Advanced Alicki-Fannes-Winter method for energy-constrained quantum systems and its use

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Abstract

We describe an universal method for quantitative continuity analysis of entropic characteristics of energy-constrained quantum systems and channels. It gives asymptotically tight continuity bounds for basic characteristics of quantum systems of wide class (including multi-mode quantum oscillators) and channels between such systems under the energy constraint.

The main application of the proposed method is the advanced version of the uniform finite-dimensional approximation theorem for basic capacities of energyconstrained quantum channels.

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1 Introduction

Quantitative continuity analysis of characteristics of quantum systems and channels is important for different tasks of quantum information theory.¹ This is confirmed by a number of works devoted to this question [1, 2, 3, 4, 14, 20, 26, 34, 37, 39, 40, 43, 48, 50].

The first result in this direction is the famous Fannes' continuity bound (estimate for variation) for the von Neumann entropy in a finite dimensional quantum system used essentially in the proofs of many theorems in quantum information theory [14]. An optimized version of Fannes's continuity bound obtained by Audenaert in [2] states that

$$|H(\rho) - H(\sigma)| \le \varepsilon \ln(d-1) + h_2(\varepsilon), \quad \varepsilon = \frac{1}{2} \|\rho - \sigma\|_1,$$

for any states ρ and σ in a *d*-dimensional Hilbert space provided that $\varepsilon \leq 1 - 1/d$.

Another important result is the Alicki-Fannes continuity bound for the quantum conditional entropy obtained in [1] by using the elegant geometric method. This continuity bound is also used essentially in applications, in particular, it allows to prove uniform continuity of the squashed entanglement E_{sq} (one of the basic entanglement measures) on the set of all states of a finite-dimensional bipartite system (in fact, it is the necessity to prove the continuity of E_{sq} that motivated the research by Alicki and Fannes, c.f.[9]). The method used in [1] was then improved by different authors (c.f.[21, 43]). The optimal version of this method was proposed and used by Winter in [50] to obtain tight continuity bound for the quantum conditional entropy and for the relative entropy of entanglement. In fact, this method (in what follows we will call it the Alicki-Fannes-Winter method, briefly, the AFW-method) is quite universal, it gives uniform continuity bound for any bounded function f on the set $\mathfrak{S}(\mathcal{H})$ of quantum states which is *locally almost affine* in the following sense

$$-a(p) \le f(p\rho + (1-p)\sigma) - pf(\rho) - (1-p)f(\sigma) \le b(p),$$
(1)

for arbitrary states ρ and σ in $\mathfrak{S}(\mathcal{H})$ and any $p \in (0,1)$, where a(p) and b(p) are nonnegative functions on (0,1) vanishing as $p \to 0^+$. In quantum information theory the following classes of functions satisfying this condition are widely used:

- real linear combinations of marginal entropies of a state of a composite quantum system and their compositions with quantum channels and operations;
- basic characteristics of a quantum channels and operations (the output entropy, the entropy exchange, the mutual and coherent informations);

¹A very noncomplete list of *this and previous years* papers which results are based on using this or that continuity bound is the following [5, 8, 10, 12, 13, 22, 23, 25, 32, 33, 35, 41, 42, 44, 49].

• relative entropy distances from a state to a given convex set of states.

In particular, the AFW-method shows that any locally almost affine bounded function on $\mathfrak{S}(\mathcal{H})$ is uniformly continuous on $\mathfrak{S}(\mathcal{H})$.

The AFW-method can be used regardless of the dimension of the underlying Hilbert space \mathcal{H} under the condition that f is a bounded function on the whole set of states. But in analysis of infinite-dimensional quantum systems we often deal with functions which are bounded only on the sets of states with bounded energy, i.e. states ρ satisfying the inequality

$$\operatorname{Tr} H \rho \le E,$$
 (2)

where H is a positive operator – Hamiltonian of a quantum system associated with the space \mathcal{H} [16, 17, 40, 47, 50].

Winter was the first who proposed a way for quantitative continuity analysis of characteristics of infinite-dimensional quantum systems under the energy constraint (2). In [50] he obtained asymptotically tight continuity bounds for the von Neumann entropy and for the quantum conditional entropy under the energy constraint by using the two-step approach based on the AFW-method combined with finite-dimensional approximation of states with bounded energy. Winter's approach was used in [37] to obtain asymptotically tight continuity bounds for the quantum conditional mutual information under the energy constraint on one subsystem.

Application of Winter's method to any function f possessing property (1) on the set of states with bounded energy is limited by the approximation step, since it requires special estimates depending on this function. An attempt to obtain an universal continuity bound for functions on the set of states with bounded energy was made in [39], where the method using initial purification of states followed by the standard AFW-technique is proposed. This method allows to obtain continuity bounds for various characteristics of quantum systems and channels under different forms of energy constraints [39, 40]. It plays a central role in the proof of the uniform finite-dimensional approximation theorem for basic capacities of infinite-dimensional energy-constrained quantum channels [38].

The main drawback of the universal continuity bound proposed in [39] is its nonaccuracy: the main term of the upper bound for $|f(\rho) - f(\sigma)|$ depends on $\sqrt{\varepsilon}$, where $\varepsilon = \frac{1}{2} \|\rho - \sigma\|_1$. This is a corollary of the initial purification of the states ρ and σ .

The main aim of this paper is to propose universal continuity bounds for any function f possessing property (1) on the set of states with bounded energy which would be tight or close-to-tight (asymptotically, for large energy bound). Our method is close to Winter's two-step approach mentioned before but uses completely different approximation step based on the special property of quantum states with bounded energy stated in Lemma 3 in [40]. This approximation step exploits only property (1) of a function f (via the AFW-technique) and does not require anything else. As a result the whole two-step method becomes quite universal and accurate (Theorem 1).

The arguments used in the construction of universal continuity bound can be applied

to show existence of appropriate infinite-dimensional extensions for characteristics of finite-dimensional n-partite quantum systems and channels (Theorem 2).

The paper is organized as follows.

The essence of the proposed method is described in Section 3.1 in full generality (Theorems 1 and 2). Then, in Section 3.2, the specification of this method to the case of a multi-mode quantum oscillator is considered (Corollary 1).

In Section 4 we apply the general results of Section 3 to concrete characteristics of quantum system and channels. In particular, we essentially improve the continuity bounds for the quantum mutual information, the coherent information and the output Holevo quantity of a quantum channel under the input energy constraint previously obtained in [39, 40].

The main application of the proposed method is the advanced version of the uniform finite-dimensional approximation theorem for basic capacities of energy-constrained quantum channels presented in Section 5. Efficiency of the obtained estimates of the ε -sufficient input dimensions for all the basic capacities is confirmed by numerical calculations with the one-mode quantum oscillator in the role of the input system.

2 Preliminaries

2.1 Basic notations

Let \mathcal{H} be a separable infinite-dimensional Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded operators on \mathcal{H} with the operator norm $\|\cdot\|$ and $\mathfrak{T}(\mathcal{H})$ the Banach space of all trace-class operators on \mathcal{H} with the trace norm $\|\cdot\|_1$. Let $\mathfrak{S}(\mathcal{H})$ be the set of quantum states (positive operators in $\mathfrak{T}(\mathcal{H})$ with unit trace) [16, 30, 46].

Denote by $I_{\mathcal{H}}$ the identity operator on a Hilbert space \mathcal{H} and by $\mathrm{Id}_{\mathcal{H}}$ the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$.

The von Neumann entropy of a quantum state $\rho \in \mathfrak{S}(\mathcal{H})$ is defined by the formula $H(\rho) = \operatorname{Tr} \eta(\rho)$, where $\eta(x) = -x \ln x$ for x > 0 and $\eta(0) = 0$. It is a concave lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$ taking values in $[0, +\infty]$ [16, 27, 45]. The von Neumann entropy satisfies the inequality

$$H(p\rho + (1-p)\sigma) \le pH(\rho) + (1-p)H(\sigma) + h_2(p)$$
 (3)

valid for any states ρ and σ in $\mathfrak{S}(\mathcal{H})$ and $p \in (0,1)$, where $h_2(p) = \eta(p) + \eta(1-p)$ is the binary entropy [30, 46].

The quantum relative entropy for two states ρ and σ in $\mathfrak{S}(\mathcal{H})$ is defined as

$$H(\rho \| \sigma) = \sum \langle i | \rho \ln \rho - \rho \ln \sigma | i \rangle,$$

where $\{|i\rangle\}$ is the orthonormal basis of eigenvectors of the state ρ and it is assumed that $H(\rho || \sigma) = +\infty$ if supp ρ is not contained in supp σ [16, 27].²

²The support supp ρ of a state ρ is the closed subspace spanned by the eigenvectors of ρ corresponding to its positive eigenvalues.

The quantum conditional entropy

$$H(X|Y)_{\rho} = H(\rho) - H(\rho_Y)$$

of a state ρ in $\mathfrak{S}(\mathcal{H}_{XY})$ with finite marginal entropies is essentially used in analysis of quantum systems [16, 46]. The quantum conditional entropy can be extended to the set of all states ρ with finite $H(\rho_X)$ by the formula

$$H(X|Y) = H(\rho_X) - H(\rho \| \rho_X \otimes \rho_Y)$$
(4)

proposed in [24]. This extension possesses all basic properties of the quantum conditional entropy valid in finite dimensions [24, 36]. In particular, it is concave and satisfies the inequality

$$|H(X|Y)_{\rho}| \le H(\rho_X)$$

for arbitrary state ρ in $\mathfrak{S}(\mathcal{H}_{XY})$ with finite $H(\rho_X)$.

The quantum mutual information of a state ρ of a bipartite quantum system XY is defined as

$$I(X:Y)_{\rho} = H(\rho \| \rho_X \otimes \rho_Y) = H(\rho_X) + H(\rho_Y) - H(\rho),$$
(5)

where the second formula is valid if $H(\rho)$ is finite [28].

The quantum conditional mutual information (QCMI) of a state ρ of a tripartite finite-dimensional system XYZ is defined as

$$I(X:Y|Z)_{\rho} \doteq H(\rho_{XZ}) + H(\rho_{YZ}) - H(\rho) - H(\rho_{Z}).$$
(6)

This quantity plays important role in quantum information theory [15, 46], its nonnegativity is a basic result well known as strong subadditivity of von Neumann entropy [29]. If system Z is trivial then (6) coincides with (5).

In infinite dimensions formula (6) may contain the uncertainty " $\infty - \infty$ ". Nevertheless the conditional mutual information can be defined for any state ρ in $\mathfrak{S}(\mathcal{H}_{XYZ})$ by the expression

$$I(X:Y|Z)_{\rho} = \sup_{P_X} \left[I(X:YZ)_{Q_X \rho Q_X} - I(X:Z)_{Q_X \rho Q_X} \right], \quad Q_X = P_X \otimes I_{YZ}, \quad (7)$$

where the supremum is over all finite rank projectors $P_X \in \mathfrak{B}(\mathcal{H}_X)$ and it is assumed that $I(X:Y')_{Q_X\rho Q_X} = \lambda I(X:Y')_{\lambda^{-1}Q_X\rho Q_X}$, where $\lambda = \operatorname{Tr} Q_X \rho$ [36].

Expression (7) defines the lower semicontinuous nonnegative function on the set $\mathfrak{S}(\mathcal{H}_{XYZ})$ coinciding with the r.h.s. of (6) for any state ρ at which it is well defined and possessing all basic properties of the quantum conditional mutual information valid in finite dimensions [36, Th.2]. In particular,

$$I(X:Y|Z)_{\rho} \le 2\min\{H(\rho_X), H(\rho_Y), H(\rho_{XZ}), H(\rho_{YZ})\}$$
(8)

for arbitrary state ρ in $\mathfrak{S}(\mathcal{H}_{XYZ})$ and

$$I(XR:Y|Z)_{\rho} \ge I(X:Y|Z)_{\rho} \tag{9}$$

for arbitrary state ρ in $\mathfrak{S}(\mathcal{H}_{XYZR})$, where R is any quantum system.

2.2 The set of quantum states with bounded energy

Let H_A be a positive (semi-definite) operator on a Hilbert space \mathcal{H}_A and $\mathcal{D}(H_A)$ its domain.³ We will assume that

$$\operatorname{Tr} H_A \rho = \begin{cases} \sup_n \operatorname{Tr} P_n H_A \rho & \text{if } \operatorname{supp} \rho \subseteq \operatorname{cl}(\mathcal{D}(H_A)) \\ +\infty & \text{otherwise} \end{cases}$$
(10)

for any positive operator $\rho \in \mathfrak{T}(\mathcal{H}_A)$, where $\operatorname{cl}(\mathcal{D}(H_A))$ is the closure of $\mathcal{D}(H_A)$ and P_n is the spectral projector of H_A corresponding to the interval [0, n]. For any bounded function f we will assume that $\operatorname{Tr} f(H_A)$ is a trace over the Hilbert space $\operatorname{cl}(\mathcal{D}(H_A))$.

Let E_0 be the infimum of the spectrum of H_A and $E \ge E_0$. Then

$$\mathfrak{C}_{H_A,E} = \{\rho \in \mathfrak{S}(\mathcal{H}_A) \,|\, \mathrm{Tr}H_A \rho \le E\}$$

is a closed convex subset of $\mathfrak{S}(\mathcal{H}_A)$. If H_A is treated as Hamiltonian of a quantum system A then $\mathfrak{C}_{H_A,E}$ is the set of states with the mean energy not exceeding E.

