

A hybrid scheme for prime factorization and its experimental implementation using IBM quantum processor

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We report a quantum-classical hybrid scheme for factorization of bi-prime numbers (which are odd and square-free) using IBM's quantum processors. The hybrid scheme proposed here involves both classical optimization techniques and adiabatic quantum optimization techniques, and is build by extending a previous scheme of hybrid factorization [Pal et al., *Pramana* 92, 26 (2019) and Xu et al., *Phys. Rev. Lett.* 108, 130501 (2012)]. The quantum part of the scheme is very general in the sense that it can be implemented using any quantum computing architecture. Here, as an example, we experimentally implement our scheme for prime factorization using IBM's QX4 quantum processor and have factorised 35.

I. INTRODUCTION

It is well known that prime factorization is a computationally difficult problem and the security of the RSA-type classical cryptographic systems derive from this difficulty [1]. Although, various RSA-type schemes for public key cryptography are still in use, the confidence in the security provided by RSA-type cryptosystem has been considerably weakened since the pioneering work of Shor [2–5]. Specifically, in [2], Shor showed that the factorization of bi-primes can be performed in polynomial time if a scalable quantum computer can be built. In other words, building of a scalable quantum computer would imply the end of RSA-type classical cryptosystem and a set of other classical cryptosystems, too. This fundamental importance of the factorisation problem and the benefit of implementing it using a quantum computer have led to a set of schemes for prime factorization, mainly experimental [6–10]. Initial implementations were based on nuclear magnetic resonance (NMR) and mostly followed Shor's original algorithm. For example, 15 was factorised using an NMR-based 7-qubit quantum computer [6]. Interestingly, largest number that was factored using Shor's algorithm until 2012 was 21 and a 10 qubit quantum computer was used for the purpose [11]. Due to the fact that considerably large quantum registers are required in Shor's original algorithm, now it's not usually used in its original form [6]. Though, Shor's algorithm, in principle, guarantees factorization of a bi-prime number in polynomial time, the requirement of very large quantum registers restricted its applicability in factorising relatively large bi-primes. This fact motivated researchers to look for alternative approaches. One such approach is to use a hybrid scheme, variants of this approach are reported in Refs. [12–14]. These variants are quite close to each other and each of them require relatively less quantum resources than that required in Shor's original approach.

Here, it would be apt to note that hybrid schemes refer to those schemes, where part of the factorisation task is done classically to reduce the requirement of quantum resources which are costly.

In 2008, Peng et al. [12], devised an algorithm utilizing adiabatic quantum computing, on the basis of the work of Farhi et al. [15] and demonstrated factorization of 21 using 3-qubits. Furthermore, in 2012, Xu et al., have improved the scheme by solving some equations mathematically. They have further demonstrated the beauty and power of the factorisation algorithms of this class by factorizing 143 using a 4-qubit NMR quantum processor [16]. Two years later, Dattani and Bryans established that Xu et al., had actually factored 3599, 11663, and 56153, but could not recognize that. In the work of Dattani and Bryans, classical resources were used for partially simplifying a set of bit-wise factoring equations which allowed them to reduce the quantum overhead for a set of numbers [17]. In 2019, Pal et al., have demonstrated a hybrid scheme for factorization of 551 using 3-qubit system resources [13]. The progress has been continuing and very recently (in 2017), factorization of 35 was demonstrated using a single solid spin system under ambient conditions by Xu et al., [18]. Furthermore, a set of relatively large numbers have recently been experimentally factorized. Specifically, combining the concepts of quantum annealing and computational algebraic geometry, a new approach for quantum factorization is developed by Dridi and Alghassi [19] and the same has been successfully used to factorize all bi-primes up to 200099 using the D-Wave 2X processor [20] and the experimental factorization of 291311 is performed by Li et al., [14] using a hybrid adiabatic quantum algorithm.

