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Detection of entanglement for multipartite quantum states

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Abstract We study genuine tripartite entanglement and multipartite entanglement of arbitrary n-partite quantum states by using the representations with generalized Pauli operators of a density matrices. While the usual Bloch representation of a density matrix uses three types of generators in the special unitary Lie algebra $\mathfrak{su}(d)$, the representation with generalized Pauli operators has one uniformed type of generators and it simplifies computation. In this paper, we take the advantage of this simplicity to derive useful and operational criteria to detect genuine tripartite entanglement. We also obtain a sufficient criterion to detect entanglement for multipartite quantum states in arbitrary dimensions. The new method can detect more entangled states than previous methods as backed by detailed examples.

Keywords Genuine entanglement · Correlation tensor · Generalized Pauli operators

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1 Introduction

Quantum entanglement is a key resource in quantum information with wide applications in entanglement swapping [1], quantum cryptography [2] and quantum secure communication [3]. The genuine multipartite entanglement (GME) stands out with significant properties [4,5]. Thus measuring and detection of genuine multipartite entanglement for given states has been an important task in quantum computation.

A lot of methods have been presented in detecting entanglement and genuine entanglement [6–9]. For tripartite quantum states, sufficient conditions to test entanglement of three-qubit states in the vicinity of the GHZ, the W states and the PPT entangled states were found in [10]. A sufficient criterion for the entanglement of tripartite systems based on local sum uncertainty relations was proposed in [11]. The sufficient conditions for judging genuine tripartite entanglement were presented by using partial transposition and realignment of density matrices in [12]. Yang et al [13] derived a criterion for detecting genuine tripartite entanglement based on quantum Fisher information. By using the Bloch representation of density matrices and the norms of correlation tensors, the genuine tripartite entangled criteria were presented in [14, 15]. The authors in [16] studied the separability criteria in tripartite and four-partite quantum system by the matrix method. The separability criteria for four-partite quantum system based on the upper bound of Bloch vectors were discussed in [17]. For higher dimensional quantum system, Chen et al [18] presented a generalized partial separability criterion of multipartite quantum systems in arbitrary dimensions. The separable criteria and k-separable criteria for general n-partite quantum states were given in [19–21].

Many of these methods have used the Bloch representation of the density matrix, which has become more complex as dimension of the quantum system increases. This is partly due to the fact that the Bloch representation is relied on the Gell-Mann basis of the special unitary Lie algebra $\mathfrak{su}(d)$ which has three kinds of basis elements: upper, diagonal and lower matrices. In view of this, perhaps using another well-known basis of the Lie algebra $\mathfrak{su}(d)$: the Weyl basis to study quantum entanglement will likely simplify some of the criteria, as the latter consists of uniformed basis elements. In Ref. [22], the authors showed that the principal basis matrix plays an essential role in the representation theory of the Yangian $Y(\mathfrak{sl}(3))$ which has a close relation with the study of entangled states in quantum information (see also for recent applications [23]).

In this paper, we study entanglement of multipartite quantum systems by using the representation with generalized Pauli operators, and we obtain several better criteria in detecting the GME than previously available tests. The paper is organized as follows. In section 2, after reviewing the representation with generalized Pauli operators of the quantum state, we construct matrices by using the correlation tensors and derive the criteria to detect entanglement and genuine tripartite entanglement. By detailed example, our results can detect more genuine entangled states. In section 3, we obtain the entanglement theorem for arbitrary n-partite quantum systems. Conclusions are given in section 4.

2 Genuine entanglement for tripartite quantum state

We first consider the GME for tripartite states. Let E_{ij} be the $d \times d$ unit matrix with the only nonzero entry 1 at the position (i, j), and let ω be a fixed d-th primitive root of unity. By means of the division with remainder, for certain d_s and $u_s \in \{0, \cdots, d_s^2 - 1\}$, there exists unique integers i and j such that $u_s = d_s i + j$ $(0 \le i, j \le d_s - 1)$, then the generalized Pauli operators of the sth

d_s-dimensional Hilbert space $H_s^{d_s}$ are given by

$$A_{u_s}^{(s)} = A_{d_s i+j}^{(s)} = \sum_{m=0}^{d_s-1} \omega^{im} E_{m,m+j}, \tag{1}$$

where $\omega^{d_s} = 1$. The basis obeys the algebraic relation:

$$A_{d_s i+j}^{(s)} A_{d_s k+l}^{(s)} = \omega^{jk} A_{d_s(i+k)+(j+l)}^{(s)},$$

then $(A_{d_s i+j}^{(s)})^\dagger = \omega^{ij} A_{d_s(d_s-i)+(d_s-j)}^{(s)}$, so $tr(A_{d_s i+j}^{(s)}(A_{d_s k+l}^{(s)})^\dagger) = \delta_{ik}\delta_{jl}d_s$ [24], where $\dagger$ stands for conjugate transpose. Denote by $\|\cdot\|$ the norm of a (column) complex vector, i.e. $\|v\| = \sqrt{v^\dagger v}$. The trace norm (Ky Fan norm) of a retangular matrix $A \in \mathbb{C}^{m\times n}$ is defined as $\|A\|_{tr} = \sum \sigma_i = tr\sqrt{AA^\dagger}$, where σ_i are the singular values of A and $\|A\|_{tr} \leq \sqrt{\min\{m,n\}}\|A\|$ for any matrix A. Clearly $\|A\|_{tr} = \|A^\dagger\|_{tr}$.