It is well known that the von Neumann entropy is continuous on the set $\mathfrak{C}_{H_A,E}$ for any $E > E_0$ if (and only if) the Hamiltonian H_A satisfies the condition

$$\operatorname{Tr} e^{-\lambda H_A} < +\infty \quad \text{for all } \lambda > 0 \tag{11}$$

and that the maximal value of the entropy on this set is achieved at the *Gibbs state* $\gamma_A(E) \doteq e^{-\lambda(E)H_A}/\text{Tr}e^{-\lambda(E)H_A}$, where the parameter $\lambda(E)$ is determined by the equality $\text{Tr}H_A e^{-\lambda(E)H_A} = E \text{Tr}e^{-\lambda(E)H_A}$ [45]. Condition (11) implies that H_A is an unbounded operator having discrete spectrum of finite multiplicity. It can be represented as follows

$$H_A = \sum_{k=0}^{+\infty} E_k |\tau_k\rangle \langle \tau_k|, \qquad (12)$$

where $\{\tau_k\}_{k=0}^{+\infty}$ is the orthonormal basis of eigenvectors of H_A corresponding to the nondecreasing sequence $\{E_k\}_{k=0}^{+\infty}$ of eigenvalues tending to $+\infty$.

We will use the function

$$F_{H_A}(E) \doteq \sup_{\rho \in \mathfrak{C}_{H_A,E}} H(\rho) = H(\gamma_A(E)).$$
(13)

It is easy to show that F_{H_A} is a strictly increasing concave function on $[E_0, +\infty)$ such that $F_{H_A}(E_0) = \ln m(E_0)$, where $m(E_0)$ is the multiplicity of E_0 [50].

In this paper we will assume that the Hamiltonian H_A satisfies the condition

$$\lim_{\lambda \to 0^+} \left[\operatorname{Tr} e^{-\lambda H_A} \right]^{\lambda} = 1, \tag{14}$$

which is slightly stronger than condition (11). In terms of the sequence $\{E_k\}$ of eigenvalues of H_A condition (11) means that $\lim_{k\to\infty} E_k / \ln k = +\infty$, while (14) is valid

³Sometimes we will deal with positive operators which are not densely defined.

if $\liminf_{k\to\infty} E_k / \ln^q k > 0$ for some q > 2 [39, Proposition 1]. By Lemma 1 in [39] condition (14) holds if and only if

$$F_{H_A}(E) = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty,$$
 (15)

while condition (11) is equivalent to $F_{H_A}(E) = o(E)$ as $E \to +\infty$. It is essential that condition (14) holds for the Hamiltonians of many real quantum systems [5, 39].⁴

The function

$$\bar{F}_{H_A}(E) = F_{H_A}(E + E_0) = H(\gamma_A(E + E_0))$$
(16)

is concave and nondecreasing on $[0, +\infty)$. Let \hat{F}_{H_A} be a continuous function on $[0, +\infty)$ such that

$$\hat{F}_{H_A}(E) \ge \bar{F}_{H_A}(E) \quad \forall E > 0, \quad \hat{F}_{H_A}(E) = o(\sqrt{E}) \quad \text{as} \quad E \to +\infty$$
 (17)

and

$$\hat{F}_{H_A}(E_1) < \hat{F}_{H_A}(E_2), \quad \hat{F}_{H_A}(E_1)/\sqrt{E_1} \ge \hat{F}_{H_A}(E_2)/\sqrt{E_2}$$
 (18)

for any $E_2 > E_1 > 0$. Sometimes we will additionally assume that

$$\hat{F}_{H_A}(E) = \bar{F}_{H_A}(E)(1+o(1)) \text{ as } E \to +\infty.$$
 (19)

By property (15) the role of \hat{F}_{H_A} can be played by the function \bar{F}_{H_A} provided that the function $E \mapsto \bar{F}_{H_A}(E)/\sqrt{E}$ is nonincreasing. In general case the existence of a function \hat{F}_{H_A} with the required properties is established in the following

Proposition 1. A) If the Hamiltonian H_A satisfies condition (14) then

$$\hat{F}_{H_A}^*(E) \doteq \sqrt{E} \sup_{E' \ge E} \bar{F}_{H_A}(E') / \sqrt{E'}$$

is the minimal function satisfying all the conditions in (17) and (18). B) Let

$$N_{\uparrow}[H_A](E) \doteq \sum_{k,j:E_k+E_j \le E} E_k^2 \quad and \quad N_{\downarrow}[H_A](E) \doteq \sum_{k,j:E_k+E_j \le E} E_k E_j$$

for any $E > E_0$. If

$$\exists \lim_{E \to +\infty} N_{\uparrow}[H_A](E) / N_{\downarrow}[H_A](E) = a > 1$$
(20)

then

• there is E_* such that the function $E \mapsto \overline{F}_{H_A}(E)/\sqrt{E}$ is nonincreasing for all $E \ge E_*$ and hence $\hat{F}^*_{H_A}(E) = \overline{F}_{H_A}(E)$ for all $E \ge E_*$;

•
$$\hat{F}^*_{H_A}(E) = (a-1)^{-1}(\ln E)(1+o(1)) \text{ as } E \to +\infty.$$

⁴Theorem 3 in [5] shows that $F_{H_A}(E) = O(\ln E)$ as $E \to +\infty$ provided that condition (20) holds.

Proof. Part A of the proposition is proved immediately.

By noting that $F_{H_A}(E) = \lambda(E)E + \ln \operatorname{Tr} e^{-\lambda(E)H_A}$ for any $E > E_0$, where $\lambda(E)$ is defined after (11), it is easy to show that

$$\frac{d}{dE}\left(F_{H_A}(E)/\sqrt{E}\right) = \left(\lambda(E)E - \ln \operatorname{Tr} e^{-\lambda(E)H_A}\right)/(2E\sqrt{E})$$

for any $E > E_0$. So, part B follows from Theorem 3 in [5]. \Box

Note: The condition of part B of Proposition 1 is valid for the Hamiltonians of many real quantum systems [5].

Practically, it is convenient to use functions \hat{F}_{H_A} defined by simple formulae. The example of such function \hat{F}_{H_A} satisfying all the conditions in (17),(18) and (19) in the case when A is a multimode quantum oscillator is considered in Section 3.2.

Let $\hat{F}_{H_A}^{-1}$ be the inverse function to the function \hat{F}_{H_A} defined on $[\hat{F}_{H_A}(0), +\infty)$ taking values in $[0, +\infty)$. Let d_0 be the minimal natural number such that $\ln d_0 > \hat{F}_{H_A}(0)$. The following lemma plays a basic role in this paper.

Lemma 1. Let $\bar{E} \doteq E - E_0 \geq 0$, $d \geq d_0$ and $\gamma(d) \doteq \hat{F}_{H_A}^{-1}(\ln d)$. Let B be any system. If $\bar{E} \leq \gamma(d)$ then for any state ρ in $\mathfrak{S}(\mathcal{H}_{AB})$ such that $\operatorname{rank} \rho_A > d$ and $\operatorname{Tr} H_A \rho_A \leq E$ there exist states ρ , σ_1 and σ_2 in $\mathfrak{S}(\mathcal{H}_{AB})$ and a number $t \in (0, 1]$ such that $\operatorname{rank} \rho_A \leq d$, $\operatorname{Tr} H_A \rho_A \leq E$, $\frac{1}{2} \| \rho - \rho \|_1 \leq t \leq \sqrt{\bar{E}/\gamma(d)}$, $\operatorname{Tr} H_A [\sigma_k]_A \leq E_0 + \bar{E}/t^2$, k = 1, 2, and

$$\frac{1}{1+t}\rho + \frac{t}{1+t}\sigma_1 = \frac{1}{1+t}\rho + \frac{t}{1+t}\sigma_2.$$

Proof. Take a pure state $\hat{\rho}$ in $\mathfrak{S}(\mathcal{H}_{ABR})$ such that $\hat{\rho}_{AB} = \rho$. By Lemma 3 in [40] there exists⁵ a pure state $\hat{\varrho}$ in $\mathfrak{S}(\mathcal{H}_{ABR})$ such that rank $\hat{\varrho}_A \leq d$, $\mathrm{Tr}H_A\hat{\varrho}_A \leq E$, $\frac{1}{2}\|\hat{\rho}-\hat{\varrho}\|_1 \leq \sqrt{\bar{E}/\gamma(d)}$ and

$$\|\hat{\rho} - \hat{\varrho}\|_1 \operatorname{Tr} \bar{H}_A \left[[\hat{\rho} - \hat{\varrho}]_- \right]_A \le 2\bar{E}, \quad \|\hat{\rho} - \hat{\varrho}\|_1 \operatorname{Tr} \bar{H}_A \left[[\hat{\rho} - \hat{\varrho}]_+ \right]_A \le 2\bar{E}.$$

where $[\hat{\rho} - \hat{\varrho}]_{-}$ and $[\hat{\rho} - \hat{\varrho}]_{+}$ are, respectively, the negative and positive parts of the Hermitian operator $\hat{\rho} - \hat{\varrho}$ and $\bar{H}_A = H_A - E_0 I_A$.

It is easy to see that $\hat{\sigma}_1 = t^{-1}[\hat{\rho} - \hat{\varrho}]_-$ and $\hat{\sigma}_2 = t^{-1}[\hat{\rho} - \hat{\varrho}]_+$, where $t = \frac{1}{2} \|\hat{\rho} - \hat{\varrho}\|_1$, are states in $\mathfrak{S}(\mathcal{H}_{ABR})$. Then

$$\frac{1}{1+t}\,\hat{\rho} + \frac{t}{1+t}\,\hat{\sigma}_1 = \frac{1}{1+t}\,\hat{\varrho} + \frac{t}{1+t}\,\hat{\sigma}_2.$$

Hence, the states $\rho = \text{Tr}_R \hat{\rho}$ and $\sigma_k = \text{Tr}_R \hat{\sigma}_k$, k = 1, 2, with the above defined parameter t have the required properties, since $\frac{1}{2} \|\rho - \rho\|_1 \leq t$ by monotonicity of the trace norm under a partial trace. \Box

⁵It is easy to see that the assertion of Lemma 3 in [40] remains valid with the function \bar{F}_{H_A} replaced by its upper bound \hat{F}_{H_A} .

3 The main results

3.1 General case

Many important characteristics of states of a *n*-partite finite-dimensional quantum system $\hat{X} = X_1...X_n$ have a form of a function f on the set $\mathfrak{S}(\mathcal{H}_{\hat{X}})$ satisfying the inequalities

$$-a_{f}h_{2}(p) \leq f(p\rho + (1-p)\sigma) - pf(\rho) - (1-p)f(\sigma) \leq b_{f}h_{2}(p)$$
(21)

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_{\widehat{X}})$ and any $p \in [0, 1]$, where h_2 is the binary entropy (defined after (3)) and $a_f b_f \in \mathbb{R}_+$, and the inequalities

$$-c_f^- H(\rho_A) \le f(\rho) \le c_f^+ H(\rho_A), \tag{22}$$

for any state ρ in $\mathfrak{S}(\mathcal{H}_{\widehat{X}})$, where A is some subsystem of \widehat{X} and $c_f^-, c_f^+ \in \mathbb{R}_+$. Examples of characteristics satisfying (21) and (22) are presented in Section 4.

The AFW method (proposed in the optimal form in [50] and described in a full generality in the proof of Proposition 1 in [36]) allows to show that

$$|f(\rho) - f(\sigma)| \le C\varepsilon \ln d + Dg(\varepsilon), \quad d = \dim \mathcal{H}_A, \tag{23}$$

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_{\widehat{X}})$ such that $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon$, where $C = c_f^+ + c_f^-$, $D = a_f + b_f$ and

$$g(x) \doteq (1+x)h_2\left(\frac{x}{1+x}\right) = (x+1)\ln(x+1) - x\ln x.$$
(24)

Assume now that $\hat{X} = X_1...X_n$ is a *n*-partite infinite-dimensional quantum system and *f* is a function well defined on the set of all states ρ in $\mathfrak{S}(\mathcal{H}_{\hat{X}})$ with finite energy of ρ_A satisfying conditions (21) and (22) on this set. If the Hamiltonian H_A of the system *A* satisfies condition (14) then the modification of the AFW method proposed in [39] allows to obtain the following continuity bound for this function

$$|f(\rho) - f(\sigma)| \le C\sqrt{2\varepsilon}\bar{F}_{H_A}(\bar{E}/\varepsilon) + Dg(\sqrt{2\varepsilon}), \quad \bar{E} = E - E_0,$$
(25)

which holds for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_{\widehat{X}})$ such that $\operatorname{Tr} H_A \rho_A, \operatorname{Tr} H_A \sigma_A \leq E$ and $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon$, where \overline{F}_{H_A} is the function defined in (16) and E_0 is the minimal eigenvalue of H_A . The r.h.s. of (25) tends to zero as $\varepsilon \to 0$ due to the property (15) equivalent to (14). Continuity bound (25) is obtained by purification of the states ρ and σ followed by the standard AFW technique based on the property (21) and the inequalities

$$-c_f^- F_{H_A}(E) \le f(\rho) \le c_f^+ F_{H_A}(E),$$
 (26)

valid for any state ρ in $\mathfrak{S}(\mathcal{H}_{\widehat{X}})$ such that $\operatorname{Tr} H_A \rho_A \leq E$, which follow from (22). The main drawback of continuity bound (25) is its nonaccuracy for small ε related to its dependence on $\sqrt{\varepsilon}$ (this is a corollary of the initial purification of ρ and σ).