The importance of the factorization problem and the facts that (i) quantum factorization has yet been performed using only a few potential candidates for the scalable quantum computer and (ii) hybrid schemes have potential to factorize large bi-primes using small and noisy quantum computers available today, have motivated us to modify the hybrid scheme given in [13, 16] to obtain a new scheme which can be implemented

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in another experimental platform, namely IBM Quantum processor. Specifically, the algorithm proposed here is designed for factorization of bi-primes using a Josephson-qubit based quantum computer [21, 22]. Interested readers may find a detailed user guide on how to use this computer at [21], and a lucid description of the working principle of a Josephson-qubit based quantum computer in Ref. [23]. This quantum computer was placed in the cloud in 2016. It immediately drew considerable attention of the quantum information processing community, and several quantum information tasks have already been performed using this quantum computer. Specifically, in the domain of quantum communication, properties of different quantum channels that can be used for quantum communication have been studied experimentally [24] and experimental realizations of a quantum analogue of a bank cheque [25] that is claimed to work in a banking system having a quantum network, and solving set of linear equations [26] and two-qubit quantum states using optimal resources [27], have been reported; in the field of quantum foundation, violation of multi-party Mermin inequality has been realized for 3, 4, and 5 parties [28]; an information theoretic version of uncertainty and measurement reversibility has also been implemented [29]; in the area of quantum computation, a comparison of two architectures using demonstration of an algorithm has been performed [30], a quantum permutation algorithm [31], and a quantum eigensolver method based experimental search for ground state energy for molecules of increasing size up to BeH_2 [32] have been implemented recently. Further, a non-abelian braiding of surface code defects [33] and a compressed simulation of the transverse field one-dimensional Ising interaction (realized as a four-qubit Ising chain that utilizes only two qubits) [34] have also been demonstrated. Clearly, the IBM quantum computer has already been used for the experimental realizations of various quantum information processing tasks. However, to the best of our knowledge, IBM quantum computer has not yet been used to realize Shor's algorithm or hybrid algorithms for factorization. This paper aims to address that gap.

This paper is organized as follows. Sec. I sets motivation behind choosing factorization problem, followed by detailing the gradual development of hybrid (i.e., combined classical and quantum) strategies to obtain efficient solution. In Sec. II, we revisit the general scheme for hybrid factorization. In Sec. III, we elaborate on a specific case- factorization of 35 using hybrid scheme of factorization, to be precise, we construct bit-wise equations and bit-wise matrix, optimize them to calculate unknown constants, and obtain relation between bit variables, formulate corresponding problem Hamiltonian using the procedure given in Sec. II. In Sec. IV, we illustrate experimental implementation of the quantum part of the hybrid scheme for factorizing 35 using IBM architecture. For the purpose, we use Adiabatic evolution for ground state search of the problem Hamiltonian constructed in the previous section, which

is the desired solution. In Sec. V, we show the experimental results, which reveals the unknown qubit state and hence one factor of the composite number, and ultimately We conclude in Sec. VI.

II. THEORY

Let's consider a b_n -bit number $N = \sum_{j=0}^{b_n-1} 2^j n_j$. Although, there may be various sets of factors of a composite number of bit length b_n with maximum number of possibilities equals $\lceil \frac{b_n}{2} \rceil$. Here, $\lceil x \rceil$ corresponds to a ceiling function. Acquainted with the requirement of cryptosystems, here we consider N as a distinct bi-prime. Let us assume that the two factors of N are $P = \sum_{k=0}^{b_p-1} 2^k p_k$ and $Q = \sum_{l=0}^{b_q-1} 2^l q_l$ with bit length b_p and b_q respectively, such that, either $b_n = b_p + b_q$ or $b_n = b_p + b_q - 1$. From the definition of prime factors, $p_0 = q_0 = 1$ and $p_{b_p-1} = q_{b_q-1} = 1$. Thus, identity $N = PQ$ becomes,

$$\sum_{j=0}^{b_n-1} 2^j n_j = \sum_{k=0}^{b_p-1} 2^k p_k \sum_{l=0}^{b_q-1} 2^l q_l. \quad (1)$$

As follows from the above equation either $b_n = b_p + b_q$ or $b_n = b_p + b_q - 1$. The hybrid scheme for factorization can be divided into the following steps.

- Formulating multiplication table for P and Q .
- Constructing bit-wise equations from multiplication table.
- Simplifying bit-wise equations using classical computation.
- Constructing bit-wise Hamiltonian and hence problem Hamiltonian.
- Obtaining unitaries corresponding to adiabatic evolution starting from given Hamiltonian to the problem Hamiltonian.
- Decomposition of a given unitary using gates available in IBM Clifford library.
- Obtaining solution of problem Hamiltonian using adiabatic quantum computation.

