# MDS Codes with Euclidean and Hermitian Hulls of Flexible Dimensions and Their Applications to EAQECCs 

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#### Abstract

The hull of a linear code is the intersection of itself with its dual code with respect to certain inner product. Both Euclidean and Hermitian hulls are of theorical and practical significance. In this paper, we construct several new classes of MDS codes via (extended) generalized Reed-Solomon (GRS) codes and determine their Euclidean or Hermitian hulls. Specifically, four new classes of MDS codes with Hermitian hulls of flexible dimensions and six new classes of MDS codes with Euclidean hulls of flexible dimensions are constructed. For the former, we further construct four new classes of entanglementassisted quantum error-correcting codes (EAQECCs) and four new classes of MDS EAQECCs of length $n>q+1$. For the latter, we also give some examples on Euclidean self-orthogonal and one-dimensional Euclidean hull MDS codes.


Keywords: hulls, entanglement-assisted quantum error-correcting codes, generalized Reed-Solomon codes, extended generalized Reed-Solomon codes

2010 MSC: 94B05, 81p70

## 1. Introduction

Let $\mathcal{C}$ be a linear code over a finite filed. Denote the dual code of $\mathcal{C}$ with respect to certain inner product by $\mathcal{C}^{\perp}$, such as the usual Euclidean inner product or classical Hermitian inner product. The hull of $\mathcal{C}$ is defined by the linear code $\mathcal{C} \cap \mathcal{C}^{\perp}$, denoted by $\operatorname{Hull}(\mathcal{C})$, which was first introduced by Assmus et al. 22] to classify finite projective planes. Over the years, numerous studies have shown that the hull of linear codes plays a very important role in coding theory. On one hand, the hull of linear codes is closely related to the complexity of algorithms for computing the automorphism group of a linear code [25] and for checking permutation equivalence of two linear codes [26, 34]. In general, these algorithms are really efficient when the dimension of the hull is small. Some magnificent results on linear codes with small hulls were proposed in [6, 21, 28, 29, 33, 37] by using tools such as Gaussian sums, algebraic function fields, partial difference sets and so on.

On the other hand, the hull of linear codes has important applications in the construction of socalled entanglement-assisted quantum error-correcting codes (EAQECCs). EAQECCs were introduced

[^0]by Burn et al. [3] and were rapidly developed by other scholars. By all accounts, the introduction of EAQECCs is regarded as a milestone in the development of coding theory. Customarily, we denote $[[n, k, d ; c]]_{q}$ as an EAQECC, which can encode $k$ logical qubits into $n$ physical qubits with the help of $c$ pairs of maximally entangled states over $\mathbb{F}_{q}$ and can correct up to $\left\lfloor\frac{d-1}{2}\right\rfloor$ qubit-errors. Different from classical quantum error-correcting codes (QECCs), EAQECCs can be constructed by any linear code, while QECCs can only be constructed by special linear codes with certain self-orthogonality or satisfying certain dual containing condition [4, 32]. Similar to classical linear codes, for EAQECCs, people are also willing to construct MDS EAQECCs, i.e., EAQECCs that reach the quantum Singleton bound [1].

However, what needs to be emphasized is the difficulty of the computation of the number of c. Fortunately, in 2018, Guenda et al. 15] described some celebrated relationships between the number of $c$ and the dimension of the hull of a linear code, where the hull was considered under both the Euclidean inner product and Hermitian inner product. After this, people worked to determine dimensions of hulls of different linear codes, and constructed new EAQECCs and MDS EAQECCs (e.g., see $8,11,14,15,19,20,22,23,24,27,30,31,36$ and references therein). We can summarize some outstanding works on this topic as follows. In 14], Guenda et al. completely determined all possible $q$-ary MDS EAQECCs of length $n \leq q+1$ via the $\ell$-intersection pair of linear codes, which implies that the construction of $q$-ary MDS EAQECCs of length $n>q+1$ will be the main theme of our researches. In [11], (extended) generalized Reed-Solomon (GRS) codes with Euclidean and Hermitian hulls of arbitrary dimensions were discussed by Fang et al.. As applications, some good MDS EAQECCs with flexible parameters were obtained. Recently, in 35], Wang et al. constructed some MDS EAQECCs based on GRS codes with Euclidean hulls of flexible dimensions and these MDS EAQECCs no longer need to use a fixed $c$. In [7], Chen proved that if an $[n, k]_{q^{2}}$ Hermitian selforthogonal code exists, then $[n, k]_{q^{2}}$ linear codes with Hermitian hulls of arbitrary dimensions exist. Based on this consequence, a large number of MDS EAQECCs can be directly derived.

Inspired and motivated by these works, in this paper, we study (extended) GRS codes and determine their Euclidean or Hermitian hulls. Moreover, using those MDS codes with Hermitian hulls of flexible dimensions, we construct four new classes of EAQECCs and four new classes of MDS EAQECCs with flexible parameters. The lengths of these MDS EAQECCs are all greater than $q+1$. For reference, we list the parameters of some known MDS EAQECCs and the new ones in Table 1 Besides, as concrete examples of the Euclidean case, some Euclidean self-orthogonal and one-dimensional Euclidean hull MDS codes are given.

The rest of this paper is organized as follows. In Section 2, we review some basic notations and results on (extended) GRS codes and hulls. In Section 3, we construct several new classes of MDS codes with Euclidean or Hermitian hulls of flexible dimensions. Section 4 constructs some new families of EAQECCs and MDS EAQECCs of length $n>q+1$. And finally, Section 5 concludes this paper.

Table 1: Some known constructions of MDS EAQECCs of length $n>q+1$

| Parameters | Constraints | Ref. |
| :---: | :---: | :---: |
| $\left[\left[\frac{q^{2}-1}{t}, \frac{q^{2}-1}{t}-2 d+t+2, d ; t\right]\right]_{q}$ | $q$ odd, $t$ odd, $t \geq 3, t \mid q+1, \frac{(t-1)(q-1)}{t}+2 \leq d \leq \frac{(t+1)(q-1)}{t}-2$ | [8] |
| $\left[\left[q^{2}+1, q^{2}+1-q-l, q+1 ; q-l\right]\right]_{q}$ | $q=p^{m} \geq 3,0 \leq l \leq q$ | [11] |
| $\left[\left[t r^{z}, t r^{z}-k-l, k+1 ; k-l\right]\right]_{q}$ | $q=p^{m} \geq 3, r=p^{e}, e \mid m, 1 \leq t \leq r, 1 \leq z \leq 2 \frac{m}{e}-1,1 \leq k \leq\left\lfloor\frac{n-1+q}{q+1}\right\rfloor, 0 \leq l \leq k$ | [11] |
| $\left[\left[t r^{z}+1, t r^{z}+1-k-l, k+1 ; k-l\right]\right]_{q}$ | $q=p^{m} \geq 3, r=p^{e}, e \mid m, 1 \leq t \leq r, 1 \leq z \leq 2 \frac{m}{e}-1,1 \leq k \leq\left\lfloor\frac{n-1+q}{q+1}\right\rfloor, 0 \leq l \leq k-1$ | [11] |
| $\left[\left[t n^{\prime}, t n^{\prime}-k-l, k+1 ; k-l\right]_{q}\right.$ | $q=p^{m} \geq 3, n^{\prime} \mid\left(q^{2}-1\right), 1 \leq t \leq \frac{q-1}{n_{1}}, n_{1}=\frac{n^{\prime}}{\operatorname{gcd}\left(n^{\prime}, q+1\right)}, 1 \leq k \leq\left\lfloor\frac{n+q}{q+1}\right\rfloor, 0 \leq l \leq k-1$ | [11] |
| $\left[\left[t n^{\prime}+1, t n^{\prime}+1-k-l, k+1 ; k-l\right]\right]_{q}$ | $q=p^{m} \geq 3, n^{\prime} \mid\left(q^{2}-1\right), 1 \leq t \leq \frac{q-1}{n_{1}}, n_{1}=\frac{n^{\prime}}{\operatorname{gcd}\left(n^{\prime}, q+1\right)}, 1 \leq k \leq\left\lfloor\frac{n+q}{q+1}\right\rfloor, 0 \leq l \leq k$ | [11] |
| $\left[\left[\frac{q^{2}-1}{t}, \frac{q^{2}-1}{t}-4 q m+4 m^{2}+3,2 m(q-1) ;(2 m-1)^{2}\right]\right]_{q}$ | $q \geq 3, t \mid q^{2}-1,1 \leq m \leq\left\lfloor\frac{q+1}{4 t}\right\rfloor$ | [19] |
| $\left[\left[\frac{q^{2}+1}{t}, \frac{q^{2}+1}{t}-4 q m+4 q+4 m^{2}-8 m+3,2 q(m-1)+2 ; 4(m-1)^{2}+1\right]\right]_{q}$ | $q \geq 7, t \mid q^{2}+1,2 \leq m \leq\left\lfloor\frac{q+1}{4 t}\right\rfloor$ | [19] |
| $[[l h+m r, l h+m r-2 d+c, d+1 ; c]]_{q}$ | $\begin{gathered} s\|q+1, t\| q-1, l=\frac{q^{2}-1}{s}, m=\frac{q^{2}-1}{t}, 1 \leq h \leq \frac{s}{2}, \\ 2 \leq r \leq \frac{t}{2}, c=h-1,1 \leq d \leq \min \left\{\frac{s+h}{2} \cdot \frac{q+1}{s}+2, \frac{q+1}{2}+\frac{q-1}{t}-1\right\} \end{gathered}$ | [20] |
| $\left[\left[1+(2 e+1) \frac{q^{2}-1}{2 s+1}, 1+(2 e+1) \frac{q^{2}-1}{2 s+1}-2 k+c, k+1 ; c\right]\right]_{q}$ | $0 \leq e \leq s-1,(2 s+1) \mid q+1, c=2 e+1,1 \leq k \leq(s+1+e) \frac{q+1}{2 s+1}-1$ | [20] |
| $\left[\left[1+(2 e+1) \frac{q^{2}-1}{2 s}, 1+(2 e+1) \frac{q^{2}-1}{2 s}-2 k+c, k+1 ; c\right]\right]_{q}$ | $0 \leq e \leq s-2,2 s \mid q+1, c=2 e+2,1 \leq k \leq(s+1+e) \frac{q+1}{2 s}-1$ | [20] |
| $\left[\left[1+(2 e+1) \frac{q^{2}-1}{2 s}, 1+(2 e+1) \frac{q^{2}-1}{2 s}-2 k+c, k+1 ; c\right]\right]_{q}$ | $0 \leq e \leq s-1,2 s \mid q+1, c=2 e+1,1 \leq k \leq(s+e) \frac{q+1}{2 s}-2$ | [20] |
| ${ }_{[ }[n, n-k-h, k+1 ; k-h]_{q}$ | $q>3, m>1, m \mid q, 1<k \leq\left\lfloor\frac{n}{2}\right\rfloor, n+k>m+1,1 \leq n \leq m, 1 \leq h \leq n-m+k-1$ | [22] |
| $[[n, n-k-h, k+1 ; k-h]]_{q}$ | $\begin{gathered} q>3, m>1, m \mid q, 1<k \leq\left\lfloor\frac{n}{2}\right\rfloor \\ 2 n-k-1<m<2 n-1,1 \leq n \leq m, 1 \leq h \leq 2 n-m-1 \end{gathered}$ | [22] |
| $[[n, n-k-l, k+1 ; k-l]]_{q}$ | $q+1<n<2(q-1), n-q<k<\left\lfloor\frac{n}{2}\right\rfloor, 1 \leq l \leq k+q-n$ | [24] |
| $\left[\left[\frac{q+1}{7}(q-1), \frac{q+1}{7}(q-1)+5-2 d, d ; 3\right]\right]_{q}$ | $d \leq \frac{n+2}{2}, \frac{5(q+1)}{7} \leq d \leq \frac{6(q+1)}{7}-2$ | [27] |
| $\left[\left[\frac{q+1}{7}(q-1), \frac{q+1}{7}(q-1)+7-2 d, d ; 5\right]\right]_{q}$ | $d \leq \frac{n+2}{2}, \frac{6(q+1)}{7} \leq d \leq q$ | [27] |
| $\left[\left[\frac{q+1}{7}(q-1), \frac{q+1}{7}(q-1)+9-2 d, d ; 7\right]\right]_{q}$ | $q$ odd, $d \leq \frac{n+2}{2}, d=\frac{8(q+1)}{7}-1$ | [27] |
| $\left[\left[\frac{q+1}{4}(q-1), \frac{q+1}{4}(q-1)+4-2 d, d ; 2\right]_{q}\right.$ | $q$ odd, $d \leq \frac{n+2}{2}, \frac{3(q+1)}{4} \leq d \leq q$ | [27] |
| $\left[\left[\frac{q+1}{4}(q-1), \frac{q+1}{4}(q-1)+6-2 d, d ; 4\right]\right]_{q}$ | $q$ odd, $d \leq \frac{n+2}{2}, q+1 \leq d \leq \frac{5(q+1)}{4}-1$ | [27] |
| $\left[\left[\frac{q+1}{6}(q-1), \frac{q+1}{6}(q-1)+4-2 d, d ; 2\right]\right]_{q}$ | $q$ odd, $d \leq \frac{n+2}{2}, \frac{4(q+1)}{6} \leq d \leq \frac{5(q+1)}{6}-1$ | [27] |
| $\left[\left[\frac{q+1}{6}(q-1), \frac{q+1}{6}(q-1)+6-2 d, d ; 4\right]\right]_{q}$ | $q$ odd, $d \leq \frac{n+2}{2}, \frac{5 q+1}{6} \leq d \leq q$ | [27] |
| $\left[\left[q^{2}+1, q^{2}-2 \delta, 2 \delta+2 ; 2 \delta+1\right]\right]_{q}$ | $q$ odd, $s=\frac{n}{2}, r \mid q-1, r \nmid q+1,0 \leq \delta \leq \frac{(r-1)(s-1)}{r}$ | [30] |
| $\left[\left[q^{2}+1, q^{2}-2 \delta-1,2 \delta+3 ; 2 \delta+2\right]_{q}\right.$ | $q=2^{m}(m \geq 1), \mu=\frac{n-r}{2}, r \mid q-1, r \nmid q+1,0 \leq \delta \leq \frac{\mu-1}{r}$ | [30] |
| $\left[\left[q^{2}+1, q^{2}-2 \delta, 2 \delta+2 ; 2 \delta+1\right]\right]_{q}$ | $q=2^{m}(m \geq 1), r \mid q-1, r \nmid q+1,0 \leq \delta \leq \frac{(r-1)(n-2)}{2 r}$ | [30] |
| $\left[\left[q^{2}+1, q^{2}-4(m-1)(q-m-1), 2(m-1) q+2 ; 4(m-1)^{2}+1\right]\right]_{q}$ | $q \geq 5,2 \leq m \leq \frac{q-1}{2}$ | [31] |
| $[[n, n-k-l, k+1 ; k-l]]_{q}$ | $q=p^{m} \geq 3,1 \leq k \leq q-1, q^{2}-k \leq n \leq q^{2}, 0 \leq l \leq n+k-q^{2}$ | new |
| $[[n+1, n+1-k-l, k+1 ; k-l]]_{q}$ | $q=p^{m} \geq 3,1 \leq k \leq q-1, q^{2}-k+1 \leq n \leq q^{2}, 0 \leq l \leq n+k-q^{2}-1$ | new |
| $[[n+1, n-1-q-l, q+1 ; q-l]]_{q}$ | $q=p^{m} \geq 3, q^{2}-q \leq n \leq q^{2}, 0 \leq l \leq n+q-q^{2}$ | new |
| $[[m(q-1)+1, m(q-1)+1-k-l, k+1 ; k-l]]_{q}$ | $q=p^{m} \geq 3,2 \leq m \leq q, 1 \leq k \leq m-1,0 \leq l \leq k-1$ | new |