In the following theorem we present more accurate continuity bound for a function f on the sets of states in $\mathfrak{S}(\mathcal{H}_{\widehat{X}})$ with finite energy of ρ_A satisfying conditions (21) and (22). We will assume that H_A is a positive operator on \mathcal{H}_A satisfying condition (14) with the minimal eigenvalue $E_0 \ge 0$, F_{H_A} is any continuous function on \mathbb{R}_+ satisfying conditions (17) and (18), d_0 is the minimal natural number such that $\ln d_0 > \hat{F}_{H_A}(0)$ and $\gamma(d) = \hat{F}_{H_4}^{-1}(\ln d)$ for any $d \ge d_0.^6$

Theorem 1. Let f be a function on the set $\{\rho \in \mathfrak{S}(\mathcal{H}_{\widehat{X}}) | \operatorname{Tr} H_A \rho_A < +\infty\}$ satisfying inequalities (21) and (22). Let $\overline{E} \doteq E - E_0 > 0$, $\varepsilon > 0$ and T = $(1/\varepsilon) \min\{1, \sqrt{\overline{E}/\gamma(d_0)}\}$. Then

$$|f(\rho) - f(\sigma)| \le C\varepsilon(1+4t) \left(\widehat{F}_{H_A}\left[\frac{\bar{E}}{(\varepsilon t)^2}\right] + \Delta\right) + D(2g(\varepsilon t) + g(\varepsilon(1+2t)))$$
(27)

for arbitrary states ρ and σ in $\mathfrak{S}(\mathcal{H}_{\hat{X}})$ s.t. $\operatorname{Tr} H_A \rho_A, \operatorname{Tr} H_A \sigma_A \leq E$ and $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon$ and any $t \in (0,T]$, where $C = c_f^+ + c_f^-$, $D = a_f + b_f$ and $\Delta = 1/d_0 + \ln 2$.

If conditions (19) and (20) hold ⁷ then the r.h.s. of (27) can be written as

$$C\varepsilon(1+4t)\left(\ln\left[\frac{\bar{E}}{(\varepsilon t)^2}\right]\frac{1+o(1)}{a-1}+\Delta\right)+D(2g(\varepsilon t)+g(\varepsilon(1+2t))),\quad\varepsilon\to0^+$$

If, in addition, both estimates in (26) are asymptotically tight for large E in the following sense

$$\lim_{E \to +\infty} \left[\frac{\inf_{\rho \in \mathfrak{C}_{H_A E}} f(\rho)}{F_{H_A}(E)} + c_f^- \right] = \lim_{E \to +\infty} \left[c_f^+ - \frac{\sup_{\rho \in \mathfrak{C}_{H_A E}} f(\rho)}{F_{H_A}(E)} \right] = 0, \quad (28)$$

where $\mathfrak{C}_{H_A,E} = \{ \rho \in \mathfrak{S}(\mathcal{H}_{\widehat{X}}) | \operatorname{Tr} H_A \rho_A \leq E \}$ and F_{H_A} is the function defined in (13), then continuity bound (27) with optimal t is asymptotically tight for large $E.^{8}$

Remark 1. Since the function \hat{F}_{H_A} satisfies condition (17) and (18), the r.h.s. of (27) (denoted by $\mathbb{CB}_t(\bar{E}, \varepsilon \mid C, D)$ in what follows) is a nondecreasing function of ε and \overline{E} tending to zero as $\varepsilon \to 0^+$ for any given \overline{E} , C, D and $t \in (0, T]$.

Remark 2. The "free" parameter t can be used to optimize continuity bound (27) for given values of E and ε .

Proof. Let $B = \widehat{X} \setminus A$. By Lemma 1 for any $d > d_0$ such that $\overline{E} \leq \gamma(d)$ there exist states $\rho, \varsigma, \alpha_k, \beta_k, k = 1, 2$, in $\mathfrak{S}(\mathcal{H}_{AB})$ and numbers $p, q \leq \sqrt{\bar{E}}/\gamma(d)$ such that rank ϱ_A , rank $\varsigma_A \leq d$, $\operatorname{Tr} H_A \varrho_A$, $\operatorname{Tr} H_A \varsigma_A \leq E$, $\frac{1}{2} \| \rho - \varrho \|_1 \leq p$, $\frac{1}{2} \| \sigma - \varsigma \|_1 \leq q$, $\operatorname{Tr} \bar{H}_A[\alpha_k]_A \leq \bar{E}/p^2$, $\operatorname{Tr} \bar{H}_A[\beta_k]_A \leq \bar{E}/q^2$, k = 1, 2, and

$$(1-p')\rho + p'\alpha_1 = (1-p')\varrho + p'\alpha_2, \quad (1-q')\sigma + q'\beta_1 = (1-q')\varsigma + q'\beta_2, \quad (29)$$

⁶The function $\hat{F}_{H_A}^*$ defined in Proposition 1 can be used in the role of \hat{F}_{H_A} .

⁷By Proposition 1 this holds, in particular, if $\hat{F}_{H_A} = \hat{F}^*_{H_A}$. ⁸A continuity bound $\sup_{x,y\in S_a} |f(x) - f(y)| \leq B_a(x,y)$ depending on a parameter *a* is called

asymptotically tight for large a if $\limsup_{a \to +\infty} \sup_{x,y \in S_a} \frac{|f(x) - f(y)|}{B_a(x,y)} = 1.$

where $\bar{H}_A = H_A - E_0 I_A$, $p' = \frac{p}{1+p}$ and $q' = \frac{q}{1+q}$. If rank $\rho_A \leq d$ we assume that $\rho = \rho$ and do not introduce the states α_k . Similar assumption holds if rank $\sigma_A \leq d$.

The function f is defined on all the states ρ , ς , α_1 , α_2 , β_1 , β_2 , since the function $\text{Tr}H_A(\cdot)$ (defined in (10)) is finite at their marginal states corresponding to the subsystem A.

By using the first relation in (29) and inequality (21) it is easy to show that

$$(1 - p')(f(\rho) - f(\varrho)) \le p'(f(\alpha_2) - f(\alpha_1)) + (a_f + b_f)h_2(p')$$

and

$$(1 - p')(f(\varrho) - f(\rho)) \le p'(f(\alpha_1) - f(\alpha_2)) + (a_f + b_f)h_2(p').$$

These inequalities imply that

$$|f(\varrho) - f(\rho)| \le p |f(\alpha_2) - f(\alpha_1)| + (a_f + b_f)g(p).$$
(30)

Similarly, by using the second relation in (29) and inequality (21) we obtain

$$|f(\varsigma) - f(\sigma)| \le q |f(\beta_2) - f(\beta_1)| + (a_f + b_f)g(q).$$
(31)

Since $\operatorname{Tr} \bar{H}_A[\alpha_k]_A \leq \bar{E}/p^2$ and $\operatorname{Tr} \bar{H}_A[\beta_k]_A \leq \bar{E}/q^2$, k = 1, 2, it follows from (26) that

$$|f(\alpha_2) - f(\alpha_1)| \le (c_f^+ + c_f^-) \widehat{F}_{H_A}(\bar{E}/p^2)$$
(32)

and

$$|f(\beta_2) - f(\beta_1)| \le (c_f^+ + c_f^-) \widehat{F}_{H_A}(\bar{E}/q^2).$$
(33)

Since $p, q \leq y \doteq \sqrt{\overline{E}/\gamma(d)}$ and the function $E \mapsto \widehat{F}_{H_A}(E)/\sqrt{E}$ is non-increasing, we have

$$x\widehat{F}_{H_A}(\bar{E}/x^2) \le y\widehat{F}_{H_A}(\bar{E}/y^2) = \sqrt{\bar{E}/\gamma(d)}\,\widehat{F}_{H_A}(\gamma(d)) = \sqrt{\bar{E}/\gamma(d)}\ln d,$$

x = p, q, where the last equality follows from the definition of $\gamma(d)$.

Thus, it follows from (30)-(33) and the monotonicity of the function g(x) that

$$|f(\varrho) - f(\rho)|, |f(\varsigma) - f(\sigma)| \le C\sqrt{\bar{E}/\gamma(d)}\ln d + Dg\left(\sqrt{\bar{E}/\gamma(d)}\right), \qquad (34)$$

where $C = c_f^+ + c_f^-$ and $D = a_f + b_f$.

Since $\operatorname{rank} \varrho_A \leq d$ and $\operatorname{rank} \varsigma_A \leq d$, the supports of both states ϱ_A and ς_A are contained in some 2*d*-dimensional subspace of \mathcal{H}_A . By the triangle inequality we have

$$\|\varrho - \varsigma\|_1 \le \|\varrho - \rho\|_1 + \|\varsigma - \sigma\|_1 + \|\rho - \sigma\|_1 \le 2\varepsilon + 4\sqrt{\bar{E}}/\gamma(d).$$

So, by using the standard AFW method one can show that⁹

$$|f(\varrho) - f(\varsigma)| \le \left(2\sqrt{\bar{E}/\gamma(d)} + \varepsilon\right) C \ln(2d) + Dg\left(2\sqrt{\bar{E}/\gamma(d)} + \varepsilon\right).$$
(35)

⁹It is easy to see that all the states used in this method are contained in the domain of f.

It follows from (34) and (35) that

$$|f(\rho) - f(\sigma)| \leq \left(4\sqrt{\bar{E}/\gamma(d)} + \varepsilon\right) C \ln d + \left(2\sqrt{\bar{E}/\gamma(d)} + \varepsilon\right) C \ln 2 + Dg\left(2\sqrt{\bar{E}/\gamma(d)} + \varepsilon\right) + 2Dg\left(\sqrt{\bar{E}/\gamma(d)}\right).$$
(36)

If $t \in (0, T]$ then, since the sequence $\gamma(d)$ is increasing, there is a natural number $d_* > d_0$ such that $\gamma(d_*) > \overline{E}/(\varepsilon t)^2 \ge \overline{E}$ but $\gamma(d_* - 1) \le \overline{E}/(\varepsilon t)^2$. It follows that

$$\sqrt{\bar{E}/\gamma(d_*)} \le \varepsilon t \le 1$$
 and $\ln(d_*-1) = \widehat{F}_{H_A}(\gamma(d_*-1)) \le \widehat{F}_{H_A}(\bar{E}/(\varepsilon t)^2),$

where the first condition in (18) was used. Since $\ln d_* \leq \ln(d_* - 1) + 1/(d_* - 1) \leq \ln(d_* - 1) + 1/d_0$, inequality (36) with $d = d_*$ implies continuity bound (27).

If conditions (19) and (20) hold then it follows from part B of Proposition 1 that $\hat{F}_{H_A}(E) = (a-1)^{-1} \ln(E)(1+o(1))$ as $E \to +\infty$. This implies the asymptotic representation of the r.h.s. of (27) and the following relation

$$\hat{F}_{H_A}(\bar{E}/(\varepsilon t)^2) = \left(F_{H_A}(E) - 2(a-1)^{-1}\ln(\varepsilon t)\right)(1+o(1)) \quad \text{as} \ E \to +\infty.$$
(37)

Assume that both estimates in (26) are asymptotically tight for large E. Then for any $\delta > 0$ there exists $E_{\delta} > 0$ such that for any $E > E_{\delta}$ the set $\mathfrak{C}_{H_A,E}$ contains states ρ and σ such that $|f(\rho) - f(\sigma)| \ge (C - \delta)F_{H_A}(E)$. Since $\frac{1}{2}\|\rho - \sigma\|_1 \le 1$, it follows that for any $\varepsilon > 0$ the set $\mathfrak{C}_{H_A,E}$ contains states ρ_{ε} and σ_{ε} such that¹⁰

$$\frac{1}{2} \|\rho_{\varepsilon} - \sigma_{\varepsilon}\|_{1} \le \varepsilon \quad \text{and} \quad |f(\rho_{\varepsilon}) - f(\sigma_{\varepsilon})| \ge \varepsilon (C - \delta) F_{H_{A}}(E).$$
(38)

It follows from (37) that the r.h.s. of (27) with $t = \varepsilon$ has the form $C\varepsilon F_{H_A}(E) + R(\varepsilon, E)$, where $R(\varepsilon, E)$ is a finite function such that $R(\varepsilon, E)/(\varepsilon F_{H_A}(E))$ tends to zero as (ε, E) tends to $(0, +\infty)$. So, it is easy to show that (38) implies the asymptotical tightness of the continuity bound (27) for large E. \Box

In Theorem 1 it is assumed that the function f is defined on the set of all states ρ in $\mathfrak{S}(\mathcal{H}_{\widehat{X}})$, where $\widehat{X} = X_1...X_n$, such that $\operatorname{Tr} H_A \rho_A < +\infty$, but often we deal with functions originally defined and satisfying the inequalities (21) and (22) only on the subset

$$\mathfrak{S}_{\mathrm{f}}(\mathcal{H}_{\widehat{X}}) \doteq \left\{ \rho \in \mathfrak{S}(\mathcal{H}_{\widehat{X}}) \mid \mathrm{rank}\rho_{X_{i}} < +\infty, \ i = \overline{1, n} \right\}$$
(39)

of $\mathfrak{S}(\mathcal{H}_{\widehat{X}})$. This is the case when we want to construct a characteristic of a infinitedimensional *n*-partite quantum system by using its finite-dimensional version. Thus, the question arises about the extension of a function defined on $\mathfrak{S}_{\mathrm{f}}(\mathcal{H}_{\widehat{X}})$ to larger subsets of $\mathfrak{S}(\mathcal{H}_{\widehat{X}})$, in particular, to the set of all states $\rho \in \mathfrak{S}(\mathcal{H}_{\widehat{X}})$ such that $\mathrm{Tr}H_A\rho_A < +\infty$. The technique used in the proof of Theorem 1 gives a partial solution of this question.

¹⁰This can be shown by using the states $\rho_k = \frac{k}{n}\rho + (1 - \frac{k}{n})\sigma$, k = 0, 1, ..., n, for sufficiently large n.

We begin with the following simple but useful observation.