In the following, we illustrate the hybrid scheme for factorization with an example of $N = 35$. We would also report experimentally obtained factors using this scheme. Experimental implementation of quantum part of the scheme, i.e., constrained minimization using adiabatic evolution has been done in 5-qubit IBM quantum processor involving superconducting qubits.

A. Simplification of bit-wise constraint equation

For the purpose of bit-wise comparison of coefficients of both sides, we need to rewrite the above equation by introducing new index $c = k + l$. In terms of this new index c modified equation becomes

$$\sum_{j=0}^{b_n-1} 2^j n_j = \sum_{c=0}^{b_p+b_q-2} 2^c \sum_{l=c_{min}}^{c_{max}} p_{c-l} q_l. \quad (2)$$

Here, $c_{min} = \max(0, c - lp - 1)$ and $c_{max} = \min(c, lq - 1)$. Further, term $p_{c-l} q_l$ can be broken as $p_{c-l} q_l + C_{c-1,c} = s_{c,l} + 2C_{c,c+1}$. Here, $C_{c-1,c}$ is the carry from $(c-1)^{th}$ column to c^{th} column and $C_{c,c+1}$ is the carry from $(c)^{th}$ column to $(c+1)^{th}$ column. Such a decomposition allows us an easy understanding of the construction of the multiplication table. In order to get simplified bit-wise factoring equations, without any loss of generality, we now add all carries in the given column. So the bit-wise factoring equation takes following form

$$\sum_{l=c_{min}}^{c_{max}} p_{c-l} q_l + C_c - 2C_{c+1} = n_j. \quad (3)$$

Here, cumulative carry $C_c = \sum_{c_{min}}^{c_{max}} C_{c-1,c}$ and n_j is the bit value of number N for the j^{th} order (power) of the base value in the binary system.

B. Constraint optimization using classical resources

Consider the following equation,

$$\sum_{j=0}^{b_n-1} 2^j n_j = \sum_{c=0}^{b_p+b_q-2} 2^c n_c + 2C_{c+1} - C_c \quad (4)$$

and equation

$$\sum_{j=0}^{b_n-1} 2^j n_j = \sum_{c=0}^{b_p+b_q-2} 2^c n_c + C_{b_p+b_q-1} \quad (5)$$

obtained after putting the value of

$$\sum_{l=c_{min}}^{c_{max}} p_{c-l} q_l$$

from Eq.(3) into Eq.(2). Writing Eq. (5) in such a way allows us to calculate values of carry $C_{b_p+b_q-1}$. A direct comparison of the L.H.S. with R.H.S. reveals the values of the carry $C_{b_p+b_q-1}$. Also, as there is no incoming carry to the first column, we set $C_0 = 0$. Moreover, bit-wise equation for $c = 0$ and $c = b_p + b_q - 2$, i.e., $1 + C_0 = 1 - 2C_1$ and $C_{b_p+b_q-2} + 1 = n_{b_p+b_q-2} + 2C_{b_p+b_q-1}$ give C_1 and $C_{b_p+b_q-2}$. Next we obtain the following equality by rewriting the bit-wise equation.

$$\max[C_{c+1}] = \lfloor \max\left(\frac{1}{2} \sum_{l=c_{min}}^{c_{max}} p_{c-l} q_l + \frac{1}{2} C_i\right) - \frac{n_i}{2} \rfloor. \quad (6)$$

This equality can be used to calculate the upper bounds over C_{j+1} , and this upper bound can be iteratively used to obtain upper bound on next C value. The bit equation obtained under these constrain, further allows us to simplify complete set of bit equations with minimum number of independent parameters. For the two cases, namely, Case A : when $b_n = b_p + b_q$ and Case B : when $b_n = b_p + b_q - 1$ the values of the carry are 1 and 0 respectively.

