

CHARACTERIZATION OF QUANTUM ENTANGLEMENT VIA A HYPERCUBE OF SEGRE EMBEDDINGS

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ABSTRACT. A particularly simple description of separability of quantum states arises naturally in the setting of complex algebraic geometry, via the Segre embedding. This is a map describing how to take products of projective Hilbert spaces. In this paper, we show that for pure states of n particles, the corresponding Segre embedding may be described by means of a directed hypercube of dimension $(n - 1)$, where all edges are bipartite-type Segre maps. Moreover, we describe the image of the original Segre map via the intersections of images of the $(n - 1)$ edges whose target is the last vertex of the hypercube. This purely algebraic result is then transferred to physics. For each of the last edges of the Segre hypercube, we introduce an observable which measures geometric separability and is related to the trace of the squared reduced density matrix. As a consequence, the hypercube approach allows to measure entanglement, naturally relating bipartitions with q -partitions for any $q \geq 1$. We test our observables against well-known states, showing that these provide well-behaved and fine measures of entanglement.

1. INTRODUCTION

Quantum entanglement is at the heart of quantum physics, with crucial roles in quantum information theory, superdense coding and quantum teleportation among others. An important problem in entanglement theory is to obtain separability criteria. While there is a clear definition of separability, in general it is difficult to determine whether a given state is entangled or separable. A refinement of this problem is to quantify entanglement on a given entangled state. This is a broadly open problem, in the sense that there is not a unique established way of measuring entanglement, which might depend on the initial set-up and applications one has in mind. There is, however, a general consensus on the desirable properties of a good entanglement measure [?, ?]. Directly attached to measuring entanglement is the notion of *maximally entangled state*, central in teleportation protocols.

Many different entanglement measures and the corresponding notions of maximal entanglement have been proposed. Methods range from using Bell inequalities, looking at inequalities in the larger scheme of entanglement witnesses, spin squeezing inequalities, entropic inequalities, the measurement of nonlinear properties of the quantum state or the approximation of positive maps. We refer to the exhaustive reviews [?, ?, ?] for the basic aspects of entanglement including its history, characterization, measurement, classification and applications.

A particularly simple description of separability of quantum states arises naturally in the setting of complex algebraic geometry. In this setting, pure multiparticle states are identified with points in the complex projective space \mathbb{P}^N , the set of lines of the complex space \mathbb{C}^{N+1} that go through the origin. Entanglement is then understood via the categorical product of

projective spaces: the *Segre embedding*. This is a map of complex algebraic varieties

$$(1.1) \quad \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \xrightarrow{\quad} \mathbb{P}^{2^n-1},$$

whose image, called the *Segre variety*, is described in terms of a family of homogeneous quadratic polynomial equations in 2^n variables, where n is the number of particles. The points of the Segre variety correspond precisely to separable states (see for instance [?, ?]).

In this paper, we exploit this geometric viewpoint. There are numerous approaches to entanglement via Segre varieties that are related to the present work [?, ?, ?, ?, ?, ?, ?]. The hypercube viewpoint presented here connects the notions of bipartite and q -partite in a geometric way. We describe the image of (1.1) by means of the intersection of all Segre varieties defined via bipartite-type Segre maps

$$(1.2) \quad \mathbb{P}^{2^\ell-1} \times \mathbb{P}^{2^{n-\ell}-1} \xrightarrow{\quad} \mathbb{P}^{2^n-1}, \text{ for all } 1 \leq \ell \leq n-1.$$

Specifically, we show that a state is q -partite if and only if it lies in q of the images of Segre maps of the form (1.2). We do this after showing that the Segre embedding (1.1) may be decomposed in various equivalent ways, leading to a hypercube of dimension $(n-1)$ whose edges are bipartite-type Segre maps. In this framework all the information about the separability of the state is contained in the last applications (1.2) of the hypercube.

While the above results are extremely precise and intuitive, they are purely algebraic. In order to build a bridge from geometry to physics, we introduce a family of observables $\{\mathcal{J}_{n,\ell}\}$, with $1 \leq \ell \leq n-1$, which allow us to detect when a given n -particle state belongs to each of the images of the bipartite-type Segre maps (1.2). As a consequence, we obtain that a state is q -partite if and only if at least q of the observables $\mathcal{J}_{n,\ell}$ vanish on this state, completely identifying the sub-partitions of the system. Each of these observables are related to the trace of the squared reduced density matrix, also known as the bipartite non-extensive Tsallis entropy with entropic index two [?, ?]. They are always positive on entangled states and our first applications indicate that they provide well-behaved and fine measures of entanglement.

We briefly explain the contents of this paper. We begin with a warm-up Section 2 where we detail the theory and results for the well-understood settings of two- and three-particle states. In Section 3 we develop the geometric aspects of the paper. In particular, we describe the Segre hypercube and prove Theorem 3.6 on geometric decomposability. The main result of Section 4 is Theorem 4.3, where we match geometric decomposability with our family of observables related to non-extensive entropic measures. The two theorems are combined in Section 5, where we study entanglement measures for pure multiparticle states of spin- $\frac{1}{2}$ and apply our observables on various well-known multiparticle states.

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2. WARM-UP: TWO AND THREE PARTICLE STATES

We begin with the well-understood example of two particle entanglement. The initial set-up consists in two particles which can be shared between two different observers, A and B , that can perform quantum measures to each of the particles. A general pure state for two spin- $\frac{1}{2}$ particles can be written as

$$(2.1) \quad |\psi\rangle_{AB} = z_0 |00\rangle + z_1 |01\rangle + z_2 |10\rangle + z_3 |11\rangle,$$

where z_i are complex numbers that satisfy the normalization condition $\sum |z_i|^2 = 1$. The labels AB , which will be often omitted, indicate that A is acting on the first particle while B is acting on the second, so

$$|ij\rangle := |i\rangle_A \otimes |j\rangle_B, \text{ for } i, j \in \{0, 1\}.$$

Entanglement is a property that can be inferred from statistical properties of different measurements by the observers of the system. In particular, we can compute the sum of the expected values of A measuring the spin of the state $|\psi\rangle$ in the different directions. With this idea in mind, we define:

$$\mathcal{J}_{A \otimes B}(\psi) := 2 - \left(\sum_{i=0}^3 |\langle \psi | \sigma_i \otimes \mathbb{I}_2 | \psi \rangle|^2 \right),$$

where σ_i , for $i = 1, 2, 3$, denote the Pauli matrices, $\sigma_0 = \mathbb{I}_2$ is the identity matrix of size two and \otimes denotes the Kronecker product. Using the expression (2.1) for $|\psi\rangle$ we obtain

$$\mathcal{J}_{A \otimes B}(\psi) = 4|z_0 z_3 - z_1 z_2|^2.$$

It is well-known that the state $|\psi\rangle$ is entangled if and only if

$$z_0 z_3 - z_1 z_2 = 0.$$

Therefore we find that $|\psi\rangle$ is a product state if and only if $\mathcal{J}_{A \otimes B}(\psi) = 0$. When $\mathcal{J}_{A \otimes B}(\psi) > 0$ we have an entangled state, which is considered to be maximally entangled when the observable reaches its maximum value at $\mathcal{J}_{A \otimes B}(\psi) = 1$.

The measure given by $\mathcal{J}_{A \otimes B}(\psi)$ may also be interpreted in terms of the density matrix operator. If ρ_A denotes the reduced density matrix for the subsystem A , then

$$\mathcal{J}_{A \otimes B}(\psi) = 2(1 - \text{Tr}_A \rho_A^2).$$

The characterization of entanglement has a simple geometric interpretation. Note first that a general pure state for a single spin- $\frac{1}{2}$ particle may be written as

$$z_0 |0\rangle + z_1 |1\rangle \text{ where } |z_0|^2 + |z_1|^2 = 1.$$

In particular, such a state is determined by the pair of complex numbers (z_0, z_1) and the normalization condition ensures $(z_0, z_1) \neq (0, 0)$. This allows one to consider the corresponding equivalence class $[z_0 : z_1]$ in the complex projective line \mathbb{P}^1 . This space is defined as the set of lines of \mathbb{C}^2 that go through the origin. The class $[z_0 : z_1]$ denotes the the set of all points $(z'_0, z'_1) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that there is a non-zero complex number λ with $(z_0, z_1) = \lambda(z'_0, z'_1)$. Note that, by construction, we have $[z_0 : z_1] = [\lambda z_0 : \lambda z_1]$ for all $\lambda \in \mathbb{C}^*$.

In summary, every one particle state of spin- $\frac{1}{2}$ defines a unique point in \mathbb{P}^1 . Conversely, given a point $[z_0 : z_1] \in \mathbb{P}^1$ we may choose a representative (z_0, z_1) such that $|z_0|^2 + |z_1|^2 = 1$ and so it determines a unique pure state $z_0 |0\rangle + z_1 |1\rangle$ up to a global phase which does not affect any state measurements.

This one-to-one correspondence between pure states and points in the projective space generalizes analogously to several particles: pure states for n particles of spin- $\frac{1}{2}$ correspond to points in $\mathbb{P}^{2^n - 1}$. This correspondence is just a way of describing the projectivization of the Hilbert space of quantum states.