## 2. Preliminaries

Let $q$ be a prime power and $\mathbb{F}_{q}$ be the finite field with $q$ elements. For any positive integer $n$, $\mathbb{F}_{q}^{n}$ can be seen as an $n$-dimensional vector space over $\mathbb{F}_{q}$. Then a $k$-dimensional subspace of $\mathbb{F}_{q}^{n}$ with minimum distance $d$ is just a linear code $\mathcal{C}$, denoted by $[n, k, d]_{q}$. A linear code $\mathcal{C}$ is called an MDS code if $d=n-k+1$. Now, we review some basic notations and results on (extended) GRS codes and hulls.

Firstly, for any two vectors $\mathbf{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ and $\mathbf{y}=\left(y_{1}, y_{2}, \ldots, y_{n}\right)$ of $\mathbb{F}_{q}^{n}$, we can define different inner products between them. Specifically, the Euclidean inner product between $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$
\langle\mathbf{x}, \mathbf{y}\rangle_{E}=\sum_{i=1}^{n} x_{i} y_{i}
$$

If $\mathcal{C}$ is a linear code of length $n$ over $\mathbb{F}_{q}$, then the Euclidean dual code of $\mathcal{C}$, denoted by $\mathcal{C}^{\perp_{E}}$, can be described as the set

$$
\mathcal{C}^{\perp_{E}}=\left\{\mathbf{x} \in \mathbb{F}_{q}^{n}:\langle\mathbf{x}, \mathbf{y}\rangle_{E}=0, \text { for all } \mathbf{y} \in \mathcal{C}\right\}
$$

The Hermitian inner product between $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$
\langle\mathbf{x}, \mathbf{y}\rangle_{H}=\sum_{i=1}^{n} x_{i} y_{i}^{q}
$$

If $\mathcal{C}$ is a linear code of length $n$ over $\mathbb{F}_{q^{2}}$, then the Hermitian dual code of $\mathcal{C}$, denoted by $\mathcal{C}^{\perp_{H}}$, can be similarly described as the set

$$
\mathcal{C}^{\perp_{H}}=\left\{\mathbf{x} \in \mathbb{F}_{q^{2}}^{n}:\langle\mathbf{x}, \mathbf{y}\rangle_{H}=0, \text { for all } \mathbf{y} \in \mathcal{C}\right\}
$$

Then as said before, we define the Euclidean hull (resp. Hermitian hull) of $\mathcal{C}$ as $\mathcal{C} \cap \mathcal{C}^{\perp_{E}}$ (resp. $\mathcal{C} \cap \mathcal{C}^{\perp_{H}}$ ), denoted by $\operatorname{Hull}_{E}(\mathcal{C})$ (resp. $\operatorname{Hull}_{H}(\mathcal{C})$ ). It is well known that $\mathcal{C}$ is an Euclidean (resp. Hermitian) self-orthogonal code if $\operatorname{Hull}_{E}(\mathcal{C})=\mathcal{C}$ (resp. $\left.\operatorname{Hull}_{H}(\mathcal{C})=\mathcal{C}\right)$. More generally, for a positive integer $l$, if $\operatorname{dim}\left(\operatorname{Hull}_{E}(\mathcal{C})\right)=l\left(\right.$ resp. $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right)=l$, we call $\mathcal{C}$ a $l$-dimensional Euclidean (resp. Hermitian) hull code.

Denote $\mathbb{F}_{q}^{*}=\mathbb{F}_{q} \backslash\{0\}$. Choose $n$ distinct elements $a_{1}, a_{2}, \cdots, a_{n}$ from $\mathbb{F}_{q}$ and $n$ nonzero elements $v_{1}, v_{2}, \cdots, v_{n}$ from $\mathbb{F}_{q}^{*}$. As special MDS codes, GRS codes and extended GRS codes can be defined as follows. Set $\mathbf{a}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and $\mathbf{v}=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$. The GRS code of length $n$ associated to a and $\mathbf{v}$, denoted by $\operatorname{GRS}(\mathbf{a}, \mathbf{v})$, is defined by

$$
\operatorname{GRS}(\mathbf{a}, \mathbf{v})=\left\{\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right)\right): f(x) \in \mathbb{F}_{q}[x] \text { and } \operatorname{deg}(f(x)) \leq k-1\right\}
$$

where the elements $a_{1}, a_{2}, \ldots, a_{n}$ are called the code locators of $\operatorname{GRS}(\mathbf{a}, \mathbf{v})$ and $v_{1}, v_{2}, \ldots, v_{n}$ are called the column multipliers of $\operatorname{GRS}(\mathbf{a}, \mathbf{v})$.

With a practical technique, the extended GRS code of length $n+1$ associated to a and $\mathbf{v}$, denoted by $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$, can be derived. Specifically, the definition of $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$ is

$$
\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)=\left\{\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right), f_{k-1}\right): f(x) \in \mathbb{F}_{q}[x] \text { and } \operatorname{deg}(f(x)) \leq k-1\right\}
$$

where $f_{k-1}$ is the coefficient of $x^{k-1}$ in $f(x)$.
For our purposes, considering both (extended) GRS codes and hulls, some basic results need to be introduced. To this end, for $0 \leq i \leq n$, we denote

$$
\begin{equation*}
u_{i}=\prod_{1 \leq j \leq n, j \neq i}\left(a_{i}-a_{j}\right)^{-1} \tag{1}
\end{equation*}
$$

which will appear frequently in this paper and is critical to our constructions. Then the coming results can help us calculate the dimension of the hull of a GRS code or an extended GRS code.

Lemma 1. ([5]) Considering the Euclidean inner product over $\mathbb{F}_{q}$, the following statements hold.
(1) A codeword $\boldsymbol{c}=\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right)\right)$ of $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v})$ is contained in $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v})^{\perp_{E}}$ if and only if there exists a polynomial $g(x) \in \mathbb{F}_{q}[x]$ with $\operatorname{deg}(g(x)) \leq n-k-1$ such that

$$
\left(v_{1}^{2} f\left(a_{1}\right), v_{2}^{2} f\left(a_{2}\right), \ldots, v_{n}^{2} f\left(a_{n}\right)\right)=\left(u_{1} g\left(a_{1}\right), u_{2} g\left(a_{2}\right), \ldots, u_{n} g\left(a_{n}\right)\right)
$$

(2) A codeword $\boldsymbol{c}=\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right), f_{k-1}\right)$ of $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$ is contained in $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}$, $\infty)^{\perp_{E}}$ if and only if there exists a polynomial $g(x) \in \mathbb{F}_{q}[x]$ with $\operatorname{deg}(g(x)) \leq n-k$ such that

$$
\left(v_{1}^{2} f\left(a_{1}\right), v_{2}^{2} f\left(a_{2}\right), \ldots, v_{n}^{2} f\left(a_{n}\right), f_{k-1}\right)=\left(u_{1} g\left(a_{1}\right), u_{2} g\left(a_{2}\right), \ldots, u_{n} g\left(a_{n}\right),-g_{n-k}\right)
$$

where $g_{n-k}$ is the coefficient of $x^{n-k}$ in $g(x)$.
Lemma 2. ([G]]) Considering the Hermitian inner product over $\mathbb{F}_{q^{2}}$, the following statements hold.
(1) A codeword $\boldsymbol{c}=\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right)\right)$ of $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v})$ is contained in $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v})^{\perp_{H}}$ if and only if there exists a polynomial $g(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(g(x)) \leq n-k-1$ such that

$$
\left(v_{1}^{q+1} f^{q}\left(a_{1}\right), v_{2}^{q+1} f^{q}\left(a_{2}\right), \ldots, v_{n}^{q+1} f^{q}\left(a_{n}\right)\right)=\left(u_{1} g\left(a_{1}\right), u_{2} g\left(a_{2}\right), \ldots, u_{n} g\left(a_{n}\right)\right)
$$

(2) A codeword $\boldsymbol{c}=\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right), f_{k-1}\right)$ of $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$ is contained in $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}$, $\infty)^{\perp_{H}}$ if and only if there exists a polynomial $g(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(g(x)) \leq n-k$ such that

$$
\left(v_{1}^{q+1} f^{q}\left(a_{1}\right), v_{2}^{q+1} f^{q}\left(a_{2}\right), \ldots, v_{n}^{q+1} f^{q}\left(a_{n}\right), f_{k-1}^{q}\right)=\left(u_{1} g\left(a_{1}\right), u_{2} g\left(a_{2}\right), \ldots, u_{n} g\left(a_{n}\right),-g_{n-k}\right)
$$

where $g_{n-k}$ is the coefficient of $x^{n-k}$ in $g(x)$.
For a given $l$-dimensional Hermitian hull linear code, by Corollary 2.2 of [7], linear codes with Hermitian hulls of flexible dimensions can be obtained in an explicit way. For convenience, we equivalently write it in the following form.