C. Construction of problem Hamiltonian

In 2008, Peng et al. developed a framework for solving factorization problem using quantum adiabatic algorithm. For the purpose, they formulated the factorization problem as a minimization problem by constructing a function $f(p, q) = (N - pq)$. The solution of this equation should reveal the values of classical variables p and q . They further suggested that any corresponding quantum approach to the minimization problem must involve finding the ground state of the Hamiltonian, which can be considered to be of the form $H = f(p, q)|p, q\rangle\langle p, q|$, where $f(p, q)$ is the ground state eigenvalue and $|p, q\rangle$ is the corresponding product state of states $|p\rangle$ and $|q\rangle$. The problem Hamiltonian for the factorization problem takes the form $H = (N I_{2^n} - PQ)^2$, where $P = \sum_i 2^i A_i$, $Q = \sum_i 2^i A_i$ and $A_i = \frac{1 - \sigma_{iz}}{2}$ with eigenvalues 0 and 1 for eigenstates $|0\rangle$ and $|1\rangle$, respectively. Furthermore, Xu et al. [16] have used another approach to construct a Hamiltonian which uses relatively less quantum resources than that used in Ref. [12] but still uses more resources than used by Pal et al. in Ref. [13]. In this article, in order to construct the problem Hamiltonian we have used the same approach as was used by Xu et al. in Ref. [16] i.e., to begin with we have transformed the classical bit variable p_i and q_i into operators such that $p_i \rightarrow A_i$ and $q_i \rightarrow A_{i+b_k-2}$.

D. Quantum adiabatic algorithm for ground state search

Quantum adiabatic algorithm states that, during the evolution of a quantum system under a slowly varying (as per adiabaticity condition [35]) time dependent Hamiltonian $H(t)$, system remains in the same eigenstate in which the system is prepared initially [36]. Given a problem, adiabatic quantum computation typically involves encoding the solution to the problem in the ground-state of the final Hamiltonian. A suitable initial Hamiltonian $H_i(0)$ is chosen for which ground-state can be prepared easily and evolved to the final Hamiltonian $H_f(T)$. Then the Hamiltonian of the system is slowly varied such that the system stays in the ground state of the instantaneous Hamiltonian. The instantaneous Hamiltonian can be designed as an interpolation (linear or nonlinear) between the initial Hamiltonian and the final Hamiltonian [35]. For the linear

interpolation parameter $s = \frac{t}{T}$, such that $0 \leq s \leq 1$, where t is the evolution time and $T = |\frac{\max(\frac{dH(s)}{ds})}{\epsilon \Delta^2 / \hbar}|$, is the total evolution time. The adiabatic theorem guarantees that system will evolve to the ground state of the final Hamiltonian with probability $1 - \epsilon^2$, and the transformed Hamiltonian would become

$$H(s) = (1 - s) H_i + s H_f. \quad (7)$$

Considering the Hamiltonian as piecewise constant Hamiltonian with M pieces the time (t) can be rewritten as $t = \frac{m}{M} T$, where $0 \leq m \leq M$ and the Hamiltonian for the m^{th} piece is

$$H_m = (1 - \frac{m}{M}) H_i + (\frac{m}{M}) H_f. \quad (8)$$

The unitary evolution $U_m = \exp(-i H_m \delta t)$ governed by the corresponding Hamiltonian H_m , where t is the duration of m^{th} piece of evolution. Unitary operation for the total evolution is $U = \prod_{m=1}^M U_m$.

III. FACTORIZATION OF 35 USING IBM'S 5-QUBIT QUANTUM PROCESSOR

In order to demonstrate the method, we take the example of 35. As mentioned in Sec. II there are two possible cases for the choice of the bit-length of the bi-prime factors, we start with the case $b_n = b_p + b_q$, thus $b_p = 3$ and $b_q = 3$. Although, one can start with any of the two cases and in case of obtaining an inconsistent solution in the first case, consistent solution will be guaranteed in the other case. We start by obtaining the multiplication table (see Tab. I) for the composite number $N = 35$.