Lemma 1 *Let $H_s^{d_s}$ denote the s^{th} d_s-dimensional Hilbert space. For a quantum state $\rho_1 \in H_1^{d_1}$, ρ_1 can be expressed as $\rho_1 = \frac{1}{d_1}\sum_{u_1=0}^{d_1^2-1} t_{u_1} A_{u_1}^{(1)}$, where $A_0^{(1)} = I_{d_1}$, $t_{u_1} = tr(\rho_1 (A_{u_1}^{(1)})^\dagger)$ are complex coefficients. Let $T^{(1)}$ be the column vector with entries t_{u_1} for $u_1 \neq 0$, we have*

$$\|T^{(1)}\|^2 \leq d_1 - 1. \tag{2}$$

Proof Since $tr(\rho_1^2) \leq 1$, we have

$$tr(\rho_1^2) = tr(\rho_1\rho_1^\dagger) = \frac{1}{d_1}(1 + \|T^{(1)}\|^2) \leq 1,$$

namely, $\|T^{(1)}\|^2 \leq d_1 - 1$. $\square$

For a state $\rho_{12} \in H_1^{d_1} \otimes H_2^{d_2}$, ρ_{12} has the generalized Pauli operators representation:

$$\rho_{12} = \frac{1}{d_1 d_2}\sum_{u_1=0}^{d_1^2-1}\sum_{u_2=0}^{d_2^2-1} t_{u_1,u_2} A_{u_1}^{(1)} \otimes A_{u_2}^{(2)} \tag{3}$$

where $A_0^{(s)} = I_{d_s}(s = 1, 2)$, the coefficients $t_{u_1,u_2} = tr(\rho_{12}(A_{u_1}^{(1)})^\dagger \otimes (A_{u_2}^{(2)})^\dagger)$ are complex numbers. Let $T^{(1)}$, $T^{(2)}$, $T^{(12)}$ be the vectors with entries $t_{u_1,0}$, t_{0,u_2}, t_{u_1,u_2} for $u_1, u_2 \neq 0$.

Lemma 2 *Let $\rho_{12} \in H_1^{d_1} \otimes H_2^{d_2}$ be a mixed state, we have $\|T^{(12)}\|^2 \leq d_1 d_2(1 - \frac{1}{d_1^2} - \frac{1}{d_2^2}) + 1$.*

Proof For a pure state ρ_{12}, we have $tr(\rho_{12}^2) = 1$, namely

$$tr(\rho_{12}^2) = tr(\rho_{12}\rho_{12}^\dagger) = \frac{1}{d_1 d_2}(1 + \|T^{(1)}\|^2 + \|T^{(2)}\|^2 + \|T^{(12)}\|^2) = 1. \tag{4}$$

By using $tr(\rho_1^2) = tr(\rho_2^2)$, we have $\frac{1}{d_1}(1 + \|T^{(1)}\|^2) = \frac{1}{d_2}(1 + \|T^{(2)}\|^2)$, where ρ_1 and ρ_2 are the reduced density operators on $H_1^{d_1}$ and $H_2^{d_2}$, respectively. Then

$$\frac{1}{d_1^2}(1 + \|T^{(1)}\|^2) + \frac{1}{d_2^2}(1 + \|T^{(2)}\|^2) = \frac{1}{d_1 d_2}(2 + \|T^{(1)}\|^2 + \|T^{(2)}\|^2).$$

Using the above two equations we obtain that

$$\begin{aligned}\|T^{(12)}\|^2 &= d_1d_2 - 1 - \|T^{(1)}\|^2 - \|T^{(2)}\|^2 \\ &= d_1d_2 + 1 - [\frac{d_2}{d_1}(1 + \|T^{(1)}\|^2) + \frac{d_1}{d_2}(1 + \|T^{(2)}\|^2)].\end{aligned}$$

By $\|T^{(1)}\|^2 \geq 0$, $\|T^{(2)}\|^2 \geq 0$, we have

$$\begin{aligned}\|T^{(12)}\|^2 &\leq d_1d_2 + 1 - (\frac{d_2}{d_1} \cdot 1 + \frac{d_1}{d_2} \cdot 1) \\ &= d_1d_2(1 - \frac{1}{d_1^2} - \frac{1}{d_2^2}) + 1,\end{aligned} \tag{5}$$

If ρ is a mixed state, then $\rho = \sum_i p_i\rho_i$ is a convex sum of pure states, $\sum_i p_i = 1$. Then $\|T^{(12)}(\rho)\| \leq \sum_i p_i\|T^{(12)}(\rho_i)\| \leq \sqrt{d_1d_2(1 - \frac{1}{d_1^2} - \frac{1}{d_2^2}) + 1}$. $\square$

A general tripartite state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3}$ can be written in terms of the generalized Pauli operators:

$$\rho = \frac{1}{d_1d_2d_3} \sum_{u_1=0}^{d_1^2-1} \sum_{u_2=0}^{d_2^2-1} \sum_{u_3=0}^{d_3^2-1} t_{u_1,u_2,u_3} A_{u_1}^{(1)} \otimes A_{u_2}^{(2)} \otimes A_{u_3}^{(3)}, \tag{6}$$

where $A_{u_f}^{(f)}$ stands for that the tensor operator with A_{u_f} acting on the space $H_f^{d_f}$, $A_0^{(f)} = I_{d_f}$, $t_{u_1,u_2,u_3} = tr(\rho(A_{u_1}^{(1)})^\dagger \otimes (A_{u_2}^{(2)})^\dagger \otimes (A_{u_3}^{(3)})^\dagger)$ are the complex coefficients.