For our two-particle case, since the state $|\psi\rangle$ introduced in (2.1) is determined by the normalized set of complex numbers (z_0, z_1, z_2, z_3) we obtain a point $[\psi] := [z_0 : z_1 : z_2 : z_3]$ in

the complex projective space \mathbb{P}^3 , the set of lines in \mathbb{C}^4 going through the origin. Interestingly for the study of entanglement, there is a map

$$f_{A \otimes B} : \mathbb{P}_A^1 \times \mathbb{P}_B^1 \longrightarrow \mathbb{P}_{AB}^3,$$

called the *Segre embedding*, defined by the products of coordinates

$$[a_0 : a_1], [b_0 : b_1] \mapsto [a_0 b_0 : a_0 b_1 : a_1 b_0 : a_1 b_1].$$

The Segre map is the categorical product of projective spaces, describing how to take products on projective Hilbert spaces. The image of this map

$$\Sigma_{A \otimes B} := \text{Im}(f_{A \otimes B})$$

is called the *Segre variety*. This is a complex algebraic variety of dimension 2 and is given by the set of points $[z_0 : z_1 : z_2 : z_3] \in \mathbb{P}^3$ satisfying the single quadratic polynomial equation

$$z_0 z_3 - z_1 z_2 = 0.$$

Therefore product states correspond precisely to points in the Segre variety and

$$\mathcal{J}_{A \otimes B}(\psi) = 0 \iff [\psi] \in \Sigma_{A \otimes B}.$$

As a consequence, $|\psi\rangle_{AB}$ is a product state if and only if its corresponding point $[\psi]$ in \mathbb{P}^3 lies in the Segre variety $\Sigma_{A \otimes B}$.

The entanglement of three particle states is also well-understood in the literature [?]. However, this case already exhibits some non-trivial facts that arise in the geometric interpretation of entanglement. We briefly review this case before discussing the general set-up.

A general pure state for three spin- $\frac{1}{2}$ particles can be written as

$$(2.2) \quad \begin{aligned} |\psi\rangle_{ABC} &= z_0 |000\rangle + z_1 |001\rangle + z_2 |010\rangle + z_3 |011\rangle + \\ &+ z_4 |100\rangle + z_5 |101\rangle + z_6 |110\rangle + z_7 |111\rangle \end{aligned}.$$

As in the two particle case, z_i are complex numbers that satisfy the normalization condition $\sum |z_i|^2 = 1$. Now we have a third observer, C , in addition to A and B , so

$$|ijk\rangle := |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \text{ for } i, j, k \in \{0, 1\}.$$

We will first measure bipartitions: the separability of this state into states of the form

$$|\varphi\rangle_A \otimes |\varphi'\rangle_{BC} \text{ or } |\varphi\rangle_{AB} \otimes |\varphi'\rangle_C.$$

For the first case, we define the observable

$$\begin{aligned} \mathcal{J}_{A \otimes BC}(\psi) &:= 2 - \left(\sum_{i=0}^3 |\langle \psi | \sigma_i \otimes \mathbb{I}_4 | \psi \rangle|^2 \right) = \\ &= 4 \{ |z_0 z_5 - z_1 z_4|^2 + |z_0 z_6 - z_2 z_4|^2 + |z_0 z_7 - z_3 z_4|^2 + \\ &+ |z_1 z_6 - z_2 z_5|^2 + |z_1 z_7 - z_3 z_5|^2 + |z_2 z_7 - z_3 z_6|^2 \}, \end{aligned}$$

where the last equality will be proven in Section 4. Let us for now interpret this expression geometrically. Our state $|\psi\rangle$ corresponds to a point in \mathbb{P}^7 and bipartitions of type $A \otimes BC$ are geometrically characterized by the Segre embedding

$$f_{A \otimes BC} : \mathbb{P}_A^1 \otimes \mathbb{P}_{BC}^3 \longrightarrow \mathbb{P}_{ABC}^7$$

defined by sending the tuples $[a_0 : a_1]$, $[b_0 : b_1 : b_2 : b_3]$ to the point of \mathbb{P}^7 given by

$$[a_0 b_0 : a_0 b_1 : a_0 b_2 : a_0 b_3 : a_1 b_0 : a_1 b_1 : a_1 b_2 : a_1 b_3].$$

Specifically, the state $|\psi\rangle$ can be written as $|\varphi\rangle_A \otimes |\varphi'\rangle_{BC}$ if and only if the corresponding point $[\psi] = [z_0 : \dots : z_7] \in \mathbb{P}^7$ lies in the Segre variety

$$\Sigma_{A \otimes BC} := \text{Im}(f_{A \otimes BC}).$$

The equations defining $\Sigma_{A \otimes BC}$, having set coordinates $[z_0 : \dots : z_7]$ of \mathbb{P}^7 , are given by the vanishing of all 2×2 minors of the matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 & z_7 \end{pmatrix}.$$

Therefore we see that

$$\mathcal{J}_{A \otimes BC}(\psi) = 0 \iff [\psi] \in \Sigma_{A \otimes BC}.$$

For the second case, we define:

$$\begin{aligned} \mathcal{J}_{AB \otimes C}(\psi) &:= 2 - \left(\frac{1}{2} \sum_{i,j=0}^3 |\langle \psi | \sigma_i \otimes \sigma_j \otimes I_2 | \psi \rangle|^2 \right) = \\ &= 4 \{ |z_0 z_3 - z_1 z_2|^2 + |z_0 z_5 - z_1 z_4|^2 + |z_0 z_7 - z_1 z_6|^2 + \\ &\quad + |z_2 z_5 - z_3 z_4|^2 + |z_2 z_7 - z_3 z_6|^2 + |z_4 z_7 - z_5 z_6|^2 \}, \end{aligned}$$

where again, the last equality is detailed in Section 4. Bipartitions of type $AB \otimes C$ are now geometrically characterized by the Segre embedding

$$f_{AB \otimes C} : \mathbb{P}_{AB}^3 \otimes \mathbb{P}_C^1 \longrightarrow \mathbb{P}_{ABC}^7.$$

The state $|\psi\rangle$ can be written as $|\varphi\rangle_{AB} \otimes |\varphi'\rangle_C$ if and only if the corresponding point $[\psi] = [z_0 : \dots : z_7] \in \mathbb{P}^7$ lies in the Segre variety

$$\Sigma_{AB \otimes C} := \text{Im}(f_{AB \otimes C}).$$

In this case, the Segre variety $\Sigma_{AB \otimes C}$ is given by all points $[z_0 : \dots : z_7] \in \mathbb{P}^7$ such that all 2×2 minors of the matrix

$$\begin{pmatrix} z_0 & z_1 \\ z_2 & z_3 \\ z_4 & z_5 \\ z_6 & z_7 \end{pmatrix}$$

vanish. Therefore we may conclude that

$$\mathcal{J}_{AB \otimes C}(\psi) = 0 \iff [\psi] \in \Sigma_{AB \otimes C}.$$

We may now ask about separability of the state $|\psi\rangle$ in a totally decomposed form

$$|\varphi\rangle_A \otimes |\varphi'\rangle_B \otimes |\varphi''\rangle_C.$$

It turns out that the above defined observables are sufficient in order to address this question. This is easily seen using the geometric characterization of entanglement, as we next explain. The total separability of the state $|\psi\rangle_{ABC}$ is geometrically characterized by the generalized Segre embedding

$$f_{A \otimes B \otimes C} : \mathbb{P}_A^1 \times \mathbb{P}_B^1 \times \mathbb{P}_C^1 \longrightarrow \mathbb{P}_{ABC}^7$$

defined by sending the tuples $[a_0 : a_1], [b_0 : b_1], [c_0 : c_1]$ to the point in \mathbb{P}^7 given by

$$[a_0 b_0 c_0 : a_0 b_0 c_1 : a_0 b_1 c_0 : a_0 b_1 c_1 : a_1 b_0 c_0 : a_1 b_0 c_1 : a_1 b_1 c_0 : a_1 b_1 c_1].$$

The state $|\psi\rangle$ is separable as $A \otimes B \otimes C$ if and only if its associated point $[\psi] \in \mathbb{P}^7$ lies in the generalized Segre variety given by

$$\Sigma_{A \otimes B \otimes C} := \text{Im}(f_{A \otimes B \otimes C}).$$

The above generalized Segre embedding factors in two equivalent ways:

$$f_{A \otimes B \otimes C} = f_{A \otimes BC} \circ (\mathbb{I}_A \times f_{B+C}) = f_{AB \otimes C} \circ (f_{A \otimes B} \times \mathbb{I}_C),$$

so we have a commutative square

$$\begin{array}{ccc} \mathbb{P}_A^1 \times \mathbb{P}_B^1 \times \mathbb{P}_C^1 & \xrightarrow{f_{A \otimes B} \times \mathbb{I}_C} & \mathbb{P}_{AB}^3 \times \mathbb{P}_C^1 \\ \mathbb{I}_A \times f_{B+C} \downarrow & & \downarrow f_{AB \otimes C} \\ \mathbb{P}_A^1 \times \mathbb{P}_{BC}^3 & \xrightarrow{f_{A \otimes BC}} & \mathbb{P}_{ABC}^7 \end{array} .$$

We will show (Theorem 3.6) that the Segre variety $\Sigma_{A \otimes B \otimes C}$ agrees with the intersection

$$\Sigma_{A \otimes B \otimes C} = \Sigma_{A \otimes BC} \cap \Sigma_{AB \otimes C}.$$

In particular, we have

$$\mathcal{J}_{A \otimes BC}(\psi) = 0 \text{ and } \mathcal{J}_{AB \otimes C}(\psi) = 0 \iff [\psi] \in \Sigma_{A \otimes B \otimes C}.$$

In summary, the observables $\mathcal{J}_{AB \otimes C}$ and $\mathcal{J}_{A \otimes BC}$ determine separability of any three particle state in the ABC order. Of course, one can also measure separability for the orders BAC and CAB by consistently taking into account permutations of the chosen Hilbert bases, as we next illustrate.