The bit-wise equations obtained from the multiplica-

| $\begin{matrix} \mathbf{j} \rightarrow \\ \mathbf{l} \downarrow \end{matrix}$ | 5 | 4 | 3 | 2 | 1 | 0 |
|---|-----------|------------------------|-----------|-----------|-------|-------|
| | | | | 1 | p_1 | 1 |
| | | | | 1 | q_1 | 1 |
| 0 | | | | 1 | p_1 | 1 |
| 1 | | | q_1 | $p_1 q_1$ | q_1 | |
| 2 | | 1 | p_1 | 1 | | |
| Carry | $c_{4,5}$ | $c_{3,4}$ $c_{2,4}$ | $c_{2,3}$ | $c_{1,2}$ | | |
| Cumulative Carry C_c | C_5 | C_4 | C_3 | C_2 | C_1 | C_0 |
| n_j | 1 | 0 | 0 | 0 | 1 | 1 |

Table I. Bit-wise multiplication table for 35. Columns in the table correspond to parameter c introduced to get simplified bit-wise equations (see Eq. (5)) while rows correspond to l values (see Eq. (4)). Bit values n_j for $j = 0: 5$ are provided in the last row of the table.

tion table (Table I) are:

$$\begin{aligned} 1 + C_0 &= 1, \\ p_1 + q_1 + C_1 - 1 &= 2C_2, \\ 1 + p_1 q_1 + 1 + C_2 - 0 &= 2C_3, \\ p_1 + q_1 + C_3 - 0 &= 2C_4, \\ 1 + C_4 - 0 &= 2C_5, \\ C_5 &= 1. \end{aligned}$$

We then optimize above set of equations using the constrain optimum condition given in Sec. II B. The carries thus obtained are $C_0 = 0, C_1 = 0, C_2 = 0, C_3 = 1, C_4 = 1, C_5 = 1$. This leaves us with only one bit equation, i.e., $p_1 + q_1 = 1$. We then construct the operators corresponding to bit values p_1 and q_1 as discussed in II C. Thus, the operators corresponding to the bit values p_1 and q_1 are $P = Q = \sum_i A_i = A_1$ and for $A_1 = \frac{1 - \sigma_{1z}}{2}$. Now the problem Hamiltonian becomes

$$\begin{aligned} H_p &= (P_1 + Q_1 - 1)^2 \\ &= \left(\frac{1 - \sigma_z}{2} + \frac{1 - \sigma_z}{2} - 1 \right)^2 \\ &= \sigma_z^2 \\ &= I. \end{aligned}$$

We now use quantum adiabatic evolution for finding the ground state of the final Hamiltonian H_p starting from the ground state of the easily initializable Hamiltonian in the IBM's QX_4 processor, i.e., $H_i = J \sigma_z$, in the units of \hbar and $J \approx 2\pi \times 10^6$ rad/sec. The ground state of the initial Hamiltonian H_i is $|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$ and ground state of the final Hamiltonian H_f is degenerate and are $|0\rangle$ and $|1\rangle$ with eigenvalues 1. We decided to adiabatically evolve the Hamiltonian in 8 steps. During each step the Hamiltonian can be considered as piecewise constant and the corresponding unitary operators can be obtained using the formula $U_m = \exp(-i H_m \Delta t)$, where H_m is the Hamiltonian in the m^{th} step and Δt is the duration of the step.

We first optimize the number of steps to check the adiabaticity condition being satisfied by the given set of Hamiltonians i.e., we check if the ground state of the initial Hamiltonian reaches the ground state of the final Hamiltonian without anticrossing as shown in the Fig. 1.

A. Decomposition of Unitaries

We now decompose the unitaries obtained in Sec. III for each of the 8 steps into the single-qubit Clifford+T gates. In actual implementation of Adiabatic evolution of the system from the ground state of the initial Hamiltonian to the ground state of the final Hamiltonian. The actual decomposition for a general unitary is shown in

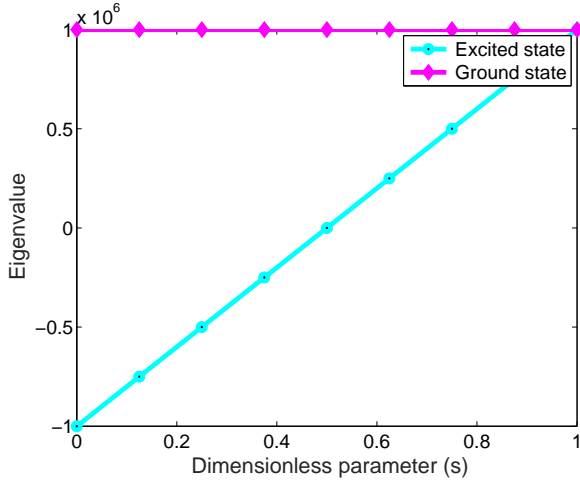


Figure 1. (Color online) Simulation shows no anticrossing between two states of the qubit q[0] of the IBM's QX4 quantum processor during adiabatic evolution for time chosen $T = 10\mu s$ in 8 steps. The Hamiltonian used are $H_i = J\sigma_z$ and $H_f = JI$.

