

# Stationary Distribution Convergence of the Offered Waiting Processes for $GI/GI/1 + GI$ Queues in Heavy Traffic

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## Abstract

A result of Ward and Glynn [31] asserts that the sequence of scaled offered waiting time processes of the  $GI/GI/1 + GI$  queue converges weakly to a reflected Ornstein-Uhlenbeck process (ROU) in the positive real line, as the traffic intensity approaches one. As a consequence, the stationary distribution of a ROU process, which is a truncated normal, should approximate the scaled stationary distribution of the offered waiting time in a  $GI/GI/1 + GI$  queue; however, no such result has been proved. We prove the aforementioned convergence, and the convergence of the moments, in heavy traffic, thus resolving a question left open in [31]. In comparison to Kingman's classical result [16] showing that an exponential distribution approximates the scaled stationary offered waiting time distribution in a  $GI/GI/1$  queue in heavy traffic, our result confirms that the addition of customer abandonment has a non-trivial effect on the queue stationary behavior.

*Keywords: Customer Abandonment; Heavy Traffic; Stationary Distribution Convergence*

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# 1 Introduction

There is a long history of studying queueing systems with abandonments, beginning with the early work of Palm [24] in the late 1930's. One common objective is to understand the long time asymptotic behavior of such systems, which is governed by the stationary distribution (assuming existence and uniqueness). However, except in special cases, the models of interest are too complex to analyze directly. Instead, some researchers have examined the heavy traffic limits of these systems, and developed analytically tractable diffusion approximations (through process-level convergence results). The question often left open is whether or not the stationary distribution of the diffusion does indeed arise as the heavy traffic limit of the sequence of stationary distributions of the relevant queueing system with abandonment. Our objective in this paper is to answer this question for one of the most fundamental models, the single server queue, operating under the FIFO service discipline, with generally distributed patience times; that is, the  $GI/GI/1 + GI$  queue.

Our asymptotic analysis relies heavily on past work that has developed heavy traffic approximations for the  $GI/GI/1 + GI$  queue using the offered waiting time process. The offered waiting time process, introduced in [2], tracks the amount of time an infinitely patient customer must wait for service. Its heavy traffic limit when the abandonment distribution is left unscaled is a reflected Ornstein-Uhlenbeck process (see Proposition 1, which re-states a result from [31] in the setting in this paper), and its heavy traffic limit when the abandonment distribution is scaled through its hazard rate is a reflected nonlinear diffusion (see [27]). However, those results are not enough to conclude that the stationary distribution of the offered waiting time process converges, which is the key to establishing the limit behavior of the stationary abandonment probability and mean queue-length. Those limits were conjectured in [27], and shown through simulation to provide good approximations. However, the proof of those limits was left as an open question. In this paper, we focus on the case when the abandonment distribution is left unscaled (i.e., the heavy traffic scaling studied in [31]).

When the system loading factor is less than one, since the  $GI/GI/1 + GI$  queue is dominated by the  $GI/GI/1$  queue, the much earlier results of [16, 17] for the  $GI/GI/1$  queue can be used to establish the weak convergence of the sequence of stationary distributions for the  $GI/GI/1 + GI$  queue in heavy traffic. The difficulty arises because, in contrast to the  $GI/GI/1$  queue, the  $GI/GI/1 + GI$  queue can have a stationary distribution when the system loading factor equals or exceeds 1 (see [2]). *Our main contribution in this paper is to establish both the convergence of the sequence of stationary distributions and the sequence of stationary moments of the offered waiting time in heavy traffic, irrespective of whether the system loading factor approaches 1 from above or below.*

An informed reader would recall [13] and [6], which establish the validity of the heavy traffic stationary approximation for a generalized Jackson network, without customer abandonment. The proof of the former paper [13] relied on certain exponential integrability

assumptions on the primitives of the network and as a result a form of exponential ergodicity was established. The latter paper [6] provided an alternative proof assuming the weaker square integrability conditions that are commonly used in heavy traffic analysis. Our analysis is inspired by the methodology developed in the latter work [6]. However, the main difficulty in extending their methodology to the current model is that the known regulator mapping under customer abandonment is only *locally* Lipschitz (that is, the Lipschitz constant depends on time parameter) whereas the proofs of both [13] and [6] critically rely upon the global Lipschitz property of the associated regulator mapping. More precisely, the approaches in [13] and [6] make use of the global Lipschitz continuity property of an associated regulator (Skorokhod) mapping to help convert the given moment bound of primitives (the inter-arrival and service times) to the bound of the key performance measures (the waiting time or queue-length processes). Such a property is not available for the model under study.

