.

In Fig. 1, the l_1 norm of coherence of Bell-diagonal states ρ_B in base γ_1 as a function of c_1 and c_2 is depicted. When $c_1 = c_2 = 0$, the coherence reach minimal value 0. As $|c_1|$ and $|c_2|$ increase, the coherence increase. When $|c_1| = 1$ or $|c_2| = 1$, the coherence obtain maximum value. Similar situation also appear in the coherence in bases γ_2 and γ_3.

Next, we denote the sum of the l_1 norm of coherence of Bell-diagonal states ρ_B in bases γ_1, γ_2, γ_3 by $C_{l_1}(\rho_B)_\gamma$, i. e

$$C_{l_1}(\rho_B)_\gamma = C_{l_1}(\rho_B)_{\gamma_1} + C_{l_1}(\rho_B)_{\gamma_2} + C_{l_1}(\rho_B)_{\gamma_3}. \tag{34}$$

In Fig. 2, we plot the surfaces [46] of the sum of the l_1 norm of coherence $C_{l_1}(\rho_B)_\gamma$ of Bell-diagonal states ρ_B in bases γ_1, γ_2, γ_3 in (a), (b), and (c). It show that the surface of the sum of the coherence is tetrahexahedron. As the sum increase, its volume expand, i. e. $|c_1|$, $|c_2|$, $|c_3|$ increase simultaneously.

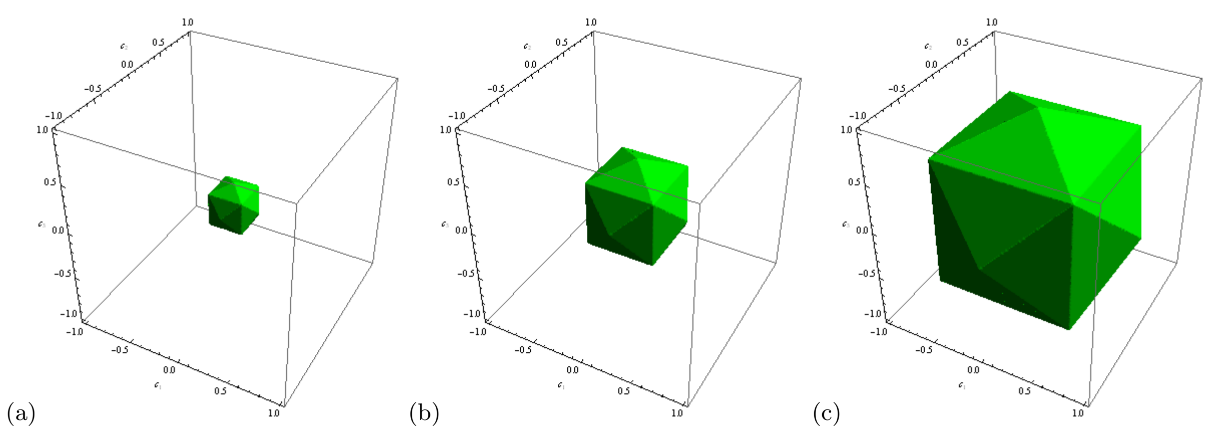

FIG. 2: Surfaces of constant of the sum of the l_1 norm of coherence $C_{l_1}(\rho_B)_\gamma$ for Bell-diagonal states ρ_B in bases γ_1, γ_2, γ_3 : (a) $C_{l_1}(\rho_B)_\gamma = 0.5$; (b) $C_{l_1}(\rho_B)_\gamma = 1$; $C_{l_1}(\rho_B)_\gamma = 2$.