Eq. (11) and the exact values of θ s for each step are given in Tab. II. In this stage, we need to be specific to the quantum processor to be used, as the available gate library and the ease with which different gates can be realized in a particular implementation/architecture are different. Here, we are interested in implementing the proposed scheme using an IBM QX4 processor, which

restricts us to decompose the unitaries in terms of the available Clifford gate library of IBM Quantum Experience (QE). To be specific, to implement our scheme in IBM's QX4, any general unitary which we require to implement as part of our scheme (circuit), has to be decomposed in terms of the Clifford+T gates. In general, a single-qubit unitary obtained with the chosen initial and obtained final Hamiltonians are of the form, $U = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Each element in the given unitary is a complex number. Thus, $a = r_1 \exp(i\theta_1)$ and, $b = r_2 \exp(i\theta_2)$ hence corresponding to each unitary we have two matrices corresponding to r and θ values i.e., $R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$ and $\Theta = \begin{pmatrix} \exp(i\theta_1) & 0 \\ 0 & \exp(i\theta_2) \end{pmatrix}$ such that

$$U = R \cdot \Theta. \quad (9)$$

The values of θ are given in Tab. II. For the case of 35, R matrices for all unitaries are identity, so $U = \Theta$. The IBM Clifford+T gate library consists of following single-qubit unitaries:

- (i) The Pauli gates: I, X, Y, and Z
- (ii) General gates: $U_1(\theta)$, $U_2(\phi, \lambda)$, and $U_3(\theta, \phi, \lambda)$.
- (iii) Phase Gates: $S(S^\dagger)$, $T(T^\dagger)$, and
- (iv) Other Gates: H (Hadamard)

In what follows, we will use phase gate and Pauli X -gate to construct unitaries to be implemented in our experiments. The decomposition of Θ in Clifford+T gate library can be obtained as

$$\begin{aligned} \Theta &= \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\theta_2) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\theta_1) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= U_1(\theta_2) \cdot X \cdot U_1(\theta_1) \cdot X. \end{aligned} \quad (10)$$

Combining R and Θ matrices, the unitary U_m for m^{th} step can be written as

$$U_m = U_1^m(\theta_1^m) \cdot X \cdot U_1^m(\theta_2^m) \cdot X, \quad (11)$$

where, θ_1^m and θ_2^m are the angles in the m^{th} unitary. Thus, the total unitary for quantum part of the hybrid factorization scheme is $U = \prod_{m=8}^{m=1} U_m$. Here, it is important to mention that gates in the unitary have been applied in the right to left order. The values for θ_1^m and θ_2^m for different steps are provided in Tab. II.

IV. EXPERIMENTS

This experiment has been performed on an open access 5-qubit quantum processor placed on cloud by IBM

corporation. In particular, we have used the architecture of IBM's QX4 (IBM Q 5 Tenerife)[21], which consists of superconducting Transmon qubits [37]. A schematic diagram of this architecture and description of the architecture can be found in [27, 38] and references therein. The basic gate library used for the single-

| m | r_1 | r_2 | θ_1^m | θ_2^m |
|---|-------|-------|--------------|--------------|
| 1 | 1 | 1 | -1.2500 | 0.9375 |
| 2 | | 1 | -1.2500 | 0.6250 |
| 3 | 1 | 1 | -1.2500 | 0.3125 |
| 4 | 1 | 1 | -1.2500 | 0 |
| 5 | 1 | 1 | -1.2500 | -0.3125 |
| 6 | 1 | 1 | -1.2500 | -0.6250 |
| 7 | 1 | 1 | -1.2500 | -0.9375 |
| 8 | 1 | 1 | -1.2500 | -1.2500 |

Table II. The r and θ values for unitaries in each step of adiabatic evolution.