Lemma 3. ([ $[7])$ Let $\mathcal{C}$ be an $[n, k]_{q^{2}}$ linear code with $l$-dimensional Hermitian hull. Then there exists an $[n, k]_{q^{2}}$ linear code with $l^{\prime}$-dimensional Hermitian hull for nonnegative integer $l^{\prime}$ satisfying $0 \leq l^{\prime} \leq l$.

In particular, according to the proof of Lemma 3 in [7], for the MDS case and $l=k-1$, we can precisely derive the following corollary.

Corollary 4. Let $\mathcal{C}$ be an $[n, k]_{q^{2}} M D S$ code with $(k-1)$-dimensional Hermitian hull. Then there exists an $[n, k]_{q^{2}} M D S$ code with $l^{\prime}$-dimensional Hermitian hull for nonnegative integer $l^{\prime}$ satisfying $0 \leq l^{\prime} \leq k-1$.

Finally, we make some conventions. As readers may have noticed, throughout this paper,

- when we talk about Euclidean inner product or Euclidean hull, the finite field that matches is always $\mathbb{F}_{q}$;
- when we talk about Hermitian inner product or Hermitian hull, the finite field that matches is always $\mathbb{F}_{q^{2}}$.

For the multiplication sign " $\prod_{i=a}^{b}$." over $\mathbb{F}_{q}$, we make the following agreement:

- if $a \leq b$, then the operation is performed according to the standard multiplication over $\mathbb{F}_{q}$;
- if $a>b$, then the result of this operation is always 1 , where 1 is the unit element of $\mathbb{F}_{q}$.

In addition, we need to emphasize the following notations:

- $\lfloor x\rfloor$ denotes the largest integer less than or equal to $x$;
- $\lceil x\rceil$ denotes the smallest integer greater than or equal to $x$;
- $[a, b]$ denotes a set of $x$ satisfying $a \leq x \leq b$, where $a \leq b$.


## 3. Constructions

In this section, we present several new classes of MDS codes via (extended) GRS codes, whose Euclidean hulls or Hermitian hulls are entirely determined. We also introduce a new method to construct $[n, k]_{q^{2}}$ MDS codes with $(k-1)$-dimensional Hermitian hull. For Euclidean cases, some Euclidean self-orthogonal and one-dimensional Euclidean hull MDS codes are given as examples.

### 3.1. Construction $A$ for $M D S$ codes with Hermitian hulls of flexible dimensions

Denote $\mathbb{F}_{q^{2}}=\left\{a_{1}, a_{2}, \cdots, a_{n}, a_{n+1}, \cdots, a_{q^{2}}\right\}$. It is clear that $a_{i} \neq a_{j}$ for any $1 \leq i \neq j \leq q^{2}$. Note that

$$
\prod_{1 \leq j \leq q^{2}, j \neq i}\left(a_{i}-a_{j}\right)^{-1}=\prod_{x \in \mathbb{F}_{q^{*}}^{*}} x=-1
$$

then $u_{i}$ defined as Eq. (11) can be further denoted by

$$
\begin{equation*}
u_{i}=\prod_{1 \leq j \leq n, j \neq i}\left(a_{i}-a_{j}\right)^{-1}=-\prod_{j=n+1}^{q^{2}}\left(a_{i}-a_{j}\right) \tag{2}
\end{equation*}
$$

Based on the basic fact above, three new classes of MDS codes with Hermitian hulls of flexible dimensions can be constructed as follows.

Theorem 5. Let $q=p^{m} \geq 3$, for $1 \leq k \leq q-1$, if $q^{2}-k \leq n \leq q^{2}$, then there exists an $[n, k]_{q^{2}} M D S$ code with $l$-dimensional Hermitian hull, where $0 \leq l \leq n+k-q^{2}$.

Proof. Let notations be the same as before. Denote $s=n+k-q^{2}-l$. Take $\mathbf{a}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and $\mathbf{v}=\left(v_{1}, v_{2}, \ldots, v_{s}, 1, \ldots, 1\right)$, where $v_{i}^{q+1} \neq 1$ for all $1 \leq i \leq s$. Note that $\left(q^{2}-1\right) \nmid(q+1)$ for any $q$ satisfying $q \geq 3$, then such $\mathbf{v}$ does exist. We now consider the Hermitian hull of the $q^{2}$-ary $[n, k]$ MDS code $\mathcal{C}=\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v})$.

For any codeword

$$
\boldsymbol{c}=\left(v_{1} f\left(a_{1}\right), \ldots, v_{s} f\left(a_{s}\right), f\left(a_{s+1}\right), \ldots, f\left(a_{n}\right)\right) \in \operatorname{Hull}_{H}(\mathcal{C})
$$

by the result (1) of Lemma 2, there exists a polynomial $g(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(g(x)) \leq n-k-1$ such that

$$
\begin{align*}
& \left(v_{1}^{q+1} f^{q}\left(a_{1}\right), \ldots, v_{s}^{q+1} f^{q}\left(a_{s}\right), f^{q}\left(a_{s+1}\right), \ldots, f^{q}\left(a_{n}\right)\right)  \tag{3}\\
= & \left(u_{1} g\left(a_{1}\right), \ldots, u_{s} g\left(a_{s}\right), u_{s+1} g\left(a_{s+1}\right), \ldots, u_{n} g\left(a_{n}\right)\right) .
\end{align*}
$$

On one hand, from the last $n-s$ coordinates of Eq. (3) and Eq. (2), we have

$$
f^{q}\left(a_{i}\right)=u_{i} g\left(a_{i}\right)=-\prod_{j=n+1}^{q^{2}}\left(a_{i}-a_{j}\right) g\left(a_{i}\right), s+1 \leq i \leq n
$$

It follows that $f^{q}(x)=-\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) g(x)$ has at least $n-s$ distinct roots. Recall that $s=$ $n+k-q^{2}-l$ and $1 \leq k \leq q-1$, then

$$
\begin{aligned}
& \operatorname{deg}\left(f^{q}(x)\right) \leq q(k-1) \leq q^{2}-k-1=n-s-l-1 \leq n-s-1 \\
& \operatorname{deg}\left(\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) g(x)\right) \leq\left(q^{2}-n\right)+(n-k-1)=q^{2}-k-1 \leq n-s-1
\end{aligned}
$$

Hence, we can conclude that $f^{q}(x)=-\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) g(x)$ from the fact $n-s-1<n-s$. Moreover, $\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) \mid f^{q}(x)$.

On the other hand, from the first $s$ coordinates of Eq. (3), we have

$$
v_{i}^{q+1} f^{q}\left(a_{i}\right)=u_{i} g\left(a_{i}\right)=f^{q}\left(a_{i}\right), 1 \leq i \leq s
$$

For any $1 \leq i \leq s$, since $v_{i}^{q+1} \neq 1$, we have $f\left(a_{i}\right)=0$. Therefore, $f(x)$ can be written as

$$
f(x)=h(x) \prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) \prod_{i=1}^{s}\left(x-a_{i}\right)
$$

where $h(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(h(x)) \leq n+k-q^{2}-s-1$. It deduces that $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right) \leq n+k-q^{2}-s$.
Conversely, let $f(x)$ be a polynomial of form $h(x) \prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) \prod_{i=1}^{s}\left(x-a_{i}\right)$, where $h(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(h(x)) \leq n+k-q^{2}-s-1$. Take $g(x)=-\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right)^{-1} f^{q}(x)$, then $g(x)$ is a polynomial in $\mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(g(x)) \leq q(k-1)-\left(q^{2}-n\right) \leq n-k-1$. Moreover, by Eq. (21), we have

$$
\begin{aligned}
& \left(v_{1}^{q+1} f^{q}\left(a_{1}\right), \ldots, v_{s}^{q+1} f^{q}\left(a_{s}\right), f^{q}\left(a_{s+1}\right), \ldots, f^{q}\left(a_{n}\right)\right) \\
= & \left(u_{1} g\left(a_{1}\right), \ldots, u_{s} g\left(a_{s}\right), u_{s+1} g\left(a_{s+1}\right), \ldots, u_{n} g\left(a_{n}\right)\right) .
\end{aligned}
$$

According to the result (1) of Lemma 2, the vector

$$
\left(v_{1} f\left(a_{1}\right), \ldots, v_{s} f\left(a_{s}\right), f\left(a_{s+1}\right), \ldots, f\left(a_{n}\right)\right) \in \operatorname{Hull}_{H}(\mathcal{C})
$$

It deduces that $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right) \geq n+k-q^{2}-s$.
In summary, $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right)=n+k-q^{2}-s=l$. This completes the proof.
Theorem 6. Let $q=p^{m} \geq 3$, for $1 \leq k \leq q-1$, if $q^{2}-k+1 \leq n \leq q^{2}$, then there exists an $[n+1, k]_{q^{2}}$ MDS code with l-dimensional Hermitian hull, where $0 \leq l \leq n+k-q^{2}-1$.

Proof. Denote $s=n+k-q^{2}-l-1$ and let other notations be the same as Theorem 5. We now consider the Hermitian hull of the $q^{2}$-ary $[n+1, k] \operatorname{MDS}$ code $\mathcal{C}=\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$.

For any codeword

$$
\boldsymbol{c}=\left(v_{1} f\left(a_{1}\right), \ldots, v_{s} f\left(a_{s}\right), f\left(a_{s+1}\right), \ldots, f\left(a_{n}\right), f_{k-1}\right) \in \operatorname{Hull}_{H}(\mathcal{C})
$$

by the result (2) of Lemma 2, there exists a polynomial $g(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(g(x)) \leq n-k$ such that

$$
\begin{align*}
& \left(v_{1}^{q+1} f^{q}\left(a_{1}\right), \ldots, v_{s}^{q+1} f^{q}\left(a_{s}\right), f^{q}\left(a_{s+1}\right), \ldots, f^{q}\left(a_{n}\right), f_{k-1}^{q}\right)  \tag{4}\\
= & \left(u_{1} g\left(a_{1}\right), \ldots, u_{s} g\left(a_{s}\right), u_{s+1} g\left(a_{s+1}\right), \ldots, u_{n} g\left(a_{n}\right),-g_{n-k}\right) .
\end{align*}
$$

On one hand, from the last $n-s+1$ coordinates of Eq. (4) and Eq. (2), we have

$$
f^{q}\left(a_{i}\right)=u_{i} g\left(a_{i}\right)=-\prod_{j=n+1}^{q^{2}}\left(a_{i}-a_{j}\right) g\left(a_{i}\right), s+1 \leq i \leq n \text { and } f_{k-1}^{q}=-g_{n-k}
$$

It follows that $f^{q}(x)=-\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) g(x)$ has at least $n-s$ distinct roots. Recall that $s=$ $n+k-q^{2}-l-1$ and $1 \leq k \leq q-1$, then

$$
\begin{aligned}
& \operatorname{deg}\left(f^{q}(x)\right) \leq q(k-1) \leq q^{2}-k=n-s-l-1 \leq n-s-1 \\
& \operatorname{deg}\left(\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) g(x)\right) \leq\left(q^{2}-n\right)+(n-k)=q^{2}-k \leq n-s-1
\end{aligned}
$$

Hence, we can conclude that $f^{q}(x)=-\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) g(x)$ from the fact $n-s-1<n-s$. Moreover, $\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) \mid f^{q}(x)$. Now, we determine the value of $f_{k-1}$. If $f_{k-1} \neq 0$, since $f^{q}(x)=-\prod_{j=n+1}^{q^{2}}(x-$ $\left.a_{j}\right) g(x)$ and $f_{k-1}^{q}=-g_{n-k}$, we have $q(k-1)=\left(q^{2}-n\right)+(n-k)$, which contradicts to $1 \leq k \leq q-1$. Hence, $f_{k-1}=0$ and $\operatorname{deg}(f(x)) \leq k-2$.

On the other hand, similar to the proof of Theorem 5, it follows that $f\left(a_{i}\right)=0$ for any $1 \leq i \leq s$ from the first $s$ coordinates of Eq. (4). Therefore, $f(x)$ can be written as

$$
f(x)=h(x) \prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) \prod_{i=1}^{s}\left(x-a_{i}\right)
$$

where $h(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(h(x)) \leq n+k-q^{2}-s-2$. It deduces that $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right) \leq n+k-q^{2}-s-1$.

