Constructions of unextendible entangled bases

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Abstract We provide several constructions of special unextendible entangled bases with fixed Schmidt number k (SUEBk) in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ for $2 \leq k \leq d \leq d'$. We generalize the space decomposition method in Guo [Phys. Rev. A 94, 052302 (2016)], by proposing a systematic way of constructing new SUEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ for $2 \leq k < d \leq d'$ or $2 \leq k = d < d'$. In addition, we give a construction of a (pqdd' - p(dd' - N))-number SUEBpk in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd'}$ from an N-number SUEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ for $p \leq q$ by using permutation matrices. We also connect a (d(d'-1) + m)-number UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ with an unextendible partial Hadamard matrix $H_{m \times d}$ with m < d, which extends the result in [Quantum Inf. Process. **16**(3), 84 (2017)].

Keywords Unextendible entangled basis \cdot Schmidt number \cdot Permutation matrix \cdot Hadamard matrix

1 Introduction

In 1999, Bennett *et al.* found that unextendible product basis (UPB), a set of incomplete orthonormal product states whose complementary space has no product states, also can displays nonlocality [1]. Consequently, the notion of unextendible basis has been explored extensively. It is shown that UPB

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also can be used to construct bound entangled states [2–6]. In 2009, Bravyi and Smolin introduced the concept of unextendible maximally entangled bases (UMEB), which is a set of incomplete orthonormal maximally entangled states whose complementary space has no maximally entangled states. UMEBs can be used to construct examples of states for which one-copy entanglement of assistance (EoA) is strictly smaller than the asymptotic, and can be used to find quantum channels that are unital but not convex mixtures of unitary operations [7, 8]. Recently, Guo et al. [9, 10] proposed the concept of special unextendible entangled bases with fixed Schmidt number k (SUEBk), which extends the definitions of both UPB and UMEB. An SUEBk is a set of incomplete orthonormal special entangled states with Schmidt number k in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ $(d \leq d')$, i.e., with the same Schmidt coefficients, whose complementary subspace has no special entangled states with Schmidt number k. An SUEBk is a UPB when k = 1 and is a UMEB when k = d.

One of the main topics in this field is the existence of different unextendible bases [7, 9, 11–15]. Bravyi and Smolin gave a 6-number UMEB in $\mathbb{C}^3 \otimes \mathbb{C}^3$ and a 12-number UMEB in $\mathbb{C}^4 \otimes \mathbb{C}^4$ [7]. Wang et al. showed that there are UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^d$ except for d = p or 2p, where p is a prime and $p \equiv 3$ (mod 4) [11]. For d < d', Li et al. showed that UMEBs exist in any $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ and gave explicit constructions [12]. There are several recursive constructions for UMEBs, such as a construction of a $(pqd^2 - p(d^2 - N))$ -number UMEB in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$ $(p \leq q)$ from an N-number UMEB in $\mathbb{C}^d \otimes \mathbb{C}^d$, and a construction of a $(pqd^2 - d(pq - N))$ -number UMEB in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd}$ $(p \leq q)$ from an N-number UMEB in $\mathbb{C}^p \otimes \mathbb{C}^q$ [13–15]. It was shown that SUEBks exist in any bipartite system when $2 \leq k < d \leq d'$ [9].

In this paper, we develop more constructions of SUEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ for $2 \leq k \leq d \leq d'$. In Section 3, we provide a systematic way of constructing SUEBks from a special entangled basis with Schmidt number k, and show that there are more new SUEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ for $2 \leq k < d \leq d'$ or $2 \leq k = d < d'$. In Section 4, we give a recursive construction of a (pqdd' - p(dd' - N))-number SUEBpk in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd'}$ from an N-number SUEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ for $p \leq q$ by using permutation matrices. Especially, it generates a (pqdd' - p(dd' - N))-number UMEB in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd'}$ from an N-number UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ for $p \leq q$. In Section 5, we show that there is a (d(d' - 1) + m)-number UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ if there exists an unextendible partial Hadamard matrix $H_{m \times d}$. Our constructions generalize and improve most results in [8, 9, 11–16], and provide new SUEBks and UMEBs.

2 Definition and preliminary

Assume that $1 \leq k \leq d \leq d'$ in this paper. Let [n] denote the set $\{1, 2, \dots, n\}$, $[n]^*$ denote the set $\{0, 1, \dots, n-1\}$, and $\{a, a, \dots, a\}_k$ denote a multiset of k numbers of a. The Schmidt decomposition of a pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$ $[17]: |\psi\rangle = \sum_{i=0}^{k-1} \lambda_i |i\rangle |i'\rangle$, where $\{|i\rangle\}$ and $\{|i'\rangle\}$ are orthonormal sets of \mathbb{C}^d and $\mathbb{C}^{d'}$, respectively. Then the Schmidt number of $|\psi\rangle$, denoted by $Sr(|\psi\rangle)$, is k. If all the Schmidt coefficients are $\{\frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}}, \cdots, \frac{1}{\sqrt{k}}\}_k$, then $|\psi\rangle$ is called a special entangled state with Schmidt number k. A special entangled state with Schmidt number k is a product state when k = 1; and it is a maximally entangled state when k = d.