In the following we will construct some matrices out of the expansion coefficients of the density matrix ρ in (6). For $f, g, h \in \{1, 2, 3\}$, set

$$N^{f|gh} = diag\{N_1^{f|gh}, N_2^{f|gh}, \cdots, N_{d_f^2-1}^{f|gh}\}, \tag{7}$$

where $N_i^{f|gh} = [t_{i,u_g,u_h}]$ is a $(d_g^2 - 1) \times (d_h^2 - 1)$ matrix, $i = 1, 2, \cdots, d_f^2 - 1$. For example, when $\rho \in H_1^2 \otimes H_2^2 \otimes H_3^3$, $N^{2|13} = diag\{N_1^{2|13}, N_2^{2|13}, N_3^{2|13}\}$, where

$$N_i^{2|13} = \begin{bmatrix} t_{1,i,1} & t_{1,i,2} & \cdots & t_{1,i,8} \\ t_{2,i,1} & t_{2,i,2} & \cdots & t_{2,i,8} \\ t_{3,i,1} & t_{3,i,2} & \cdots & t_{3,i,8} \end{bmatrix}, i = 1, 2, 3.$$

Theorem 1 *For a biseparable tripartite pure state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3}$ and $f \neq g \neq h \in \{1, 2, 3\}$, we have*
(i) if ρ is separable under bipartite partition $f|gh$, then

$$\|N^{f|gh}\|_{tr} \leq \sqrt{(d_f^2 - 1) \cdot min\{d_g^2 - 1, d_h^2 - 1\}(d_f - 1)[d_gd_h(1 - \frac{1}{d_g^2} - \frac{1}{d_h^2}) + 1]}$$

(ii) if ρ is separable under bipartite partition $g|fh$, then

$$\|N^{f|gh}\|_{tr} \leq \sqrt{(d_f^2 - 1)(d_g - 1)[d_fd_h(1 - \frac{1}{d_f^2} - \frac{1}{d_h^2}) + 1]}$$

(iii) if ρ is separable under bipartite partition $h|fg$, then

$$\|N^{f|gh}\|_{tr} \leq \sqrt{(d_f^2 - 1)(d_h - 1)[d_fd_g(1 - \frac{1}{d_f^2} - \frac{1}{d_g^2}) + 1]}$$

Proof (i) If the tripartite pure state ρ is separable under the bipartition $f|gh$, then it can be expressed as

$$\rho_{f|gh} = \rho_f \otimes \rho_{gh}, \tag{8}$$

where

$$\rho_f = \frac{1}{d_f} \sum_{u_f=0}^{d_f^2-1} t_{u_f} A_{u_f}^{(f)}, \tag{9}$$

$$\rho_{gh} = \frac{1}{d_g d_h} \sum_{u_g=0}^{d_g^2-1} \sum_{u_h=0}^{d_h^2-1} t_{u_g,u_h} A_{u_g}^{(g)} \otimes A_{u_h}^{(h)}. \tag{10}$$

Let $T^{(f)}$, $T^{(g)}$, $T^{(h)}$ and $T^{(gh)}$ be the vectors with entries t_{u_f}, $t_{u_g,0}$, t_{0,u_h} and t_{u_g,u_h} for $u_f, u_g, u_h \neq 0$. Then

$$N^{f|gh} = \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_{d_f^2-1} \end{bmatrix} \otimes \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1(d_h^2-1)} \\ t_{21} & t_{22} & \cdots & t_{2(d_h^2-1)} \\ \vdots & \vdots & & \vdots \\ t_{(d_g^2-1)1} & t_{(d_g^2-1)2} & \cdots & t_{(d_g^2-1)(d_h^2-1)} \end{bmatrix}. \tag{11}$$

It follows from Lemma 1 and Lemma 2 that

$$\begin{aligned} \|N^{f|gh}\|_{tr} &= \| \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_{d_f^2-1} \end{bmatrix} \|_{tr} \| \begin{bmatrix} t_{11} & \cdots & t_{1(d_h^2-1)} \\ \vdots & & \vdots \\ t_{(d_g^2-1)1} & \cdots & t_{(d_g^2-1)(d_h^2-1)} \end{bmatrix} \|_{tr} \\ &\leq \sqrt{(d_f^2-1)}\|T^{(f)}\| \cdot \sqrt{min\{d_g^2-1, d_h^2-1\}}\|T^{(gh)}\| \\ &\leq \sqrt{(d_f^2-1) \cdot min\{d_g^2-1, d_h^2-1\}(d_f-1)[d_g d_h(1-\frac{1}{d_g^2}-\frac{1}{d_h^2})+1]}. \end{aligned}$$

(ii) If the tripartite pure state ρ is separable under the bipartition $g|fh$, then it can be expressed as

$$\rho_{g|fh} = \rho_g \otimes \rho_{fh}, \tag{12}$$

where

$$\rho_g = \frac{1}{d_g} \sum_{u_g=0}^{d_g^2-1} t_{u_g} A_{u_g}^{(g)}, \tag{13}$$

$$\rho_{fh} = \frac{1}{d_f d_h} \sum_{u_f=0}^{d_f^2-1} \sum_{u_h=0}^{d_h^2-1} t_{u_f,u_h} A_{u_f}^{(f)} \otimes A_{u_h}^{(h)}. \tag{14}$$

Then, we have $N_i^{f|gh} = T^{(g)} \cdot (T_i^{(fh)})^t$, and

$$\begin{aligned} \|N^{f|gh}\|_{tr} &= \|T^{(g)}\| \cdot \sum_{i=1}^{d_f^2-1} \|T_i^{(fh)}\| \leq \sqrt{d_f^2-1}\|T^{(g)}\|\|T^{(fh)}\| \\ &\leq \sqrt{(d_f^2-1)(d_g-1)[d_f d_h(1-\frac{1}{d_f^2}-\frac{1}{d_h^2})+1]} \end{aligned}$$

where $T_i^{(fh)}$ is the vector with entries t_{i,u_h}, $i = 1, \cdots, d_f^2 - 1$, t stands for transpose.