Conversely, let $f(x)$ be a polynomial of form $h(x) \prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) \prod_{i=1}^{s}\left(x-a_{i}\right)$, where $h(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(h(x)) \leq n+k-q^{2}-s-2$. Take $g(x)=-\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right)^{-1} f^{q}(x) \in \mathbb{F}_{q^{2}}[x]$, then $g(x)$ is a polynomial in $\mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(g(x)) \leq q(k-2)-\left(q^{2}-n\right) \leq n-k-1$. Moreover, by Eq. (2), we have

$$
\begin{aligned}
& \left(v_{1}^{q+1} f^{q}\left(a_{1}\right), \ldots, v_{s}^{q+1} f^{q}\left(a_{s}\right), f^{q}\left(a_{s+1}\right), \ldots, f^{q}\left(a_{n}\right), 0\right) \\
= & \left(u_{1} g\left(a_{1}\right), \ldots, u_{s} g\left(a_{s}\right), u_{s+1} g\left(a_{s+1}\right), \ldots, u_{n} g\left(a_{n}\right), 0\right) .
\end{aligned}
$$

According to the result (2) of Lemma 2, the vector

$$
\left(v_{1} f\left(a_{1}\right), \ldots, v_{s} f\left(a_{s}\right), f\left(a_{s+1}\right), \ldots, f\left(a_{n}\right), 0\right) \in \operatorname{Hull}_{H}(\mathcal{C})
$$

It deduces that $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right) \geq n+k-q^{2}-s-1$.
In summary, $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right)=n+k-q^{2}-s-1=l$. This completes the proof.

In particular, considering $k=q$, one more new classes of MDS codes with Hermitian hulls of more flexible dimensions can be obtained from the following way.

Theorem 7. Let $q=p^{m} \geq 3$, for any $q^{2}-q \leq n \leq q^{2}$, there exists an $[n+1, q]_{q^{2}}$ MDS code with $l$-dimensional Hermitian hull, where $0 \leq l \leq n+q-q^{2}$.

Proof. Denote $s=n+q-q^{2}-l$ and let other notations be the same as before. We now consider the Hermitian hull of the $q^{2}$-ary $[n+1, q]$ MDS code $\mathcal{C}=\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$.

For any codeword

$$
\boldsymbol{c}=\left(v_{1} f\left(a_{1}\right), \ldots, v_{s} f\left(a_{s}\right), f\left(a_{s+1}\right), \ldots, f\left(a_{n}\right), f_{q-1}\right) \in \operatorname{Hull}_{H}(\mathcal{C})
$$

by the result (2) of Lemma 2, there exists a polynomial $g(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(g(x)) \leq n-q$ such that

$$
\begin{align*}
& \left(v_{1}^{q+1} f^{q}\left(a_{1}\right), \ldots, v_{s}^{q+1} f^{q}\left(a_{s}\right), f^{q}\left(a_{s+1}\right), \ldots, f^{q}\left(a_{n}\right), f_{q-1}^{q}\right)  \tag{5}\\
= & \left(u_{1} g\left(a_{1}\right), \ldots, u_{s} g\left(a_{s}\right), u_{s+1} g\left(a_{s+1}\right), \ldots, u_{n} g\left(a_{n}\right),-g_{n-q}\right) .
\end{align*}
$$

On one hand, from the last $n-s+1$ coordinates of Eq. (5) and Eq. (2), we have

$$
f^{q}\left(a_{i}\right)=u_{i} g\left(a_{i}\right)=-\prod_{j=n+1}^{q^{2}}\left(a_{i}-a_{j}\right) g\left(a_{i}\right), s+1 \leq i \leq n, \text { and } f_{q-1}^{q}=-g_{n-q}
$$

Note that

$$
\begin{aligned}
& \operatorname{deg}\left(f^{q}(x)\right) \leq q(q-1)=q^{2}-q \\
& \operatorname{deg}\left(\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) g(x)\right) \leq\left(q^{2}-n\right)+(n-q)=q^{2}-q .
\end{aligned}
$$

Since $f_{q-1}^{q}=-g_{n-q}$, then $\operatorname{deg}\left(f^{q}(x)\right)=q^{2}-q$, i.e., $f_{q-1} \neq 0$ is equivalent to $\operatorname{deg}\left(\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) g(x)\right)=$ $q^{2}-q$, i.e., $g_{n-q} \neq 0$. It follows that

$$
\operatorname{deg}\left(f^{q}(x)+\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) g(x)\right) \leq q^{2}-q-1 \leq n-s-l-1 \leq n-s-1
$$

for any possible $f(x)$. Hence, we can conclude that $f^{q}(x)=-\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) g(x)$ from the fact $n-s-1<n-s$. Moreover, $\prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) \mid f^{q}(x)$.

On the other hand, taking a similar manner to Theorem6, $f(x)$ can be written as

$$
f(x)=h(x) \prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) \prod_{i=1}^{s}\left(x-a_{i}\right)
$$

where $h(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(h(x)) \leq n+q-q^{2}-s-1$. It deduces that $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right) \leq n+q-q^{2}-s$.
Conversely, let $f(x)$ be a polynomial of form $h(x) \prod_{j=n+1}^{q^{2}}\left(x-a_{j}\right) \prod_{i=1}^{s}\left(x-a_{i}\right)$, where $h(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(h(x)) \leq n+q-q^{2}-s-1$. Take $g(x)=-\prod_{i=n+1}^{q^{2}}\left(x-a_{i}\right)^{-1} f^{q}(x) \in \mathbb{F}_{q^{2}}[x]$, then $g(x)$ is a polynomial in $\mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(g(x)) \leq q(q-1)-\left(q^{2}-n\right)=n-q$. Moreover, by Eq. (2), we have

$$
\begin{aligned}
& \left(v_{1}^{q+1} f^{q}\left(a_{1}\right), \ldots, v_{s}^{q+1} f^{q}\left(a_{s}\right), f^{q}\left(a_{s+1}\right), \ldots, f^{q}\left(a_{n}\right), f_{q-1}^{q}\right) \\
= & \left(u_{1} g\left(a_{1}\right), \ldots, u_{s} g\left(a_{s}\right), u_{s+1} g\left(a_{s+1}\right), \ldots, u_{n} g\left(a_{n}\right),-g_{n-q}\right) .
\end{aligned}
$$

According to the result (2) of Lemma 2 the vector

$$
\left(v_{1} f\left(a_{1}\right), \ldots, v_{s} f\left(a_{s}\right), f\left(a_{s+1}\right), \ldots, f\left(a_{n}\right), f_{q-1}\right) \in \operatorname{Hull}_{H}(\mathcal{C})
$$

It deduces that $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right) \geq n+q-q^{2}-s$.
In summary, $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right)=n+q-q^{2}-s=l$. This completes the proof.
Remark 1. (1) Discuss the range of lengths in Theorems 5, 6 and 7 as follows:

- For Theorem 55: It follows from $1 \leq k \leq q-1$ that $n \geq q^{2}-k \geq q^{2}-q+1$. Since $q \geq 3$, we have $n>q+1$;
- For Theorem 6: It follows from $1 \leq k \leq q-1$ that $n \geq q^{2}-k+1 \geq q^{2}-q+2$. Since $q \geq 3$, we have $n>q+1$;
- For Theorem 7; It follows from $q \geq 3$ that $n \geq q^{2}-q>q+1$.

Hence, lengths in Theorems 5, 6 and 7 are at least $q^{2}-q+1$ and must be greater than $q+1$.
(2) Note that the following facts:

- the dimension $l$ of the Hermitian hull can take $n+k-q^{2}$ in Theorem 55:
- the length $n+1$ of the MDS codes can take $q^{2}+1$ in Theorem 6;
- the dimension $k$ of the MDS codes can and only can take $q$ in Theorem 7.

Hence, MDS codes constructed by Theorems 5, 6 and 7 will not be exactly the same as each other.

### 3.2. Construction $B$ for MDS codes with Hermitian hulls of arbitrary dimensions

In this subsection, we construct a new classes of MDS codes with Hermitian hulls of arbitrary dimensions. To this end, we need the following Lemma.

Lemma 8. Let $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$ be an extended $G R S$ code associated to $\mathbf{a}$ and $\mathbf{v}$ over $\mathbb{F}_{q^{2}}$. Let $a_{i}$ and $v_{i}$ be the $i$-th elements of $\mathbf{a}$ and $\mathbf{v}$, respectively, where $1 \leq i \leq n$. Assume that there exists a monic polynomial $h(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(h(x)) \leq q+n-(q+1) k$ such that

$$
\begin{equation*}
\lambda u_{i} h\left(a_{i}\right)=v_{i}^{q+1}, 1 \leq i \leq n \tag{6}
\end{equation*}
$$

where $\lambda \in \mathbb{F}_{q^{2}}^{*}$. If $\operatorname{deg}(h(x))=q+n-(q+1) k$ and $\lambda=-1$ do not hold at the same time, then $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$ is a $(k-1)$-dimensional Hermitian hull MDS code of length $n$.

Proof. Multiplying both sides of Eq. (6) by $f^{q}\left(a_{i}\right)$, we have

$$
\begin{equation*}
\lambda u_{i} h\left(a_{i}\right) f^{q}\left(a_{i}\right)=v_{i}^{q+1} f^{q}\left(a_{i}\right), 1 \leq i \leq n \tag{7}
\end{equation*}
$$

where $\lambda \in \mathbb{F}_{q^{2}}^{*}$.
Let $g(x)=\lambda h(x) f^{q}(x)$ and substitute $g\left(a_{i}\right)$ into Eq. (7), then

$$
\operatorname{deg}(g(x)) \leq q+n-(q+1) k+q(k-1)=n-k
$$

and

$$
u_{i} g\left(a_{i}\right)=v_{i}^{q+1} f^{q}\left(a_{i}\right), 1 \leq i \leq n
$$

We discuss the relationship between $f_{k-1}$ and $g_{n-k}$ in two ways.

- Case 1: When $\operatorname{deg}(f(x))<k-1$, we have $\operatorname{deg}(f(x)) \leq k-2$ and $f_{k-1}=0$. Note that, in this case,

$$
\operatorname{deg}(g(x)) \leq q+n-(q+1) k+q(k-2)=n-k-q<n-k
$$

which implies $g_{n-k}=0$. Hence, $f_{k-1}^{q}=0=-g_{n-k}$ and the fact has nothing to do with $\lambda$.

- Case 2: When $\operatorname{deg}(f(x))=k-1$, we have $f_{k-1} \neq 0$ and

$$
\operatorname{deg}(g(x)) \leq q+n-(q+1) k+q(k-1)=n-k .
$$

Note that $\operatorname{deg}(g(x))=n-k$ if and only if $\operatorname{deg}(h(x))=q+n-(q+1) k$.

- Subcase 2.1: When $\operatorname{deg}(h(x))<q+n-(q+1) k$, we still have $\operatorname{deg}(g(x))<n-k$ and $g_{n-k}=0$. Then $f_{k-1}^{q} \neq-g_{n-k}$ for the fact $f_{k-1} \neq 0$.
- Subcase 2.2: When $\operatorname{deg}(h(x))=q+n-(q+1) k$, i.e., $\operatorname{deg}(g(x))=n-k$ and $g_{n-k} \neq 0$. Note that $h(x)$ is a monic polynomial over $\mathbb{F}_{q^{2}}$, then the coefficients of the highest degree of $g(x)$ and $\lambda h(x) f^{q}(x)$ are $g_{n-k}$ and $\lambda f_{k-1}^{q}$, respectively. Since $g(x)=\lambda h(x) f^{q}(x)$, we have $f_{k-1}^{q}=\lambda g_{n-k}$. Therefore, it is easy to see that $f_{k-1}^{q}=-g_{n-k}$ if and only if $\lambda=-1$.