Let $\mathcal{M}_{d \times d'}$ be the space of all $d \times d'$ complex matrices equipped with an inner product defined by $(A, B) = \operatorname{Tr}(A^{\dagger}B)$. There is a one-to-one relation between the space $\mathbb{C}^{d} \otimes \mathbb{C}^{d'}$ and the space $\mathcal{M}_{d \times d'}$ [10, 18]:

$$\begin{aligned} |\psi_i\rangle &= \sum_{k,l} a_{k,l}^{(i)} |k\rangle |l'\rangle \in \mathbb{C}^d \otimes \mathbb{C}^{d'} \iff A_i = (a_{k,l}^{(i)}) \in \mathcal{M}_{d \times d'}, \\ Sr(|\psi_i\rangle) &= \operatorname{rank}(A_i), \ \langle \psi_i | \psi_j \rangle = \operatorname{Tr}(A_i^{\dagger} A_j), \end{aligned}$$
(1)

where $\{|k\rangle\}$ and $\{|l'\rangle\}$ are the standard computational bases of \mathbb{C}^d and $\mathbb{C}^{d'}$, respectively. A $d \times d'$ matrix is called a *k*-singular-value-1 matrix if its nonzero singular values are $\{1, 1, 1, \dots, 1\}_k$. Then $|\psi_i\rangle$ is a special entangled state with Schmidt number k if and only if $\sqrt{k}A_i$ is a k-singular-value-1 matrix. Specially, when $d = d', |\psi_i\rangle$ is a maximally entangled state if and only if $\sqrt{d}A_i$ is a unitary matrix [13].

Definition 1 [18] A set of special entangled states with Schmidt number $k \{|\psi_i\rangle\}_{i=1}^{dd'}$ of $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ is called a special entangled basis with Schmidt number k (SEBk) if $\langle \psi_i | \psi_j \rangle = \delta_{ij}$.

Definition 2 [9, 10] A set of special entangled states with Schmidt number k $\{|\psi_i\rangle : i \in [n], n < dd'\}$ of $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ is called an *n*-number special unextendible entangled basis with Schmidt number k (SUEBk) if

- (i) $\langle \psi_i | \psi_j \rangle = \delta_{ij};$
- (ii) If $\langle \psi_i | \phi \rangle = 0$ for all $i \in [n]$, then $|\phi\rangle$ can not be a special entangled state with Schmidt number k.

Note that the condition (ii) in Definition 2.2 is a bit weaker than that in [9, 10], where it states that "if $\langle \psi_i | \phi \rangle = 0$ for all $i \in [n]$, then $Sr(|\psi\rangle) \neq k$ ". Specially, an SEBk is a product basis and an SUEBk is a UPB when k = 1; an SEBk is a maximally entangled basis (MEB) and an SUEBk is a UMEB when k = d [1, 6–8]. Analogous to SEBks and SUEBks, we give the definitions regarding to k-singular-value-1 matrices.

Definition 3 A set of k-singular-value-1 matrices $\{A_i\}_{i=1}^{dd'}$ of $\mathcal{M}_{d\times d'}$ is called a k-singular-value-1 Hilbert-Schmidt basis (SV1Bk) if $\operatorname{Tr}(A_i^{\dagger}A_i) = k\delta_{ij}$.

Definition 4 A set of k-singular-value-1 matrices $\{A_i : i \in [n], n < dd'\}$ of $\mathcal{M}_{d \times d'}$ is called an *n*-number unextendible k-singular-value-1 Hilbert-Schmidt basis (USV1Bk) if

- (i) $\operatorname{Tr}(A_i^{\dagger}A_j) = k\delta_{ij};$
- (ii) If $\text{Tr}(A_i^{\dagger}B) = 0$ for all $i \in [n]$, then B can not be a k-singular-value-1 matrix.

Due to the one-to-one relation, $\{|\psi_i\rangle\}$ is an SEBk if and only if $\{A_i\}$ is an SV1Bk; and $\{|\psi_i\rangle\}$ is an SUEBk if and only if $\{A_i\}$ is a USV1Bk.

Lemma 1 [18] If k|dd', then there is an SEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, and consequently there is an SV1Bk in $\mathcal{M}_{d \times d'}$.

Let \mathcal{L} denote a subspace of $\mathcal{M}_{d \times d'}$, \mathcal{L}^{\perp} denote the complementary space of \mathcal{L} , and \oplus denote the direct sum. Inspired by Lemmas 2 and 3 of [13], we get the following two lemmas.

Lemma 2 Let $\mathcal{M}_{d\times d'} = \mathcal{L} \oplus \mathcal{L}^{\perp}$. If there is an SV1Bk $\{A_i\}$ in \mathcal{L} and there is no k-singular-value-1 matrix in \mathcal{L}^{\perp} , then $\{A_i\}$ is a USV1Bk in $\mathcal{M}_{d\times d'}$.

Lemma 3 Let $\mathcal{M}_{d\times d'} = \mathcal{L} \oplus \mathcal{L}^{\perp}$. If there is an SV1Bk $\{A_i\}$ in \mathcal{L} and a USV1Bk $\{B_i\}$ in \mathcal{L}^{\perp} , then $\{A_i\} \cup \{B_i\}$ is a USV1Bk in $\mathcal{M}_{d\times d'}$.