Lemma 2. Let f be a function on the set $\mathfrak{S}_{f}(\mathcal{H}_{\widehat{X}})$ (defined in (39)) satisfying inequalities (21) and (22). Then there exists an extension of this function to the set $\mathfrak{C}_{A}^{*} \doteq \{\rho \in \mathfrak{S}(\mathcal{H}_{\widehat{X}}) | \operatorname{rank} \rho_{A} < +\infty\}$ satisfying the same inequalities, which is uniformly continuous on the set $\mathfrak{C}_{A}^{k} \doteq \{\rho \in \mathfrak{S}(\mathcal{H}_{\widehat{X}}) | \operatorname{rank} \rho_{A} < k\}$ for any natural k. This extension (also denoted by f) satisfies on the set \mathfrak{C}_{A}^{k} continuity bound (23) with dreplaced by 2k.

Proof. Let ρ and σ be any states in $\mathfrak{C}_k = \mathfrak{C}_A^k \cap \mathfrak{S}_f(\mathcal{H}_{\widehat{X}})$ for any given k. Since there is a 2k-dimensional subspace of \mathcal{H}_A containing the supports of both states ρ_A and σ_A , the standard AFW technique shows that inequality (23) holds for the states ρ and σ with d replaced by 2k. It follows that the function f is uniformly continuous on \mathfrak{C}_k . Since the set \mathfrak{C}_k is dense in \mathfrak{C}_A^k , the function f has a unique uniformly continuous extension to the set \mathfrak{C}_A^k satisfying continuity bound (23).

Since the extensions of f to the sets \mathfrak{C}_A^k and \mathfrak{C}_A^l agree with each other for any k and l (this follows from their uniqueness), the function f has an extension to the set $\mathfrak{C}_A^* = \bigcup_{k \in \mathbb{N}} \mathfrak{C}_A^k$ with the required properties. \Box

Theorem 2. Let f be a function on the set $\mathfrak{S}_{f}(\mathcal{H}_{\widehat{X}})$ (defined in (39)) satisfying inequalities (21) and (22). If H_{A} is a positive operator on \mathcal{H}_{A} satisfying condition (14) then for any $E \geq E_{0}$ the function f has a unique uniformly continuous extension to the set $\mathfrak{C}_{H_{A},E} = \{\rho \in \mathfrak{S}(\mathcal{H}_{\widehat{X}}) | \operatorname{Tr} H_{A} \rho_{A} \leq E\}$ satisfying the same inequalities and continuity bound (27) (obtained by means of any appropriate function $\hat{F}_{H_{A}}$).¹¹

Proof. By Lemma 2 the function f has an extension to the set \mathfrak{C}_A^* satisfying inequalities (21) and (22). Let \hat{F}_{H_A} be a function satisfying conditions (17) and (18), for example, the function $\hat{F}_{H_A}^*$ described in Proposition 1. Let ρ and σ be any states in

$$\mathfrak{C}^*_{H_A,E} = \{ \rho \in \mathfrak{S}(\mathcal{H}_{\widehat{X}}) \mid \mathrm{Tr}H_A\rho_A \leq E, \, \mathrm{rank}\rho_A < +\infty \}$$

By repeating the arguments from the proof of Theorem 1 one can prove inequality (27) for these states. It implies that the function f is uniformly continuous on the set $\mathfrak{C}^*_{H_A,E}$. Since the set $\mathfrak{C}^*_{H_A,E}$ is dense in $\mathfrak{C}_{H_A,E}$, the function f has a unique uniformly continuous extension to the set $\mathfrak{C}_{H_A,E}$ satisfying inequalities (21) and (22) with the same parameters and continuity bound (27). \Box

Example 1. Consider the function

$$P(\rho) = I(B:R)_{\rho} - I(E:R)_{\rho} = H(\rho_B) - H(\rho_{BR}) - H(\rho_E) + H(\rho_{ER})$$
(40)

on the set $\mathfrak{S}_{f}(\mathcal{H}_{BER})$, where B, E and R are any quantum systems.

Since $P(\rho) = H(R|E)_{\rho} - H(R|B)_{\rho}$ for any state ρ in $\mathfrak{S}_{f}(\mathcal{H}_{BER})$, by using concavity of the quantum conditional entropy and inequality (3) it is easy to show that the function P satisfies the inequality (21) on the set $\mathfrak{S}_{f}(\mathcal{H}_{BER})$ with $a_{f} = b_{f} = 1$.

¹¹We do not assume that H_A is a densely defined operator. The quantity $\text{Tr}H_A\rho$ for any ρ in $\mathfrak{S}(\mathcal{H}_A)$ is defined according to the rule (10).

The monotonicity of the quantum mutual information and upper bound (8) with trivial Z imply that

$$\max\{I(B:R)_{\rho}, I(E:R)_{\rho}\} \le I(BE:R)_{\rho} \le 2H(\rho_{BE})$$

for any state $\rho \in \mathfrak{S}_{f}(\mathcal{H}_{BER})$. So, the function P satisfies inequality (22) with A = BEand $c_{f}^{-} = c_{f}^{+} = 2$ on the set $\mathfrak{S}_{f}(\mathcal{H}_{BER})$. By Theorem 2 for any positive operator H_{BE} on \mathcal{H}_{BE} satisfying condition (14) and any $E > E_{0}$ there exists a unique uniformly continuous extension of the function P to the set $\{\rho \in \mathfrak{S}(\mathcal{H}_{BER}) | \operatorname{Tr} H_{BE} \rho_{BE} \leq E\}$ satisfying inequality (21) with $a_{f} = b_{f} = 1$ and inequality (22) with A = BE and $c_{f}^{-} = c_{f}^{+} = 2$. Note that the r.h.s. of (40) is not well defined on the above set.

The function P will be used in Section 4.5 for deriving continuity bound for the privacy of energy constrained quantum channels.

3.2 The case when A is the ℓ -mode quantum oscillator

Assume now that the system A (involved in (22)) is the ℓ -mode quantum oscillator with the frequencies $\omega_1, ..., \omega_\ell$. The Hamiltonian of this system has the form

$$H_{A} = \sum_{i=1}^{\ell} \hbar \omega_{i} a_{i}^{*} a_{i} + E_{0} I_{A}, \quad E_{0} = \frac{1}{2} \sum_{i=1}^{\ell} \hbar \omega_{i}, \quad (41)$$

where a_i and a_i^* are the annihilation and creation operators of the *i*-th mode [16]. Note that this Hamiltonian satisfies condition (20) with $a = 1 + 1/\ell$ [5, 6].

It is shown in [37, Section III.B] that in this case the function $F_{H_A}(E)$ defined in (13) is bounded above by the function

$$F_{\ell,\omega}(E) \doteq \ell \ln \frac{E + E_0}{\ell E_*} + \ell, \quad E_* = \left[\prod_{i=1}^{\ell} \hbar \omega_i\right]^{1/\ell}, \tag{42}$$

and that upper bound (42) is ε -sharp for large E. So, the function

$$\bar{F}_{\ell,\omega}(E) \doteq F_{\ell,\omega}(E+E_0) = \ell \ln \frac{E+2E_0}{\ell E_*} + \ell,$$
(43)

is a upper bound on the function $\overline{F}_{H_A}(E) \doteq F_{H_A}(E + E_0)$ satisfying all the conditions in (17),(18) and (19). The second condition in (18) follows from Lemma 5 in [40]. By using the function $\overline{F}_{\ell,\omega}$ in the role of function \hat{F}_{H_A} in Theorem 1 we obtain the following

Corollary 1. Let f be a function on the set $\{\rho \in \mathfrak{S}(\mathcal{H}_{\widehat{X}}) | \operatorname{Tr} H_A \rho_A < +\infty\}$ satisfying (21) and (22), where A is the ℓ -mode quantum oscillator with the frequencies $\omega_1, ..., \omega_\ell$. Let $E > E_0$, $\varepsilon > 0$ and $T_* = (1/\varepsilon) \min\{1, \sqrt{\overline{E}/E_0}\}$, where $\overline{E} = E - E_0$. Then

$$|f(\rho) - f(\sigma)| \le C\varepsilon(1+4t) \left(\ell \ln \frac{\bar{E}/(\varepsilon t)^2 + 2E_0}{\ell E_*} + \ell + \Delta^* \right) + D(2g(\varepsilon t) + g(\varepsilon(1+2t)))$$

$$(44)$$

$$\leq C\varepsilon(1+4t)\left(F_{\ell,\omega}(E) - 2\ell\ln(\varepsilon t) + \Delta^*\right) + D(2g(\varepsilon t) + g(\varepsilon(1+2t)))$$

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_{\widehat{X}})$ such that $\operatorname{Tr} H_A \rho_A \leq E$, $\operatorname{Tr} H_A \sigma_A \leq E$ and $\frac{1}{2} \|\rho - \sigma\|_1 \leq \varepsilon$ and any $t \in (0, T_*]$, where $C = c_f^- + c_f^+$, $D = a_f + b_f$ and $\Delta^* = e^{-\ell} + \ln 2$.

If both relations in (28) hold then continuity bound (44) with optimal t is asymptotically tight for large E.

Proof. All the assertions of the corollary directly follow from Theorem 1. It suffices to note that in this case d_0 is the minimal natural number not less than x^{ℓ} , where $x = 2E_0 e/(\ell E_*) \ge e$, and hence

$$\gamma(d_0) \doteq \bar{F}_{\ell,\omega}^{-1}(\ln d_0) = (\ell/e)E_*\sqrt[\ell]{d_0} - 2E_0 \le (\ell/e)E_*x\sqrt[\ell]{1+e^{-\ell}} - 2E_0 \le E_0$$

4 Applications

4.1 Basic examples

In this section we apply Theorem 1 to the basic entropic quantities: the von Neumann entropy, the quantum conditional entropy and the quantum conditional mutual information (QCMI). Asymptotically tight continuity bounds for the entropy and for the conditional entropy under the energy constraint have been obtained by Winter [50]. Asymptotically tight continuity bounds for the QCMI under the energy constraint on one of the subsystems has been obtained in [37] by using Winter's technique. The aim of this section is to show that our technique also gives asymptotically tight continuity bounds for these quantities without any claim of their superiority. In fact, the continuity bounds obtained by Winter's technique are slightly better than their analogues presented below, in particular, they hold under condition (11) (which is weaker than condition (14)) any do not require a function \hat{F}_{H_A} with properties (17) and (18). The main advantage of the method proposed in this paper is its universality, consisting, in particular, in possibility to obtain continuity bounds under different forms of energy constraint (see Example 4 and Remark 3 below).

In all the below examples f is a function on the set of states of infinite-dimensional composite system XY... satisfying the inequality (22) for a particular subsystem A of XY... and some c_f^-, c_f^+ and the inequality (21) for some a_f, b_f on the sets of all states ρ in $\mathfrak{S}(\mathcal{H}_{XY...})$ with finite $\operatorname{Tr} H_A \rho_A$, where H_A is a positive operator on \mathcal{H}_A satisfying condition (14) with the minimal eigenvalue E_0 . If H_A is not densely defined on \mathcal{H}_A we define the quantity $\operatorname{Tr} H_A \rho$ for any $\rho \in \mathfrak{S}(\mathcal{H}_A)$ according to the rule (10). We will assume that \hat{F}_{H_A} is a continuous function on \mathbb{R}_+ satisfying conditions (17) and (18). Denote by $\mathbb{CB}_t(\bar{E}, \varepsilon | C, D)$ the expression in the r.h.s. of (27).

Example 2. Let $f(\rho) = H(\rho)$ be the von Neumann entropy of a state ρ of a single quantum system X. This function satisfies inequality (21) with $a_f = 0$, $b_f = 1$ and inequality (22) with A = X and $c_f^+ = 1$, $c_f^- = 0$. So, for given $E > E_0$ and $\varepsilon > 0$ Theorem 1 implies that

$$|H(\rho) - H(\sigma)| \le \mathbb{CB}_t(\bar{E}, \varepsilon \,|\, 1, 1), \qquad \bar{E} = E - E_0, \tag{45}$$

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_A)$ such that $\operatorname{Tr} H_A \rho, \operatorname{Tr} H_A \sigma \leq E$ and $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon$ and any $t \in (0, T]$, where T is defined in Theorem 1. If conditions (19) and (20) hold then the last assertion of Theorem 1 shows that continuity bound (45) with optimal t is asymptotically tight for large E. This is true if A is the ℓ -mode quantum oscillator (and $\hat{F}_{H_A} = \bar{F}_{\ell,\omega}$). In this case (45) holds with $\mathbb{CB}_t(\bar{E}, \varepsilon \mid 1, 1)$ replaced by the expression in the r.h.s. of (44) with C = D = 1 for any $t \in (0, T_*]$, where $T_* = (1/\varepsilon) \min\{1, \sqrt{E/E_0}\}$.

Example 3. Let $f(\rho)$ be the extended quantum conditional entropy $H(X|Y)_{\rho}$ defined in (4). The extended conditional entropy satisfies inequality (21) with $a_f = 0$ and $b_f = 1$ and inequality (22) with A = X and $c_f^- = c_f^+ = 1$ [24, 36, 39]. So, for given $E > E_0$ and $\varepsilon > 0$ Theorem 1 implies that

$$H(A|Y)_{\rho} - H(A|Y)_{\sigma}| \le \mathbb{CB}_t(\bar{E}, \varepsilon \mid 2, 1), \qquad \bar{E} = E - E_0, \tag{46}$$

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_{AY})$ such that $\operatorname{Tr} H_A \rho_A, \operatorname{Tr} H_A \sigma_A \leq E$ and $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon$ and any $t \in (0, T]$, where T is defined in Theorem 1. If conditions (19) and (20) hold then continuity bound (46) with optimal t is asymptotically tight for large E. This is true if A is the ℓ -mode quantum oscillator (and $\hat{F}_{H_A} = \bar{F}_{\ell,\omega}$). In this case (46) holds with $\mathbb{CB}_t(\bar{E}, \varepsilon \mid 2, 1)$ replaced by the expression in the r.h.s. of (44) with C = 2, D = 1 for any $t \in (0, T_*]$, where $T_* = (1/\varepsilon) \min\{1, \sqrt{\bar{E}/E_0}\}$.