In summary, when $\operatorname{deg}(h(x))=q+n-(q+1) k$ and $\lambda=-1$ do not hold at the same time, $f_{k-1}^{q}=-g_{n-k}$ follows if and only if $\operatorname{deg}(f(x))<k-1$, if and only if $f_{k-1}=-g_{n-k}=0$. Hence, by the result (2) of Lemma 2, for any polynomial $f(x) \in \mathbb{F}_{q^{2}}[x]$, we have

$$
\mathbf{c}=\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right), 0\right) \in \operatorname{Hull}_{H}\left(\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)\right), \text { where } \operatorname{deg}(f(x)) \leq k-2
$$

and
$\mathbf{c}=\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right), f_{k-1}\right) \notin \operatorname{Hull}_{H}\left(\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)\right)$, where $\operatorname{deg}(f(x))=k-1$.

Clearly, we can conclude that $\operatorname{dim}\left(\operatorname{Hull}_{H}\left(\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)\right)\right)=k-1$. The desired result follows.
Moreover, a criterion for an extended GRS code being Hermitian self-orthogonal can be derived. We write it in the following corollary and one can finish the proof in a similar manner to Lemma 8 ,

Corollary 9. Let $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$ be an extended GRS code associated to $\mathbf{a}$ and $\mathbf{v}$ over $\mathbb{F}_{q^{2}}$. Let $a_{i}$ and $v_{i}$ be the $i$-th elements of $\mathbf{a}$ and $\mathbf{v}$, respectively, where $1 \leq i \leq n$. If there exists a monic polynomial $h(x) \in \mathbb{F}_{q^{2}}[x]$ with $\operatorname{deg}(h(x))=q+n-(q+1) k$ such that

$$
\begin{equation*}
-u_{i} h\left(a_{i}\right)=v_{i}^{q+1}, 1 \leq i \leq n, \tag{8}
\end{equation*}
$$

then $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$ is a Hermitian self-orthogonal MDS code of length $n$.
As we all know, if $\operatorname{Hull}_{H}(\mathcal{C})=\mathcal{C}$, i.e., $\operatorname{dim}(\mathcal{C})=\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right)=k$, then $\mathcal{C}$ is called a Hermitian self-orthogonal code. For a unified description, we introduce the definition of Hermitian almost selforthogonal codes as follows.

Definition 10. For a linear code $\mathcal{C}$ of dimension $k$, if $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right)=k-1$, then we call $\mathcal{C}$ a Hermitian almost self-orthogonal code. In particular, if $\mathcal{C}$ is Hermitian almost self-orthogonal and $M D S$, then we call $\mathcal{C}$ a Hermitian almost self-orthogonal MDS code.

Now, we consider a kind of decompositions of $\mathbb{F}_{q^{2}}$, which was first introduced in [13]. Assume $n=m(q-1), 1 \leq m \leq q$. Set

$$
\begin{align*}
I_{t} & =\left\{x \cdot \omega^{t-1}: x \in \mathbb{F}_{q}^{*}\right\}  \tag{9}\\
& =\left\{\omega^{t-1}, \omega^{q+1+t-1}, \cdots, \omega^{(q+1) i+t-1}, \cdots, \omega^{(q+1)(q-2)+t-1}\right\}
\end{align*}
$$

where $1 \leq t \leq q$. Denote

$$
\mathcal{I}=\bigcup_{t=1}^{m} I_{t}=\left\{a_{1}, a_{2}, \cdots, a_{n}\right\}
$$

where $a_{i}=\omega^{(q+1) i+t-1}$ for $1 \leq i \leq n$. From [13], $u_{i}$ can be written as certain concrete form for different $q$. We rephrase the result in Lemma 11 ,

Lemma 11. ([13]) Let notations be the same as before.
(1) If $q$ is even, then

$$
\begin{align*}
u_{i} & =(q-1)^{-1} a_{i}^{2-q} \omega^{(t-1)(m-2)+\frac{m(m+1)}{2}-(q+1) i_{0}} \\
& =(q-1)^{-1} \omega^{(t-1)(m-q)+\frac{m(m+1)}{2}-(q+1)\left[(q-2) i+i_{0}\right]} \tag{10}
\end{align*}
$$

for some integer $i_{0}$.
(2) If $q$ is odd, then

$$
\begin{align*}
u_{i} & =(q-1)^{-1} a_{i}^{2-q} \omega^{(t-1)(m-2)+\frac{(m-q-1)(m-1)}{2}-(q+1) i_{0}} \\
& =(q-1)^{-1} \omega^{(t-1)(m-q)+\frac{(m-q-1)(m-1)}{2}-(q+1)\left[(q-2) i+i_{0}\right]} \tag{11}
\end{align*}
$$

for some integer $i_{0}$.
Theorem 12. Let $q$ be a prime power and $2 \leq m \leq q$. Then there exists an $[m(q-1)+1, k]_{q^{2}}$ Hermitian almost self-orthogonal $M D S$ code, where $1 \leq k \leq m-1$.

Proof. Let notations be the same as before. We continue our proof in the following two cases.
Case 1: When $q$ is even, set $\lambda=(q-1) \omega^{-\frac{m(m+1)}{2}}$. By Eq. (10), $u_{i}$ can be written as

$$
\begin{aligned}
u_{i} & =(q-1)^{-1} \omega^{(t-1)(m-q)+\frac{m(m+1)}{2}-(q+1)\left[(q-2) i+i_{0}\right]} \\
& =\lambda^{-1} \omega^{(t-1)(m-q)-(q+1)\left[(q-2) i+i_{0}\right]}
\end{aligned}
$$

for some integer $i_{0}$. Let $h(x)=x^{q-m}$, then according to the form of $a_{i}$, we have

$$
\begin{aligned}
\lambda u_{i} h\left(a_{i}\right) & =\omega^{(t-1)(m-q)-(q+1)\left[(q-2) i+i_{0}\right]} \cdot \omega^{(t-1)(q-m)+(q+1)(q-m) i} \\
& =\omega^{-(q+1)\left[(m-2) i+i_{0}\right]}
\end{aligned}
$$

It is obviously that $\lambda u_{i} h\left(a_{i}\right) \in \mathbb{F}_{q}^{*}$ and there exists $v_{i} \in \mathbb{F}_{q^{2}}^{*}$ such that $v_{i}^{q+1}=\lambda u_{i} h\left(a_{i}\right)$ for each $1 \leq i \leq n$. Set $\mathbf{v}=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$.

Note that $\left\lfloor\frac{n+m}{q+1}\right\rfloor=\left\lfloor\frac{m(q+1)-m}{q+1}\right\rfloor=m-1 \geq 1$ for $2 \leq m \leq q$. Therefore, for $1 \leq k \leq\left\lfloor\frac{n+m}{q+1}\right\rfloor=m-1$, we can consider the extended GRS code $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$ associated to $\mathbf{a}$ and $\mathbf{v}$, where $\mathbf{a}$ and $\mathbf{v}$ are defined as above. Furthermore, it is easy to see that

$$
\operatorname{deg}(h(x))=q-m \leq q+n-(q+1) k
$$

and the equation holds if and only if $k=m-\frac{m}{q+1}$, which contradicts to the fact that $k$ is an integer. It follows that $\operatorname{deg}(h(x))<q+n-(q+1) k$, thus, according to Lemma 8 and Definition 10, $\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$ is an $[m(q-1)+1, k]_{q^{2}}$ Hermitian almost self-orthogonal MDS code.

Case 2: When $q$ is odd, set $\lambda=(q-1) \omega^{\frac{(q-m+1)(m-1)}{2}}$. Taking a similar manner to Case 1 above, let $h(x)=x^{q-m}$ again, then we can also obtain an $[m(q-1)+1, k]_{q^{2}}$ Hermitian almost self-orthogonal MDS code, where $1 \leq k \leq m-1$.

Combining Case 1 and Case 2, the desired result follows.
Remark 2. One of the most critical steps in the proof of Theorem 12 is to show that $\operatorname{deg}(h(x))<$ $q+n-(q+1) k$. In fact, by Lemma88, we can also prove $\lambda \neq-1$. And a simple calculation shows that $q \geq 3$ is required if we finish the proof of Theorem 12 in this way. Therefore, this specifical example shows us that the new method introduced in Lemma 8 is flexible for using and that it may produce a small difference when we use it from different views.

Applying Corollary 4 we can draw the following result.

Corollary 13. Let $q$ be a prime power and $2 \leq m \leq q$. Then for $1 \leq k \leq m-1$, there exists an $[m(q-1)+1, k]_{q^{2}}$ MDS code with l-dimensional Hermitian hull, where $0 \leq l \leq k-1$.

Note that all MDS codes in Corollary 13 have type $[m(q-1)+1, k]_{q^{2}}$. As far as we know, for a similar form of length, there are some constructions of Hermitian self-orthogonal MDS codes in previous studies. We list them in Table 2 According to Lemma 3. MDS codes with l-dimensional Hermitian hull for each $0 \leq l \leq k$ can be derived from these Hermitian self-orthogonal MDS codes. Therefore, it is necessary to compare these results with our construction. In Remark 3, we discuss their differences in detail.

Table 2: Some known Hermitian self-orthogonal MDS codes

| Class | $q$ | $n$ | $k$ | Ref. |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $q \geq 3$ | $\begin{gathered} n=r \frac{q^{2}-1}{2 s+1}+1 \\ (2 s+1) \mid(q+1), 1 \leq r \leq 2 s+1 \end{gathered}$ | $1 \leq k \leq(s+1) \frac{q+1}{2 s+1}-1$ | [10] |
| 2 | $q \geq 3$ | $\begin{gathered} n=r \frac{q^{2}-1}{2 s}+1, \\ 2 s \mid(q+1), 2 \leq r \leq 2 s \end{gathered}$ | $1 \leq k \leq(s+1) \frac{q+1}{2 s}-1$ | [10] |
| 3 | $q \geq 3$ | $\begin{gathered} n=t n^{\prime}+1,1 \leq t \leq \frac{q-1}{n_{1}}, \\ n_{1}=\frac{n^{\prime}}{\operatorname{gcd}\left(n^{\prime}, q+1\right)}, n^{\prime} \mid\left(q^{2}-1\right) \end{gathered}$ | $1 \leq k \leq\left\lfloor\frac{n+q}{q+1}\right\rfloor$ | [11] |
| 4 | $\begin{aligned} 0 & \leq r \leq q \\ q+1 & \equiv r(\bmod 2 r) \end{aligned}$ | $n=r(q-1)+1$ | $k \leq \frac{q-1+r}{2}$ | [18] |
| 5 | $t \geq 1, q \equiv-1(\bmod 2 t+1)$ | $\begin{gathered} n=1+\frac{r\left(q^{2}-1\right)}{2 t+1} \\ 0 \leq r \leq 2 t+1, \operatorname{gcd}(r, q)=1 \end{gathered}$ | $k \leq \frac{t+1}{2 t+1} \cdot q-\frac{t}{2 t+1}$ | [17] |
| 6 | Arbitrary | $\begin{gathered} n=\frac{q^{2}-1}{t}+1, \\ t=2 w+1, t \mid(q+1) \end{gathered}$ | $2 \leq k \leq \frac{t+1}{2 t}(q-1)$ | [16] |

Remark 3. From the perspective of length and dimension, we can easily illustrate that the MDS codes with Hermitian hulls of arbitrary dimensions constructed by Theorem 12 are new in general. Recall that our $M D S$ codes have type $[m(q-1)+1, k]_{q^{2}}$, where $2 \leq m \leq q$ and $1 \leq k \leq m-1$.
(1) Comparisons of lengths:

- For Class 3:

Taking $n^{\prime}=q-1$, if $q$ is even, then $\operatorname{gcd}(q-1, q+1)=1$ and $n_{1}=q-1$; if $q$ is odd, then $\operatorname{gcd}(q-1, q+1)=2$ and $n_{1}=\frac{q-1}{2}$. Hence, Class 3 can generate $[t(q-1)+1, k]_{q^{2}}$ Hermitian self-orthogonal MDS codes, where $q$ is even and $t=1 ;[t(q-1)+1, k]_{q^{2}}$ Hermitian self-orthogonal MDS codes, where $q$ is odd and $1 \leq t \leq 2$. However, $2 \leq m \leq q$, thus, most lengths in our construction are new.

- For Classes 4 and 5:

Note that more conditions are required for $q$ or $r$, which confirms our assertion.

- For Class 6:

Rewrite $n=\frac{q+1}{t} \cdot(q-1)+1$, where $t=2 w+1$ is odd and $t \mid(q+1)$. Clearly, the value set
of $\frac{q+1}{t}$ is narrow relative to the value set of $m$ in our construction. In fact, all even $t$ and $t \nmid(q+1)$ are not suitable to Class 6.