3 Constructions of SUEBks from SEBks

In this section, we introduce several constructions of SUEBks from SEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

In the following notations, each \mathcal{L} defines a subspace of $\mathcal{M}_{d \times d'}$ which consists of all matrices having zero entries in the specified places. The subscripts of $\mathcal{L}^{(i)}$ denote the size of the submatrices consisting of stars for $i \in [5]$, and the size of the bottom right submatrix for i = 6. Let

$$\mathcal{L}_{d\times(d'-i)}^{(1)} = \begin{pmatrix} * \cdots * 0 \cdots 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * \cdots * 0 & \cdots & 0 \end{pmatrix}_{d\times d'}, \quad \mathcal{L}_{d\times i}^{(2)} = \begin{pmatrix} 0 \cdots 0 * \cdots * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 \cdots 0 * \cdots * \end{pmatrix}_{d\times d'}, ,$$

$$\mathcal{L}_{(d-i)\times d'}^{(3)} = \begin{pmatrix} * \cdots * \\ \vdots & \ddots & \vdots \\ 0 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 \cdots & 0 \end{pmatrix}_{d\times d'}, \quad \mathcal{L}_{i\times d'}^{(4)} = \begin{pmatrix} 0 \cdots 0 \\ \vdots & \ddots & \vdots \\ 0 \cdots & 0 \\ * \cdots * \\ \vdots & \ddots & \vdots \\ * \cdots * \end{pmatrix}_{d\times d'}, ,$$

$$\mathcal{L}_{i\times d'}^{(4)} = \begin{pmatrix} 0 \cdots 0 \\ \vdots & \ddots & \vdots \\ 0 \cdots & 0 \\ * \cdots * \\ \vdots & \ddots & \vdots \\ * \cdots * \end{pmatrix}_{d\times d'}, ,$$

$$\mathcal{L}_{i\times d'}^{(4)} = \begin{pmatrix} 0 \cdots 0 \\ \vdots & \ddots & \vdots \\ * \cdots * \end{pmatrix}_{d\times d'}, ,$$

$$\mathcal{L}_{i\times d'}^{(4)} = \begin{pmatrix} 0 \cdots 0 \\ \vdots & \ddots & \vdots \\ * \cdots * \end{pmatrix}_{d\times d'}, ,$$

then $\mathcal{M}_{d \times d'}$ has direct-sum decompositions $\mathcal{L}_{d \times (d'-i)}^{(1)} \oplus \mathcal{L}_{d \times i}^{(2)}, \mathcal{L}_{(d-i) \times d'}^{(3)} \oplus \mathcal{L}_{i \times d'}^{(4)}$ and $\mathcal{L}_{(d-i)\times(d'-t)}^{(5)} \oplus \mathcal{L}_{i\times t}^{(6)}$. We apply Lemmas 1 and 2 to the following four cases.

(1) k|d. For any $i \in [k-1]$ satisfying $d' - i \ge k$, use decomposition

$$\mathcal{M}_{d\times d'} = \mathcal{L}_{d\times (d'-i)}^{(1)} \oplus \mathcal{L}_{d\times i}^{(2)}$$

Since k|d, there is an SV1Bk in $\mathcal{L}_{d\times(d'-i)}^{(1)}$ from Lemma 1. As the rank of any matrix in $\mathcal{L}_{d \times i}^{(2)}$ is no more than i < k, then there is no k-singular-value-1 matrix in $\mathcal{L}_{d\times i}^{(2)}$. Thus there is a d(d'-i)-number USV1Bk in $\mathcal{M}_{d\times d'}$ by Lemma 2.

By similar arguments, we have the following three cases.

(2) $k \nmid d$. Write d = sk + r such that 0 < r < k. For any $t \in [k - r]^*$ satisfying $d' - t \ge k$, use

$$\mathcal{M}_{d\times d'} = \mathcal{L}_{sk\times (d'-t)}^{(5)} \oplus \mathcal{L}_{r\times t}^{(6)}.$$

Then there is an sk(d'-t)-number USV1Bk in $\mathcal{M}_{d\times d'}$.

(3) k|d'. For any $i \in [k-1]$ satisfying $d-i \geq k$, use decomposition

$$\mathcal{M}_{d\times d'} = \mathcal{L}_{(d-i)\times d'}^{(3)} \oplus \mathcal{L}_{i\times d'}^{(4)}.$$

Then there is a d'(d-i)-number USV1Bk in $\mathcal{M}_{d\times d'}$.

(4) $k \nmid d'$. Write d' = sk + r such that 0 < r < k. For any $t \in [k - r]^*$ satisfying $d-t \geq k$, use

$$\mathcal{M}_{d\times d'} = \mathcal{L}_{(d-t)\times sk}^{(5)} \oplus \mathcal{L}_{t\times r}^{(6)}$$

Then there is an sk(d-t)-number USV1Bk in $\mathcal{M}_{d\times d'}$.

For any k, d and d' satisfying the conditions of any of the above four cases, we can construct SUEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ from SEBks. However, if k = d = d' or k = 1, none of the four cases are satisfied. So we can not use this method to construct UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^d$ or UPBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$. We summarize them in the following theorem.

Theorem 1 When $2 \le k < d \le d'$ or $2 \le k = d < d'$, there are four different classes of SUEBks as the cases (1)-(4) above.

Example 1 There is a 28-number SUEB4 in $\mathbb{C}^6 \otimes \mathbb{C}^7$.