Consider the following well-known states and their corresponding points in the projective space \mathbb{P}^7 :

$$\begin{array}{ll} |\text{Sep}\rangle = |000\rangle & [\text{Sep}] = [1 : 0 : 0 : 0 : 0 : 0 : 0 : 0] \\ |B_1\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle) & [B_1] = [1 : 0 : 0 : 1 : 0 : 0 : 0 : 0] \\ |B_2\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |101\rangle) & [B_2] = [1 : 0 : 0 : 0 : 0 : 0 : 1 : 0 : 0] \\ |B_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) & [B_3] = [1 : 0 : 0 : 0 : 0 : 0 : 0 : 1 : 0] \\ |W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) & [W] = [0 : 1 : 1 : 0 : 1 : 0 : 0 : 0] \\ |\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) & [\text{GHZ}] = [1 : 0 : 0 : 0 : 0 : 0 : 0 : 1] \end{array}$$

These can be taken as representatives for the six existing equivalence classes of three particle states under SLOCC-equivalence. The state $|\text{Sep}\rangle$ is obviously separable and $|\text{GHZ}\rangle$ and $|W\rangle$ are the only genuinely entangled states. Moreover, these two entangled states represent two different equivalence classes of entanglement [?]. The former state, named after [?], is maximally entangled with respect to most entanglement measures existing in the literature

and its one-particle reduced density matrices are all maximally mixed. The remaining states $|B_i\rangle$ are 2-partite (depending on the order of the Hilbert basis). We have the following table:

ABC	Sep	B_1	B_2	B_3	W	GHZ
$\mathcal{J}_{A\otimes BC}$	0	1	1	0	$\frac{8}{9}$	1
$\mathcal{J}_{AB\otimes C}$	0	0	1	1	$\frac{8}{9}$	1
\mathcal{J}	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{8}{9}$	1

Here the labels ABC indicate we are measuring entanglement of the states with the fixed order ABC and

$$\mathcal{J} := \frac{1}{2}(\mathcal{J}_{A\otimes BC} + \mathcal{J}_{AB\otimes C})$$

is the average measure. In particular, we see that while B_1 and B_3 are bipartite in this order, the state B_2 is classified as entangled. Note however that if we measure entanglement with respect to the order ACB , the roles of B_2 and B_3 are exchanged and we obtain the following table.

ACB	Sep	B_1	B_2	B_3	W	GHZ
$\mathcal{J}_{A\otimes BC}$	0	1	0	1	$\frac{8}{9}$	1
$\mathcal{J}_{AB\otimes C}$	0	0	1	1	$\frac{8}{9}$	1
\mathcal{J}	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{8}{9}$	1

The states $|\text{Sep}\rangle$, $|W\rangle$ and $|\text{GHZ}\rangle$ are invariant under permutations of the basis ABC and so the values of the observables always remain unchanged. Note as well that $|W\rangle$ exhibits less entanglement than $|\text{GHZ}\rangle$ with respect to the above measures, in agreement with the existing entanglement measures.

3. HYPERCUBE OF SEGRE EMBEDDINGS

This section is purely mathematical. Given an integer $n \geq 2$, we consider the generalized Segre embedding

$$\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \xrightarrow{\binom{n}{}} \mathbb{P}^{2^n - 1}$$

and introduce the notion of q -decomposability of a point $z \in \mathbb{P}^{2^n - 1}$ for any integer $1 < q \leq n$. We show that q -decomposability is detected by looking at all the Segre embeddings of the type

$$\mathbb{P}^{2^\ell - 1} \times \mathbb{P}^{2^{n-\ell} - 1} \longrightarrow \mathbb{P}^{2^n - 1}, \text{ for } 1 \leq \ell \leq n - 1,$$

which accommodate as edges of a directed hypercube of Segre embeddings. Let us first review some basic definitions and constructions.

The *complex projective space* \mathbb{P}^N is the set of lines in the complex space \mathbb{C}^{N+1} passing through the origin. It may be described as the quotient

$$\mathbb{P}^N := \frac{\mathbb{C}^{N+1} - \{0\}}{z \sim \lambda z}, \lambda \in \mathbb{C}^*.$$

A point $z \in \mathbb{P}^N$ will be denoted by its homogeneous coordinates

$$z = [z_0 : \cdots : z_N]$$

where, by definition, there is always at least an integer i such that $z_i \neq 0$, and for any $\lambda \in \mathbb{C}^*$ we have

$$[z_0 : \cdots : z_N] = [\lambda z_0 : \cdots : \lambda z_N].$$

Definition 3.1. Given positive integers k and ℓ , the *Segre embedding* $f_{k,\ell}$ is the map

$$f_{k,\ell} : \mathbb{P}^k \times \mathbb{P}^\ell \longrightarrow \mathbb{P}^{(k+1)(\ell+1)-1}$$

defined by sending a pair of points

$$a = [a_0 : \cdots : a_k] \in \mathbb{P}^k \text{ and } b = [b_0 : \cdots : b_\ell] \in \mathbb{P}^\ell$$

to the point of $\mathbb{P}^{(k+1)(\ell+1)-1}$ whose homogeneous coordinates are the pairwise products of the homogeneous coordinates of a and b :

$$f_{k,\ell}(a, b) = [\cdots : z_{ij} : \cdots] \text{ with } z_{ij} := a_i b_j,$$

where we take the lexicographical order.

The Segre embedding is injective, but not surjective in general. The image of $f_{k,\ell}$ is called the *Segre variety* and is denoted by

$$\Sigma_{k,\ell} := \text{Im}(f_{k,\ell}) = \left\{ [\cdots : z_{ij} : \cdots] \in \mathbb{P}^{(k+1)(\ell+1)-1}, \begin{array}{l} z_{ij} z_{i'j'} - z_{i'j} z_{ij'} = 0, \\ \forall i \neq i', j \neq j' \end{array} \right\}.$$

In other words, $\Sigma_{k,\ell}$ is given by the zero locus of all the 2×2 minors of the matrix

$$\begin{pmatrix} z_{00} & \cdots & z_{0\ell} \\ \vdots & \ddots & \vdots \\ z_{k0} & \cdots & z_{k\ell} \end{pmatrix}.$$

A combinatorial argument shows that there is a total of

$$\xi_{k,\ell} := \binom{k+1}{2} \cdot \binom{\ell+1}{2} = \frac{k \cdot (k+1) \cdot \ell \cdot (\ell+1)}{4}$$

minors of size 2×2 in such a matrix.

The above construction generalizes to products of more than two projective spaces of arbitrary dimensions as follows. Given positive integers k_1, \dots, k_n , let

$$N(k_1, \dots, k_n) := (k_1 + 1) \cdots (k_n + 1) - 1.$$

For $1 \leq j \leq n$, let $[a_0^j : \cdots : a_{k_j}^j]$ denote coordinates of \mathbb{P}^{k_j} .

Definition 3.2. The *generalized Segre embedding*

$$f_{k_1, \dots, k_n} : \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_n} \longrightarrow \mathbb{P}^{N(k_1, \dots, k_n)}$$

is defined by letting

$$f_{k_1, \dots, k_n}([\cdots : a_{i_1}^1 : \cdots], \dots, [\cdots : a_{i_n}^n : \cdots]) := [\cdots : z_{i_1 \dots i_n} : \cdots]$$

where

$$z_{i_1 \dots i_n} = a_{i_1}^1 \cdots a_{i_n}^n$$

and the lexicographical is assumed. Denote the *generalized Segre variety* by

$$\Sigma_{k_1, \dots, k_n} := \text{Im}(f_{k_1, \dots, k_n}).$$

It follows from the definition that every generalized Segre embedding may be written as compositions of maps of the form

$$\mathbb{I}_m \times f_{k,\ell} \times \mathbb{I}_{m'}$$

for certain values of m , k , ℓ and m' , where \mathbb{I}_m denotes the identity map of \mathbb{P}^m . These compositions may be arranged in a directed $(n-1)$ -dimensional hypercube, where the initial vertex is $\mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_n}$ and the final vertex is $\mathbb{P}^{N(k_1, \dots, k_n)}$. Note that the $(n-1)$ final edges of the hypercube (those edges whose target is the final vertex $\mathbb{P}^{N(k_1, \dots, k_n)}$) are given by Segre embeddings of bipartite-type

$$f_{N(k_1, \dots, k_j), N(k_{j+1}, \dots, k_n)} : \mathbb{P}^{N(k_1, \dots, k_j)} \times \mathbb{P}^{N(k_{j+1}, \dots, k_n)} \longrightarrow \mathbb{P}^{N(k_1, \dots, k_n)},$$

where $1 \leq j \leq n-1$.

Example 3.3. For the generalized Segre embedding $f_{1,1,1,1}$ characterizing entanglement of 4 particle states of spin- $\frac{1}{2}$, we obtain a cube with commutative faces

$$\begin{array}{ccccc}
 \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\mathbb{I} \times \mathbb{I} \times f_{1,1}} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 & & \\
 \downarrow f_{1,1} \times \mathbb{I} \times \mathbb{I} & \searrow \mathbb{I} \times f_{1,1} \times \mathbb{I} & \downarrow \mathbb{I} \times f_{1,3} & & \\
 & \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^1 & \xrightarrow{\mathbb{I} \times f_{3,1}} & \mathbb{P}^1 \times \mathbb{P}^7 & \\
 & \downarrow f_{1,3} \times \mathbb{I} & \downarrow f_{1,1} \times \mathbb{I} & \downarrow f_{1,7} & \\
 \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{f_{3,1} \times \mathbb{I}} & \mathbb{P}^3 \times \mathbb{P}^3 & & \\
 & \downarrow \mathbb{I} \times f_{1,1} & \searrow f_{3,3} & & \\
 & \mathbb{P}^7 \times \mathbb{P}^1 & \xrightarrow{f_{7,1}} & \mathbb{P}^{15} &
 \end{array}$$

We next state a general decomposability result for arbitrary products of projective spaces. Since our interest lies in spin- $\frac{1}{2}$ particle systems, for the sake of simplicity we will restrict to the case where the initial spaces are projective lines. For any integer $m \geq 1$, let

$$N_m := N(1, \overset{(m)}{\dots}, 1) = 2^m - 1.$$

We will consider the decompositions associated to the generalized Segre embedding

$$\mathbb{P}^1 \times \overset{(n)}{\dots} \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{N_n}.$$

Definition 3.4. Let $n \geq 2$ and $1 < q \leq n$ be integers. We will say that a point $z \in \mathbb{P}^{N_n}$ is q -decomposable if and only if there exist positive integers m_1, \dots, m_q with $m_1 + \dots + m_q = n$ such that

$$z \in \Sigma_{N_{m_1}, \dots, N_{m_q}}.$$

Note that every q -decomposable point is also $(q-1)$ -decomposable. If z is not 2-decomposable, we will say that it is *indecomposable*.