## (2) Comparisons of dimensions:

- For Classes 1 and 2:

Take $m=\frac{r(q+1)}{2 s+1}$ and $\frac{r(q+1)}{2 s}$ in Class 1 and Class 2, respectively. Then the corresponding ranges of the dimension in our construction should be $1 \leq k \leq \frac{r(q+1)}{2 s+1}-1$ and $\frac{r(q+1)}{2 s}-1$. Considering the ranges of r, almost half of our MDS codes can take larger dimensions in general.

Of course, the restrictions of $2(s+1) \mid(q+1)$ and $2 s \mid(q+1)$ may also limit the lengths of Classes 1 and 2 in some cases. For example, MDS codes of lengths 25, 41, 57 and 73 over $\mathbb{F}_{3^{4}}$ can not be constructed from Class 1 and all possible $M D S$ codes over $\mathbb{F}_{q}$ with even $q$ can not be constructed from Class 2.

Remark 4. The length of MDS codes in Corollary 4 has form $m(q-1)+1$. On one hand, since $2 \leq m \leq q$, then the length of MDS codes are always $m(q-1)+1 \geq 2(q-1)+1 \geq q+1$. And only when $q=2$, the equation holds. On the other hand, $m(q-1)+1 \leq q(q-1)+1=q^{2}-q+1$. According to Remark 1, the length of MDS codes from Theorems 5, 6 and 7 is at least $q^{2}-q+1$. Hence, almost all MDS codes from Corollary 4 are new with respect to MDS codes from Theorems 5, 6 and 7,

Example 14. For some different $q$, we give some concrete examples of Hermitian almost self-orthogonal MDS codes from Theorem 12, For clarity, we list them in Table 3.

Table 3: Some examples of new Hermitian almost self-orthogonal MDS codes from Theorem 12

| $q$ | $m$ | $n$ | $k$ | $q$ | $m$ | $n$ | $k$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 3 | 10 | $[1,2]$ | 4 | 4 | 13 | $[1,3]$ |
| 8 | 5 | 36 | $[1,4]$ | 8 | 6 | 43 | $[1,5]$ |
| 8 | 7 | 50 | $[1,6]$ | 8 | 8 | 57 | $[1,7]$ |
| 25 | 17 | 409 | $[1,16]$ | 25 | 18 | 433 | $[1,17]$ |
| 25 | 21 | 505 | $[1,20]$ | 25 | 22 | 529 | $[1,21]$ |
| 27 | 11 | 287 | $[1,10]$ | 27 | 12 | 313 | $[1,11]$ |
| 27 | 23 | 599 | $[1,22]$ | 27 | 24 | 625 | $[1,23]$ |

### 3.3. Construction $C$ for MDS codes with Euclidean hulls of flexible dimensions

In this construction, we consider the Euclidean inner product over $\mathbb{F}_{q}$. Note that $(q-1) \nmid 2$ for any $q>3$, then there exists $v_{i} \in \mathbb{F}_{q}^{*}$ such that $v_{i}^{2} \neq 1$. By similar ways to Theorems [5] 6] and (7) the dimension of the Euclidean hull of corresponding MDS codes can be determined. We present these results in Theorem 15 and omit the proof.

Theorem 15. Let $q=p^{m}>3$. The following statements hold.
(1) For $1 \leq k \leq\left\lfloor\frac{q}{2}\right\rfloor$, if $q-k \leq n \leq q$, then there exists an $[n, k]_{q} M D S$ code with $l$-dimensional Euclidean hull, where $0 \leq l \leq n+k-q$.
(2) For $1 \leq k \leq\left\lfloor\frac{q}{2}\right\rfloor$, if $q-k+1 \leq n \leq q$, then there exists an $[n+1, k]_{q} M D S$ code with $l$-dimensional Euclidean hull, where $0 \leq l \leq n+k-q-1$.
(3) For odd $q$ and any $\frac{q-1}{2} \leq n \leq q$, then there exists an $\left[n+1, \frac{q+1}{2}\right]_{q} M D S$ code with $l$-dimensional Euclidean hull, where $0 \leq l \leq n-\frac{q-1}{2}$.

Besides, another two classes of MDS codes with Euclidean hulls of flexible dimensions can be exactly construted. To this end, we need to introduce a fact. Denote $\mathbb{F}_{q}=\left\{a_{1}, a_{2}, \cdots, a_{n}, a_{n+1}, \cdots, a_{q}\right\}$. Clearly, these $q$ elements are distinct. With similar reasoning as Eq. (2), we can further express $u_{i}$ as

$$
\begin{equation*}
u_{i}=\prod_{1 \leq j \leq n, j \neq i}=\left(a_{i}-a_{j}\right)^{-1}=-\prod_{j=n+1}^{q}\left(a_{i}-a_{j}\right) \tag{12}
\end{equation*}
$$

over $\mathbb{F}_{q}$.
Theorem 16. Let $q=p^{m}$ be a prime power. The following statements hold.
(1) For $1 \leq k \leq\left\lfloor\frac{q}{2}\right\rfloor$, if $\left\lceil\frac{q}{2}\right\rceil \leq n \leq \min \left\{q-k,\left\lceil\frac{q}{2}\right\rceil+k\right\}$, then there exists an $[n, k] M D S$ code with $l$-dimensional Euclidean hull, where $0 \leq l \leq n-\left\lceil\frac{q}{2}\right\rceil$.
(2) For $1 \leq k \leq\left\lfloor\frac{q+1}{2}\right\rfloor$, if $\left\lceil\frac{q+1}{2}\right\rceil \leq n \leq \min \left\{q-k+1,\left\lceil\frac{q+1}{2}\right\rceil+k-1\right\}$, then there exists an $[n+1, k]$ $M D S$ code with $l$-dimensional Euclidean hull, where $0 \leq l \leq n-\left\lceil\frac{q+1}{2}\right\rceil$ and $l \neq n-\frac{q+1}{2}$.
(3) If $q$ is odd, for $1 \leq k \leq \frac{q+1}{2}$, if $\frac{q+1}{2} \leq n \leq \min \left\{q-k+1, \frac{q+1}{2}+k-1\right\}$, then there exists an $[n+1, k]$ MDS code with $\left(n-\frac{q+1}{2}+1\right)$-dimensional Euclidean hull.

Proof. (1) Let notations be the same as before. Denote $s=q-n-k+l$. Take $\mathbf{a}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and $\mathbf{v}=\left(v_{1}, v_{2}, \ldots, v_{s}, v_{s+1}, \ldots, v_{n}\right)$, where $v_{i}=\prod_{j=n+1}^{n+s}\left(a_{i}-a_{j}\right)$ for any $1 \leq i \leq n$. We now consider the Euclidean hull of the $q$-ary $[n, k] \operatorname{MDS}$ code $\mathcal{C}=\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v})$.

For any codeword

$$
\boldsymbol{c}=\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right)\right) \in \operatorname{Hull}_{E}(\mathcal{C})
$$

by the result (1) of Lemma [1, there exists a polynomial $g(x) \in \mathbb{F}_{q}[x]$ with $\operatorname{deg}(g(x)) \leq n-k-1$ such that

$$
\left(v_{1}^{2} f\left(a_{1}\right), v_{2}^{2} f\left(a_{2}\right), \ldots, v_{n}^{2} f\left(a_{n}\right)\right)=\left(u_{1} g\left(a_{1}\right), u_{2} g\left(a_{2}\right), \ldots, u_{n} g\left(a_{n}\right)\right)
$$

which implies that $v_{i}^{2} f\left(a_{i}\right)=u_{i} g\left(a_{i}\right)$ for $1 \leq i \leq n$. Since

$$
v_{i}^{2} f\left(a_{i}\right)=\prod_{j=n+1}^{n+s}\left(a_{i}-a_{j}\right)^{2} f\left(a_{i}\right)
$$

and by Eq. (12),

$$
u_{i} g\left(a_{i}\right)=-\prod_{j=n+1}^{q}\left(a_{i}-a_{j}\right) g\left(a_{i}\right)
$$

it follows that

$$
\prod_{j=n+1}^{n+s}\left(a_{i}-a_{j}\right) f\left(a_{i}\right)=-\prod_{j=n+s+1}^{q}\left(a_{i}-a_{j}\right) g\left(a_{i}\right)
$$

for $1 \leq i \leq n$. Note that

$$
\begin{aligned}
& \operatorname{deg}\left(\prod_{j=n+1}^{n+s}\left(x-a_{j}\right) f(x)\right) \leq s+k-1=q-n+l-1 \leq n-1 \\
& \operatorname{deg}\left(-\prod_{j=n+s+1}^{q}\left(x-a_{j}\right) g(x)\right) \leq(q-n-s)+(n-k-1)=q-s-k-1=n-l-1 \leq n-1
\end{aligned}
$$

Hence, we can derive that

$$
\prod_{j=n+1}^{n+s}\left(x-a_{j}\right) f(x)=-\prod_{j=n+s+1}^{q}\left(x-a_{j}\right) g(x)
$$

from the fact $n-1<n$. Moreover, since $\left(\prod_{j=n+1}^{n+s}\left(x-a_{j}\right), \prod_{j=n+s+1}^{q}\left(x-a_{j}\right)\right)=1$, we have

$$
\prod_{j=n+s+1}^{q}\left(x-a_{j}\right) \mid f(x)
$$

Therefore, $f(x)$ can be written as

$$
f(x)=h(x) \prod_{j=n+s+1}^{q}\left(x-a_{j}\right)
$$

where $\operatorname{deg}(h(x)) \leq k-1-(q-n-s)=n-q+k+s-1$. It deduces that $\operatorname{dim}\left(\operatorname{Hull}_{E}(\mathcal{C})\right) \leq n-q+k+s$.
Conversely, let $f(x)$ be a polynomial of form $h(x) \prod_{j=n+s+1}^{q}\left(x-a_{j}\right)$, where $h(x) \in \mathbb{F}_{q}[x]$ and $\operatorname{deg}(h(x)) \leq n-q+k+s-1$. Take

$$
g(x)=-\prod_{j=n+s+1}^{q}\left(x-a_{j}\right)^{-1} \prod_{j=n+1}^{n+s}\left(x-a_{j}\right) f(x)=-h(x) \prod_{j=n+1}^{n+s}\left(x-a_{j}\right)
$$

then $g(x)$ is a polynomial in $\mathbb{F}_{q}[x]$ with $\operatorname{deg}(g(x)) \leq(n-q+k+s-1)+s \leq n-k-1$. Moreover, by Eq. (12), we have

$$
\left(v_{1}^{2} f\left(a_{1}\right), v_{2}^{2} f\left(a_{2}\right), \ldots, v_{n}^{2} f\left(a_{n}\right)\right)=\left(u_{1} g\left(a_{1}\right), u_{2} g\left(a_{2}\right), \ldots, u_{n} g\left(a_{n}\right)\right)
$$

According to the result (1) of Lemma the vector

$$
\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right)\right) \in \operatorname{Hull}_{E}(\mathcal{C})
$$

It deduces that $\operatorname{dim}\left(\operatorname{Hull}_{E}(\mathcal{C})\right) \geq n-q+k+s$.
In summary, we have $\operatorname{dim}\left(\operatorname{Hull}_{E}(\mathcal{C})\right)=n-q+k+s=l$. This completes the proof.
(2) Denote $s=q-n-k+l+1$ and let other notations be the same as before. We now consider the Euclidean hull of the $q$-ary $[n+1, k]$ MDS code $\mathcal{C}=\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$.