Since $4 \nmid 6$, we can get a 28-number SUEB4 in $\mathbb{C}^6 \otimes \mathbb{C}^7$ from Case (2) with t = 0.

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle), \ |\psi_2\rangle &= \frac{1}{2}(|01\rangle + |12\rangle + |23\rangle + |34\rangle), \\ |\psi_3\rangle &= \frac{1}{2}(|02\rangle + |13\rangle + |24\rangle + |35\rangle), \ |\psi_4\rangle &= \frac{1}{2}(|03\rangle + |14\rangle + |25\rangle + |36\rangle), \end{aligned}$$

Table 1: Our results about SUEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, $2 \leq k \leq d \leq d'$.

	Condition	No. of SUEBk
	$k d$ $d = sk + r, r \in [k-1]$ $k d'$ $d' = sk + r, r \in [k-1]$	$\begin{array}{l} d(d'-i),i\in [k-1],d'-i\geq k\\ sk(d'-t),t\in [k-r]^*,d'-t\geq k\\ d'(d-i),i\in [k-1],d-i\geq k\\ sk(d-t),t\in [k-r]^*,d-t\geq k \end{array}$
$ \psi_5 angle$:	$=\frac{1}{2}(04\rangle+ 15\rangle+ 26\rangle+ 30\rangle$	$\rangle), \ \psi_6\rangle = \frac{1}{2}(05\rangle + 16\rangle + 20\rangle + 31\rangle),$
$ \psi_7 angle$:	$=\frac{1}{2}(06\rangle+ 10\rangle+ 21\rangle+ 32\rangle$	$\rangle), \ \psi_8\rangle = \frac{1}{2}(00\rangle - 11\rangle + 22\rangle - 33\rangle),$
$ \psi_9 angle$ =	$=\frac{1}{2}(01\rangle- 12\rangle+ 23\rangle- 34\rangle$), $ \psi_{10}\rangle = \frac{1}{2}(02\rangle - 13\rangle + 24\rangle - 35\rangle),$
$ \psi_{11} angle$:	$=\frac{1}{2}(03\rangle - 14\rangle + 25\rangle - 36\rangle$	$\rangle), \psi_{12}\rangle = \frac{1}{2}(04\rangle - 15\rangle + 26\rangle - 30\rangle),$
$ \psi_{13} angle$:	$=\frac{1}{2}(05\rangle - 16\rangle + 20\rangle - 31\rangle$	$\rangle), \psi_{14}\rangle = \frac{1}{2}(06\rangle - 10\rangle + 21\rangle - 32\rangle),$
$ \psi_{15}\rangle$:	$=\frac{1}{2}(00\rangle+ 11\rangle- 22\rangle- 33\rangle$	$\rangle), \ \psi_{16}\rangle = \frac{1}{2}(01\rangle + 12\rangle - 23\rangle - 34\rangle),$
$ \psi_{17} angle$:	$=\frac{1}{2}(02\rangle+ 13\rangle- 24\rangle- 35\rangle$	$\rangle), \ \psi_{18}\rangle = \frac{1}{2}(03\rangle + 14\rangle - 25\rangle - 36\rangle),$
$ \psi_{19} angle$:	$=\frac{1}{2}(04\rangle+ 15\rangle- 26\rangle- 30\rangle$	$\rangle), \psi_{20}\rangle = \frac{1}{2}(05\rangle + 16\rangle - 20\rangle - 31\rangle),$
$ \psi_{21}\rangle$:	$=\frac{1}{2}(06\rangle+ 10\rangle- 21\rangle- 32\rangle$	$\rangle), \psi_{22}\rangle = \frac{1}{2}(00\rangle - 11\rangle - 22\rangle + 33\rangle),$
$ \psi_{23}\rangle$:	$=\frac{1}{2}(01\rangle - 12\rangle - 23\rangle + 34\rangle$	$\rangle), \psi_{24}\rangle = \frac{1}{2}(02\rangle - 13\rangle - 24\rangle + 35\rangle),$
$ \psi_{25}\rangle$:	$=\frac{1}{2}(03\rangle - 14\rangle - 25\rangle + 36\rangle$	$\rangle), \psi_{26}\rangle = \frac{1}{2}(04\rangle - 15\rangle - 26\rangle + 30\rangle),$
$ \psi_{27}\rangle$:	$=\frac{1}{2}(05\rangle - 16\rangle - 20\rangle + 31\rangle$	$\rangle), \psi_{28}\rangle = \frac{1}{2}(06\rangle - 10\rangle - 21\rangle + 32\rangle).$

Remark 1 Propositions 1 and 2 in [9] belong to our Case (4); Propositions 3 and 4 in [9] belong to our Case (2); Propositions 5 and 6 in [9] belong to our Case (1) and Case (3). But we give more constructions than those in [9], which can be easily seen from the constructions of SUEB4s in $\mathbb{C}^7 \otimes \mathbb{C}^{12}$. In fact, Case (3) provides a 72-number SUEB4, a 60-number SUEB4 and a 48-number SUEB4 in $\mathbb{C}^7 \otimes \mathbb{C}^{12}$, while [9] only gives a 48-number SUEB4 in $\mathbb{C}^7 \otimes \mathbb{C}^{12}$. Also, Our constructions cover all of the results in [8, 12, 13, 16]. See Table 1.