Points that are q -decomposable will correspond precisely to q -partite states and indecomposable points will correspond to entangled states.

Example 3.5. A point z in \mathbb{P}^3 is 2-decomposable if and only if $z \in \Sigma_{1,1}$. Otherwise it is indecomposable. A point z in \mathbb{P}^7 is 2-decomposable if and only if $z \in \Sigma_{3,1} \cup \Sigma_{1,3}$. It is 3-decomposable if and only if $z \in \Sigma_{1,1,1}$. The following result shows that z is actually

3-decomposable if and only if $z \in \Sigma_{3,1} \cap \Sigma_{1,3}$, so that it is 2-decomposable in every possible way. In physical terms, it just says that a state is 3-partite if and only if it is 2-partite when considering both types of bipartitions.

Theorem 3.6 (Generalized Decomposability). *Let $n \geq 2$ and $1 < q \leq n$ be integers. A point $z \in \mathbb{P}^{N_n}$ is q -decomposable if and only if it lies in at least $q-1$ different Segre varieties of the form $\Sigma_{N_\ell, N_{n-\ell}}$, with $1 \leq \ell \leq n-1$.*

For the particular extreme cases we have that a point $z \in \mathbb{P}^{N_n}$ is:

$$\begin{cases} \text{Indecomposable} & \iff z \notin \Sigma_{N_\ell, N_{n-\ell}}, \text{ for all } 1 \leq \ell \leq n-1. \\ n\text{-decomposable} & \iff z \in \Sigma_{N_\ell, N_{n-\ell}}, \text{ for all } 1 \leq \ell \leq n-1. \end{cases}$$

We refer to the Appendix for the proof. This result will be essential in the next section, where we give a general method for measuring decomposability. Indeed, Theorem 3.6 asserts that decomposability is entirely determined by the Segre varieties $\Sigma_{N_\ell, N_{n-\ell}}$ for all $1 \leq \ell \leq n-1$. Note that this family of varieties is the one arising when looking at the $(n-1)$ edges whose target is the last vertex of the $(n-1)$ -dimensional hypercube and corresponds precisely to the family of all possible bipartitions of \mathbb{P}^{N_n} .

This result will translate into taking $(n-1)$ measures of a given n -particle state, in order to detect the level of decomposability of its associated point in the projective space.

Example 3.7. In view of Theorem 3.6, a point z in \mathbb{P}^{15} is:

$$\begin{cases} \text{indecomposable} & \iff z \notin \Sigma_{7,1} \cup \Sigma_{1,7} \cup \Sigma_{3,3} \\ \text{2-decomposable} & \iff z \in \Sigma_{7,1} \cup \Sigma_{1,7} \cup \Sigma_{3,3} \\ \text{3-decomposable} & \iff z \in (\Sigma_{1,7} \cap \Sigma_{7,1}) \cup (\Sigma_{1,7} \cap \Sigma_{3,3}) \cup (\Sigma_{7,1} \cap \Sigma_{3,3}). \\ \text{4-decomposable} & \iff z \in \Sigma_{7,1} \cap \Sigma_{1,7} \cap \Sigma_{3,3} \end{cases}$$

4. OPERATORS CONTROLLING EDGES OF THE HYPERCUBE

Given an n -particle state, in this section we define $(n-1)$ observables which measure its entanglement. Let us first fix some notation. We will denote by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and by \mathbb{I}_k the identity matrix of size k .

The *Kronecker product* of two matrices $\mathcal{A} = (a_{ij}) \in \text{Mat}_{k \times k}$ and $\mathcal{B} \in \text{Mat}_{n \times n}$ is the matrix of size $kn \times kn$ given by:

$$\mathcal{A} \otimes \mathcal{B} := \begin{pmatrix} a_{11}\mathcal{B} & \cdots & a_{1k}\mathcal{B} \\ \vdots & \ddots & \vdots \\ a_{k1}\mathcal{B} & \cdots & a_{kk}\mathcal{B} \end{pmatrix}.$$

The *Hermitian product* of two complex vectors $u = (u_0, \dots, u_k)$ and $v = (v_0, \dots, v_k)$ is

$$u \cdot v = \sum_{i=0}^k \overline{u_i} v_i.$$

Also, let

$$\|u\|^2 := u \cdot u = \sum_{i=0}^k \bar{u}_i u_i.$$

For α a complex number, we denote $|\alpha| := \|\alpha\| = \sqrt{\bar{\alpha} \cdot \alpha}$ its absolute value.

We will use the *Lagrange identity*, which states that

$$\|u\|^2 \cdot \|v\|^2 - |u \cdot v|^2 = \sum_{i=0}^{k-1} \sum_{j=i+1}^k |u_i v_j - u_j v_i|^2.$$

Note that in many references, the term on the right side of the identity is often written in the equivalent form $|u_i \bar{v}_j - u_j \bar{v}_i|^2$ instead of $|u_i v_j - u_j v_i|^2$.

In order to describe n -particle states we fix an ordered basis of $N_n = 2^n - 1$ linearly independent vectors of the corresponding Hilbert space

$$|i_1 \cdots i_n\rangle = |i_1\rangle_{\mathcal{O}_1} \otimes \cdots \otimes |i_n\rangle_{\mathcal{O}_n}$$

where $\mathcal{O}_1, \dots, \mathcal{O}_n$ denote the different observers and $\{i_1, \dots, i_n\} \in \{0, 1\}$, since we are in the spin- $\frac{1}{2}$ case. Using this basis, the coordinates for a general pure state for n particles will be written as

$$|\psi\rangle_{\mathcal{O}_1 \cdots \mathcal{O}_n} = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N_n} \end{pmatrix},$$

where z_i are complex numbers satisfying the normalization condition

$$\sum_{i \geq 0} |z_i|^2 = 1.$$

Likewise, we will write:

$$\langle \psi | = (\bar{z}_0, \bar{z}_1, \dots, \bar{z}_{N_n}).$$

Given such a state, we may consider its class $[\psi] \in \mathbb{P}^{N_n}$ by taking its homogeneous coordinates

$$[\psi] = [z_0 : \cdots : z_{N_n}],$$

where we recall that $[z_0 : \cdots : z_{N_n}] = [\lambda z_0 : \cdots : \lambda z_{N_n}]$ for any $\lambda \in \mathbb{C}^*$.

For all $\ell = 1, \dots, n-1$, define the following observable acting on n -particle states:

$$\mathcal{J}_{n,\ell}(\psi) := 2 - \left(\frac{1}{2^{\ell-1}} \sum_{i_1, \dots, i_\ell=0}^1 |\langle \psi | \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_\ell} \otimes \mathbb{I}_{2^{n-\ell}} | \psi \rangle|^2 \right).$$

The purpose of this section is to show that $\mathcal{J}_{n,\ell}(\psi) = 0$ if and only if the class $[\psi]$ in the projective space \mathbb{P}^{N_n} lies in the Segre variety $\Sigma_{N_\ell, N_{n-\ell}}$.

The next two examples detail the computations of the observables introduced in Section 2 for the cases of two and three particles respectively. Note that we used a slightly different notation, namely:

2 particles: $\mathcal{J}_{A \otimes B} \equiv \mathcal{J}_{2,1}$ and $\Sigma_{A \otimes B} \equiv \Sigma_{1,1}$.

3 particles: $\mathcal{J}_{A \otimes BC} \equiv \mathcal{J}_{3,1}$, $\Sigma_{A \otimes BC} \equiv \Sigma_{1,3}$, $\mathcal{J}_{AB \otimes C} \equiv \mathcal{J}_{3,2}$, and $\Sigma_{AB \otimes C} \equiv \Sigma_{3,1}$.

Example 4.1. For 2-particle states we have a single observable

$$\mathcal{J}_{2,1}(\psi) = 2 - \left(\sum_{i=0}^3 |\langle \psi | \sigma_i \otimes \mathbb{I}_2 | \psi \rangle|^2 \right).$$

Define vectors $\mathcal{A}_0 = (z_0, z_1)$ and $\mathcal{A}_1 = (z_2, z_3)$, so that $\langle \psi | = (\mathcal{A}_0, \mathcal{A}_1)$. Then we have

$$\begin{aligned} \mathcal{J}_{2,1}(\psi) &= 2 - (|\mathcal{A}_0 \cdot \mathcal{A}_0 + \mathcal{A}_1 \cdot \mathcal{A}_1|^2 + |\mathcal{A}_0 \cdot \mathcal{A}_1 + \mathcal{A}_1 \cdot \mathcal{A}_0|^2 \\ &\quad + |\mathcal{A}_0 \cdot \mathcal{A}_1 - \mathcal{A}_1 \cdot \mathcal{A}_0|^2 + |\mathcal{A}_0 \cdot \mathcal{A}_0 - \mathcal{A}_1 \cdot \mathcal{A}_1|^2) = \\ &= 1 - \left(|\bar{z}_0 z_0 + \bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{z}_3 z_3|^2 + |\bar{z}_0 z_2 + \bar{z}_1 z_3 + \bar{z}_2 z_0 + \bar{z}_3 z_1|^2 + \right. \\ &\quad \left. + |-\bar{z}_0 z_2 - \bar{z}_1 z_3 + \bar{z}_2 z_0 + \bar{z}_3 z_1|^2 + |\bar{z}_0 z_0 + \bar{z}_1 z_1 - \bar{z}_2 z_2 - \bar{z}_3 z_3|^2 \right) = \\ &= 1 - \left(|\bar{z}_0 z_2 + \bar{z}_1 z_3 + \bar{z}_2 z_0 + \bar{z}_3 z_1|^2 + |-\bar{z}_0 z_2 - \bar{z}_1 z_3 + \bar{z}_2 z_0 + \bar{z}_3 z_1|^2 + \right. \\ &\quad \left. + |\bar{z}_0 z_0 + \bar{z}_1 z_1 - \bar{z}_2 z_2 - \bar{z}_3 z_3|^2 \right) = 4 |z_0 z_3 - z_1 z_2|^2. \end{aligned}$$

Therefore $\mathcal{J}_{2,1}(\psi) = 0$ if and only if $[\psi] \in \Sigma_{1,1}$.