(iii) Using similar method, if ρ is separable under the bipartition $h|fg$, we have $N_i^{f|gh} = T_i^{(fg)} \cdot (T^{(h)})^t$, $i = 1, \cdots, d_f^2 - 1$ and

$$\|N^{f|gh}\|_{tr} \leq \sqrt{(d_f^2 - 1)(d_h - 1)[d_f d_g (1 - \frac{1}{d_f^2} - \frac{1}{d_g^2}) + 1]}.$$

□

Now we consider genuine tripartite entanglement. A mixed state is said to be genuine multipartite entangled if it cannot be written as a convex combination of biseparable states. Let $T(\rho) = \frac{1}{3}(\|N^{1|23}\|_{tr} + \|N^{2|13}\|_{tr} + \|N^{3|12}\|_{tr})$, we define

$$\begin{aligned} Q_1 = \mathrm{Max}\{ & \sqrt{(d_1^2 - 1) \cdot min\{d_2^2 - 1, d_3^2 - 1\}(d_1 - 1)[d_2 d_3 (1 - \frac{1}{d_2^2} - \frac{1}{d_3^2}) + 1]}, \\ & \sqrt{(d_1^2 - 1)(d_2 - 1)[d_1 d_3 (1 - \frac{1}{d_1^2} - \frac{1}{d_3^2}) + 1]}, \\ & \sqrt{(d_1^2 - 1)(d_3 - 1)[d_1 d_2 (1 - \frac{1}{d_1^2} - \frac{1}{d_2^2}) + 1]}\}. \end{aligned}$$

$$\begin{aligned} Q_2 = \mathrm{Max}\{ & \sqrt{(d_2^2 - 1) \cdot min\{d_1^2 - 1, d_3^2 - 1\}(d_2 - 1)[d_1 d_3 (1 - \frac{1}{d_1^2} - \frac{1}{d_3^2}) + 1]}, \\ & \sqrt{(d_2^2 - 1)(d_1 - 1)[d_2 d_3 (1 - \frac{1}{d_2^2} - \frac{1}{d_3^2}) + 1]}, \\ & \sqrt{(d_2^2 - 1)(d_3 - 1)[d_2 d_1 (1 - \frac{1}{d_2^2} - \frac{1}{d_1^2}) + 1]}\}. \end{aligned}$$

$$\begin{aligned} Q_3 = \mathrm{Max}\{ & \sqrt{(d_3^2 - 1) \cdot min\{d_1^2 - 1, d_2^2 - 1\}(d_3 - 1)[d_1 d_2 (1 - \frac{1}{d_1^2} - \frac{1}{d_2^2}) + 1]}, \\ & \sqrt{(d_3^2 - 1)(d_1 - 1)[d_3 d_2 (1 - \frac{1}{d_3^2} - \frac{1}{d_2^2}) + 1]}, \\ & \sqrt{(d_3^2 - 1)(d_2 - 1)[d_3 d_1 (1 - \frac{1}{d_3^2} - \frac{1}{d_1^2}) + 1]}\}. \end{aligned}$$

We have the following theorem.

Theorem 2 *A mixed state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes H_3^{d_3}$ is genuine tripartite entangled if $T(\rho) > \frac{1}{3}(Q_1 + Q_2 + Q_3)$.*

Proof If ρ is a biseparable, one has $\rho=\sum_i o_i\rho_i^1\otimes\rho_i^{23}+\sum_j r_j\rho_j^2\otimes\rho_j^{13}+\sum_k s_k\rho_k^3\otimes\rho_k^{12}$ with $0\le o_i,r_j,s_k\le 1$ and $\sum_i o_i+\sum_j r_j+\sum_k s_k=1$. By Theorem 1, we have that

$$\begin{aligned}
&T(\rho)=\frac{1}{3}(\|N^{1|23}(\rho)\|_{tr}+\|N^{2|13}(\rho)\|_{tr}+\|N^{3|12}(\rho)\|_{tr})\\
=&\frac{1}{3}[\|N^{1|23}(\sum_i o_i\rho_i^1\otimes\rho_i^{23}+\sum_j r_j\rho_j^2\otimes\rho_j^{13}+\sum_k s_k\rho_k^3\otimes\rho_k^{12})\|_{tr}+\|N^{2|13}(\sum_i o_i\rho_i^1\otimes\rho_i^{23}\\
+&\sum_j r_j\rho_j^2\otimes\rho_j^{13}+\sum_k s_k\rho_k^3\otimes\rho_k^{12})\|_{tr}+\|N^{3|12}(\sum_i o_i\rho_i^1\otimes\rho_i^{23}+\sum_j r_j\rho_j^2\otimes\rho_j^{13}+\sum_k s_k\rho_k^3\otimes\rho_k^{12})\|_{tr}]\\
\le&\frac{1}{3}[\sum_i o_i\|N^{1|23}(\rho_i^1\otimes\rho_i^{23})\|_{tr}+\sum_j r_j\|N^{1|23}(\rho_j^2\otimes\rho_j^{13})\|_{tr}+\sum_k s_k\|N^{1|23}(\rho_k^3\otimes\rho_k^{12})\|_{tr}\\
&+\sum_i o_i\|N^{2|13}(\rho_i^1\otimes\rho_i^{23})\|_{tr}+\sum_j r_j\|N^{2|13}(\rho_j^2\otimes\rho_j^{13})\|_{tr}+\sum_k s_k\|N^{2|13}(\rho_k^3\otimes\rho_k^{12})\|_{tr}\\
&+\sum_i o_i\|N^{3|12}(\rho_i^1\otimes\rho_i^{23})\|_{tr}+\sum_j r_j\|N^{3|12}(\rho_j^2\otimes\rho_j^{13})\|_{tr}+\sum_k s_k\|N^{3|12}(\rho_k^3\otimes\rho_k^{12})\|_{tr}]\\
\le&\frac{1}{3}[(\sum_i o_iQ_1+\sum_j r_jQ_1+\sum_k s_kQ_1)+(\sum_i o_iQ_2+\sum_j r_jQ_2+\sum_k s_kQ_2)+(\sum_i o_iQ_3+\sum_j r_jQ_3+\sum_k s_kQ_3)\\
=&\frac{1}{3}[(\sum_i o_i+\sum_j r_j+\sum_k s_k)Q_1+(\sum_i o_i+\sum_j r_j+\sum_k s_k)Q_2+(\sum_i o_i+\sum_j r_j+\sum_k s_k)Q_3\\
=&\frac{1}{3}(Q_1+Q_2+Q_3)
\end{aligned}\tag{15}$$