In connection to the aforementioned technical issue, the studies of [33, 34] extend the works of [13] and [6] to a wider range of stochastic processing networks, e.g., the multiclass queueing network and the resource-sharing network, by relaxing the requirement of the aforementioned Lipschitz continuity. However, their study in [33, 34] deals with networks that have heavy-traffic limits satisfying the *linear* dynamic complementarity problem, i.e., the state process depends on the “free process” (and the regulating process as well) linearly. Therefore, their results do not apply to our  $GI/GI/1 + GI$  model directly, as the resulting heavy-traffic limit is a reflected Ornstein-Uhlenbeck process and the state process of this limit (i.e.,  $V(\cdot)$  in (3) below) depends on the “free process” (i.e., the drifted Brownian motion in (3)) in a nonlinear manner. Nevertheless, their hydrodynamics approach is adapted to establish a key property, i.e., the uniform moment stability of the offered waiting time process (see Section 4), in our paper.

A closely related paper is that of Huang and Gurvich [14], which studies the Poisson arrival case (i.e.,  $M/GI/1 + GI$  queue) and shows the associated Brownian model is accurate uniformly over a family of patience distributions and universally in the heavy-traffic regime. For instance, Section EC.3.1 therein corresponds to the critically loaded regime, as considered in this paper. Their approach is based on the generator comparison methodology, and owing to the Poisson arrivals, it is enough to consider a one-dimensional process with a simple generator, whereas with general arrival processes, one needs to consider a two-dimensional process (tracking, e.g., the residual arrival times) and correspondingly more complicated generator.

In comparison to results for many-server queues, the process-level convergence result for the  $GI/GI/N + GI$  queue in the quality-and-efficiency-driven regime was established in [19] when the hazard rate is not scaled, and in [26], under the assumption of exponential service times, when the hazard rate is scaled. Neither paper establishes the convergence of the stationary distributions. That convergence is shown under the assumption that the abandonment distribution is exponential and the service time distribution is phase type in

[10]. The question remains open for the fully general  $GI/GI/N + GI$  setting. There has been some progress made in this direction in [15], which establishes the convergence of the sequence of stationary distributions under fluid scaling in the aforementioned fully general  $GI/GI/N + GI$  setting.

The remainder of this paper is organized as follows. We conclude this section with a summary of our mathematical notation. In Section 2, we set up the model assumptions and recall the known process-level convergence results for the  $GI/GI/1 + GI$  queue. In Section 3, we state our main result, that gives the convergence of the stationary distribution of the offered waiting time process, and its moments. To prove the main result, we first obtain bounds on the moments of the scaled state process that are uniform in the heavy traffic scaling parameter ( $n$ ) in Section 4. The proofs of lemmas in this section are technically involved, and are delayed to Section 6. Lastly, we use uniform moment bounds established in Section 4 to prove our main result in Section 5.

**Notation and Terminology.** Use the symbol “ $\equiv$ ” to stand for equality by definition. The set of positive integers is denoted by  $\mathbb{N}$  and denote  $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ . Let  $\mathbb{R}$  represent the real numbers  $(-\infty, \infty)$  and  $\mathbb{R}_+$  the non-negative real line  $[0, \infty)$ . For  $x, y \in \mathbb{R}$ ,  $x \vee y \equiv \max\{x, y\}$  and  $x \wedge y \equiv \min\{x, y\}$ . The function  $e(\cdot)$  represents the identity map; that is,  $e(t) = t$  for all  $t \in \mathbb{R}_+$ . For  $t \in \mathbb{R}_+$  and a real-valued function  $f$ , define  $\|f\|_t \equiv \sup_{0 \leq s \leq t} |f(s)|$ . Let  $D(\mathbb{R}) \equiv D(\mathbb{R}_+, \mathbb{R})$  be the space of right-continuous functions  $f : \mathbb{R}_+ \rightarrow \mathbb{R}$  with left limits, endowed with the Skorokhod  $J_1$ -topology (see, for example, [3]). Lastly, the symbol “ $\Rightarrow$ ” stands for the weak convergence; we make this explicit for stochastic processes in  $D(\mathbb{R})$ , otherwise, it is used for weak convergence for a sequence of random variables.

## 2 The Model and Known Results

The  $GI/GI/1+GI$  model having FIFO service is built from three independent i.i.d. sequences of nonnegative random variables  $\{u_i, i \geq 2\}$ ,  $\{v_i, i \geq 1\}$ ,  $\{d_i, i \geq 1\}$ , that are representing inter-arrival times, service times, and patience times, respectively, and are defined on a common probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . At time 0, the previous arrival to the system occurred at time  $t_0^n < 0$ , so that  $|t_0^n|$  represents the time elapsed since the last arrival in the  $n$ -th system. We let  $u_1$  be the random variable representing the remaining time conditioned on  $|t_0^n|$  time units having passed; that is,

$$\mathbb{P}(u_1 > x) = \mathbb{P}(u_2 > x | u_2 > |t_0^n|).$$

We let  $F(\cdot)$  represent the distribution function associated with the patience time  $d_1$  and, consistent with [2], assume  $F(\cdot)$  is proper. The system primitives are assumed to satisfy:

(A1) For some  $p \in (2, \infty)$ ,  $\mathbb{E}[u_2^p + v_2^p] < \infty$ , and  $F'(0) \in (0, \infty)$ .