Example 4.2. For 3-particle states we have two observables

$$\begin{aligned} \mathcal{J}_{3,1}(\psi) &:= 2 - \left(\sum_{i=0}^3 |\langle \psi | \sigma_i \otimes I_4 | \psi \rangle|^2 \right) \text{ and} \\ \mathcal{J}_{3,2}(\psi) &:= 2 - \left(\frac{1}{2} \sum_{i,j=0}^3 |\langle \psi | \sigma_i \otimes \sigma_j \otimes I_2 | \psi \rangle|^2 \right). \end{aligned}$$

Write $z = (\mathcal{A}_0, \mathcal{A}_1)$ where $\mathcal{A}_0 = (z_0, \dots, z_3)$ and $\mathcal{A}_1 = (z_4, \dots, z_7)$. Then we have

$$\begin{aligned} \mathcal{J}_{3,1}(\psi) &= 2 - (|\bar{\mathcal{A}}_0 \mathcal{A}_0 + \bar{\mathcal{A}}_1 \mathcal{A}_1|^2 + |\bar{\mathcal{A}}_0 \mathcal{A}_0 + \bar{\mathcal{A}}_1 \mathcal{A}_1|^2 + |\bar{\mathcal{A}}_0 \mathcal{A}_1 + \bar{\mathcal{A}}_1 \mathcal{A}_0|^2 + \\ &\quad + |-\bar{\mathcal{A}}_0 \mathcal{A}_1 + \bar{\mathcal{A}}_1 \mathcal{A}_0|^2 + |\bar{\mathcal{A}}_0 \mathcal{A}_0 - \bar{\mathcal{A}}_1 \mathcal{A}_1|^2) = \\ &= 1 - (|\bar{\mathcal{A}}_0 \mathcal{A}_1 + \bar{\mathcal{A}}_1 \mathcal{A}_0|^2 + |-\bar{\mathcal{A}}_0 \mathcal{A}_1 + \bar{\mathcal{A}}_1 \mathcal{A}_0|^2 + |\bar{\mathcal{A}}_0 \mathcal{A}_0 - \bar{\mathcal{A}}_1 \mathcal{A}_1|^2) = \\ &= 4 \{ |z_0 z_5 - z_1 z_4|^2 + |z_0 z_6 - z_2 z_4|^2 + |z_0 z_7 - z_3 z_4|^2 + \\ &\quad + |z_1 z_6 - z_2 z_5|^2 + |z_1 z_7 - z_3 z_5|^2 + |z_2 z_7 - z_3 z_6|^2 \}. \end{aligned}$$

Note that the numbers $z_i z_j - z'_j z'_i$ correspond to the minors describing the zero locus of $\Sigma_{1,3}$. Indeed, this is determined by the vanishing of all the 2×2 minors of the matrix

$$\begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 & z_7 \end{pmatrix}.$$

Therefore $\mathcal{J}_{3,1}(\psi) = 0$ if and only if $[\psi] \in \Sigma_{1,3}$.

Likewise, writing $z = (\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ with $\mathcal{A}_0 = (z_0, z_1)$, $\mathcal{A}_1 = (z_2, z_3)$, $\mathcal{A}_2 = (z_4, z_5)$ and $\mathcal{A}_3 = (z_6, z_7)$ we easily obtain

$$\begin{aligned} \mathcal{J}_{3,2}(\psi) &= 4 \{ |z_0 z_3 - z_1 z_2|^2 + |z_0 z_5 - z_1 z_4|^2 + |z_0 z_7 - z_1 z_6|^2 + \\ &\quad + |z_2 z_5 - z_3 z_4|^2 + |z_2 z_7 - z_3 z_6|^2 + |z_4 z_7 - z_5 z_6|^2 \}. \end{aligned}$$

Note that the numbers $z_i z_j - z'_j z'_i$ correspond to the minors describing the zero locus of $\Sigma_{3,1}$. Indeed, this is determined by the vanishing of all the 2×2 minors of the matrix

$$\begin{pmatrix} z_0 & z_1 \\ z_2 & z_3 \\ z_4 & z_5 \\ z_6 & z_7 \end{pmatrix}.$$

Therefore $\mathcal{J}_{3,2}(\psi) = 0$ if and only if $[\psi] \in \Sigma_{3,1}$.

Returning to the general setting, note that the measures $\mathcal{J}_{n,\ell}(\psi)$ are related to the trace of the squared density matrix for a given partition of the system. Indeed, the density matrix for any physical state is a Hermitian operator and therefore can be written in terms of σ_i as

$$\rho = \sum_{i_1, \dots, i_n \in \{1,2,3\}} c_{i_1, \dots, i_\ell} \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_\ell} \otimes \mathbb{I}_{2^{n-\ell}} + d_{i_{\ell+1}, \dots, i_n} \mathbb{I}_{2^\ell} \otimes \sigma_{i_{\ell+1}} \otimes \cdots \otimes \sigma_{i_n},$$

where we write explicitly the partition of the system in A and B . In this notation, the reduced density matrix of the system A reads

$$\rho_A = 2^{n-\ell} c_{i_1, \dots, i_\ell} \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_\ell}.$$

We may now compute expectation values for our operators:

$$\langle \psi | \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_\ell} \otimes \mathbb{I}_{2^{n-\ell}} | \psi \rangle = \text{Tr}(\rho \cdot (\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_\ell} \otimes \mathbb{I}_{2^{n-\ell}})),$$

and using the identity

$$\text{Tr}((\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}) \cdot (\sigma_{j_1} \otimes \cdots \otimes \sigma_{j_n})) = 2^n \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_n j_n}$$

we obtain

$$c_{i_1, \dots, i_\ell} = 2^n \langle \psi | \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_\ell} \otimes \mathbb{I}_{2^{n-\ell}} | \psi \rangle.$$

We now compute the trace of the squared density matrix:

$$\begin{aligned} \text{Tr}_A \rho_A^2 &= 2^{2(n-\ell)} c_{i_1, \dots, i_\ell} c_{j_1, \dots, j_\ell} \text{Tr}((\sigma_{i_1} \otimes \cdots \otimes \sigma_{i_n}) \cdot (\sigma_{j_1} \otimes \cdots \otimes \sigma_{j_n})) = \\ &= 2^\ell \langle \psi | \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_\ell} \otimes \mathbb{I}_{2^{n-\ell}} | \psi \rangle^2 \end{aligned}$$

which leads to the identity

$$\mathcal{J}_{n,\ell}(\psi) = 2(1 - \text{Tr}_A \rho_A^2).$$

The main result of this section relates the operators $\mathcal{J}_{n,\ell}$ with the minors of order two that define Segre varieties. In particular, our operators are directly related to the entanglement measures studied in [?].

Theorem 4.3. *Let $|\psi\rangle$ be a pure n -particle state and let $1 \leq \ell \leq n-1$. Then*

$$\mathcal{J}_{n,\ell}(\psi) = 4 \sum_I |M_I|^2,$$

where the sum runs over all 2×2 minors M_I determining the zero locus of $\Sigma_{N_\ell, N_{n-\ell}}$. In particular,

$$\mathcal{J}_{n,\ell}(\psi) = 0 \iff [\psi] \in \Sigma_{N_\ell, N_{n-\ell}}.$$

Proof. Write $(z_0, \dots, z_{N_n}) = (\mathcal{A}_0, \dots, \mathcal{A}_{N_\ell})$, where

$$\mathcal{A}_j = (z_{j(N_{n-\ell}+1)}, \dots, z_{j(N_{n-\ell}+1)+N_{n-\ell}})$$

for $j = 0, \dots, N_\ell$, so that each \mathcal{A}_j is tuple with $N_{n-\ell} + 1$ components. With this notation, we have

$$\mathcal{J}_{n,\ell}(\psi) = 4 \sum_{j=0}^{N_\ell-1} \sum_{k>j}^{N_\ell} (|\mathcal{A}_j|^2 \cdot |\mathcal{A}_k|^2 - |\mathcal{A}_j \cdot \mathcal{A}_k|^2).$$

Applying the Lagrange identity we obtain

$$\mathcal{J}_{n,\ell}(\psi) = 4 \sum_{j=0}^{N_\ell-1} \sum_{k>j}^{N_\ell} \sum_{s=0}^{N_\ell-1} \sum_{t>s}^{N_\ell} |\mathcal{A}_{j,s} \cdot \mathcal{A}_{k,t} - \mathcal{A}_{j,t} \cdot \mathcal{A}_{k,s}|^2,$$

where $\mathcal{A}_{i,j}$ denotes the j -th component of the tuple \mathcal{A}_i . It now suffices to note that the numbers

$$\mathcal{A}_{j,s} \mathcal{A}_{k,t} - \mathcal{A}_{j,t} \mathcal{A}_{k,s}$$

correspond to the minors M_I . Indeed, the zero locus of $\Sigma_{N_\ell, N_{n-\ell}}$ is determined by the vanishing of the 2×2 minors of the matrix

$$\begin{pmatrix} \mathcal{A}_{0,0} & \cdots & \mathcal{A}_{0,N_{n-\ell}} \\ \vdots & & \vdots \\ \mathcal{A}_{N_\ell,0} & \cdots & \mathcal{A}_{N_\ell,N_{n-\ell}} \end{pmatrix}.$$

□

5. PHYSICAL INTERPRETATION

Given an integer $n \geq 1$, we fix an ordered basis of $N_n = 2^n - 1$ linearly independent vectors of the Hilbert space of pure n -particle states of spin- $\frac{1}{2}$

$$|i_1 \cdots i_n\rangle_{\mathcal{O}_1 \cdots \mathcal{O}_n} = |i_1\rangle_{\mathcal{O}_1} \otimes \cdots \otimes |i_n\rangle_{\mathcal{O}_n},$$

where $\mathcal{O}_1, \dots, \mathcal{O}_n$ denote the different observers. We will omit the labels of the observers, but one should note that, in the following, the chosen basis always has the same fixed order unless stated otherwise.