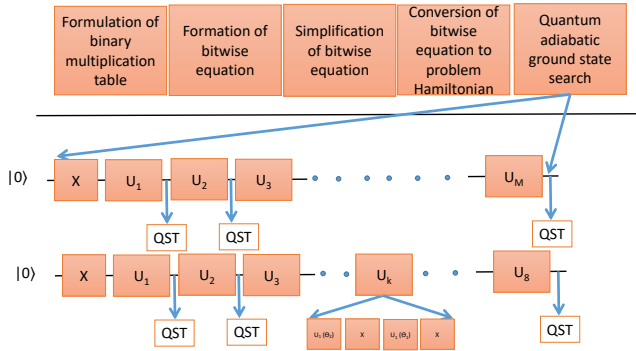


Figure 2. (Color online) Schematic procedure of complete scheme for hybrid factorization. Top trace shows the steps of the scheme. Second trace shows circuit for obtaining ground state of the final Hamiltonian H_f starting from H_i through adiabatic evolution. The trace below that shows quantum circuit for implementing quantum adiabatic evolution part on IBM's QX4 processor for factorization of 35, by initializing the qubit system to the ground state i.e., $|1\rangle$ of the initial Hamiltonian $J\sigma_z$ and the lowest trace shows the gate decomposition in IBM's gate library of k^{th} unitary U_k .

qubit gates are H, Pauli operators X, Y, Z, parametric gates U_1 , U_2 , and U_3 . The operator U_1 depends on single parameter θ , operator U_2 depends on two parameters θ, ϕ , and operator U_3 depends on three parameters θ, ϕ, λ . We chose qubit q[0] for implementation of quantum adiabatic evolution of ground state search. We initialize the system to the ground state of the initial Hamiltonian by applying the X gate (σ_x) to the qubit q[0]. To evolve this state adiabatically we apply a set of eight unitaries, in their decomposed form as shown in the previous section. In order to extract the probabilities of the final state we perform quantum state tomography after each step. The directly measured observable in IBM processor are $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ which allow us to calculate $\langle Z \rangle$. This is sufficient to reveal the probabilities p_0 and p_1 . In order to measure real and imaginary part of the coherence term, we have used the method described in our earlier work [39] i.e., we have applied, H gate followed by Z-measurement to reveal real part and $S^\dagger H$ followed by Z-measurement for measuring its imaginary part.

V. RESULTS

The QST results, are shown in Fig. 3 and Fig. 4. To measure the final state we have the corresponding measurement operator in the Clifford library. The ground state of the final Hamiltonian provides us the solution to our problem, in our case, we would obtain probability of p_0 and p_1 for the states $|0\rangle$ and $|1\rangle$, respectively, after performing the experiment 8192 times (i.e., the maximum number of runs that one can select from the interface provided by IBM QE). The tomography

results reveal that after the full adiabatic evolution of the system in 8 steps, system is in the ground state of the final Hamiltonian. Since the Hamiltonian at this point is degenerate, consequently, solution of the problem Hamiltonian are any of the two states $|0\rangle$ and $|1\rangle$. If we consider $|0\rangle$ as the solution, then the corresponding classical bit value would be $p_1 = 0$ leading to the first factor of the composite number 35 in the binary system as $1p_11 = 101$ and consequently, in decimal system as $P = \sum_k 2^k p_k = 5$. The conjugate bit value by using the identity $p_1 + q_1 = 1$ is $q_1 = 1$ and the corresponding prime factor in the binary system is $1q_11 = 111$, and consequently the number, in the decimal system would be $Q = \sum_l 2^l q_l = 7$. The two factors can also be obtained in the same way if we consider $|1\rangle$ as the solution of the ground state search of the adiabatic evolution.