Consequently, if $T(\rho)>\frac{1}{3}(Q_1+Q_2+Q_3)$, ρ is genuine tripartite entangled. □

Next we consider the permutational invariant state ρ, i.e. $\rho=\rho^p=p\rho p^\dagger$ for any permutation p of the qudits. A biseparable permutational invariant state can be written as $\rho=\sum_i p_i\rho_i^1\otimes\rho_i^{23}+\sum_j r_j\rho_j^2\otimes\rho_j^{13}+\sum_k s_k\rho_k^3\otimes\rho_k^{12}$, where $0<p_i,r_j,s_k\le 1$. Set $d_1=d_2=d_3=d$, we have the following corollary.

Corollary 1 *If a permutational invariant mixed state is biseparable, then we have*

$$T(\rho)=\frac{1}{3}(\|N^{1|23}\|_{tr}+\|N^{2|13}\|_{tr}+\|N^{3|12}\|_{tr})\le J_1.$$

Therefore if $T(\rho)>J_1$, ρ is genuine tripartite entangled. Here

$$J_1=\frac{(d-1)^2\sqrt{(d-1)(d^2-1)}+2(d^2-1)\sqrt{d-1}}{3}.$$

Example 1 Consider the mixed three-qubit W state,

$$\rho=\frac{1-x}{8}I_8+x|W\rangle\langle W|,\quad 0\le x\le 1,\tag{16}$$

where $|W\rangle=\frac{1}{\sqrt{3}}(|001\rangle+|010\rangle+|100\rangle)$ and I_8 is the 8×8 identity matrix. Let $f_1(x)=T(\rho)-J_1=5x-(2+\sqrt{3})$, using Corollary 1 we have ρ is genuine entangled if $f_1(x)>0$, i.e. $0.7464<x\le 1$. Theorem 2 in [15] implies that ρ is genuine entangled if $g_1(x)=3.26x-\frac{6+\sqrt{3}}{3}>0$, i.e. $0.791<x\le 1$. Our corollary can detect more genuine entangled, see the comparison in Fig. 1.

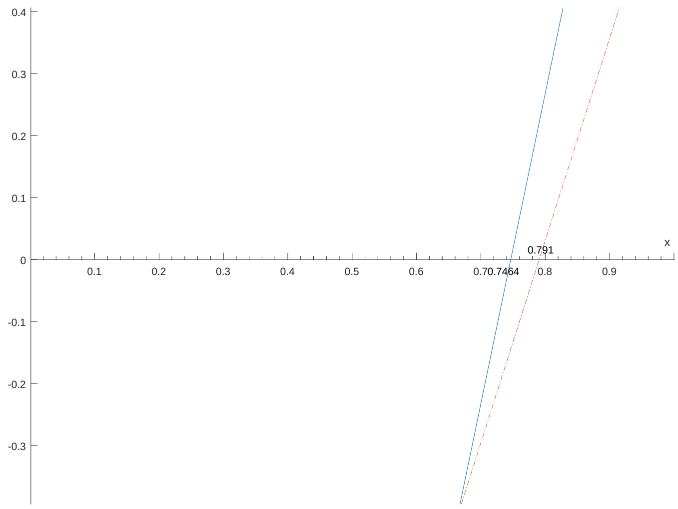

Fig. 1 $f_1(x)$ from our result (solid straight line), $g_1(x)$ from Theorem 2 in [15](dash-dot straight line).

3 Entanglement for multipartite quantum state

Now we consider entanglement of n-partite quantum systems. Let $\{A^{(s)}_{u_s}\}$ $(u_s = 0, \cdots, d_s^2 - 1)$ be the generalized Pauli operators of the sth d_s-dimensional Hilbert space $H_s^{d_s}$. Any quantum state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n}$ has the generalized Pauli operators representation:

$$\rho = \frac{1}{d_1 d_2 \cdots d_n} \sum_{s=1}^{n} \sum_{u_s=0}^{d_s^2-1} t_{u_1,u_2,\cdots,u_n} A^{(1)}_{u_1} \otimes A^{(2)}_{u_2} \otimes \cdots \otimes A^{(n)}_{u_n} \tag{17}$$

where $A_0^{(s)} = I_{d_s}(s = 1, \cdots, n)$, $t_{u_1,u_2,\cdots,u_n} = tr(\rho(A^{(1)}_{u_1})^\dagger \otimes (A^{(2)}_{u_2})^\dagger \otimes \cdots \otimes (A^{(n)}_{u_n})^\dagger)$ are complex coefficients. Let $T^{(l_1 \cdots l_k)}$ be the vectors with entries $t_{u_{l_1},\cdots,u_{l_k},\cdots,0}$, $u_{l_1}, \cdots, u_{l_k} \neq 0$ and $1 \le l_1 < \cdots < l_k \le n$. We have

$$\|T^{(1)}\|^2 = \sum_{u_1=1}^{d_1^2-1} t_{u_1,\cdots,0} t^*_{u_1,\cdots,0},$$
$$\cdots,$$
$$\|T^{(l_1\cdots l_k)}\|^2 = \sum_{s=1}^{k} \sum_{u_{l_s}=1}^{d_{l_s}^2-1} t_{u_{l_1}\cdots u_{l_k}\cdots 0} t^*_{u_{l_1}\cdots u_{l_k}\cdots 0},$$
$$\cdots,$$
$$\|T^{(12\cdots n)}\|^2 = \sum_{s=1}^{n} \sum_{u_s=1}^{d_s^2-1} t_{u_1\cdots u_n} t^*_{u_1\cdots u_n},$$

where $*$ represents the conjugate. Set

$$A_1 = \|T^{(1)}\|^2 + \cdots + \|T^{(n)}\|^2,$$
$$A_2 = \|T^{(12)}\|^2 + \cdots + \|T^{((n-1),n)}\|^2,$$

$$\cdots,$$
$$A_n = \|T^{(1\cdots n)}\|^2.$$

Lemma 3 *Let $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n}$ $(n \geq 2)$ be a n-partite pure quantum state. Then*

$$\|T^{(12\cdots n)}\|^2 \leq \frac{d_1 \cdots d_n(n-1-\sum\limits_{s=1}^{n}\frac{1}{d_s^2})+1}{n-1}. \tag{18}$$