Definition 5.1. Let $1 \leq q \leq n$ be an integer. An n -particle state $|\psi\rangle$ is said to be q -partite if it can be written as

$$|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_q\rangle,$$

where $|\psi_i\rangle$ are n_i -particle states, with $n_i > 0$ and $n_1 + \cdots + n_q = n$.

1-partite states are called *entangled*, while n -partite states are called *separable*. Physically, separable states are those that are uncorrelated. A product state can thus be easily prepared in a local way: each observer \mathcal{O}_i produces the state $|\psi_i\rangle$ and the measurement outcomes for each observer do not depend on the outcomes for the other observers.

A basic observation is that a state is separable if and only if it lies in the generalized Segre variety of Definition 3.2 (see for instance [?]). Likewise, a state is q -partite if and only if its corresponding projective point on \mathbb{P}^{N_n} lies in a Segre variety of the form

$$\Sigma_{N_{m_1}, \dots, N_{m_q}}$$

with m_1, \dots, m_q positive integers such that $m_1 + \dots + m_q = n$. So q -partite states correspond geometrically to the q -decomposable points of Definition 3.4. Combining the results of the previous two sections we have that an n -particle state $|\psi\rangle$ is q -partite if and only if there are $q - 1$ indices $\ell_1, \dots, \ell_{q-1}$ with $1 \leq \ell_i \leq n - 1$ and $\ell_i \neq \ell_j$ such that $\mathcal{J}_{n,\ell_i}(\psi) = 0$. Indeed, from Theorem 3.6 we know that $|\psi\rangle$ is q -partite if and only if its corresponding point $[\psi]$ in \mathbb{P}^{N_n} lies in at least $q - 1$ different Segre varieties of the form $\Sigma_{N_\ell, N_{n-\ell}}$, with $1 \leq \ell \leq n - 1$. Moreover, from Theorem 4.3 we know that $[\psi] \in \Sigma_{N_\ell, N_{n-\ell}}$ if and only if $\mathcal{J}_{n,\ell}(\psi) = 0$.

Note that, a priori, given an n -particle state, one would have to take $(n - 1)!$ measures of bipartite type to completely determine its q -decomposability (namely, for each possible bipartition, check further bipartitions recursively). The hypercube approach tells us that it suffices to take $(n - 1)$ measures, corresponding to the operators $\{\mathcal{J}_{n,\ell}\}$ in order to determine completely its q -decomposability.

In the remaining of the section, we study the behaviour of the observables $\mathcal{J}_{n,\ell}$ in some particular cases of interest. We first introduce the average observable acting on n -particle states:

$$\mathcal{J}(\psi) := \frac{1}{n-1} \sum_{\ell=1}^{n-1} \mathcal{J}_{n,\ell}(\psi).$$

Note that this is a measure of entanglement, in the sense that it does not increase under LOCC. Indeed, assume that a state $|\psi\rangle$ is transformed, using LOCC, into an ensemble $\{p_i, |\varphi_i\rangle\}$ with $\sum_i p_i = 1$. Recall that $\mathcal{J}_{n,\ell}(\psi) = 2(1 - \text{Tr}_A \rho_A^2)$ for a certain partition of the system into $A \otimes B$, and so we have the well-known inequalities

$$\mathcal{J}_{n,\ell}(\psi) \geq \sum_i p_i \mathcal{J}_{n,\ell}(\varphi_i)$$

(see for instance [?], [?]). Therefore we obtain

$$\begin{aligned} \mathcal{J}(\psi) &= \frac{1}{n-1} \sum_{\ell=1}^{n-1} \mathcal{J}_{n,\ell}(\psi) \geq \frac{1}{n-1} \sum_{\ell=1}^{n-1} \sum_i p_i \mathcal{J}_{n,\ell}(\varphi_i) = \\ &= \sum_i p_i \frac{1}{n-1} \sum_{\ell=1}^{n-1} \mathcal{J}_{n,\ell}(\varphi_i) = \sum_i p_i \mathcal{J}(\varphi_i). \end{aligned}$$

The two- and three- particle states discussed in Section 2 which are invariant under permutations of the Hilbert basis ($|\text{Sep}\rangle$, $|\text{EPS}\rangle$, $|\text{GHS}\rangle$, $|W\rangle$) allow for natural generalizations to the n -particle case. We study their entanglement measures.

Note first that the separable n -particle state

$$|\text{Sep}_n\rangle := |0 \overset{(n)}{\dots} 0\rangle$$

corresponds to the point in \mathbb{P}^{N_n} given by

$$[\text{Sep}_n] = [1 : 0 : \dots : 0]$$

and so one easily verifies that $\mathcal{J}(\text{Sep}_n) = 0$.

The *Schrödinger n -particle state* is a superposition of two maximally distinct states

$$|S_n\rangle := \frac{1}{\sqrt{2}} \left(|0 \overset{(n)}{\dots} 0\rangle + |1 \overset{(n)}{\dots} 1\rangle \right).$$

It generalizes the two-particle state $|\text{EPS}\rangle$ and the three-particle state $|\text{GHZ}\rangle$ and it corresponds to the point in \mathbb{P}^{N_n} given by

$$[S_n] = [1 : 0 : \dots : 0 : 1].$$

One easily computes $\mathcal{J}_{n,\ell}(S_n) = 1$ for all $1 \leq \ell \leq n$ and so $\mathcal{J}(S_n) = 1$. For $n > 3$, it's not clear that the Schrödinger state exhibits maximal entanglement [?]. This observation agrees with our measures for $\mathcal{J}_{n,\ell}$, as we will see below.

We now consider a generalization of the W state for three particles, to the case of n -particles. For each fixed integer $0 \leq k \leq n$, these states are constructed by adding all permutations of generators of the form

$$|1\rangle \otimes \dots \otimes |1\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle$$

with k states $|1\rangle$ and $n - k$ states $|0\rangle$, together with a global normalization constant. These are clearly invariant with respect to permutations of the basis. Denote such states by

$$|D_{n,k}\rangle = \binom{n}{k}^{-\frac{1}{2}} \sum_{\text{permut}} |1\rangle \otimes \dots \otimes |1\rangle \otimes |0\rangle \otimes \dots \otimes |0\rangle.$$

These are known as *Dicke states* [?]. Note that $|D_{n,0}\rangle = |\text{Sep}_n\rangle$ and $|D_{3,1}\rangle = |W_3\rangle$. In the case of four particles we have

$$\begin{aligned} |D_{4,1}\rangle &= \frac{1}{\sqrt{4}} (|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle), \\ |D_{4,2}\rangle &= \frac{1}{\sqrt{6}} (|1100\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle + |0011\rangle), \\ |D_{4,3}\rangle &= \frac{1}{\sqrt{4}} (|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle). \end{aligned}$$

Their corresponding points in \mathbb{P}^{15} are

$$\begin{aligned} |D_{4,1}\rangle &= [0 : 1 : 1 : 0 : 1 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 0 : 0 : 0], \\ |D_{4,2}\rangle &= [0 : 0 : 0 : 1 : 0 : 1 : 1 : 0 : 0 : 1 : 1 : 0 : 1 : 0 : 0], \\ |D_{4,3}\rangle &= [0 : 0 : 0 : 0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 1 : 0 : 1 : 1]. \end{aligned}$$

We obtain the following table for the observables $\mathcal{J}_{4,\ell}$:

	$ D_{4,1}\rangle$	$ D_{4,2}\rangle$	$ D_{4,3}\rangle$
$\mathcal{J}_{4,1}$	$\frac{3}{4}$	1	$\frac{3}{4}$
$\mathcal{J}_{4,2}$	1	1	1
$\mathcal{J}_{4,3}$	$\frac{3}{4}$	1	$\frac{3}{4}$
\mathcal{J}	$\frac{5}{6}$	1	$\frac{5}{6}$

Note $\mathcal{J}(D_{4,2}) = 1$ as is the case for the state $|S_4\rangle$. For more than four particles we obtain values > 1 for the observables $\mathcal{J}_{n,\ell}$. For instance, in the five particle case, we have:

	$ D_{5,1}\rangle$	$ D_{5,2}\rangle$	$ D_{5,3}\rangle$	$ D_{5,4}\rangle$
$\mathcal{J}_{5,1}$	$\frac{16}{25}$	$\frac{24}{25}$	$\frac{24}{25}$	$\frac{16}{25}$
$\mathcal{J}_{5,2}$	$\frac{24}{25}$	$\frac{27}{25}$	$\frac{27}{25}$	$\frac{24}{25}$
$\mathcal{J}_{5,3}$	$\frac{24}{25}$	$\frac{27}{25}$	$\frac{27}{25}$	$\frac{24}{25}$
$\mathcal{J}_{5,4}$	$\frac{16}{25}$	$\frac{24}{25}$	$\frac{24}{25}$	$\frac{16}{25}$
\mathcal{J}	$\frac{4}{5}$	$\frac{51}{50}$	$\frac{51}{50}$	$\frac{4}{5}$

In particular, we see that

$$\mathcal{J}(D_{5,2}) = \mathcal{J}(D_{5,3}) > \mathcal{J}(S_5) = 1.$$

For higher particle states the same pattern repeats itself, with the middle states

$$|D_{n,\lfloor \frac{n}{2} \rfloor}\rangle \text{ and } |D_{n,\lceil \frac{n}{2} \rceil}\rangle$$

always exhibiting the largest entanglement as well as symmetries of the tables in both directions.