In Eq. (25), let $c_1 = c_2 = c_3 = \frac{4p}{3} - 1$, where $0 \leq p \leq 1$,Bell-diagonal states ρ_B turn into Werner state

$$\rho_W = \begin{pmatrix} \frac{p}{3} & 0 & 0 & 0 \\ 0 & -\frac{p}{3}+\frac{1}{2} & \frac{2p}{3}-\frac{1}{2} & 0 \\ 0 & \frac{2p}{3}-\frac{1}{2} & -\frac{p}{3}+\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{p}{3} \end{pmatrix}. \tag{35}$$

We denoted the l_1 norm of coherence of Werner states ρ_W in bases $\gamma_1, \gamma_2, \gamma_3$ by $C_{l_1}(\rho_W)_{\gamma_1}, C_{l_1}(\rho_W)_{\gamma_2}, C_{l_1}(\rho_W)_{\gamma_3}$ respectively. By Eqs. (26), (31), (33), we find that $C_{l_1}(\rho_W)_{\gamma_1} = C_{l_1}(\rho_W)_{\gamma_2} = C_{l_1}(\rho_W)_{\gamma_3} = |\frac{4p}{3} - 1|$.

In Eq. (25), let $c_1 = \frac{4F-1}{3}, c_2 = -\frac{4F-1}{3}, c_3 = \frac{4F-1}{3}$, where $0 \leq F \leq 1$,Bell-diagonal states ρ_B turn into isotropic state

$$\rho_{iso} = \begin{pmatrix} \frac{F}{3}+\frac{1}{6} & 0 & 0 & \frac{2F}{3}-\frac{1}{6} \\ 0 & \frac{1}{3}-\frac{F}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3}-\frac{F}{3} & 0 \\ \frac{2F}{3}-\frac{1}{6} & 0 & 0 & \frac{F}{3}+\frac{1}{6} \end{pmatrix}. \tag{36}$$

We denoted the l_1 norm of coherence of isotropic states ρ_{iso} in bases $\gamma_1, \gamma_2, \gamma_3$ by $C_{l_1}(\rho_{iso})_{\gamma_1}, C_{l_1}(\rho_{iso})_{\gamma_2}, C_{l_1}(\rho_{iso})_{\gamma_3}$ respectively. By Eqs. (26), (31), (33), we find that $C_{l_1}(\rho_{iso})_{\gamma_1} = C_{l_1}(\rho_{iso})_{\gamma_2} = C_{l_1}(\rho_{iso})_{\gamma_3} = |\frac{4F-1}{3}|$.

V. SUMMARY

In this work, we studied the l_1 norm of coherence of quantum states in mutually unbiased bases. We have found the sum of squared l_1 norm of coherence of the mixed state single qubit is less than two. We have obtained the l_1 norm of coherence of three classes of X states in nontrivial mutually unbiased bases for 4-dimensional Hilbert space is equal. We have proposed "autotensor of mutually unbiased basis(AMUB)" by the tensor of mutually unbiased bases, and given the level surface[46] of constant the sum of the l_1 norm of coherence of Bell-diagonal states in AMUB. We have found the l_1 norm of coherence of Werner states and isotropic states in AMUB is equal respectively.

[1] P. K. Jha, M. Mrejen, J. Kim, C. Wu, Y. Wang, Y. V. Rostovtsev, and X. Zhang, Phys. Rev. Lett. **116**, 165502 (2016).
[2] E. Bagan, J. A. Bergou, S. S. Cottrell, and M. Hillery, Phys. Rev. Lett. **116**, 160406 (2016).
[3] P. Kammerlander and J. Anders, Sci. Rep. **6**, 22174 (2016).
[4] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[5] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
[6] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, UK, 1995).
[7] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).
[8] R. Demkowicz-Dobrzański and L. Maccone, Phys. Rev. Lett. **113**, 250801 (2014).
[9] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photonics **5**, 222 (2011).
[10] M. Lostaglio, D. Jennings, and T. Rudolph, Nat. Commun. **6**, 6383 (2015).
[11] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Phys. Rev. X **5**, 021001 (2015).
[12] H. Vazquez, R. Skouta, S. Schneebeli, M. Kamenetska, R. Breslow, L. Venkataraman, and M. S. Hybertsen, Nat. Nanotechnol. **7**, 663 (2012).
[13] O. Karlström, H. Linke, G. Karlström, and A. Wacker, Phys. Rev. B **84**, 113415 (2011).
[14] A. Misra, U. Singh, S. Bhattacharya, and A. K. Pati, Phys. Rev. A **93**, 052335 (2016).