The tightness of (46) can be easily shown by using the last assertion of Theorem 1. Indeed, to prove the first relation in (28) one can take any purification of the Gibbs state $\gamma_A(E)$ in $\mathfrak{S}(\mathcal{H}_{AY})$ in the role of ρ . To show validity of the second relation in (28) one can take $\rho = \gamma_A(E) \otimes \sigma$, where σ is any state in $\mathfrak{S}(\mathcal{H}_Y)$.

Example 4. Let $f(\rho)$ be the extended QCMI $I(X:Y|Z)_{\rho}$ defined by the expression (7). We will consider it as a function on the set of states of the extended system XYZR (which coincides with XYZ if the system R is trivial). The extended QCMI satisfies inequality (21) with $a_f = 1$ and $b_f = 1$ [40]. It satisfies inequality (22) with any of the subsystems

X, Y, XR, YR, XZ, YZ, XZR, YZR(47)

in the role of system A and $c_f^- = 0$, $c_f^+ = 2$ in all the cases. This follows from the nonnegativity of the QCMI, the upper bounds (8) and the monotonicity of QCMI expressed by inequality (9). So, for given $E > E_0$ and $\varepsilon > 0$ Theorem 1 implies that

$$|I(X:Y|Z)_{\rho} - I(X:Y|Z)_{\sigma}| \le \mathbb{CB}_t(\bar{E},\varepsilon \mid 2,2), \qquad \bar{E} = E - E_0, \tag{48}$$

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_{XYZR})$ such that $\operatorname{Tr} H_A \rho_A$, $\operatorname{Tr} H_A \sigma_A \leq E$ and $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon$ and any $t \in (0, T]$, where A is one of the subsystems in (47) and T is defined in Theorem 1. If conditions (19) and (20) hold then continuity bound (48) with optimal t is asymptotically tight for large E (for any choice of A). This is true if A is the ℓ -mode quantum oscillator (and $\hat{F}_{H_A} = \bar{F}_{\ell,\omega}$). In this case (48) holds with $\mathbb{CB}_t(\bar{E}, \varepsilon \mid 2, 2)$ replaced by the expression in the r.h.s. of (44) with C = D = 2 for any $t \in (0, T_*]$, where $T_* = (1/\varepsilon) \min\{1, \sqrt{E/E_0}\}$.

The tightness of (48) can be easily shown by using the last assertion of Theorem 1. Indeed, assume that A = X and that the systems Z and R are trivial, so that $f(\rho) = I(A:Y)_{\rho}$. Then to show the validity of first and the second relations in (28) one can take, respectively, any product state $\rho_A \otimes \sigma_Y$ in $\mathfrak{S}(\mathcal{H}_{AY})$ such that $\operatorname{Tr} H_A \rho_A \leq E$ and any purification of the Gibbs state $\gamma_A(E)$ in $\mathfrak{S}(\mathcal{H}_{AY})$ in the role of ρ .

Remark 3. The possibility to take any of the systems in (47) in the role of system A in continuity bound (48) means possibility to obtain continuity bound for the QCMI under different forms of energy constraint. It is this feature that implies efficiency of the proposed technique, it will be used essentially in the following sections.

4.2 Tight continuity bound for the QCMI at the output of a local energy-constrained channel

Let Φ be a quantum channel from a system A to a system B (completely positive trace preserving linear map from $\mathfrak{T}(\mathcal{H}_A)$ into $\mathfrak{T}(\mathcal{H}_B)$), C and D any systems. In this subsection we obtain tight continuity bound for the function

$$\rho \mapsto I(B:D|C)_{\Phi \otimes \mathrm{Id}_{CD}(\rho)}$$

on $\mathfrak{S}(\mathcal{H}_{ACD})$ under the energy constraint on ρ_A which essentially refines the continuity bound for this function obtained in [40] by using the method from [39].¹²

Assume that H_A is the Hamiltonian of a quantum system A with the minimal energy E_0 satisfying condition (14) and \hat{F}_{H_A} is any function on \mathbb{R}_+ satisfying conditions (17) and (18). Denote by $\mathbb{CB}_t(\bar{E}, \varepsilon | C, D)$ the expression in the r.h.s. of (27).

Proposition 2. Let $\Phi : A \to B$ be a quantum channel, C and D be any systems, $E > E_0$ and $\varepsilon > 0$. Then

$$|I(B:D|C)_{\Phi\otimes \mathrm{Id}_{CD}(\rho)} - I(B:D|C)_{\Phi\otimes \mathrm{Id}_{CD}(\sigma)}| \le \mathbb{CB}_t(\bar{E},\varepsilon \,|\, 2,2)$$

$$\tag{49}$$

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_{ACD})$ such that $\operatorname{Tr} H_A \rho_A$, $\operatorname{Tr} H_A \sigma_A \leq E$ and $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon$ and any $t \in (0, T]$, where $\overline{E} = E - E_0$ and $T = T(\overline{E}, \varepsilon)$ is defined in Theorem 1.

If conditions (19) and (20) hold then continuity bound (49) with optimal t is asymptotically tight for large E. This is true, in particular, if A is the ℓ -mode quantum oscillator. In this case (49) holds with $\mathbb{CB}_t(\bar{E}, \varepsilon \mid 2, 2)$ replaced by the expression in the r.h.s. of (44) with C = D = 2 for any $t \in (0, T_*]$, where $T_* = (1/\varepsilon) \min\{1, \sqrt{E/E_0}\}$.

 $^{^{12}\}mathrm{We}$ assume that $I\left(B\!:\!D|C\right)$ is the extended QCMI defined by (7).

Proof. By the Stinespring theorem a quantum channel $\Phi: A \to B$ can be represented as

$$\Phi(\rho) = \text{Tr}_E V_\Phi \rho V_\Phi^*,\tag{50}$$

where V_{Φ} is an isometry from \mathcal{H}_A into \mathcal{H}_{BE} (\mathcal{H}_E is a separable Hilbert space) [16]. Continuity bound (49) can be obtained from continuity bound (48) with A = XR, where X = B and R = E, by identifying \mathcal{H}_A and H_A with the subspace $V_{\Phi}\mathcal{H}_A$ of \mathcal{H}_{BE} and the operator $V_{\Phi}H_AV_{\Phi}^*$ on \mathcal{H}_{BE} correspondingly.¹³

If conditions (19) and (20) hold then the tightness of continuity bound (49) follows from the tightness of continuity bound (48) in the case A = X. \Box

4.3 Tight continuity bounds for the mutual information and coherent information of energy constrained channels

In analysis of information properties of a channel Φ between finite-dimensional quantum systems A and B the quantities

$$I(\Phi, \rho) = H(\rho) + H(\Phi(\rho)) - H(\Phi, \rho)$$
(51)

and

$$I_c(\Phi,\rho) = H(\Phi(\rho)) - H(\Phi,\rho), \qquad (52)$$

where ρ is a state in $\mathfrak{S}(\mathcal{H}_A)$ and $H(\Phi, \rho)$ is the entropy exchange, are widely used. They are called, respectively, the *mutual information* and the *coherent information* of a quantum channel Φ at a state ρ [16, 46].

In infinite dimensions these quantities are well defined for any input state ρ with finite entropy by the expressions

$$I(\Phi, \rho) = I(B:R)_{\Phi \otimes \mathrm{Id}_R(\hat{\rho})},\tag{53}$$

and

$$I_c(\Phi,\rho) = I(B:R)_{\Phi \otimes \mathrm{Id}_R(\hat{\rho})} - H(\rho), \tag{54}$$

where $\mathcal{H}_R \cong \mathcal{H}_A$ and $\hat{\rho}$ is a pure state in $\mathfrak{S}(\mathcal{H}_{AR})$ such that $\hat{\rho}_A = \rho$ [19].

For any quantum channel $\Phi: A \to B$ the inequalities

$$0 \le I(\Phi, \rho) \le 2H(\rho),\tag{55}$$

and

$$-H(\rho) \le I_c(\Phi,\rho) \le H(\rho).$$
(56)

hold for any state ρ in $\mathfrak{S}(\mathcal{H}_A)$ with finite entropy. They follow from the expressions (53) and (54) and the well known properties of the quantum mutual information.

¹³Since in general the operator $H_{BE} \doteq V_{\Phi} H_A V_{\Phi}^*$ is not densely defined on \mathcal{H}_{BE} , the quantity $\operatorname{Tr} H_{BE} \rho$ for a state ρ in $\mathfrak{S}(\mathcal{H}_{BE})$ is defined according to the rule (10).

Continuity bound for the function $\rho \to I(\Phi, \rho)$ under the energy constraint on the input states was obtained in [39, Corollary 6]. It allows to prove uniform continuity of the function $\rho \to I(\Phi, \rho)$ on the set of states ρ such that $\text{Tr}H_A\rho \leq E$ for any $E > E_0$ provided that the Hamiltonian H_A satisfies the condition (14). The main drawback of that continuity bound is its nonaccuracy for small $\varepsilon = \frac{1}{2} \|\rho - \sigma\|_1$ connected with its dependence on $\sqrt{\varepsilon}$.

The technique proposed in Section 3 allows to essentially refine the continuity bound obtained in [39]. To apply this technique we will need the following lemma, in which it is assumed that $I(\Phi, \rho)$ and $I_c(\Phi, \rho)$ are defined by formulae (53) and (54).

Lemma 3. Let $\Phi : A \to B$ be an arbitrary quantum channel. Then

$$0 \le I(\Phi, p\rho + (1-p)\sigma) - pI(\Phi, \rho) - (1-p)I(\Phi, \sigma) \le 2h_2(p)$$
(57)

and

$$-h_2(p) \le I_c(\Phi, p\rho + (1-p)\sigma) - pI_c(\Phi, \rho) - (1-p)I_c(\Phi, \sigma) \le h_2(p)$$
(58)

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_A)$ with finite entropy and $p \in (0, 1)$.

Proof. The first inequality in (57) means concavity of the function $\rho \to I(\Phi, \rho)$ defined in (53). It is proved in [19] (by using the well known concavity of this function in the finite-dimensional case).

All the other inequalities in (57) and (58) are easily proved provided that the channel Φ has finite-dimensional output. Indeed, in this case $H(\Phi(\rho))$ is finite for any input state ρ . So, if $H(\rho)$ is finite then the entropy exchange $H(\Phi, \rho)$ (coinciding with the output entropy of a complementary channel) is also finite by the triangle inequality [16]. Hence, for any state ρ with finite entropy $H(\rho)$ the quantities $I(\Phi, \rho)$ and $I_c(\Phi, \rho)$ are well defined by formulae (51) and (52). So, in this case (58) and the second inequality in (57) follow from the concavity of the entropy and inequality (3).

If Φ is an arbitrary quantum channel then there is a sequence $\{\Phi_n\}$ of channels with finite dimensional output strongly converging to the channel Φ , i.e. $\lim_{n\to+\infty} \Phi_n(\rho) = \Phi(\rho)$ for any input state ρ . Proposition 10 in [36] implies that

$$\lim_{n \to +\infty} I(\Phi_n, \rho) = I(\Phi, \rho) \quad \text{and} \quad \lim_{n \to +\infty} I_c(\Phi_n, \rho) = I_c(\Phi, \rho)$$
(59)

for any input state ρ with finite entropy. It was mentioned before that (58) and the second inequality in (57) hold with $\Phi = \Phi_n$ for all n. So, it follows from (59) that these inequalities hold for the channel Φ as well. \Box

Assume that H_A is the Hamiltonian of a system A with the minimal energy E_0 satisfying condition (14) and \hat{F}_{H_A} is a function on \mathbb{R}_+ satisfying conditions (17) and (18). Denote by $\mathbb{CB}_t(\bar{E}, \varepsilon | C, D)$ the expression in the r.h.s. of (27).

Proposition 3. Let $\Phi: A \to B$ be a quantum channel, $E > E_0$ and $\varepsilon > 0$. Then

$$|I(\Phi,\rho) - I(\Phi,\sigma)| \le \mathbb{CB}_t(\bar{E},\varepsilon \,|\, 2,2),\tag{60}$$

$$|I_c(\Phi,\rho) - I_c(\Phi,\sigma)| \le \mathbb{CB}_t(\bar{E},\varepsilon \,|\, 2,2)$$
(61)

for any states ρ and σ in $\mathfrak{S}(\mathcal{H}_A)$ such that $\operatorname{Tr} H_A \rho$, $\operatorname{Tr} H_A \sigma \leq E$ and $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon$ and any $t \in (0, T]$, where $\overline{E} = E - E_0$ and $T = T(\overline{E}, \varepsilon)$ is defined in Theorem 1.

If conditions (19) and (20) hold then continuity bounds (60) and (61) with optimal t is asymptotically tight for large E. This is true, in particular, if A is the ℓ -mode quantum oscillator. In this case (60) and (61) hold for any $t \in (0, T_*]$ with the r.h.s. replaced by

$$2\varepsilon(1+4t)\left(F_{\ell,\omega}(E) - 2\ell\ln(\varepsilon t) + e^{-\ell} + \ln 2\right) + 4g(\varepsilon t) + 2g(\varepsilon(1+2t)), \qquad (62)$$

where $F_{\ell,\omega}$ is the function defined in (42) and $T_* = (1/\varepsilon) \min\{1, \sqrt{\overline{E}/E_0}\}.$

If Φ is an antidegradable channel (cf.[11, 46]) then (60) and (61) hold with $\mathbb{CB}_t(\bar{E}, \varepsilon \mid 1, 2)$ in the r.h.s. and the first factor 2 in (62) can be removed.

If Φ is degradable channel (cf.[11, 46]) then (61) holds with $\mathbb{CB}_t(\bar{E}, \varepsilon | 1, 2)$ in the r.h.s. and the first factor 2 in (62) can be removed.