Proof It's enough to show the lemma for a pure state ρ, where we have $tr(\rho^2)=1$ and $tr(\rho_{l_1}^2) = tr(\rho_{l_2\cdots l_n}^2)$ for any distinct indices $l_1, \ldots, l_n \in \{1, 2, \cdots, n\}$. Here ρ_{l_1} and $\rho_{l_2\cdots l_n}$ are the reduced states for the subsystem $H_{l_1}^{d_{l_1}}$ and $H_{l_2}^{d_{l_2}} \otimes \cdots \otimes H_{l_n}^{d_{l_n}}$. Therefore, we have

$$tr(\rho^2) = \frac{1}{d_1 d_2 \cdots d_n}(1 + A_1 + \cdots + A_n) = 1, \tag{19}$$

and

$$\frac{1}{d_{l_1}}(1+\|T^{(l_1)}\|^2) = \frac{1}{d_{l_2}\cdots d_{l_n}}(1+\|T^{(l_2)}\|^2 + \cdots + \|T^{(l_n)}\|^2 + \cdots + \|T^{(l_2\cdots l_n)}\|^2). \tag{20}$$

Since $\sum\limits_{l_1=1}^{n} \frac{1}{d_{l_1}} tr(\rho_{l_1}^2) = \sum\limits_{l_1=1}^{n} \frac{1}{d_{l_1}} tr(\rho_{l_2\cdots l_n}^2)$, we get that

$$\sum_{l_1=1}^{n} \frac{1}{d_{l_1}^2}(1+\|T^{(l_1)}\|^2) = \frac{1}{d_1\cdots d_n}[n + (n-1)A_1 + (n-2)A_2 + \cdots + A_{n-1}].$$

Therefore,

$$A_1 = \frac{d_1\cdots d_n}{n-1}\sum_{s=1}^{n}\frac{1}{d_s^2}(1+\|T^{(s)}\|^2) - \frac{n}{n-1} - \frac{n-2}{n-1}A_2 - \frac{n-3}{n-1}A_3 - \cdots - \frac{1}{n-1}A_{n-1}. \tag{21}$$

Substituting (21) into (19), we get

$$\begin{aligned} A_n =& d_1\cdots d_n - 1 - \frac{1}{n-1}\left(d_1\cdots d_n\sum_{s=1}^{n}\frac{1}{d_s^2}(1+\|T^{(s)}\|^2) - n\right) - \frac{1}{n-1}A_2 \\ &- \frac{2}{n-1}A_3 - \cdots \frac{n-2}{n-1}A_{n-1} \\ \leq& \frac{d_1\cdots d_n(n-1-\sum\limits_{s=1}^{n}\frac{1}{d_s^2})+1}{n-1} \end{aligned} \tag{22}$$

□

Let ρ be a n-partite state $\rho \in H_1^{d_1} \otimes H_2^{d_2} \otimes \cdots \otimes H_n^{d_n}$ represented as (17), for real number α, β and distinct indices $l_1, \ldots, l_n \in \{1, 2, \cdots, n\}$, set

$$N^{l_1\cdots l_{k-1}|l_k\cdots l_n} = \alpha S_0^{l_1\cdots l_{k-1}|l_k} + \beta S^{l_1\cdots l_{k-1}|l_k\cdots l_n}, \tag{23}$$

for $k-1 = 1, 2, \cdots, [n/2]$, the smallest integer less or equal to $n/2$. Let $T^{(l_1\cdots l_k)}$ be the $(d_{l_1}^2 - 1)\cdots(d_{l_k}^2-1)$-dimensional column vector with entries $t_{u_{l_1}\cdots u_{l_k}\cdots 0}$ associated with the generalized Pauli operators representation of ρ, and define $S_0^{l_1\cdots l_{k-1}|l_k}$ to be the block matrix $S_0^{l_1\cdots l_{k-1}|l_k} =$

$[S^{l_1\cdots l_{k-1}|l_k} \quad O_{l_1\cdots l_{k-1}}]$, where $S^{l_1\cdots l_{k-1}l_k} = ([t_{u_{l_1}\cdots u_{l_k}\cdots 0}])$ is the $\prod_{s=1}^{k-1}(d_{l_s}^2-1)\times(d_{l_k}^2-1)$ matrix and $O_{l_1\cdots l_{k-1}}$ is the $\prod_{s=1}^{k-1}(d_{l_s}^2-1)\times[\prod_{s=k}^{n}(d_{l_s}^2-1)-(d_{l_k}^2-1)]$ zero matrix, and $S^{l_1\cdots l_{k-1}|l_k\cdots l_n} = [t_{u_1,\cdots,u_n}]$ to be a $\prod_{s=1}^{k-1}(d_{l_s}^2-1)\times\prod_{s=k}^{n}(d_{l_s}^2-1)$ matrix. For example, when $\rho\in H_1^2\otimes H_2^2\otimes H_3^2\otimes H_4^3$, $N^{13|24}=\alpha S_0^{13|2}+\beta S^{13|24}$, where