[15] J. Aberg, Phys. Rev. Lett. **113**, 150402 (2014).
[16] V. Narasimhachar and G. Gour, Nat. Commun. **6**, 7689 (2015).
[17] P. Ćwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, Phys. Rev. Lett. **115**, 210403 (2015).
[18] S. Lloyd, J. Phys: Conf. Ser. **302**, 012037 (2011).
[19] C.-M. Li, N. Lambert, Y.-N. Chen, G.-Y. Chen, and F. Nori, Sci. Rep. **2**, 885 (2012).
[20] S. F. Huelga and M. B. Plenio, Contemp. Phys. **54**, 181 (2013).
[21] F. Levi, and F. Mintert, New J. Phys. **16**, 033007 (2014).
[22] M. B. Plenio and S. F. Huelga, New J. Phys. **10**, 113019 (2008).
[23] P. Rebentrost, M. Mohseni, and A. Aspuru-Guzik, J. Phys. Chem. B **113**, 9942 (2009).
[24] L.-H. Shao, Z. Xi, H. Fan and Y. Li, Phys. Rev. A **91** 042120 (2015).
[25] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[26] A. E. Rastegin, Phys. Rev. A **93** 032136 (2016).
[27] E. Chitambar and G. Gour, Phys. Rev. A **94** 052336 (2016).
[28] J. Ma, B. Yadin, D. Girolami, V. Vedral, and M. Gu, Phys. Rev. Lett. **116**, 160407 (2016).
[29] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, Phys. Rev. Lett. **115**, 020403 (2015).
[30] C. Radhakrishnan, M. Parthasarathy, S. Jambulingam, and T. Byrnes, Phys. Rev. Lett. **116**, 150504 (2016).
[31] Y. Yao, X. Xiao, L. Ge, and C. P. Sun, Phys. Rev. A **92**, 022112 (2015).
[32] Z. Xi, Y. Li, and H. Fan, Sci. Rep. **5**, 10922 (2015).
[33] T. R. Bromley, M. Cianciaruso, and G. Adesso, Phys. Rev. Lett. **114**, 210401 (2015).
[34] X.-D. Yu, D.-J. Zhang, C. L. Liu, and D. M. Tong, Phys. Rev. A **93**, 060303 (2016).
[35] C. Spengler, M. Huber, S. Brierley, T. Adaktylos, B. C. Hiesmayr, Phys. Rev. A **86**, 022311 (2012).
[36] W. K. Wootters, B. D Fields, Ann. Phys. **191**, 363 (1989).
[37] D. Gottesman, Phys. Rev. A **54**, 1862 (1996).
[38] A. R. Calderbank, E. M. Rains, P.W. Shor, N. J. A. Sloane, Phys. Rev. Lett. **78**, 405 (1997).
[39] L. Vaidman, Y. Aharonov, D. Z. Albert, Phys. Rev. Lett. **58**, 1385 (1987).
[40] B.-G. Englert, Y. Aharonov, Phys. Lett. A **284**, 1 (2001).
[41] T. Durt, B.-G. Englert, I. Bengtsson, K. Zyczkowski, Int. J. Quantum Inf. **8**, 535 (2010).
[42] I. D. Ivanovic, J. Phys. A: Math. Gen. **25**, L363 (1995).
[43] J. Sánchez, Phys. Lett. A **173**, 233 (1993).
[44] L-. H Ren, H. Fan, Phys. Rev. A **90**, 052110 (2016).
[45] B. Chen, S.-M. Fei, Phys. Rev. A **88**, 034301 (2013).
[46] M. D. Lang and C. M. Caves, Phys. Rev. Lett. **105**, 15051 (2010).