Note: The r.h.s. of (60) and (61) coincide and do not depend on a channel Φ . They tend to zero as $\varepsilon \to 0$ for any given \overline{E} and t due to the second condition in (17) implying uniform continuity of the functions $\rho \mapsto I(\Phi, \rho)$ and $\rho \mapsto I_c(\Phi, \rho)$ on the set of states with bounded energy.

Proof. Continuity bounds (60) and (61) and their specifications for the case when A is the ℓ -mode quantum oscillator are derived from Theorem 1 and Corollary 1 by using (55), (56) and Lemma 3.

Assume that conditions (19) and (20) hold. To show the asymptotic tightness of continuity bound (60) assume that Φ is the noiseless channel, ρ is the Gibbs state $\gamma_A(E)$ for given $E > E_0$ and σ is any pure state in $\mathfrak{S}(\mathcal{H}_A)$ such that $\mathrm{Tr}H_A\sigma \leq E$. Then

$$I(\Phi, \rho) = 2H(\gamma_A(E)) = 2F_{H_A}(E)$$
 and $I(\Phi, \sigma) = 0.$

Thus, the asymptotic tightness of continuity bound (60) follows from the last assertion of Theorem 1.

To show the asymptotic tightness of continuity bound (61) assume that A is the one mode quantum oscillator with the frequency ω . In this case

$$\gamma_A(E) = (1 - p_E) \sum_{k=0}^{+\infty} p_E^k |\eta_k\rangle \langle \eta_k|,$$

where $p_E = (E - E_0)/(E + E_0)$, $E_0 = \frac{1}{2}\hbar\omega$, and $\{\eta_k\}_{k=0}^{+\infty}$ is the Fock basis [16].

Assume that $E \gg E_0$ and $q = p_{E'}$, where $E' = E - 4E_0$. Consider the states

$$\rho_1 = (1 - q^2) \sum_{k=1}^{+\infty} q^{2(k-1)} |\eta_{2k}\rangle \langle \eta_{2k}| \quad \text{and} \quad \rho_2 = (1 - q^2) \sum_{k=1}^{+\infty} q^{2(k-1)} |\eta_{2k-1}\rangle \langle \eta_{2k-1}|$$

and the channel $\Phi(\rho) = P\rho P + [\text{Tr}Q\rho]|\eta_0\rangle\langle\eta_0|$, where $P = \sum_{k=0}^{+\infty} |\eta_{2k}\rangle\langle\eta_{2k}|$ and $Q = I_A - P$. It is easy to show that $H(\rho_1) = H(\rho_2) \geq H(\gamma_A(E')) - \ln 2$ and that $\text{Tr}H_A\rho_1, \text{Tr}H_A\rho_2 \leq E$. So, by direct calculation we obtain

$$I_c(\Phi, \rho_1) = H(\rho_1) \ge F_{H_A}(E') - \ln 2, \quad I_c(\Phi, \rho_2) = -H(\rho_2) \le -F_{H_A}(E') + \ln 2.$$

Since in this case $F_{H_A}(E) = g((E-E_0)/(2E_0))$, these inequalities imply validity of both limit relations in (28) for the function $\rho \mapsto I_c(\Phi, \rho)$. Thus, the asymptotic tightness of continuity bound (61) follows from the last assertion of Theorem 1.

If Φ is an antidegradable channel then the r.h.s. of (55) and (56) can be replaced, respectively, by $H(\rho)$ and 0. If Φ is a degradable channel then the l.h.s. of (56) can be replaced by 0. These observations imply the last assertions of the proposition. \Box

4.4 Close-to-tight continuity bound for the output Holevo quantity of energy constrained channels

The technic proposed in Section 3 allows to essentially strengthen Proposition 7 in [40].

Let $\Phi : A \to B$ be an arbitrary quantum channel and $\{p_i, \rho_i\}$ a discrete ensemble of input states – a finite or countable collection $\{\rho_i\} \subset \mathfrak{S}(\mathcal{H}_A)$ with the corresponding probability distribution $\{p_i\}$. The output Holevo quantity of this ensemble under the channel Φ is defined as

$$\chi(\{p_i, \Phi(\rho_i)\}) \doteq \sum_i p_i H(\Phi(\rho_i) \| \Phi(\bar{\rho})) = H(\Phi(\bar{\rho})) - \sum_i p_i H(\Phi(\rho_i)),$$

where $\bar{\rho} = \sum_{i} p_i \rho_i$ is the average state of $\{p_i, \rho_i\}$ and the second formula is valid under the condition $H(\Phi(\bar{\rho})) < +\infty$. It is an important characteristic related to the classical capacity of a quantum channel [16, 46].

In analysis of continuity of the Holevo quantity we will use three measures of divergence between ensembles $\mu = \{p_i, \rho_i\}$ and $\nu = \{q_i, \sigma_i\}$ described in detail in [31, 37].

The quantity

$$D_0(\mu,\nu) \doteq \frac{1}{2} \sum_i \|p_i \rho_i - q_i \sigma_i\|_1$$

is an easily computable metric on the set of all discrete ensembles of quantum states considered as *ordered* collections of states with the corresponding probability distributions.

From the quantum information point of view it is natural to consider an ensemble of quantum states $\{p_i, \rho_i\}$ as a discrete probability measure $\sum_i p_i \delta(\rho_i)$ on the set $\mathfrak{S}(\mathcal{H})$ (where $\delta(\rho)$ is the Dirac measure concentrated at a state ρ) rather than ordered (or disordered) collection of states. If we want to identify ensembles corresponding to the same probability measure then we have to use the factorization of D_0 , i.e. the quantity

$$D_*(\mu,\nu) \doteq \inf_{\mu' \in \mathcal{E}(\mu), \nu' \in \mathcal{E}(\nu)} D_0(\mu',\nu')$$

as a measure of divergence between ensembles $\mu = \{p_i, \rho_i\}$ and $\nu = \{q_i, \sigma_i\}$, where $\mathcal{E}(\mu)$ and $\mathcal{E}(\nu)$ are the sets of all countable ensembles corresponding to the measures $\sum_i p_i \delta(\rho_i)$ and $\sum_i q_i \delta(\sigma_i)$ respectively. The factor-metric D_* coincides with the EHS-distance D_{ehs} between ensembles of quantum states proposed by Oreshkov and Cal-samiglia in [31]. It is obvious that

$$D_*(\mu,\nu) \le D_0(\mu,\nu)$$
 (63)

for any ensembles μ and ν .

We will also use the Kantorovich distance

$$D_K(\mu,\nu) = \frac{1}{2} \inf_{\{P_{ij}\}} \sum P_{ij} \|\rho_i - \sigma_j\|_1$$
(64)

between ensembles $\mu = \{p_i, \rho_i\}$ and $\nu = \{q_i, \sigma_i\}$ of quantum states, where the infimum is taken over all joint probability distributions $\{P_{ij}\}$ such that $\sum_j P_{ij} = p_i$ for all i and $\sum_i P_{ij} = q_j$ for all j. It is shown in [31] that

$$D_*(\mu,\nu) \le D_K(\mu,\nu) \tag{65}$$

for any discrete ensembles μ and ν .

In the study of infinite-dimensional quantum systems and channels the notion of generalized (continuous) ensemble defined as a Borel probability measure on the set of quantum states is widely used [16, 18]. We denote by $\mathcal{P}(\mathcal{H})$ the set of all Borel probability measures on $\mathfrak{S}(\mathcal{H})$. It contains the subset $\mathcal{P}_0(\mathcal{H})$ of discrete measures (corresponding to discrete ensembles). The average state of a generalized ensemble $\mu \in \mathcal{P}(\mathcal{H})$ is defined as the barycenter of the measure μ , that is $\bar{\rho}(\mu) = \int_{\mathfrak{S}(\mathcal{H})} \rho \mu(d\rho)$.

The Kantorovich distance (64) is extended to generalized ensembles μ and ν by the expression

$$D_{K}(\mu,\nu) = \frac{1}{2} \inf_{\Lambda \in \Pi(\mu,\nu)} \int_{\mathfrak{S}(\mathcal{H}) \times \mathfrak{S}(\mathcal{H})} \|\rho - \sigma\|_{1} \Lambda(d\rho, d\sigma),$$
(66)

where $\Pi(\mu, \nu)$ is the set of all Borel probability measures on $\mathfrak{S}(\mathcal{H}) \times \mathfrak{S}(\mathcal{H})$ with the marginals μ and ν . Since $\frac{1}{2} \|\rho - \sigma\|_1 \leq 1$ for any states ρ and σ , the Kantorovich distance (66) generates the weak convergence on the set $\mathcal{P}(\mathcal{H})$ [7].¹⁴

For an ensemble $\mu \in \mathcal{P}(\mathcal{H}_A)$ its image $\Phi(\mu)$ under a quantum channel $\Phi : A \to B$ is defined as the ensemble in $\mathcal{P}(\mathcal{H}_B)$ corresponding to the measure $\mu \circ \Phi^{-1}$ on $\mathfrak{S}(\mathcal{H}_B)$, i.e. $\Phi(\mu)[\mathfrak{S}_B] = \mu[\Phi^{-1}(\mathfrak{S}_B)]$ for any Borel subset \mathfrak{S}_B of $\mathfrak{S}(\mathcal{H}_B)$, where $\Phi^{-1}(\mathfrak{S}_B)$ is the pre-image of \mathfrak{S}_B under the map Φ . If $\mu = \{p_i, \rho_i\}$ then $\Phi(\mu) = \{p_i, \Phi(\rho_i)\}$.

For a given channel $\Phi : A \to B$ the output Holevo quantity of a generalized ensemble μ in $\mathcal{P}(\mathcal{H}_A)$ is defined as

$$\chi(\Phi(\mu)) = \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\rho) \| \Phi(\bar{\rho}(\mu))) \mu(d\rho) = H(\Phi(\bar{\rho}(\mu))) - \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\rho)) \mu(d\rho),$$

where the second formula is valid under the condition $H(\Phi(\bar{\rho}(\mu))) < +\infty$ [18].

Assume that H_A is the Hamiltonian of system A with the minimal energy E_0 satisfying condition (14) and \hat{F}_{H_A} is a function on \mathbb{R}_+ satisfying conditions (17) and (18). Denote by $\mathbb{CB}_t(\bar{E}, \varepsilon | C, D)$ the expression in the r.h.s. of (27).

¹⁴A sequence $\{\mu_n\}$ of measures weakly converges to a measure μ_0 if $\lim_{n\to\infty} \int f(\rho)\mu_n(d\rho) = \int f(\rho)\mu_0(d\rho)$ for any continuous bounded function f on $\mathfrak{S}(\mathcal{H})$ [7].

The following proposition contains continuity bound for the function $\mu \mapsto \chi(\Phi(\mu))$ under the constraint on the average energy of μ , i.e. under the condition

$$E(\mu) \doteq \operatorname{Tr} H_A \bar{\rho}(\mu) = \int \operatorname{Tr} H_A \rho \,\mu(d\rho) \le E.$$
(67)

Proposition 4. Let $\Phi: A \to B$ be a quantum channel, $E > E_0$ and $\varepsilon > 0$. Then

$$|\chi(\Phi(\mu)) - \chi(\Phi(\nu))| \le \mathbb{CB}_t(\bar{E}, \varepsilon \,|\, 2, 2) \tag{68}$$

for any ensembles μ and ν such that $E(\mu), E(\nu) \leq E$ and $D_K(\mu, \nu) \leq \varepsilon$ and any $t \in (0,T]$, where $\overline{E} = E - E_0$ and $T = T(\overline{E}, \varepsilon)$ is defined in Theorem 1.

If μ and ν are discrete ensembles then the Kantorovich metric D_K can be replaced by any of the metrics D_0 and D_* .

If conditions (19) and (20) hold then continuity bound (68) with optimal t is closeto-tight for large E up to factor 2 in the main term. This is true, in particular, if A is the ℓ -mode quantum oscillator. In this case (68) holds with $\mathbb{CB}_t(\bar{E}, \varepsilon | 2, 2)$ replaced by the expression in the r.h.s. of (44) with C = D = 2 for any $t \in (0, T_*]$, where $T_* = (1/\varepsilon) \min\{1, \sqrt{\bar{E}/E_0}\}.$

Note: The r.h.s. of (68) does not depend on a channel Φ . It tends to zero as $\varepsilon \to 0$ for any given \overline{E} and t due to the second condition in (17).

Proof. The arguments from the proof of Corollary 7 in [39] with the use of Proposition 2 in Section 4.2 (instead of Proposition 5 in [39]) implies validity of the inequality (68) any discrete ensembles μ and ν such that $E(\mu), E(\nu) \leq E$ and $D_*(\mu, \nu) \leq \varepsilon$. It follows from (63) and (65) that this inequality holds for any $\varepsilon \geq D(\mu, \nu)$, where D is either D_0 or D_K .

If μ and ν are arbitrary generalized ensembles such that $E(\mu), E(\nu) \leq E$ and $D_K(\mu, \nu) \leq \varepsilon$ then inequality (68) can be proved by repeating the arguments from the proof of Proposition 7 in [40] based on approximation of μ and ν by weakly converging sequences of discrete ensembles.

Assume that conditions (19) and (20) hold. To show that in this case continuity bound (68) with optimal t is close-to-tight for large E one should assume that Φ is the ideal channel, take an ensemble μ consisting of the single Gibbs state $\gamma_A(E)$ and ensemble ν of pure states with the average state $\gamma_A(E)$ and to repeat the arguments from the proof of the last assertion of Theorem 1.