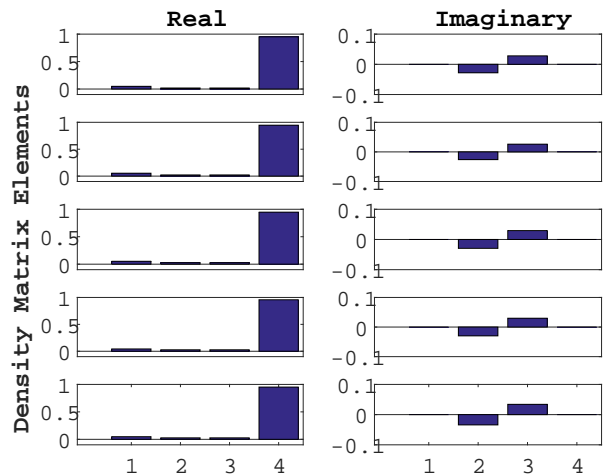


Figure 3. (Color online) Figure shows density matrices in rows, with the columns representing the real and imaginary parts of the density matrix. Top row indicates the initial state, subsequent rows represent state at even instances of applying unitaries i.e., 2, 4, 6, and 8 times, while going down. Ticks 1, 2, 3, 4 correspond to the elements $|0\rangle\langle 0|$, $|0\rangle\langle 1|$, $|1\rangle\langle 0|$, $|1\rangle\langle 1|$.

VI. CONCLUSIONS

As discussed above, factorisation of bi-prime numbers are important for hacking of RSA type cryptographic schemes and various related problems. However, factorisation of some bi-primes are not as difficult as that of the others. Specifically, for an even bi-prime, we already know that one factor is 2 and it's trivial to find the other factor. Further, there are excellent algorithms for finding square root, so factorisation of square bi-primes are easy. What we are left with is odd and square-free bi-primes which are used in cryptography. Here, we report a quantum-classical hybrid scheme for factorization of such (i.e., odd and square-free) bi-prime numbers. The scheme proposed here is hybrid in nature,

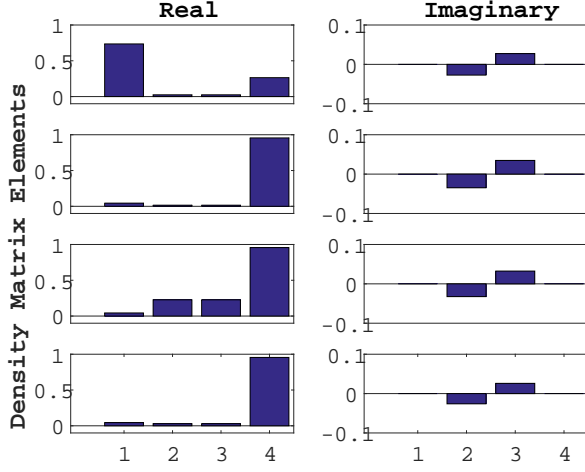


Figure 4. (Color online) Figure shows density matrices in rows, with the columns representing the real and imaginary parts of the density matrix. Rows represent state at odd instances of applying unitaries to the initial state i.e., 1, 3, 5, and 7 times, while going down. Ticks 1, 2, 3, 4 correspond to the elements $|0\rangle\langle 0|$, $|0\rangle\langle 1|$, $|1\rangle\langle 0|$, $|1\rangle\langle 1|$. The result reveals the system to be in the ground state of the final Hamiltonian.

implying that the scheme utilizes both classical optimization techniques and adiabatic quantum optimization techniques. The advantage of such hybrid schemes underlies in the fact that they require less quantum re-

sources (which are fragile and costly at the moment) in comparison with the purely quantum schemes designed for the same purpose. For, example, it's already understood that the extremely large quantum registers required in Shor's original protocol are not required in the hybrid schemes proposed later on. The same is true for our scheme as well as the schemes [13, 16] which have been extended here. Thus, in short the proposed scheme has the capability to factorise relatively large odd and square-free bi-primes using a small amount of quantum resources. To illustrate this, we have explicitly performed factorisation of 35 (which is an odd and square free bi-prime) using the smallest quantum computer available on the cloud (i.e., a five qubit quantum processor called IBM's QX4). The quantum processor used here is known to be noisy, but here we have correctly obtained the prime factors of 35 with some small signatures of noise depicting the strength of the algorithm. We conclude the paper with a hope that with the availability of larger quantum processors, larger bi-primes will be factored using this algorithm and it will be found useful in the future development of the hybrid algorithm designing.

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