4 SUEBpks from SUEBks

In this section, we give a general construction of SUEB*pks* in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd'}$ from SUEB*ks* in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, where $p \leq q$. We introduce a combinatorial object first. A permutation matrix is a square matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere. By abusing of this concept, for nonsquare matrices, we call a $p \times q$ matrix ($p \leq q$) a permutation matrix if it has exactly one entry of 1 in each row and at most one entry of 1 in each column and 0s elsewhere. Let $J_{p\times q}$ be a $p \times q$ matrix with all entries being 1, then $J_{p\times q}$ can be decomposed as $J_{p\times q} = P_0 + P_1 + \cdots + P_{q-1}$, where each P_i is a permutation matrix. For example, let $P_l = P_0T^l$, $l \in [q]^*$, where

$$P_{0} = \begin{pmatrix} 1 \ 0 \ \cdots \ 0 \ 0 \ \cdots \ 0 \\ 0 \ 1 \ \cdots \ 0 \ 0 \ \cdots \ 0 \\ \vdots \ \vdots \ \ddots \ \vdots \ \vdots \\ 0 \ 0 \ \cdots \ 1 \ 0 \ \cdots \ 0 \end{pmatrix}_{p \times q} \text{ and } T = \begin{pmatrix} 0 \ 1 \ 0 \ \cdots \ 0 \\ 0 \ 0 \ 1 \ \cdots \ 0 \\ \vdots \ \vdots \ \ddots \ \vdots \\ 0 \ 0 \ 0 \ \cdots \ 1 \\ 1 \ 0 \ 0 \ \cdots \ 0 \\ a \times a \end{pmatrix}_{q \times q}$$

Theorem 2 If there is an N-number SUEBk in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ constructed from Section 3, and k|dd', then there is a (pqdd' - p(dd' - N))-number SUEBpk in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd'}$ for $p \leq q$.

Proof Given any decomposition $J_{p \times q} = P_0 + P_1 + \cdots + P_{q-1}$. For any $l \in [q]^*$, $a \in [p]^*$, define a $p \times q$ matrix Q_l^a by

$$Q_l^a(i,j) = \begin{cases} 0 & \text{if } P_l(i,j) = 0, \\ \xi_p^{a(i-1)} & \text{if } P_l(i,j) = 1, \end{cases}$$

where $\xi_p = e^{\frac{2\pi i}{p}}$. For each matrix $M \in \mathcal{M}_{pd \times qd'}$, write it as a block matrix $M = (M_{i,j})_{p \times q}$, where each $M_{i,j}$ is a $d \times d'$ submatrix. Then let \mathcal{L}_l be a subspace of $\mathcal{M}_{pd \times qd'}$ which consists of all block matrices M with $M_{i,j} = 0$ if $P_l(i,j) = 0$. Then we have space decomposition,

$$\mathcal{M}_{pd imes qd'} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_{q-1}$$

such that dim $\mathcal{L}_l = pdd'$ for all $l \in [q]^*$.

Assume that $\{A_j\}_{j=1}^N$ is an N-number USV1Bk in $\mathcal{M}_{d \times d'}$ constructed from Section 3. Since k|dd', there is an SV1Bk in $\mathcal{M}_{d \times d'}$ by Lemma 1. Denote it by $\{B_{s,t}\}$, where $s \in [d]$ and $t \in [d']$. For any $a \in [p]^*$, $l \in [q-1]$, $s \in [d]$, $t \in [d']$ and $j \in [N]$, define

$$C_{a,l}^{s,t} = Q_l^a \otimes B_{s,t}$$
 and $C_{a,0}^j = Q_0^a \otimes A_j$.

Obviously, $C_{a,l}^{s,t} \in \mathcal{L}_l$ for any $l \in [q-1]$ and $C_{a,0}^j \in \mathcal{L}_0$. Now we show that $\{C_{a,l}^{s,t}\} \cup \{C_{a,0}^j\}$ is a (pqdd' - p(dd' - N))-number USV1Bpk in $\mathcal{M}_{pd \times qd'}$.

If the nonzero singular values of two matrices A and B are $\{1, 1, \dots 1\}_p$ and $\{1, 1, \dots 1\}_k$, respectively, then the singular values of $A \otimes B$ are $\{1, 1, \dots, 1\}_{pk}$

[19]. Thus it is easy to see that $C_{k,l}^{s,t}$ and $C_{a,0}^{j}$ are pk-singular-value-1 matrices. It is also easy to see that $\operatorname{Tr}[(C_{\tilde{a},l}^{\tilde{s},\tilde{t}})^{\dagger}C_{a,l}^{s,t}] = pk\delta_{\tilde{a}a}\delta_{\tilde{s}s}\delta_{\tilde{t}t}$ and $\operatorname{Tr}[(C_{\tilde{a},0}^{\tilde{j}})^{\dagger}C_{a,0}^{j}] = pk\delta_{\tilde{a}a}\delta_{\tilde{j}j}$, where $a, \tilde{a} \in [p]^*$; $s, \tilde{s} \in [d]$; $t, \tilde{t} \in [d']$; $j, \tilde{j} \in [N]$ and $l \in [q-1]$. It follows that $\{C_{a,l}^{s,t} : s \in [d], t \in [d'], a \in [p]^*\}$ is an SV1Bpk in \mathcal{L}_l for any $l \in [q-1]$. We assert that $\{C_{a,0}^{j}\}$ is a USV1Bpk in \mathcal{L}_0 . Given any $D = (D_{i,j})_{p \times q} \in \mathcal{L}_0$, let $D_i \triangleq D_{i,j}$ when $P_0(i,j) = 1$ for $i \in [p]$. If $\operatorname{Tr}(D^{\dagger}C_{a,0}^{j}) = 0$ for all $a \in [p]^*$ and $j \in [N]$, then

$$\operatorname{Tr}(D_1^{\dagger}A_j) + \xi_p^a \operatorname{Tr}(D_2^{\dagger}A_j) + \dots + \xi_p^{a(p-1)} \operatorname{Tr}(D_p^{\dagger}A_j) = 0.$$
⁽²⁾