For any codeword

$$
\boldsymbol{c}=\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right), f_{k-1}\right) \in \operatorname{Hull}_{E}(\mathcal{C})
$$

by the result (2) of Lemman there exists a polynomial $g(x) \in \mathbb{F}_{q}[x]$ with $\operatorname{deg}(g(x)) \leq n-k$ such that

$$
\left(v_{1}^{2} f\left(a_{1}\right), v_{1}^{2} f\left(a_{2}\right), \ldots, v_{n}^{2} f\left(a_{n}\right), f_{k-1}\right)=\left(u_{1} g\left(a_{1}\right), u_{2} g\left(a_{2}\right), \ldots, u_{n} g\left(a_{n}\right),-g_{n-k}\right)
$$

Similar to the proof of (1) above, we have

$$
\prod_{j=n+1}^{n+s}\left(a_{i}-a_{j}\right) f\left(a_{i}\right)=-\prod_{j=n+s+1}^{q}\left(a_{i}-a_{j}\right) g\left(a_{i}\right)
$$

for $1 \leq i \leq n$. Note that

$$
\begin{aligned}
& \operatorname{deg}\left(\prod_{j=n+1}^{n+s}\left(x-a_{j}\right) f(x)\right) \leq s+k-1=q-n+l \leq n-1 \\
& \operatorname{deg}\left(-\prod_{j=n+s+1}^{q}\left(x-a_{j}\right) g(x)\right) \leq(q-n-s)+(n-k)=q-s-k=n-l-1 \leq n-1
\end{aligned}
$$

Hence, we can derive that

$$
\prod_{j=n+1}^{n+s}\left(x-a_{j}\right) f(x)=-\prod_{j=n+s+1}^{q}\left(x-a_{j}\right) g(x)
$$

and $\prod_{j=n+s+1}^{q}\left(x-a_{j}\right) \mid f(x)$ for the same reasoning as (1) above. Now, we determine the value of $f_{k-1}$. If $f_{k-1} \neq 0$, then $s+k-1=(q-n-s)+(n-k)$, which contradicts to $l \neq n-\frac{q+1}{2}$. Thus $f_{k-1}=0$ and $\operatorname{deg}(f(x)) \leq k-2$. Therefore, $f(x)$ can be written as

$$
f(x)=h(x) \prod_{j=n+s+1}^{q}\left(x-a_{j}\right)
$$

where $\operatorname{deg}(h(x)) \leq k-2-(q-n-s)=n-q+k+s-2$. It deduces that $\operatorname{dim}\left(\operatorname{Hull}_{E}(\mathcal{C})\right) \leq n-q+k+s-1$.
Conversely, let $f(x)$ be a polynomial of form $h(x) \prod_{j=n+s+1}^{q}\left(x-a_{j}\right)$, where $h(x) \in \mathbb{F}_{q}[x]$ and $\operatorname{deg}(h(x)) \leq n-q+k+s-2$. Take

$$
g(x)=-\prod_{j=n+s+1}^{q}\left(x-a_{j}\right)^{-1} \prod_{j=n+1}^{n+s}\left(x-a_{j}\right) f(x)=-h(x) \prod_{j=n+1}^{n+s}\left(x-a_{j}\right)
$$

then $g(x)$ is a polynomial in $\mathbb{F}_{q}[x]$ with $\operatorname{deg}(g(x)) \leq(n-q+k+s-2)+s \leq n-k-1$. Moreover, by Eq. (12), we have

$$
\left(v_{1}^{2} f\left(a_{1}\right), v_{2}^{2} f\left(a_{2}\right), \ldots, v_{n}^{2} f\left(a_{n}\right), 0\right)=\left(u_{1} g\left(a_{1}\right), u_{2} g\left(a_{2}\right), \ldots, u_{n} g\left(a_{n}\right), 0\right)
$$

According to the result (2) of Lemma 1 the vector

$$
\left(v_{1} f\left(a_{1}\right), v_{2} f\left(a_{2}\right), \ldots, v_{n} f\left(a_{n}\right), 0\right) \in \operatorname{Hull}_{E}(\mathcal{C})
$$

It deduces that $\operatorname{dim}\left(\operatorname{Hull}_{E}(\mathcal{C})\right) \geq n-q+k+s-1$.
In summary, we have $\operatorname{dim}\left(\operatorname{Hull}_{E}(\mathcal{C})\right)=n-q+k+s-1=l$. This completes the proof.
(3) Let notations be the same as before. We now consider the $q$-ary $\left[l+\frac{q+1}{2}+1, k\right] \operatorname{MDS}$ code $\mathcal{C}=\operatorname{GRS}_{k}(\mathbf{a}, \mathbf{v}, \infty)$, where $q$ is odd. From the proof of (2) above, we can easily conclude that the dimension of the Euclidean hull of $\mathcal{C}$ is $l+1$. In other words, there exists an $[n+1, k]_{q}$ MDS code with $\left(n-\frac{q+1}{2}+1\right)$-dimensional Euclidean hull. This completes the proof.

Taking $l=k$ and $l=1$ in Theorems 15 and 16, we can obtain some Euclidean self-orthogonal and one-dimensional Euclidean hull MDS codes. In particular, for MDS $\operatorname{codes} \mathcal{C}$ with dimension $k=1$, it is easy to see that $\mathcal{C}$ is Euclidean self-orthogonal if and only if $\operatorname{dim}\left(\operatorname{Hull}_{E}(\mathcal{C})\right)=1$. Hence, we only consider the MDS codes with dimension $k \geq 2$ in the following examples.

Example 17. Taking $l=k$ in Theorem 15, we have the following Euclidean self-orthogonal MDS codes, which can also be obtained from [23].
(1) $[q, k]_{q} M D S$ codes, where $1 \leq k \leq\left\lfloor\frac{q}{2}\right\rfloor$ and $q>3$;
(2) $\left[q+1, \frac{q+1}{2}\right]_{q} M D S$ codes, where $q>3$ is odd.

Taking $l=k$ in Theorem 16, we have the following Euclidean self-orthogonal MDS codes. As far as we know, they are new.
(3) $\left[\left\lceil\frac{q}{2}\right\rceil+k, k\right]_{q}$ MDS codes, where $1 \leq k \leq\left\lfloor\frac{q}{4}\right\rfloor$ and $q \geq 4$;
(4) $\left[k+\frac{q+1}{2}, k\right]_{q} M D S$ codes, where $1 \leq k \leq\left\lfloor\frac{q+3}{4}\right\rfloor$ and $q \geq 3$ is odd.

Example 18. Let $q=p^{m}$ be a prime power, then the dimension of Euclidean hulls of the following MDS codes is at most 1 .
(1) $[q+1-k, k]_{q} M D S$ codes, where $2 \leq k \leq\left\lfloor\frac{q}{2}\right\rfloor$ and $q \geq 4$;
(2) $[q+3-k, k]_{q} M D S$ codes, where $2 \leq k \leq\left\lfloor\frac{q}{2}\right\rfloor$ and $q \geq 4$;
(3) $\left[\left\lceil\frac{q}{2}\right\rceil+1, k\right]_{q}$ MDS codes, where $2 \leq k \leq\left\lfloor\frac{q}{2}\right\rfloor-1$ and $q \geq 6$;
(4) $\left[\left\lceil\frac{q+1}{2}\right\rceil+2, k\right]_{q}$ MDS codes, where $2 \leq k \leq\left\lfloor\frac{q+1}{2}\right\rfloor-1$ and $q \geq 6$ is even;
(5) $\left[\frac{q+3}{2}, k\right]_{q}$ MDS codes, where $2 \leq k \leq \frac{q+1}{2}$ and $q \geq 3$ is odd;

Moreover, since $\operatorname{dim}\left(\operatorname{Hull}_{E}(\mathcal{C})\right)=\operatorname{dim}\left(\operatorname{Hull}_{E}\left(\mathcal{C}^{\perp_{E}}\right)\right)$, we can deduce that all Euclidean dual codes of MDS codes listed in $(1)-(5)$ are also one-dimensional Euclidean hull MDS codes.

## 4. Application to EAQECCs

### 4.1. Quantum codes

In this subsection, we introduce some basic notions about quantum codes. Let $\mathbb{C}$ be the complex field and $\mathbb{C}^{q}$ be the $q$-dimensional Hilbert space over $\mathbb{C}$. A qubit is actually a non-zero vector of $\mathbb{C}^{q}$. Denote a basis of $\mathbb{C}^{q}$ by $\left\{|a\rangle: a \in \mathbb{F}_{q}\right\}$, then a qubit $|v\rangle$ can be written as

$$
|v\rangle=\sum_{a \in \mathbb{F}_{q}} v_{a}|a\rangle
$$

where $v_{a} \in \mathbb{C}$.

Let $(\mathbb{C})^{\otimes n}\left(\cong \mathbb{C}^{q^{n}}\right)$ be the $q^{n}$-dimensional Hilbert space over $\mathbb{C}$. Similar to classical linear codes, a $q$ ary quantum code $\mathcal{Q}$ of length $n$ is a subspace of $\mathbb{C}^{q^{n}}$. Let $\left\{|\mathbf{a}\rangle=\left|a_{1}\right\rangle \otimes \cdots \otimes\left|a_{n}\right\rangle:\left(a_{1}, a_{2}, \ldots, a_{n}\right) \in \mathbb{F}_{q}^{n}\right\}$ be a basis of $\mathbb{C}^{q^{n}}$. Similarly, an $n$-qubit is a joint state of $n$ qubits in $\mathbb{C}^{q^{n}}$ and can be written as

$$
|\mathbf{v}\rangle=\sum_{\mathbf{a} \in \mathbb{F}_{q}} v_{\mathbf{a}}|\mathbf{a}\rangle
$$

where $v_{\mathbf{a}} \in \mathbb{C}$. The Hermitian inner product of any two $n$-qubits $|\mathbf{u}\rangle=\sum_{\mathbf{a} \in \mathbb{F}_{q}} u_{\mathbf{a}}|\mathbf{a}\rangle$ and $|\mathbf{v}\rangle=$ $\sum_{\mathbf{a} \in \mathbb{F}_{q}} v_{\mathbf{a}}|\mathbf{a}\rangle$ is defined by

$$
\langle\mathbf{u} \mid \mathbf{v}\rangle=\sum_{\mathbf{a} \in \mathbb{F}_{q}^{n}} u_{\mathbf{a}} \overline{v_{\mathbf{a}}} \in \mathcal{C}
$$

where $\overline{v_{\mathbf{a}}}$ is the complex conjugate of $v_{\mathbf{a}} .|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$ are said to be orthogonal if $\langle\mathbf{u} \mid \mathbf{v}\rangle=0$.
Let $\zeta_{p}$ be a complex primitive $p$-th root of unity. The actions (rules) of $X(\mathbf{a})$ and $Z(\mathbf{b})$ on $|\mathbf{v}\rangle \in$ $\mathbb{C}^{q^{n}}\left(\mathbf{v} \in \mathbb{F}_{q}^{n}\right)$ are depicted as

$$
X(\mathbf{a})|\mathbf{v}\rangle=|\mathbf{v}+\mathbf{a}\rangle \quad \text { and } \quad Z(\mathbf{b})|\mathbf{v}\rangle=\zeta_{p}^{\operatorname{tr}\left(\langle\mathbf{v}, \mathbf{b}\rangle_{E}\right)} \mid \mathbf{v}
$$

respectively, where $\operatorname{tr}(\cdot)$ is the trace function from $\mathbb{F}_{q}$ to $\mathbb{F}_{p}$. In a quantum system, the quantum errors are some unitary operators. Denote the error group by $G_{n}$, then

$$
G_{n}=\left\{\zeta_{p}^{t} X(\mathbf{a}) Z(\mathbf{b}): \mathbf{a}, \mathbf{b} \in \mathbb{F}_{q}^{n}, t \in \mathbb{F}_{p}\right\}
$$

For any error $E=\zeta_{p}^{t} X(\mathbf{a}) Z(\mathbf{b}) \in G_{n}$, we define the quantum weight of $E$ as

$$
w t_{Q}(E)=\sharp\left\{i:\left(a_{i}, b_{i}\right) \neq(0,0)\right\},
$$

where $\sharp$ denotes the number of elements in the set. A quantum code $\mathcal{Q}$ with dimension $K \geq 2$ is said to detect $d-1$ quantum errors $(d \geq 1)$, if for any pair $|\mathbf{u}\rangle$ and $|\mathbf{v}\rangle$ in $\mathcal{Q}$ with $\langle\mathbf{u} \mid \mathbf{v}\rangle=0$ and any $E \in G_{n}$ with $w t_{Q}(E) \leq d-1$, we have $\langle\mathbf{u}| E|\mathbf{v}\rangle=0$. For a $q$-ary quantum code of length $n$, dimension $K$ and minimum distance $d$, we usually denote it by $((n, K, d))_{q}$ or $[[n, k, d]]_{q}$, where $k=\log _{q} K$.