We end this section with some notable examples in the four- and five-particle cases. The state

$$|\text{HS}\rangle = \frac{1}{\sqrt{6}} (|1100\rangle + |0011\rangle + \omega |1001\rangle + \omega |0110\rangle + \omega^2 |1010\rangle + \omega^2 |0101\rangle)$$

where $\omega = e^{\frac{2\pi i}{3}}$ is a third root of unity, was conjectured to be maximally entangled by Higuchi-Sudbery [?] and it actually gives a local maximum of the average two-particle von Neumann entanglement entropy [?]. Another highly (though not maximally) entangled state, found by Brown-Stepney-Sudbery-Braunstein [?], is given by

$$|\text{BSSB}_4\rangle = \frac{1}{2} (|0000\rangle + |+\rangle \otimes |011\rangle + |1101\rangle + |-\rangle \otimes |110\rangle),$$

where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Our measures agree with these facts, as shown in the table below.

	$ S_4\rangle$	$ D_{4,2}\rangle$	$ \text{BSSB}_4\rangle$	$ \text{HS}\rangle$
$\mathcal{J}_{4,1}$	1	1	$\frac{3}{4}$	1
$\mathcal{J}_{4,2}$	1	1	$\frac{5}{4}$	$\frac{4}{3}$
$\mathcal{J}_{4,3}$	1	1	1	1
\mathcal{J}	1	1	1	$\frac{10}{9}$

In [?], a related measure of entanglement is introduced, based on vector lengths and the angles between vectors of certain coefficient matrices. While this measure is strongly related to concurrence and hence to the observables \mathcal{J} , their measure E_{avg} does not distinguish the states $|\text{BSSB}_4\rangle$ and $|\text{HS}\rangle$. In contrast, we do find that

$$1 = \mathcal{J}(\text{BSSB}_4) < \mathcal{J}(\text{HS}).$$

In [?], a highly entangled five-particle state is described as

$$|\text{BSSB}\rangle_5 = \frac{1}{2} (|000\rangle \otimes |\Phi_-\rangle + |010\rangle \otimes |\Psi_-\rangle + |100\rangle \otimes |\Phi_+\rangle + |111\rangle \otimes |\Psi_+\rangle),$$

where $|\Psi_\pm\rangle = |00\rangle \pm |11\rangle$ and $|\Phi_\pm\rangle = |01\rangle \pm |10\rangle$. Our measures give:

	$ S_5\rangle$	$ D_{5,2}\rangle$	$ \text{BSSB}_5\rangle$
$\mathcal{J}_{5,1}$	1	$\frac{24}{25}$	1
$\mathcal{J}_{5,2}$	1	$\frac{27}{25}$	$\frac{3}{2}$
$\mathcal{J}_{5,3}$	1	$\frac{27}{25}$	$\frac{5}{4}$
$\mathcal{J}_{5,4}$	1	$\frac{24}{25}$	1
\mathcal{J}	1	$\frac{51}{50}$	$\frac{19}{16}$

In particular, we see that

$$1 = \mathcal{J}(S_5) < \mathcal{J}(D_{5,2}) < \mathcal{J}(\text{BSSB}_5).$$

So far we have only considered pure states. As is well-known, the identification of quantum states with points in the complex projective space is only valid for pure states, and so the approach to entanglement via the Segre embedding is very particular to the pure setting. Indeed, recall that pure states of a single particle of spin- $\frac{1}{2}$ are identified with points in $\mathbb{C}\mathbb{P}^1$, a space diffeomorphic to the sphere S^2 , while mixed states would correspond to points in the interior of this sphere, so in the 3-dimensional ball B^3 , which has S^2 as its boundary. Still, via the *convex roof construction*, given a mixed state with density matrix ρ one may extend the entanglement measure \mathcal{J} by letting

$$\mathcal{J}(\rho) := \inf \sum p_i \mathcal{J}(\varphi_i)$$

where the infimum is taken with respect to all ensembles $\{p_i, |\varphi_i\rangle\}$ such that

$$\rho = \sum p_i |\varphi_i\rangle \langle \varphi_i| \quad \text{with} \quad \sum_i p_i = 1.$$

Any operator defined via the convex roof construction of an entanglement measure, is again an entanglement measure, in the sense that it will still be non-increasing under LOCC (see [?], [?]). However, the interpretation of this entanglement measure as well as its applicability is not clear and requires a more general framework than the one provided by the hypercube of Segre embeddings presented here.

6. SUMMARY AND CONCLUSIONS

Within a purely geometric framework, we have described entanglement of n -particle states in terms of a hypercube of bipartite-type Segre maps. For simplicity, in this paper we have restricted to spin- $\frac{1}{2}$ particles, but the geometric results generalize almost verbatim to the case of qudits. The hypercube picture allows to identify separability (or more generally, q -decomposability) in terms of a depth factor within the hypercube: the deeper a state lies in the hypercube, the more separable.

We have defined a collection of operators which measure the properties of a state in the above geometric set-up. Given an n -particle state and having fixed an ordered basis of the total Hilbert space, there are 2^{n-1} different decomposability possibilities, given by

the different ordered q -partitions, for $1 \leq q \leq n$. A standard way to characterize the decomposability of such a state would be to consider, for any possible bipartition, all of its possible bipartitions in a recursive way. This gives a total of $(n-1)!$ measures to be taken. Our hypercube approach says that it suffices to take $(n-1)$ measures, given by the operators $\mathcal{J}_{n,\ell}$, with $1 \leq \ell \leq n-1$. So as the complexity of the problem grows factorially, our solution just grows linearly on n .

The operators $\mathcal{J}_{n,\ell}$ measure the different bipartitions of the system, corresponding geometrically with the last $(n-1)$ edges of the Segre hypercube. The expected value of these operators is related with the quantum Tsallis entropy ($q=2$) of both parts of the state. The concrete values of ℓ for which $\mathcal{J}_{n,\ell}$ vanishes shows precisely in what edge of the hypercube the state belongs or, more physically, in which parts the state is separable.

To illustrate the physical interest of our approach, we have computed the value of the operators $\mathcal{J}_{n,\ell}$ for various entangled states considered in the literature. The motivation for many of them arises in quantum computing and are therefore classified from the quantum control perspective. In all cases, the proposed observables give results consistent with the expectations.

APPENDIX A. PROOF OF THE GENERALIZED DECOMPOSABILITY THEOREM

This appendix is devoted to the proof of Theorem 3.6 on geometric decomposability. Let us first consider the tripartite-type Segre embedding

$$f_{k_1, k_2, k_3} : \mathbb{P}^{k_1} \times \mathbb{P}^{k_2} \times \mathbb{P}^{k_3} \rightarrow \mathbb{P}^{N(k_1, k_2, k_3)},$$

where we recall that

$$N(k_1, \dots, k_n) := (k_1 + 1) \cdots (k_n + 1) - 1.$$

The following lemma relates the Segre varieties associated to it.

Lemma A.1. *Let k_1, k_2, k_3 be positive integers. Then*

$$\Sigma_{k_1, k_2, k_3} = \Sigma_{k_1, N(k_2, k_3)} \cap \Sigma_{N(k_1, k_2), k_3}.$$

Proof. It suffices to prove the inclusion $\Sigma_{k_1, N(k_2, k_3)} \cap \Sigma_{N(k_1, k_2), k_3} \subseteq \Sigma_{k_1, k_2, k_3}$. Recall that we have identities

$$f_{k_1, k_2, k_3} = f_{k_1, N(k_2, k_3)} \circ (\mathbb{I} \times f_{k_2, k_3}) = f_{N(k_1, k_2), k_3} \circ (f_{k_1, k_2} \times \mathbb{I}).$$

Let $a = [a_i]$, $b = [b_j]$ and $c = [c_k]$ be coordinates for \mathbb{P}^{k_1} , \mathbb{P}^{k_2} and \mathbb{P}^{k_3} respectively. We will also let $x = [x_{ij}]$ and $y = [y_{jk}]$ be coordinates for $\mathbb{P}^{N(k_1, k_2)}$ and $\mathbb{P}^{N(k_2, k_3)}$ respectively, and $z = [z_{ijk}]$ will denote coordinates for $\mathbb{P}^{N(k_1, k_2, k_3)}$.

We have:

$$\begin{aligned} (f_{k_1, k_2} \times \mathbb{I})(a, b, c) &= (x, c), \text{ with } x_{ij} := a_i b_j. \\ (\mathbb{I} \times f_{k_2, k_3})(a, b, c) &= (a, y), \text{ with } y_{jk} := b_j c_k. \\ f_{N(k_1, k_2), k_3}(x, c) &= (z), \text{ with } z_{ijk} := x_{ij} c_k. \\ f_{k_1, N(k_2, k_3)}(a, y) &= (z), \text{ with } z_{ijk} := a_i y_{jk}. \end{aligned}$$

Given a point $z = (z_{ijk}) \in \mathbb{P}^{N(k_1, k_2, k_3)}$, we claim the following:

- (1) $z \in \Sigma_{k_1, N(k_2, k_3)}$ if and only if all the 2×2 minors of the matrix

$$\begin{pmatrix} z_{000} & \cdots & z_{0k_2 k_3} \\ \vdots & & \vdots \\ z_{k_1 00} & \cdots & z_{k_1 k_2 k_3} \end{pmatrix}$$

vanish, so that $z_{ijk} \cdot z_{i'j'k'} = z_{i'jk} \cdot z_{ij'k'}$ for all $i \neq i'$ and $(j, k) \neq (j', k')$.