This is equivalent to $WY_j = 0$ for all $j \in [N]$, where

$$W = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \xi_p & \cdots & \xi_p^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_p^{p-1} & \cdots & \xi_p^{(p-1)^2} \end{pmatrix}_{p \times p} \text{ and } Y_j = \begin{pmatrix} \operatorname{Tr}(D_1^{\dagger}A_j) \\ \operatorname{Tr}(D_2^{\dagger}A_j) \\ \vdots \\ \operatorname{Tr}(D_p^{\dagger}A_j) \end{pmatrix}_{p \times 1}.$$

Since det $(W) \neq 0$, we know that $Y_j = 0$ for all $j \in [N]$. It means that $\operatorname{Tr}(D_1^{\dagger}A_j) = \operatorname{Tr}(D_2^{\dagger}A_j) = \cdots = \operatorname{Tr}(D_p^{\dagger}A_j) = 0$ for all $j \in [N]$. As $\{A_j\}_{j=1}^N$ is a USV1Bk in $\mathcal{M}_{d\times d'}$ constructed from Section 3, we have $\operatorname{rank}(D_i) < k$ for all $i \in [p]$. Since the nonzero singular values of D_i , $i \in [p]$, form all the nonzero singular values of D, D can not be a pk-singular-value-1 matrix. This shows that $\{C_{a,0}^j\}$ is a USV1Bpk in \mathcal{L}_0 .

We conclude that $\{C_{a,l}^{s,t}\} \cup \{C_{a,0}^j\}$ is a (pqdd' - p(dd' - N))-number USV1Bpk in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd'}$ by Lemma 3.

From the proof of Theorem 2, we see that if k = d and D_i , $i \in [p]$ satisfy Eq (2) for all $j \in [N]$, then for any USV1Bd $\{A_j\}_{j=1}^N$ (not necessarily from Section 3), D_i is not a *d*-singular-value-1 matrix for $i \in [p]$, and consequently Dis not a *pd*-singular-value-1 matrix. Therefore, we have the following corollary.

Corollary 1 If there is an N-number UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$, then there is a (pqdd' - p(dd' - N))-number UMEB in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd'}$ for $p \leq q$.

Our method is better than the method in [13–15] since we can choose any decomposition of $J_{p\times q}$ into permutation matrices, and choose any different SV1Bds $\{B_{s,t}\}$ in $\mathcal{M}_{d\times d'}$.

Example 2 There is a 32-number UMEB in $\mathbb{C}^4 \otimes \mathbb{C}^9$ constructed from a 4-number UMEB in $\mathbb{C}^2 \otimes \mathbb{C}^3$.

Let $\{A_i\}_{i=1}^4$ be a 4-number USV1B2 in $\mathcal{M}_{2\times 3}$ that is constructed from Section 3 Case (1) with i = 1:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 \ 1 \ 0 \\ 1 \ 0 \ 0 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 \ 1 \ 0 \\ -1 \ 0 \ 0 \end{pmatrix}$$

Let

$$B_{s,t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^s & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}^t$$

where $s \in [2]^*$, $t \in [3]^*$. It is easy to see that $\{B_{s,t}\}$ is an SV1B2 in $\mathcal{M}_{2\times 3}$. Let $J_{2\times 3} = P_0 + P_1 + P_2$, $P_l = P_0 T^l$, $l \in [3]^*$, where

$$P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

For any $l \in [3]^*$, $a \in [2]^*$, define a 2 × 3 matrix Q_l^a by

$$Q_l^a(i,j) = \begin{cases} 0 & \text{if } P_l(i,j) = 0, \\ (-1)^{a(i-1)} & \text{if } P_l(i,j) = 1. \end{cases}$$

Then for each $s \in [2]^*$, $t \in [3]^*$ and $j \in [4]$,

$$\begin{split} C^{s,t}_{0,1} &= \begin{pmatrix} 0 & B_{s,t} & 0 \\ 0 & 0 & B_{s,t} \end{pmatrix}, \\ C^{s,t}_{0,2} &= \begin{pmatrix} 0 & 0 & B_{s,t} \\ B_{s,t} & 0 & 0 \end{pmatrix}, \\ C^{j}_{0,0} &= \begin{pmatrix} A_{j} & 0 & 0 \\ 0 & A_{j} & 0 \end{pmatrix}, \\ C^{j}_{1,0} &= \begin{pmatrix} A_{j} & 0 & 0 \\ 0 & -A_{j} & 0 \end{pmatrix}. \end{split}$$

Thus $\{C_{0,1}^{s,t}, C_{1,1}^{s,t}, C_{0,2}^{s,t}, C_{1,2}^{s,t}, C_{0,0}^{j}, C_{1,0}^{j} : s \in [2]^{*}, t \in [3]^{*}, j \in [4]\}$ is a 32-number USV1B4 in $\mathcal{M}_{4\times 9}$.