Let $S$ be an abelian subgroup of $G_{n}$. Then the quantum stabilizer codes $C(S)$ can be defined by

$$
C(S)=\{|\phi\rangle: E|\phi\rangle=|\phi\rangle, \forall E \in S\}
$$

which are analogues of classical additive codes. As we mentioned before, from classical linear codes satisfying certain orthogonality, by the method (namely, CSS construction) introduced by Calderbank et al. 4] and Steane [32], one can obtain quantum stabilizer codes. However, this method fails when $S$ is a non-abelian. By extending $S$ to be a new abelian subgroup in a larger error group and assuming that both sender and receiver shared the pre-existing entangled bits, Burn et al. [3] introduced EAQECCs. In this case, EAQECCs can be derived from any classical linear codes.

### 4.2. New EAQECCs and MDS EAQECCs of length $n>q+1$

Based on the known method of constructing EAQECCs, we use MDS codes with Hermitian hulls of flexible dimensions obtained in Section 3 to obtain new EAQECCs and MDS EAQECCs. Like classical linear codes, there exists a trade-off between the parameters $n, k, d$ and $c$ of an EAQECC, called quantum Singleton bound.

Lemma 19. (Quantum Singleton bound [1]) Let $\mathcal{Q}$ be an $[[n, k, d ; c]]_{q} E A Q E C C$. If $2 d \leq n+2$, then

$$
k \leq n+c-2(d-1)
$$

Remark 5. (1) An EAQECC for which equality holds in this bound, i.e., $2 d \leq n+2$ and $k=$ $n+c-2(d-1)$, is called an MDS EAQECC.
(2) It is well known that if a classical linear code $\mathcal{C}$ is $M D S$ and $d \leq \frac{n+2}{2}$, then the EAQECC constructed by it is an MDS EAQECC.

For a matrix $M=\left(m_{i j}\right)$ over $\mathbb{F}_{q^{2}}$, we denote the conjugate transpose of $M$ by $M^{\dagger}=\left(m_{j i}^{q}\right)$. In practice, the explict method of constructing EAQECCs from a linear code with certain dimensional Hermitian hull was established by Galindo et al. in 12]. We rephrase the important result in the following.

Lemma 20. ([12]) Let $H$ be a parity check matrix of a $q^{2}$-ary $[n, k, d]$ linear code. Then there exists an $[[n, 2 k-n+c, d ; c]]_{q} E A Q E C C \mathcal{Q}$, where $c=\operatorname{rank}\left(H H^{\dagger}\right)$ is the required number of maximally entangled states.

Generally speaking, the determination of the number of $c$ is difficult. Guenda et al. [15] proposed a relationship between $c$ and $\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right)$ as follows, which simplifies the problem of calculating the number of $c$.

Lemma 21. ([15])Let $\mathcal{C}$ be a $q^{2}$-ary $[n, k, d]$ linear code and $H$ be a parity check matrix of $\mathcal{C}$. Then

$$
\begin{aligned}
\operatorname{rank}\left(H H^{\dagger}\right) & =n-k-\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right) \\
& =n-k-\operatorname{dim}\left(\operatorname{Hull}_{H}\left(\mathcal{C}^{\perp_{H}}\right)\right)
\end{aligned}
$$

Since the Hermitian dual code of an $[n, k, n-k+1]_{q^{2}}$ MDS code is an $[n, n-k, k+1]_{q^{2}}$ MDS code and $\left(\mathcal{C}^{\perp_{H}}\right)^{\perp_{H}}=\mathcal{C}$, by the result (2) of Remark 5 and Lemma 20, we can obtain the following result immediately.

Lemma 22. Let $H$ be a parity check matrix of a $q^{2}$-ary $[n, k, d]$ linear code and $l=\operatorname{dim}\left(\operatorname{Hull}_{H}(\mathcal{C})\right)$. If $k \leq\left\lfloor\frac{n}{2}\right\rfloor$, then there exists an $[[n, k-l, n-k+1 ; n-k-l]]_{q} E A Q E C C \mathcal{Q}$ and an $[[n, n-k-l, k+1 ; k-l]]_{q}$ $M D S E A Q E C C \mathcal{Q}^{\prime}$.

Now, according to Lemma 22, we can present our new constructions of $q$-ary EAQECCs and MDS EAQECCs.

Theorem 23. Let $q=p^{m} \geq 3$ be a prime power. The following statements hold.
(1) For $0 \leq k \leq q-1$, if $q^{2}-k \leq n \leq q^{2}$, then there exists an $[[n, k-l, n-k+1 ; n-k-l]]_{q}$ $E A Q E C C \mathcal{Q}$ and an $[[n, n-k-l, k+1 ; k-l]]_{q} M D S E A Q E C C \mathcal{Q}^{\prime}$, where $0 \leq l \leq n+k-q^{2}$.
(2) For $0 \leq k \leq q-1$, if $q^{2}-k+1 \leq n \leq q^{2}$, then there exists an $[[n+1, k-l, n-k+2 ; n+1-k-l]]_{q}$ $E A Q E C C \mathcal{Q}$ and an $[[n+1, n+1-k-l, k+1 ; k-l]]_{q} M D S E A Q E C C \mathcal{Q}^{\prime}$, where $0 \leq l \leq n+k-q^{2}-1$.
(3) If $q^{2}-q \leq n \leq q^{2}$, then there exists an $[[n+1, q-l, n-q+2 ; n+1-q-l]]_{q} E A Q E C C \mathcal{Q}$ and an $[[n+1, n+1-q-l, q+1 ; q-l]]_{q}$ MDS EAQECC $\mathcal{Q}^{\prime}$, where $0 \leq l \leq n+q-q^{2}$.

Example 24. The results (1), (2) and (3) of Theorem 23 can be used to obtain many EAQECCs and MDS EAQECCs. According to Remark [1, the length of these new MDS EAQECCs are always greater than $q+1$ and do not completely cover each other. As an intuitive example, we list some new MDS EAQECCs in Table 4.

Table 4: Some EAQMDS codes construted by Theorem 23 over $\mathbb{F}_{9}$

| $k$ | $l$ | EAQMDS codes | Ref. | $k$ | $l$ | EAQMDS codes | Ref. |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 3 | $[[76,65,9 ; 5]]_{9}$ | Theorem [23(1) | 8 | 3 | [ 7, 66, 9; 5] ${ }_{9}$ | Theorem 23(1) |
| 8 | 3 | $[[78,67,9 ; 5]]_{9}$ | Theorem 23(1) | 8 | 3 | $[[79,68,9 ; 5]]_{9}$ | Theorem 23(1) |
| 8 | 3 | $[[80,69,9 ; 5]]_{9}$ | Theorem 23(1) | 8 | 5 | $[[78,65,9 ; 3]]_{9}$ | Theorem 23(1) |
| 8 | 5 | $[[79,66,9 ; 3]]$ | Theorem [23(1) | 8 | 5 | [[80, 67, 9; 3] | Theorem [23(1) |
| 8 | 3 | [[81, 70, 9; 5] $]_{9}$ | Theorem 23(2) | 8 | 5 | $[[81,68,9 ; 3]]_{9}$ | Theorem 23(2) |
| 9 | 3 | ${ }^{[876,64,10 ; 6]]_{9}}$ | Theorem [23(3) | 9 | 3 | ${ }_{[[77,65,10 ; ~ 6] ~}^{9} 9$ | Theorem 23(3) |
| 9 | 3 | [[78, 66, 10; 6]] ${ }_{9}$ | Theorem [23(3) | 9 | 3 | $[[79,67,10 ; 6]]_{9}$ | Theorem 23(3) |
| 9 | 3 | [[80, 68, 10; 6] $]_{9}$ | Theorem 23(3) | 9 | 3 | $[[79,65,10 ; 4]]_{9}$ | Theorem 23(3) |
| 9 | 5 | $[[78,64,10 ; 4]]_{9}$ | Theorem [23(3) | 9 | 5 | [[79, 65, 10; 4] $]_{9}$ | Theorem [23(3) |
| 9 | 5 | $[[80,66,10 ; 4]]_{9}$ | Theorem 23(3) | 9 | 5 | $[[81,67,10 ; 4]]_{9}$ | Theorem 23(3) |

Theorem 25. Let $q=p^{m} \geq 3$ be a prime power and $n=m(q-1)$, where $2 \leq m \leq q$. Then for any $1 \leq k \leq m-1$, there exists an $[[n+1, k-l, n-k+2 ; n+1-k-l]]_{q}$ EAQECC $\mathcal{Q}$ and an $[[n+1, n+1-k-l, k+1 ; k-l]]_{q}$ MDS EAQECC $\mathcal{Q}^{\prime}$, where $0 \leq l \leq k-1$.

Example 26. According to Remark 圂, the EAQECCs and MDS EAQECCs of length $n>q+1$ constructed by Theorem 25 are new. According to Remark 4, most of these EAQECCs and MDS EAQECCs can not be obtained by Theorem [23. We list some of them in Table 圆,

## 5. Summary and concluding remarks

The main contribution of this paper is to construct several new classes of MDS codes and totally determine their Euclidean hulls (See Theorems 15 and 16) or Hermitian hulls (See Theorems [5, 6 and Corollary [13). For Hermitian cases, four new classes of $q$-ary EAQECCs and four new classes of $q$-ary MDS EAQECCs of length $n>q+1$ are further obtained (See Theorems 23) and 25). And for Euclidean cases, some new Euclidean self-orthogonal and one-dimensional Euclidean hull MDS codes are given as examples.

In particular, for convenience, MDS codes with $(k-1)$-dimensional Hermitian hull are called Hermitian almost self-orthogonal MDS codes in this paper. Some new criterions for extended GRS

Table 5: Some EAQMDS codes construted by Theorem 25 for some $q$

| $q$ | $m$ | $k$ | $l$ | EAQMDS codes | $q$ | $m$ | $k$ | $l$ | EAQMDS codes |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | 5 | 4 | 2 | $[[36,30,5 ; 2]]_{8}$ | 8 | 6 | 4 | 2 | $[[43,37,5 ; 2]]_{8}$ |
| 8 | 7 | 6 | 4 | $[[50,40,7 ; 2]]_{8}$ | 8 | 8 | 6 | 4 | $[[57,47,7 ; 2]]_{8}$ |
| 9 | 5 | 4 | 2 | $[[41,35,5 ; 2]]_{9}$ | 9 | 7 | 4 | 2 | $[[57,51,5 ; 2]]_{9}$ |
| 9 | 9 | 8 | 3 | $[[73,62,9 ; 5]]_{9}$ | 9 | 9 | 8 | 5 | $[[73,60,9 ; 3]]_{9}$ |
| 16 | 7 | 6 | 2 | $[[106,98,7 ; 4]]_{16}$ | 16 | 8 | 6 | 2 | $[[121,113,7 ; 4]]_{16}$ |
| 16 | 13 | 6 | 2 | $[[196,188,7 ; 4]]_{16}$ | 16 | 14 | 6 | 2 | $[[211,203,7 ; 4]]_{16}$ |
| 25 | 14 | 4 | 2 | $[[337,331,5 ; 2]]_{25}$ | 25 | 18 | 4 | 2 | $[[433,427,5 ; 2]]_{25}$ |
| 25 | 15 | 4 | 2 | $[[361,355,5 ; 2]]_{25}$ | 25 | 19 | 4 | 2 | $[[457,451,5 ; 2]]_{25}$ |
| 25 | 16 | 6 | 4 | $[[385,375,7 ; 2]]_{25}$ | 25 | 20 | 6 | 4 | $[[481,471,7 ; 2]]_{25}$ |
| 25 | 17 | 6 | 4 | $[[409,399,7 ; 2]]_{25}$ | 25 | 21 | 6 | 4 | $[[505,495,7 ; 2]]_{25}$ |

codes being Hermitian almost self-orthogonal MDS codes and Hermitian self-orthogonal MDS codes are presented (See Lemma 8 and Corollary (9). For future research, it would be interesting to construct more Hermitian (almost) self-orthogonal MDS codes and MDS EAQECCs of length $n>q+1$.

## Acknowledgments

This research was supported by the National Natural Science Foundation of China (Nos.U21A20428 and 12171134).

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[^0]:    ${ }^{\star}$ This research was supported by the National Natural Science Foundation of China (No.U21A20428 and 12171134).

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