The specification of (68) to the case when A is a multi-mode oscillator follows from Corollary 1. \Box

4.5 Continuity bound for the privacy of energy constrained channels

In this subsection we will use the notion of a (generalized) ensemble of quantum states and different measures of divergence between such ensembles briefly described in the previous subsection. Let $\Phi : A \to B$ be an arbitrary quantum channel with the Stinespring represententation (50). Then the channel

$$\widehat{\Phi}(\rho) = \text{Tr}_B V_{\Phi} \rho V_{\Phi}^*, \tag{69}$$

from the system A to the environment E is called complementary to the channel [16].

The privacy of a quantum channel Φ at a discrete or continuous ensemble μ of input states is defined as

$$\pi_{\Phi}(\mu) = \chi(\Phi(\mu)) - \chi(\overline{\Phi}(\mu)),$$

provided that this difference is well defined [16].¹⁵ Despite the fact that complementary channel (69) to the channel Φ depends on the Stinespring representation (50) of Φ , its output Holevo quantity $\chi(\widehat{\Phi}(\mu))$ is uniquely defined [16].

Since the continuity bound for the output Holevo quantity presented in Proposition 4 does not depend on a channel, continuity bound for the function $\mu \mapsto \pi_{\Phi}(\mu)$ under the input average energy constraint can be obtained by using the same continuity bounds for the functions $\mu \mapsto \chi(\widehat{\Phi}(\mu))$ and $\mu \mapsto \chi(\Phi(\mu))$. But direct application of Theorem 1 gives more sharp continuity bound for the function $\mu \mapsto \pi_{\Phi}(\mu)$, especially, for degradable and antidegradable channels [11, 46].

Assume that H_A is the Hamiltonian of system A with the minimal energy E_0 satisfying condition (14) and \hat{F}_{H_A} is a function on \mathbb{R}_+ satisfying conditions (17) and (18). Denote by $\mathbb{CB}_t(\bar{E}, \varepsilon | C, D)$ the expression in the r.h.s. of (27).

Proposition 5. Let $\Phi: A \to B$ be a quantum channel, $E > E_0$ and $\varepsilon > 0$. Then

$$|\pi_{\Phi}(\mu) - \pi_{\Phi}(\nu)| \le \mathbb{CB}_t(\bar{E}, \varepsilon \,|\, 4, 2) \tag{70}$$

for any ensembles μ and ν such that $E(\mu), E(\nu) \leq E$ and $D_K(\mu, \nu) \leq \varepsilon$ and any $t \in (0,T]$, where $\overline{E} = E - E_0$ and $T = T(\overline{E}, \varepsilon)$ is defined in Theorem 1.¹⁶

If μ and ν are discrete ensembles then the Kantorovich metric D_K can be replaced by any of the metrics D_0 and D_* .

If A is the ℓ -mode quantum oscillator then (70) holds for any $t \in (0, T_*]$ with the r.h.s. replaced by

$$4\varepsilon(1+4t)\left(F_{\ell,\omega}(E) - 2\ell\ln(\varepsilon t) + e^{-\ell} + \ln 2\right) + 4g(\varepsilon t) + 2g(\varepsilon(1+2t)),\tag{71}$$

where $F_{\ell,\omega}$ is the function defined in (42) and $T_* = (1/\varepsilon) \min\{1, \sqrt{\overline{E}/E_0}\}$.

If Φ is either degradable or antidegradable channel then (70) holds with $\mathbb{CB}_t(\bar{E}, \varepsilon | 2, 2)$ in the r.h.s. and the first factor 4 in (71) can be replaced by 2.

Note: The r.h.s. of (70) does not depend on a channel Φ . It tends to zero as $\varepsilon \to 0$ for any given \overline{E} and t due to the second condition in (17) implying uniform continuity of the function $\mu \mapsto \pi_{\Phi}(\mu)$ on the set of generalized ensembles with bounded average energy w.r.t. the weak convergence topology.

¹⁵The term "privacy" is used in the literature in different senses.

 $^{{}^{16}}E(\mu)$ is the average energy of an ensemble μ defined in (67).

Proof. By using Example 1 at the end of Section 3.1 and representations (50) and (69) it is easy to show that the function

$$P_{\Phi}(\rho) = I(B:R)_{\Phi \otimes \mathrm{Id}_R(\rho)} - I(E:R)_{\widehat{\Phi} \otimes \mathrm{Id}_R(\rho)}, \quad \rho \in \mathfrak{S}(\mathcal{H}_{AR}),$$

where R is any system, is well defined and satisfies inequality (21) with $a_f = b_f = 1$ and inequality (22) with $c_f^- = c_f^+ = 2$ on the set of input states $\rho \in \mathfrak{S}(\mathcal{H}_A)$ with finite energy $\text{Tr}H_A\rho$.

Hence Theorem 1 implies that

$$|P_{\Phi}(\rho) - P_{\Phi}(\sigma)| \le \mathbb{CB}_t(\bar{E}, \varepsilon \mid 4, 2)$$
(72)

for any states ρ and σ such that $\operatorname{Tr} H_A \rho$, $\operatorname{Tr} H_A \sigma \leq E$ and $t \in (0, T]$, where $E = E - E_0$.

Continuity bound (72) implies inequality (70) for discrete ensembles μ and ν such that $E(\mu), E(\nu) \leq E$ and $D_*(\mu, \nu) \leq \varepsilon$. It is sufficient to note that $\pi_{\Phi}(\{p_i, \rho_i\}) = P_{\Phi}(\hat{\rho})$ for arbitrary discrete ensemble $\{p_i, \rho_i\}$, where $\hat{\rho} = \sum_i p_i \rho_i \otimes |i\rangle \langle i|$ is the qc-state determined by some orthonormal system $\{|i\rangle\}$ in \mathcal{H}_R , and to use the arguments from the proof of Corollary 7 in [39]. It follows from (63) and (65) that inequality (70) holds provided that $D(\mu, \nu) \leq \varepsilon$, where D is either D_0 or D_K .

If μ and ν are arbitrary generalized ensembles such that $E(\mu), E(\nu) \leq E$ and $D_K(\mu, \nu) \leq \varepsilon$ then inequality (70) can be proved by repeating the arguments from the proof of Proposition 7 in [40] based on approximation of μ and ν by weakly converging sequences of discrete ensembles.

If Φ is a degradable (antidegradable) channel then the above function P_{Φ} is non-negative (non-positive). It follows that (72) holds with $\mathbb{CB}_t(\bar{E}, \varepsilon | 2, 2)$ in the r.h.s.

The assertion concerning the case when A is the ℓ -mode quantum oscillator follows from Corollary 1. \Box

5 Advanced version of the uniform finite-dimensional approximation theorem for capacities of energyconstrained channels

The uniform finite-dimensional approximation theorem for capacities of energy-constrained channels (the UFA-theorem, in what follows) obtained in [38] states, briefly speaking, that dealing with some capacity C_* we may assume (accepting arbitrarily small error ε) that all the channels have the same finite-dimensional input space – the subspace corresponding to the $m_{C_*}(\varepsilon)$ minimal eigenvalues of the input Hamiltonian.

The estimates for the ε -sufficient input dimension $m_{C_*}(\varepsilon)$ obtained in [38] for all the basic capacities (excepting C_{ea}) turned out extremely hight for small ε (see Tables 1,2 in [38]). In this section we apply the advanced AFW-method to essentially refine that estimates for $m_{C_*}(\varepsilon)$. This makes the UFA-theorem more applicable for real tasks of quantum information theory. Assume that H_A is the Hamiltonian of a quantum system A satisfying condition (14). Then H_A has the representation (12) with the orthonormal basis of eigenvectors $\{\tau_k\}_{k=0}^{+\infty}$ and the corresponding nondecreasing sequence $\{E_k\}_{k=0}^{+\infty}$ of eigenvalues tending to $+\infty$.

Let \hat{F}_{H_A} be any function on \mathbb{R}_+ satisfying conditions (17) and (18), for example, the function $\hat{F}_{H_A}^*$ defined in Proposition 1. We will use the notations $\bar{E} = E - E_0$, $\bar{E}_m = E_m - E_0$ for all m > 0 and denote by $\mathbb{CB}_t(\bar{E}, \varepsilon | C, D)$ the expression in the r.h.s. of (27). Let d_0 be the minimal natural number such that $\ln d_0 > \hat{F}_{H_A}(0)$ and m_0 a number such that $\bar{E}_m \ge \gamma(d_0) \doteq \hat{F}_{H_A}^{-1}(\ln d_0)$ for all $m \ge m_0$.

A central role in the proof of the original UFA-theorem is played by Lemma 3 in [38]. By using the results of Section 3 one can essentially strengthen this lemma.

Lemma 4. Let $\Pi_m(\rho) = P_m \rho P_m + [\operatorname{Tr}(I_A - P_m)\rho] |\tau_0\rangle \langle \tau_0|$, where P_m is the projector on the subspace \mathcal{H}_A^m corresponding to the minimal m eigenvalues $E_0, ..., E_{m-1}$ of H_A and τ_0 is any eigenvector corresponding to the eigenvalue E_0 . Let ρ be a state in $\mathfrak{S}(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_R)$ such that $\sum_{k=1}^n \operatorname{Tr} H_A \rho_{A_k} \leq nE$ and $m \geq m_0$. Then

$$\left| I(B^n : R)_{\Phi^{\otimes n} \otimes \mathrm{Id}_R(\rho)} - I(B^n : R)_{\Psi_m^{\otimes n} \otimes \mathrm{Id}_R(\rho)} \right| \le n \mathbb{F}_t(u_m, m \mid 1), \quad u_m = \sqrt{\bar{E}/\bar{E}_m}, \quad (73)$$

for any channel $\Phi: A \to B$ and any $t \in (0,1]$, where $\Psi_m = \Phi \circ \Pi_m$ and

$$\mathbb{F}_{t}(u_{m}, m | s) \doteq ((4+8t)u_{m} + 2su_{m}^{2}t^{2})\hat{F}_{H_{A}}(\bar{E}_{m}/t^{2}) \\
+ (4+8t)(1/d_{0} + \ln 2)u_{m} + 4g(tu_{m}) + 2g((2+2t)u_{m})$$
(74)

is a quantity tending to zero as $m \to +\infty$ for any given t and $s \in \{0, 1\}$.¹⁷

If $\operatorname{Tr} H_A \rho_{A_k} \leq E$ for all $k = \overline{1, n}$ and $\overline{E} \leq \overline{E}_m/t^2$ then (73) holds with $n \mathbb{F}_t(u_m, m \mid 0)$ in the right hand side. If n = 1 and $s_m \doteq \overline{E}/\overline{E}_m + \sqrt{\overline{E}/\overline{E}_m} \leq 2$ then (73) holds with $n \mathbb{CB}_{t/2}(\overline{E}, s_m \mid 2, 2)$ in the right hand side.

Proof. The assumption of the lemma implies that $H(\rho_{A_k}) < +\infty$ for $k = \overline{1, n}$.

Let E be an environment for the channel Φ , so that the Stinespring representations (50) holds with some isometry V_{Φ} from \mathcal{H}_A into \mathcal{H}_{BE} .

Following the Leung-Smith telescopic method from [26] consider the states

$$\sigma_k = \Phi^{\otimes k} \otimes \Psi_m^{\otimes (n-k)} \otimes \mathrm{Id}_R(\rho), \quad k = 0, 1, ..., n.$$

By repeating the arguments from the proof of Lemma 3 in [38] we obtain

$$|I(B^{n}:R)_{\sigma_{n}} - I(B^{n}:R)_{\sigma_{0}}| \leq \sum_{k=1}^{n} \left| I(B_{k}:R|C_{k})_{\sigma_{k}} - I(B_{k}:R|C_{k})_{\sigma_{k-1}} \right|,$$
(75)

¹⁷The function g(x) is defined in (24).

where $C_k = B^n \setminus B_k$ and $I(B_k; R|C_k)$ is the extended QCMI defined in (7). The finite entropy of the states $\rho_{A_1}, ..., \rho_{A_n}$, upper bound (8) and monotonicity of the QCMI under local channels guarantee finiteness of all the terms in (75).