Remark 2 Theorem 2 gives a unified recursive construction for SUEBks, including the special case about UMEBs when k = d in Corollary 1. In fact, Corollary 1 generalizes all the recursive constructions about UMEBs in [13–15] (see Table 2): Theorem 1 in [13, 14], is a special case when d = d' and p = q; Theorem 1 in [15] is the case when d = d'; and Theorem 2 in [15] is the case when p = q. Corollary 1 can obtain more new examples than the constructions in [15], see Example 2.

Remark 3 Corollary 1 can also provide new examples of UMEB in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ that are different from the ones constructed in Section 3. This can be easily seen from Example 2. Note that we can also get a 32-number UMEB in $\mathbb{C}^4 \otimes \mathbb{C}^9$ either from Case (1) with i = 1 or Case (4) with t = 0 in Section 3. For both cases, the Schmidt numbers of the states are no more than 1 in the complementary subspace of these UMEBs. However, there is a state with Schmidt number 2 in the complementary subspace of the UMEB in Example 2.

Table 2: Recursive constructions for UMEBs $(p \le q, d \le d')$

Condition	No. of UMEB	Reference
$\begin{array}{l} N\text{UMEB in } \mathbb{C}^d \otimes \mathbb{C}^d \\ N\text{UMEB in } \mathbb{C}^d \otimes \mathbb{C}^d \\ N\text{UMEB in } \mathbb{C}^p \otimes \mathbb{C}^q \\ N\text{UMEB in } \mathbb{C}^d \otimes \mathbb{C}^{d'} \end{array}$	$\begin{array}{l} ((qd)^2 - q(d^2 - N))\text{UMEB in } \mathbb{C}^{qd} \otimes \mathbb{C}^{qd} \\ (pqd^2 - p(d^2 - N))\text{UMEB in } \mathbb{C}^{pd} \otimes \mathbb{C}^{qd} \\ (pqd^2 - d(pq - N))\text{UMEB in } \mathbb{C}^{pd} \otimes \mathbb{C}^{qd} \\ (pqdd' - p(dd' - N))\text{UMEB in } \mathbb{C}^{pd} \otimes \mathbb{C}^{qd'} \end{array}$	[13, 14] [15] [15] This paper

5 UMEBs from partial Hadamard matrices

In [11], the authors gave a construction of UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^d$ from partial Hadamard matrices. In this section, we generalize this construction to UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ with $d \leq d'$.

A Hadamard matrix is a complex square matrix with entries in the unit circle T whose rows are pairwise orthogonal. It is called a partial Hadamard matrix when the number of rows is less than the number of columns. As in Section 4, there is a matrix decomposition $J_{d \times d'} = P_0 + P_1 + \cdots + P_{d'-1}, d \leq d'$. For any $l \in [d']^*, a \in [d]^*$, define a $d \times d'$ matrix Q_l^a by

$$Q_l^a(i,j) = \begin{cases} 0 & \text{if } P_l(i,j) = 0 \\ \xi_d^{a(i-1)} & \text{if } P_l(i,j) = 1 \end{cases}$$

Let \mathcal{L}_l be the subspace of $\mathcal{M}_{d \times d'}$ which consists of all matrices in $M_{d \times d'}$ with $m_{i,j} = 0$ if $P_l(i,j) = 0$. Then

$$\mathcal{M}_{d\times d'}=\mathcal{L}_0\oplus \mathcal{L}_1\oplus\cdots\oplus \mathcal{L}_{d'-1},$$

and dim $\mathcal{L}_l = d$ for all $l \in [d']^*$. Obviously, $Q_l^a \in \mathcal{L}_l$ for any $l \in [d']^*$. It is easy to see that $\{Q_l^a : a \in [d]^*\}$ is an SV1Bd in \mathcal{L}_l for any $l \in [d'-1]$. Let $Z_0 = \{Q_l^a : l \in [d'-1], a \in [d]^*\}$. Similar to the method in Theorem 2, if there is a USV1Bd Z_1 in \mathcal{L}_0 , then $Z_0 \cup Z_1$ is a USV1Bd in $\mathcal{M}_{d \times d'}$ by Lemma 3.

Now we construct a USV1Bd Z_1 in \mathcal{L}_0 from a partial Hadamard matrix. Given an $m \times d$ partial Hadamard matrix $H_{m \times d} = (h_{i,j})$ with m < d, define a $d \times d'$ matrix $H_y, y \in [m]$, by

$$H_y(i,j) = \begin{cases} 0 & \text{if } P_0(i,j) = 0, \\ h_{y,i} & \text{if } P_0(i,j) = 1, \end{cases}$$

and let

$$Z_1 = \{H_y\}_{y=1}^m$$

then each $H_y \in \mathcal{L}_0$ and it is a *d*-singular-value-1 matrix. Further,

$$\operatorname{Tr}(H_y^{\dagger}H_{y'}) = \sum_{l=1}^{d} \overline{h}_{y,l} h_{y',l} = d\delta_{yy'}$$

for all $y, y' \in [m]$ by the definition of Hadamard matrices. So we only need the unextendibility of Z_1 , which is equivalent to the unextendibility of the Hadamard matrix H.