$$S^{13|2}=\begin{bmatrix} t_{1,1,1,0} & t_{1,2,1,0} & t_{1,3,1,0}\\ t_{1,1,2,0} & t_{1,2,2,0} & t_{1,3,2,0}\\ t_{1,1,3,0} & t_{1,2,3,0} & t_{1,3,3,0}\\ \vdots & \vdots & \vdots\\ t_{3,1,3,0} & t_{3,2,3,0} & t_{3,3,3,0}\end{bmatrix},\quad S^{13|24}=\begin{bmatrix} t_{1,1,1,1} & t_{1,1,1,2} & \cdots & t_{1,1,1,8} & \cdots & t_{1,3,1,8}\\ t_{1,1,2,1} & t_{1,1,2,2} & \cdots & t_{1,1,2,8} & \cdots & t_{1,3,2,8}\\ t_{1,1,3,1} & t_{1,1,3,2} & \cdots & \cdot & \cdots & \cdot\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ t_{3,1,3,1} & t_{3,1,3,2} & \cdots & \cdot & \cdots & \cdot\end{bmatrix}.$$

Theorem 3 *Fix α,β as above. If the n-partite state $\rho\in H_1^{d_1}\otimes H_2^{d_2}\otimes\cdots\otimes H_n^{d_n}$ is separable under the bipartition $l_1\cdots l_{k-1}|l_k\cdots l_n$, then we have that*
(i) $\|N^{l_1|l_2\cdots l_n}\|_{tr}\le M_{l_1}$;
(ii) $\|N^{l_1\cdots l_{k-1}|l_k\cdots l_n}\|_{tr}\le M_{l_1\cdots l_{k-1}}$ $(k\ge 3)$;
where

$$M_{l_1}=\sqrt{d_{l_1}-1}\left(|\alpha|\sqrt{d_{l_2}-1}+|\beta|\sqrt{\frac{d_{l_2}\cdots d_{l_n}(n-2-\sum_{s=2}^{n}d_{l_s}^{-2})+1}{n-2}}\right),$$

$$M_{l_1\cdots l_{k-1}}=\sqrt{\frac{d_{l_1}\cdots d_{l_{k-1}}(k-2-\sum_{s=1}^{k-1}d_{l_s}^{-2})+1}{k-2}}\left(|\alpha|\sqrt{d_{l_k}-1}+|\beta|\sqrt{\frac{d_{l_k}\cdots d_{l_n}(n-k-\sum_{s=k}^{n}d_{l_s}^{-2})+1}{n-k}}\right).$$

Proof (i) If the n-partite mixed state is separable under the bipartition $l_1|l_2\cdots l_n$, it can be expressed as

$$\rho_{l_1|l_2\cdots l_n}=\sum_s p_s\rho_{l_1}^s\otimes\rho_{l_2\cdots l_n}^s,\ 0<p_s\le 1,\sum_s p_s=1, \tag{24}$$

where

$$\rho_{l_1}^s=\frac{1}{d_{l_1}}\sum_{u_{l_1}=0}^{d_{l_1}^2-1}t_{u_{l_1}}^s A_{u_{l_1}}^{(l_1)}, \tag{25}$$

$$\rho_{l_2\cdots l_n}^s=\frac{1}{d_{l_2}\cdots d_{l_n}}\sum_{q=2}^{n}\sum_{u_{l_q}=0}^{d_{l_q}^2-1}t_{u_{l_2},\cdots,u_{l_n}}^s A_{u_{l_2}}^{(l_2)}\otimes\cdots\otimes A_{u_{l_n}}^{(l_n)} \tag{26}$$

Then,

$$S^{l_1|l_2}=\sum_s p_sT_s^{(l_1)}(T_s^{(l_2)})^t,\quad S^{l_1|l_2\cdots l_n}=\sum_s p_sT_s^{(l_1)}(T_s^{(l_2\cdots l_n)})^t. \tag{27}$$

By Lemma 1 and Lemma 3, we have

$$
\begin{aligned}
\|N^{l_1|l_2\cdots l_n}\|_{tr} &\le \sum_s p_s(|\alpha|\|T_s^{(l_1)}\|\|T_s^{(l_2)}\| + |\beta|\|T_s^{(l_1)}\|\|T_s^{(l_2\cdots l_n)}\|) \\
&\le \sqrt{d_{l_1}-1}\left(|\alpha|\sqrt{d_{l_2}-1}+|\beta|\sqrt{\frac{d_{l_2}\cdots d_{l_n}(n-2-\sum\limits_{s=2}^{n}\frac{1}{d_{l_s}^2})+1}{n-2}}\right) \\
&= M_{l_1},
\end{aligned}
$$

where we have used $\|A+B\|_{tr} \le \|A\|_{tr}+\|B\|_{tr}$ for matrices A and B and $\||a\rangle\langle b|\|_{tr} = \||a\rangle\|\||b\rangle\|$ for vectors $|a\rangle$ and $|b\rangle$.
(ii) If ρ is separable under the bipartition $l_1\cdots l_{k-1}|l_k\cdots l_n$, it can be expressed as

$$
\rho_{l_1\cdots l_{k-1}|l_k\cdots l_n} = \sum_s p_s\rho^s_{l_1\cdots l_{k-1}}\otimes\rho^s_{l_k\cdots l_n}, 0<p_s\le 1, \sum_s p_s = 1, \tag{28}
$$

where

$$
\rho^s_{l_1\cdots l_{k-1}} = \frac{1}{d_{l_1}\cdots d_{l_{k-1}}}\sum_{p=1}^{k-1}\sum_{u_{l_p}=0}^{d_{l_p}^2-1} t^s_{u_{l_1},\cdots,u_{l_{k-1}}} A^{(l_1)}_{u_{l_1}}\otimes\cdots\otimes A^{(l_{k-1})}_{u_{l_{k-1}}}, \tag{29}
$$

$$
\rho^s_{l_k\cdots l_n} = \frac{1}{d_{l_k}\cdots d_{l_n}}\sum_{q=k}^{n}\sum_{u_{l_q}=0}^{d_{l_q}^2-1} t^s_{u_{l_k},\cdots,u_{l_n}} A^{(l_k)}_{u_{l_k}}\otimes\cdots\otimes A^{(l_n)}_{u_{l_n}}. \tag{30}
$$