- (2) $z \in \Sigma_{N(k_1, k_2), k_3}$ if and only if all the 2×2 minors of the matrix

$$\begin{pmatrix} z_{000} & \cdots & z_{00k_3} \\ \vdots & & \vdots \\ z_{k_1 k_2 0} & \cdots & z_{k_1 k_2 k_3} \end{pmatrix}$$

vanish, so that $z_{ijk} \cdot z_{i'j'k'} = z_{ij'k} \cdot z_{i'jk'}$ for all $(i, j) \neq (i', j')$ and $k \neq k'$.

- (3) $z \in \Sigma_{k_1, k_2, k_3}$ if and only if $z \in \Sigma_{N(k_1, k_2), k_3}$, so that $z_{ijk} = x_{ij} \cdot c_k$, and $x = (x_{ij}) \in \Sigma_{k_1, k_2}$. This last condition gives the vanishing of the 2×2 minors of the matrix

$$\begin{pmatrix} x_{00} & \cdots & x_{0k_2} \\ \vdots & & \vdots \\ x_{k_1 0} & \cdots & x_{k_1 k_2} \end{pmatrix}.$$

Therefore we have $x_{ij} \cdot x_{i'j'} = x_{ij'} \cdot x_{i'j}$ for all $i \neq i', j \neq j'$. This gives identities

$$z_{ijk} \cdot z_{i'j'k'} = x_{ij} \cdot c_k \cdot x_{i'j'} \cdot c_{k'} = x_{i'j} \cdot c_k \cdot x_{ij'} \cdot c_{k'} = z_{i'jk} \cdot z_{ij'k'}$$

for all $i \neq i', j \neq j'$ and $k \neq k'$.

Claims (1) and (2) are straightforward, while (3) follows from the identity

$$\Sigma_{k_1, k_2, k_3} = \text{Im}(f_{N(k_1, k_2), k_3} \circ (f_{k_1, k_2} \times \mathbb{I})).$$

Assume now that $z \in \Sigma_{k_1, N(k_2, k_3)} \cap \Sigma_{N(k_1, k_2), k_3}$. Then the equations in (1) and (2) are satisfied, and moreover we may write $z_{ijk} = x_{ij} \cdot c_k$. To show that $z \in \Sigma_{k_1, k_2, k_3}$ it only remains to prove that $x_{ij} \cdot x_{i'j'} = x_{ij'} \cdot x_{i'j}$ for all $i \neq i'$ and $j \neq j'$. By (1) we have

$$z_{ijk} \cdot z_{i'j'k'} = x_{ij} \cdot c_k \cdot x_{i'j'} \cdot c_{k'} = x_{i'j} \cdot c_k \cdot x_{ij'} \cdot c_{k'} = z_{i'jk} \cdot z_{ij'k'}$$

for all $i \neq i'$ and $(j, k) \neq (j', k')$. Take $k = k'$ such that $c_k \neq 0$. This gives

$$c_k^2 \cdot x_{ij} \cdot x_{i'j'} = c_k^2 \cdot x_{i'j} \cdot x_{ij'} \text{ for all } i \neq i' \text{ and } j \neq j'.$$

Since $c_k^2 \neq 0$, we obtain the desired identities. \square

In order to prove the Generalized Decomposability Theorem it will be useful to denote the cubical decomposition of

$$\mathbb{P}^1 \times \dots \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{N_n}$$

by the $(n-1)$ -dimensional hypercube whose vertices are given by tuples $v = (v_1, \dots, v_{n-1})$ where $v_i \in \{0, 1\}$, with initial vertex $v_0 = (0, \dots, 0)$ representing $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$ and final vertex $v_f = (1, \dots, 1)$ representing \mathbb{P}^{N_n} . We will call $|v| := v_1 + \dots + v_{n-1}$ the *degree of a vertex*. All edges are of the form

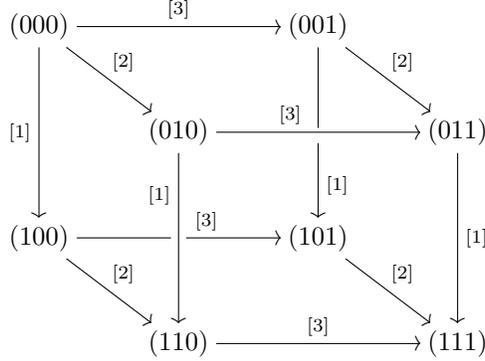
$$(v_1, \dots, v_{n-1}) \xrightarrow{[j]} (w_1, \dots, w_{n-1})$$

where $v_i = w_i$ for all $i \neq j$ and $w_j = v_j + 1 = 1$, so that $|w| = |v| + 1$. Such an edge represents a contraction of a product \times at the position j via a Segre embedding.

Example A.2. For example, the cubical representation of $f_{1,1,1}$ is

$$\begin{array}{ccc} (00) & \xrightarrow{[1]} & (10) \\ [2] \downarrow & & \downarrow [2] \\ (01) & \xrightarrow{[1]} & (11) \end{array}$$

and the cubical representation of $f_{1,1,1,1}$ is



We will say that a point $z \in \mathbb{P}^{N_n}$ lives in a vertex $v = (v_1, \dots, v_{n-1})$ if it is in the image of the map $v \rightarrow v_f$ given by the composition of all edges $[i]$ such that $v_i = 0$. It follows that z is q -decomposable if and only if it lives in some vertex v of degree $|v| = n - q$.

Lemma A.3. *Let $v \neq v'$ be two vertices of the same degree in the $(n - 1)$ -dimensional hypercube. Then there is a unique 2-dimensional face of the form*

$$\begin{array}{ccc}
 w' & \xrightarrow{[i]} & v' \\
 \downarrow [j] & & \downarrow [j] \\
 v & \xrightarrow{[i]} & w
 \end{array}$$

and if z lives in w, v and v' , then it also lives in w' .

Proof. Since $|v| = |v'|$ and $v \neq v'$, there are i, j such that $v_i = 0, v'_i = 1, v_j = 1, v'_j = 0$, and $v_k = v'_k$ for all $k \neq i, j$. Let w and w' be the vertices whose components are given by

$$w_k = \max\{v_k, v'_k\} \text{ and } w'_k = \min\{v_k, v'_k\}.$$

This gives the above commutative square. Assume now that z lives in w, v and v' . It suffices to consider two cases:

Case $j = i + 1$. In this case, the above commutative square represents morphisms of the form

$$\begin{array}{ccc}
 A \times \mathbb{P}^{k_1} \times \mathbb{P}^{k_2} \times \mathbb{P}^{k_3} \times B & \xrightarrow{\mathbb{I}_A \times f_{k_1, k_2} \times \mathbb{I}_{k_3} \times \mathbb{I}_B} & A \times \mathbb{P}^{N(k_1, k_2)} \times \mathbb{P}^{k_3} \times B \\
 \downarrow \mathbb{I}_A \times \mathbb{I}_{k_1} \times f_{k_2, k_3} \times \mathbb{I}_B & & \downarrow \mathbb{I}_A \times f_{N(k_1, k_2), k_3} \times \mathbb{I}_B \\
 A \times \mathbb{P}^{k_1} \times \mathbb{P}^{N(k_2, k_3)} \times B & \xrightarrow{\mathbb{I}_A \times f_{k_1, N(k_2, k_3)} \times \mathbb{I}_B} & A \times \mathbb{P}^{N(k_1, k_2, k_3)} \times B,
 \end{array}$$

where A and B are products of projective spaces. Therefore we can apply Lemma A.1 to conclude that z lives in w' .

Case $j < i + 1$. In this case, the above commutative square represents morphisms of the form

$$\begin{array}{ccc}
 A \times \mathbb{P}^{k_1} \times \mathbb{P}^{k_2} \times B \times \mathbb{P}^{k_3} \times \mathbb{P}^{k_4} \times C & \longrightarrow & A \times \mathbb{P}^{N(k_1, k_2)} \times B \times \mathbb{P}^{k_3} \times \mathbb{P}^{k_4} \times C \\
 \downarrow & & \downarrow \\
 A \times \mathbb{P}^{k_1} \times \mathbb{P}^{k_2} \times B \times \mathbb{P}^{N(k_3, k_4)} \times C & \longrightarrow & A \times \mathbb{P}^{N(k_1, k_2)} \times B \times \mathbb{P}^{N(k_3, k_4)} \times C
 \end{array}$$

Since z lives in w we may decompose $z = (z_A, z_{12}, z_B, z_{34}, z_C)$. Since it lives in v , we have $z_{12} \in \Sigma_{N(k_1, k_2)}$ and since it lives in v' we have $z_{34} \in \Sigma_{N(k_3, k_4)}$. It directly follows that z lives in w' . \square

Theorem A.4 (Generalized Decomposability). *Let $n \geq 2$ and $1 < q \leq n$ be integers. A point $z \in \mathbb{P}^{N_n}$ is q -decomposable if and only if it is in at least $q - 1$ different Segre varieties of the form $\Sigma_{N_\ell, N_{n-\ell}}$, with $0 \leq \ell \leq n$.*

Proof. Using the cubical representation introduced above, it suffices to show that if z lives in $q - 1$ different vertices v^1, \dots, v^{q-1} of degree $|v^i| = n - 2$, then z lives in a vertex of degree $(n - q)$. By recursively applying Lemma A.3 we find that z lives in the vertex v^* of degree $(n - q)$ whose components are $v_j^* = \min_i \{v_j^i\}$. This implies that z is q -decomposable. The converse statement is trivial. \square