Theorem 3 Given a partial Hadamard matrix $H_{m\times d}$, then $Z_0 \cup Z_1$ is a (d(d'-1)+m)-number UMEB in $\mathcal{M}_{d\times d'}$ if and only if $H_{m\times d}$ can not be extended to an $(m+1)\times d$ partial Hadamard matrix.

Example 3 There is a (5(d'-1)+3)-number UMEB in $\mathbb{C}^5 \otimes \mathbb{C}^{d'}$ for all $d' \geq 5$. Let

$$H_{3\times 5} = (h_{i,j}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & \omega & \omega^2 \\ \frac{\sqrt{5}+i}{\sqrt{6}} & i & \frac{-\sqrt{5}+i}{\sqrt{6}} & \frac{\sqrt{6}\omega^2 i + (\omega-1)i}{3} & \frac{\sqrt{6}\omega i + (\omega^2-1)i}{3} \end{pmatrix},$$

where $\omega = e^{\frac{2\pi i}{3}}$, then it is a partial Hadamard matrix that can not be extended to a 4×5 partial Hadamard matrix [11]. Let $J_{5 \times d'} = P_0 + P_1 + \cdots + P_{d'-1}$, $P_l = P_0 T^l$, $l \in [d']^*$, where

$$P_{0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}_{5 \times d'} \text{ and } T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{d \times d'}$$

For any $l \in [d']^*$, $a \in [5]^*$, define a $5 \times d'$ matrix Q_l^a by

$$Q_l^a(i,j) = \begin{cases} 0 & \text{if } P_l(i,j) = 0\\ \xi_5^{a(i-1)} & \text{if } P_l(i,j) = 1 \end{cases}$$

Let $Z_0 = \{Q_l^a, l \in [d'-1], a \in [5]^*\}$. Define a $5 \times d'$ matrix $H_y, y \in [3]$, where

$$H_y = \begin{pmatrix} h_{y,1} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & h_{y,2} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & h_{y,3} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & h_{y,4} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & h_{y,5} & 0 & \cdots & 0 \end{pmatrix}_{5 \times d'}$$

Then $Z_0 \cup \{H_1, H_2, H_3\}$ is a (5(d'-1)+3)-number USV1B5 in $\mathcal{M}_{5 \times d'}$.

Remark 4 Proposition 1 in [11] is a special case of Theorem 3 for d = d'. Theorem 3 provides new UMEBs, which can be seen from Example 3. In fact, if $d' \ge 10$, then there is a 5(d'-i)-number UMEB in $\mathbb{C}^5 \otimes \mathbb{C}^{d'}$, where $i \in [4]$; if d' = 5 + r, $r \in [4]$, then there is a 5(d'-i)-number UMEB in $\mathbb{C}^5 \otimes \mathbb{C}^{d'}$, where $i \in [r]$ [8, 12, 13, 16]. All these UMEBs have number divisible by 5. But Example 3 constructs a UMEB with number (5(d'-1)+3). The construction in Theorem 3 is also different from the constructions in Section 3 for the same reason. See Table 3 about a summary of numbers in UMEBs.

No. of UMEB	Reference
no UMEB	[7]
6	[7]
12	[7]
23	[11]
d^2	[8]
$dm, d' - m \in [d-1]$	[12]
$dm, d' - m \in [r]$	[12]
qd^2	[12, 16]
$d(d'-i), i \in [d-1]$	[13]
$d(d'-i), i \in [r]$	[13]
5(d'-1)+3	This paper
	no UMEB 6 12 23 d^2 $dm, d' - m \in [d - 1]$ $dm, d' - m \in [r]$ qd^2 $d(d' - i), i \in [d - 1]$ $d(d' - i), i \in [r]$

Table 3: Results about UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$.

6 Conclusion and discussion

We proposed three methods to construct SUEBks (UMEBs) in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$: a construction of SUEBks from SEBks; a recursive construction of SUEBpks in $\mathbb{C}^{pd} \otimes \mathbb{C}^{qd'}$ from SUEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$; and a construction of UMEBs from unextendible partial Hadamard matrices. We also give three examples: a 28-number SUEB4 in $\mathbb{C}^6 \otimes \mathbb{C}^7$; a 32-number UMEB in $\mathbb{C}^4 \otimes \mathbb{C}^9$ constructed from a 4-number UMEB in $\mathbb{C}^2 \otimes \mathbb{C}^3$; and a (5(d'-1)+3)-number UMEB in $\mathbb{C}^5 \otimes \mathbb{C}^{d'}$, respectively. Our results cover and improve most results in [8, 9, 11–16]. We hope that our results would be useful in studying quantum state tomography and cryptographic protocols with fixed Schmidt number.

Although there exist a lot of constructions of UMEBs and SUEBks, there are still many open questions. Are there UMEBs in $\mathbb{C}^d \otimes \mathbb{C}^d$ when d = p or 2p, where p is a prime and $p \equiv 3 \pmod{4}$? The minimum open case is when d = 7, which was said to have been solved in [11], but there is a mistake since k_4 can be zero in the construction. Further, what are the minimum and maximum numbers in SUEBks or UMEBs if they exist? Are there SEBks in $\mathbb{C}^d \otimes \mathbb{C}^{d'}$ when $k \nmid dd'$?

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