Then,

$$
S^{l_1\cdots l_{k-1}|l_k} = \sum_s p_s T_s^{(l_1\cdots l_{k-1})}(T_s^{(l_k)})^t, \quad S^{l_1\cdots l_{k-1}|l_k\cdots l_n} = \sum_s p_s T_s^{(l_1\cdots l_{k-1})}(T_s^{(l_k\cdots l_n)})^t. \tag{31}
$$

Similarly, we get

$$
\begin{aligned}
&\|N^{l_1\cdots l_{k-1}|l_k\cdots l_n}\|_{tr} \\
\le&\sum_s p_s(|\alpha|\|T_s^{(l_1\cdots l_{k-1})}\|\|T_s^{(l_k)}\| + |\beta|\|T_s^{(l_1\cdots l_{k-1})}\|\|T_s^{(l_k\cdots l_n)}\|) \\
\le&\sqrt{\frac{d_{l_1}\cdots d_{l_{k-1}}(k-2-\sum\limits_{s=1}^{k-1}\frac{1}{d_{l_s}^2})+1}{k-2}}[|\alpha|\sqrt{d_{l_k}-1}+|\beta|\sqrt{\frac{d_{l_k}\cdots d_{l_n}(n-k-\sum\limits_{s=k}^{n}\frac{1}{d_{l_s}^2})+1}{n-k}}] \\
=&M_{l_1\cdots l_{k-1}}.
\end{aligned}
$$

□

Example 2 Consider the quantum state $\rho\in H_1^3\otimes H_2^3\otimes H_3^2$,

$$
\rho = \frac{1-x}{18}I_{18} + x|\varphi\rangle\langle\varphi|, \tag{32}
$$

where $|\varphi\rangle = \frac{1}{\sqrt{5}}[(|10\rangle+|21\rangle)|0\rangle + (|00\rangle+|11\rangle+|22\rangle)|1\rangle]$, $0\le x\le 1$, I_{18} is the 18×18 identity matrix. By Theorem 3 (i), we can determine the range of x where ρ is surely entangled. Table1 shows that when $\alpha=0$, $\beta=1$, our criterion detects the entanglement for $0.3405 < x \le 1$, which is better than the result $0.35\le x\le 1$given in [25].

Table 1 The entanglement regions of ρ as given by Theorem 3.

	$\|N^{2\|13}\|_{tr}$	The range of entanglement
$\alpha=1,\beta=1$	$10.5292x$	$0.4852 < x \leq 1$
$\alpha=\frac{1}{2},\beta=2$	$18.4650x$	$0.3909 < x \leq 1$
$\alpha=0,\beta=1$	$9.1321x$	$0.3405 < x \leq 1$

Example 3 Consider the four-qubit state $\rho \in H_1^2 \otimes H_2^2 \otimes H_3^2 \otimes H_4^2$,

$$\rho = x|\psi\rangle\langle\psi| + \frac{1-x}{16} I_{16}, \tag{33}$$

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$, $0 \leq x \leq 1$, I_{16} is the 16×16 identity matrix. Using Theorem 3 (i) with $\alpha = 1, \beta = 1$, we set $f_2(x) = \|N^{l_1|l_2l_3l_4}\|_{tr} - (1 + \sqrt{\frac{11}{2}}) = (4+\sqrt{2})x - (1+\sqrt{\frac{11}{2}})$, ρ is not separable under the bipartition $l_1|l_2l_3l_4$ for $f_2(x) > 0$, i.e. $0.6179 < x \leq 1$, while according to Theorem 3 in [17], ρ is not separable under the bipartition $l_1|l_2l_3l_4$ for $g_2(x) = 9x^2 - 4 > 0$, i.e. $0.6667 < x \leq 1$. Fig. 2 shows that our method detects more entanglement.

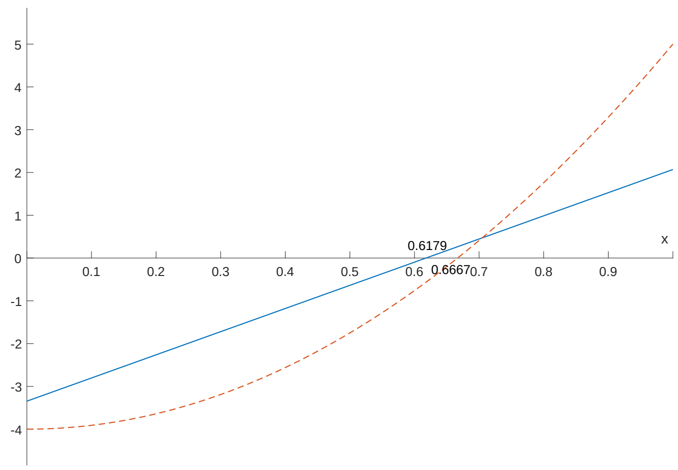

Fig. 2 $f_2(x)$ from our result (solid straight line), $g_2(x)$ from Theorem 3 in [17] (dashed curve line).

4 Conclusions

By adopting the representation with generalized Pauli operators of density matrices, we have come up with several general tests to judge genuine entanglement for tripartite quantum systems. Our approach starts with some finer upper bounds for the norms of correlation tensors by using the generalized Pauli operators presentation, then we have obtained the entanglement criteria for genuine tripartite quantum states based on certain matrices constructed by the correlation tensor of the density matrices. We also conducted conclusion to detect entanglement in arbitrary dimensional multipartite quantum states. Compared with previously available criteria, ours can detect more situations, and these are explained in details with several examples.

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