To estimate the k-th summand in the r.h.s. of (75) consider the states

$$\hat{\sigma}_k = V_{\Phi}^{\otimes n} \otimes I_R \, \varrho_k \, [V_{\Phi}^{\otimes n}]^* \otimes I_R$$

in $\mathfrak{S}(\mathcal{H}_{B^n E^n R})$, where $\varrho_k = \mathrm{Id}_A^{\otimes k} \otimes \Pi_m^{\otimes (n-k)} \otimes \mathrm{Id}_R(\rho)$, k = 0, 1, 2, ..., n. The state $\hat{\sigma}_k$ is an extension of the state σ_k for each k, i.e. $\mathrm{Tr}_{E^n} \hat{\sigma}_k = \sigma_k$. Note that $[\varrho_k]_{A_j} = \rho_{A_j}$ for $j \leq k$ and $[\varrho_k]_{A_j} = \Pi_m(\rho_{A_j})$ for j > k. Hence

$$\operatorname{Tr} H_A[\varrho_k]_{A_j} \le x_j \doteq \operatorname{Tr} H_A \rho_{A_j} \quad \text{for all } k \text{ and } j.$$
 (76)

In the proof of Lemma 3 in [38] it is shown that

$$\|\hat{\sigma}_k - \hat{\sigma}_{k-1}\|_1 \le 2\mathrm{Tr}(I_A - P_m)\rho_{A_k} + 2\sqrt{\mathrm{Tr}(I_A - P_m)\rho_{A_k}} \le 2\varepsilon_k,\tag{77}$$

where $\varepsilon_k \doteq 2\sqrt{\bar{x}_k/\bar{E}_m}, \ \bar{x}_k = x_k - E_0.$

Take any $t \in (0, 1/2]$. Let N_1 be the set of all indexes k for which $\bar{x}_k \leq \bar{E}_m/(4t^2)$ and $N_2 = \{1, ..., n\} \setminus N_1$. Let $n_i = \sharp(N_i), X_i = \frac{1}{n_i} \sum_{k \in N_i} x_k$ and $\bar{X}_i = X_i - E_0, i = 1, 2$. It follows from (75) that the left hand side of (73) do not exceed $S_1 + S_2$, where

$$S_{i} = \sum_{k \in N_{i}} |I(B_{k}:R|C_{k})_{\sigma_{k}} - I(B_{k}:R|C_{k})_{\sigma_{k-1}}|.$$

For each $k \in N_1$ we have $\varepsilon_k t \leq 1$. Since $\varepsilon_k^{-1} \sqrt{\bar{x}_k/\gamma(d_0)} = \frac{1}{2} \sqrt{\bar{E}_m/\gamma(d_0)} \geq 1/2$ for any $m \geq m_0$, continuity bound (48) with (76) and (77) imply (by the arguments used in the proof of Proposition 2) that

$$|I(B_k:R|C_k)_{\sigma_k} - I(B_k:R|C_k)_{\sigma_{k-1}}| \le \mathbb{CB}_t(\bar{x}_k,\varepsilon_k | 2,2)$$

= 4(1+4t) $\sqrt{\bar{x}_k/\bar{E}_m} \left(\hat{F}_{H_A}(\bar{E}_m/(4t^2)) + 1/d_0 + \ln 2\right)$
+4g $\left(2t\sqrt{\bar{x}_k/\bar{E}_m}\right) + 2g\left(2(1+2t)\sqrt{\bar{x}_k/\bar{E}_m}\right).$

Hence, by using the concavity of the functions \sqrt{x} and g(x) along with the monotonicity of g(x) we obtain

$$S_{1} \leq 4n_{1}(1+4t)\sqrt{\bar{X}_{1}/\bar{E}_{m}}\left(\hat{F}_{H_{A}}(\bar{E}_{m}/(4t^{2}))+1/d_{0}+\ln 2\right) + 4n_{1}g\left(2t\sqrt{\bar{X}_{1}/\bar{E}_{m}}\right)+2n_{1}g\left(2(1+2t)\sqrt{\bar{X}_{1}/\bar{E}_{m}}\right).$$
(78)

For each $k \in N_2$ the inequality $I(B_k : R | C_k) \leq I(B_k E_k : R | C_k)$ and upper bound (8) imply

$$|I(B_k:R|C_k)_{\sigma_k} - I(B_k:R|C_k)_{\sigma_{k-1}}| \le 2\max\{H([\hat{\sigma}_k]_{B_kE_k}), H([\hat{\sigma}_{k-1}]_{B_kE_k})\}$$
$$= 2\max\{H([\varrho_k]_{A_k}), H([\varrho_{k-1}]_{A_k})\} \le 2F_{H_A}(x_k),$$

where the last inequality follows from (76). Since $(n-n_2)X_1+n_2X_2 \leq nE$ and $X_1 \geq E_0$, we have $X_2 \leq n\bar{E}/n_2 + E_0$. So, by using concavity and monotonicity of the function F_{H_A} on $[E_0, +\infty)$ we obtain

$$S_2 \le \sum_{k \in N_2} 2F_{H_A}(x_k) \le 2n_2 F_{H_A}(X_2) \le 2n_2 F_{H_A}(n\bar{E}/n_2 + E_0) = 2n_2 \bar{F}_{H_A}(n\bar{E}/n_2).$$
(79)

It is easy to see that $\bar{X}_1 \leq \bar{E}$. Since $\bar{x}_k > \bar{E}_m/(4t^2)$ for all $k \in N_2$ and $(n - n_2)E_0 + \sum_{k \in N_2} \bar{x}_k + n_2 E_0 \leq \sum_{k \in N_1} x_k + \sum_{k \in N_2} x_k \leq nE$, we have $n_2/n \leq 4t^2 \bar{E}/\bar{E}_m$. So, it follows from (78),(79), concavity of the function \bar{F}_{H_A} on \mathbb{R}_+ and Lemma 1 in [38] that

$$\frac{S_1 + S_2}{n} \le 4(1+4t)\sqrt{\bar{E}/\bar{E}_m} \left(\hat{F}_{H_A}(\bar{E}_m/(4t^2)) + 1/d_0 + \ln 2\right) + 4g\left(2t\sqrt{\bar{E}/\bar{E}_m}\right) + 2g\left(2(1+2t)\sqrt{\bar{E}/\bar{E}_m}\right) + 8t^2(\bar{E}/\bar{E}_m)\bar{F}_{H_A}(\bar{E}_m/(4t^2)).$$

By replacing t by t/2 we obtain the main assertion of the lemma. The vanishing of the quantity $\mathbb{F}_t(u_m, m \mid s)$ as $m \to +\infty$ follows from the second condition in (17).

The assertion concerning the case $\operatorname{Tr} H_A \rho_{A_k} \leq E$ for all k = 1, n follows from the above proof, since in this case the set N_2 is empty. In the case n = 1 one can directly apply continuity bound (48) with trivial C by using the first inequality in (77) with k = 1, since in this case $\operatorname{Tr}(I_A - P_m)\rho_A \leq \overline{E}/\overline{E}_m$ by Lemma 5 in [38]. \Box

By using Lemma 4 one can obtain an advanced version of the UFA-theorem presented in [38]. In what follows $C_*(\Phi, H_A, E)$, where C_* is one of the capacities C_{χ} , C, \bar{Q} , Q, \bar{C}_p and C_p , denotes the corresponding capacity of a quantum channel Φ from a system A to any system B under the energy constraint determined by the Hamiltonian H_A and energy bound E (see the surveys in Section 4 in [38] and in [47]). In contrast to Theorem 1 in [38] we add the non-regularized quantum and private capacities \bar{Q} and \bar{C}_p but exclude the entanglement-assisted capacity C_{ea} , since for the latter capacity the estimates of the ε -sufficient input dimension obtained in [38] are close to tight.

Following [38] denote by $C^m_*(\Phi, H_A, E)$ the corresponding capacity of Φ obtained by block encoding used only states supported by the tensor powers of the *m*-dimensional subspace \mathcal{H}^m_A (defined in Lemma 4). It coincides with the capacity $C_*(\Phi_m, H_A, E)$ of the subchannel Φ_m of Φ corresponding to the subspace \mathcal{H}^m_A .

Assume that the Hamiltonian H_A has form (12) and \hat{F}_{H_A} is any function on \mathbb{R}_+ satisfying conditions (17) and (18).¹⁸ In the following theorem $\mathbb{CB}_t(\bar{E}, \varepsilon | 2, 2)$ is the

¹⁸By Proposition 1 such function \hat{F}_{H_A} exists if and only if the Hamiltonian H_A satisfies condition (14).

expression in the r.h.s. of (27) with C = D = 2 and $\mathbb{F}_t(u_m, m \mid s)$ is the function defined in (74) for all $m \ge m_0$ and $\mathbb{F}_t(u_m, m \mid s) = +\infty$ otherwise, where m_0 is defined before Lemma 4. We use the standard notations $\bar{E} = E - E_0$ and $\bar{E}_m = E_m - E_0$.

Theorem 3. Let C_* be one of the capacities C_{χ} , C, \overline{Q} , Q, \overline{C}_p and C_p . If the Hamiltonian H_A satisfies condition (14) and $E \geq E_0$ then for any $\varepsilon > 0$ there exists natural number $m_{C_*}(\varepsilon)$ such that

$$|C_*(\Phi, H_A, E) - C^m_*(\Phi, H_A, E)| \le \varepsilon \qquad \forall m \ge m_{C_*}(\varepsilon)$$

for arbitrary channel Φ from the system A to any system B.

The above number $m_{C_*}(\varepsilon)$ is the minimal natural number such that $E_m \ge E$ and $f_{C_*}(E, m, t) \le \varepsilon$ for at least one $t \in (0, 1]$, where

$$\begin{split} f_{C_{\chi}}(E,m,t) &= f_{\bar{Q}}(E,m,t) = \mathbb{CB}_{t/2}(\bar{E},s_m \mid 2,2), \quad s_m = \bar{E}/\bar{E}_m + \sqrt{\bar{E}/\bar{E}_m}, \\ f_C(E,m,t) &= \mathbb{F}_t(u_m,m \mid 0), \quad f_Q(E,m,t) = \mathbb{F}_t(u_m,m \mid 1), \quad u_m = \sqrt{\bar{E}/\bar{E}_m}, \\ f_{\bar{C}_p}(E,m,t) &= 2f_{C_{\chi}}(E,m,t) \quad and \quad f_{C_p}(E,m,t) = 2f_Q(E,m,t). \end{split}$$

If A is the ℓ -mode quantum oscillator with frequencies $\omega_1, ..., \omega_\ell$ then

- the sequence $\{E_k\}_{k\geq 0}$ consists of the numbers $\sum_{i=1}^{\ell} \hbar \omega_i (n_i 1/2), n_1, ..., n_{\ell} \in \mathbb{N}$ arranged in the nondecreasing order and m_0 is a number such that $E_{m_0} \geq 2E_0$;
- the quantities $\mathbb{CB}_{t/2}(\bar{E}, s_m | 2, 2)$ and $\mathbb{F}_t(u_m, m | s)$ can be defined, respectively, by the expressions

$$2s_m(1+2t)\left(\bar{F}_{\ell,\omega}(4\bar{E}/(s_m t)^2) + \Delta^*\right) + 2g(s_m(1+t)) + 4g(s_m t/2),$$

and

$$((4+8t)u_m + 2su_m^2 t^2)\bar{F}_{\ell,\omega}\left(\bar{E}_m/t^2\right) + (4+8t)u_m\Delta^* + 2g((2+2t)u_m) + 4g(tu_m),$$

where $\bar{F}_{\ell,\omega}(E)$ is the function defined in (43) and $\Delta^* = e^{-\ell} + \ln 2.$

Proof. The theorem is proved by repeating the arguments from the proof of Theorem 1 in [38] with the use of Lemma 4 instead of Lemma 3 in [38]. \Box

Example 5. Let A be the one-mode quantum oscillator with the frequency ω . In this case the Hamiltonian H_A has the spectrum $\{E_k = (k + 1/2)\hbar\omega\}_{k\geq 0}$, $F_{H_A}(E) = g(E/\hbar\omega - 1/2)$ and $\bar{F}_{1,\omega}(E) = \ln(E/\hbar\omega + 1) + 1$ [16, Ch.12]. The results of numerical calculations of $m_{C_*}(\varepsilon)$ for different values of the input energy bound E are presented

in the following tables corresponding to two values of the relative error $\varepsilon/F_{H_A}(E)$ equal respectively to 0.1 and 0.01.¹⁹

Table 1. The approximate values of $m_{C_*}(c)$ for $c = 0.11 H_A(D)$.									
$E/\hbar\omega$	$m_{C_{\chi}}(\varepsilon) = m_{\bar{Q}}(\varepsilon)$	$m_C(arepsilon)$	$m_Q(arepsilon)$	$m_{\bar{C}_p}(\varepsilon)$	$m_{C_p}(\varepsilon)$				
3	$2.4 \cdot 10^{5}$	$9.0 \cdot 10^{5}$	$9.0 \cdot 10^{5}$	$1.1 \cdot 10^{6}$	$4.2 \cdot 10^{6}$				
10	$3.7\cdot 10^5$	$1.4 \cdot 10^{6}$	$1.4 \cdot 10^{6}$	$1.6 \cdot 10^{6}$	$6.5\cdot10^6$				
100	$1.4 \cdot 10^{6}$	$5.3 \cdot 10^{6}$	$5.3 \cdot 10^{6}$	$6.3 \cdot 10^{6}$	$2.5 \cdot 10^{7}$				

Table 1. The approximate values of $m_{C_*}(\varepsilon)$ for $\varepsilon = 0.1 F_{H_A}(E)$.

Table 2. The approximate values of $m_{C_*}(\varepsilon)$ for $\varepsilon = 0.01 F_{H_A}(E)$.								
	$E/\hbar\omega$	$m_{C_{\chi}}(\varepsilon) = m_{\bar{Q}}(\varepsilon)$	$m_C(\varepsilon)$	$m_Q(\varepsilon)$	$m_{\bar{C}_p}(\varepsilon)$	$m_{C_p}(\varepsilon)$		
	3	$3.7 \cdot 10^{7}$	$1.5 \cdot 10^{8}$	$1.5 \cdot 10^{8}$	$1.6 \cdot 10^{8}$	$6.5 \cdot 10^{8}$		
	10	$5.6 \cdot 10^{7}$	$2.2 \cdot 10^{8}$	$2.2 \cdot 10^{8}$	$2.5 \cdot 10^{8}$	$1.0 \cdot 10^{9}$		
	100	$2.1 \cdot 10^{8}$	$8.4 \cdot 10^{8}$	$8.4 \cdot 10^{8}$	$9.5 \cdot 10^{8}$	$3.8 \cdot 10^{9}$		

Comparing the above tables with Tables 1 and 2 in [38] shows that Theorem 3 gives substantially smaller estimates of the ε -sufficient input dimension $m_{C_*}(\varepsilon)$ for all the capacities than Theorem 1 in [38]. It is essential that the estimates of $m_{C_*}(\varepsilon)$ given by Theorem 3 grow with increasing energy (in contrast to the estimates obtained in [38]). Since it is clear that real values of $m_{C_*}(\varepsilon)$ must grow with increasing energy, one can assume that the estimates of $m_{C_*}(\varepsilon)$ given by Theorem 3 are quite adequate. However, the question of the accuracy of these estimates remains open.

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¹⁹All the capacities $C_*(\Phi, H_A, E)$, $C_* = C_{\chi}, C, Q, \overline{Q}, \overline{C}_p, C_p$, take values in $[0, F_{H_A}(E)]$.

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