We consider a sequence of systems indexed by  $n \geq 1$  in which the arrival rates become large and service times small. By convention, we use superscript  $n$  for any processes or quantities associated with the  $n$ -th system. The arrival and service rates in the  $n$ -th system are  $\lambda^n$  and  $\mu^n$  and satisfy the following heavy traffic assumption:

$$(A2) \quad \lambda^n \equiv n\lambda, \lim_{n \rightarrow \infty} \frac{\mu^n}{n} = \lambda \in (0, \infty) \text{ and } \lim_{n \rightarrow \infty} \sqrt{n} \left( \lambda - \frac{\mu^n}{n} \right) = \theta \in \mathbb{R}.$$

The  $i$ -th arrival to the  $n$ -th system occurs at time

$$t_i^n \equiv \sum_{j=1}^i \frac{u_j}{\lambda^n} \text{ for } i \in \mathbb{N},$$

and has service time

$$v_i^n \equiv \frac{v_i}{\mu^n} \text{ for } i \in \mathbb{N},$$

and abandons without receiving service if processing does not begin by time  $t_i^n + d_i$ .

### The Offered Waiting Time Process

The offered waiting time process, first given in [2], tracks the amount of time an incoming customer at time  $t$  has to wait for service. That time depends only upon the service times of the non-abandoning customers already waiting in the queue, that is, those waiting customers whose patience time upon arrival exceeds their waiting time. For  $t \geq 0$ , the *offered waiting time* process having initial state  $V^n(0)$  has the evolution equation

$$V^n(t) = V^n(0) + \sum_{j=1}^{A^n(t)} v_j^n \mathbf{1}_{[V^n(t_j^n -) < d_j]} - \int_0^t \mathbf{1}_{[V^n(s) > 0]} ds \geq 0, \quad (1)$$

where

$$A^n(t) \equiv \max\{i \in \mathbb{N}_0 : t_i^n \leq t\} \quad (2)$$

is a delayed renewal process when  $|t_0^n| > 0$  and is a regular (non-delayed) renewal process when  $t_0^n = 0$ . The quantity  $V^n(t)$  can also be interpreted as the time needed to empty the system from time  $t$  onwards if there are no arrivals after time  $t$ , and hence it is also known as the workload at time  $t$ . The initial state  $V^n(0)$  is 0 if no job is in service and otherwise represents the total workload of all jobs that arrived prior to time 0 and that will not abandon before their service begins.

### Reflected Ornstein-Uhlenbeck Approximation.

We consider the one-dimensional reflected Ornstein-Uhlenbeck process  $V \equiv \{V(t)\}_{t \geq 0}$

$$\begin{aligned} V(t) &= V(0) + \sigma W(t) + \frac{\theta}{\lambda} t - F'(0) \int_0^t V(s) ds + L(t) \geq 0 \\ \text{subject to: } L &\text{ is non-decreasing, has } L(0) = 0 \text{ and } \int_0^\infty V(s) dL(s) = 0, \end{aligned} \quad (3)$$

where  $\{W(t) : t \geq 0\}$  denotes a one-dimensional standard Brownian motion, and the infinitesimal variance parameter is

$$\sigma^2 \equiv \lambda^{-1}(\text{var}(u_2) + \text{var}(v_2)).$$

Given  $\mathcal{F}_0$ -measurable initial condition  $V(0) \geq 0$ , the strong existence and pathwise uniqueness of solution  $(V, L)$  to the stochastic differential equation (3) hold for the data  $(V(0), W)$ , i.e., the solution is adapted to  $(\mathcal{F}_t^W \vee \mathcal{F}_0)_{t \geq 0}$  (see, e.g., [35]).

The following weak convergence result is a simple modification of Theorem 1(a) of Ward and Glynn [31] and we provide its proof in the Appendix for the sake of completeness.

**Proposition 1.** *Assuming  $\sqrt{n}V^n(0) \Rightarrow V(0)$  as  $n \rightarrow \infty$ , we have*

$$\sqrt{n}V^n \Rightarrow V \text{ in } D(\mathbb{R}) \text{ as } n \rightarrow \infty. \quad (4)$$

### 3 The Stationary Distribution Existence and Convergence Results

The weak convergence (4) motivates approximating the scaled stationary distributions for  $V^n$ , and its moments, using the stationary distribution of  $V$ , and its moments. This requires establishing the limit interchange depicted graphically in Figure 1. When the limit  $n \rightarrow \infty$  is taken first, and the limit  $t \rightarrow \infty$  is taken second, the convergence is known. More specifically, the convergence as  $n \rightarrow \infty$  was established in (4), and Proposition 1 in [30] shows

$$V(t) \Rightarrow V(\infty) \text{ as } t \rightarrow \infty$$

for  $V(\infty)$  a random variable having density

$$f(x) = \frac{b^{-1}\phi(\frac{x-m}{b})}{1 - \Phi(-\frac{m}{b})} \text{ for } x \geq 0, \quad (5)$$

where  $m \equiv \theta/(\lambda F'(0))$ ,  $b \equiv \sigma/\sqrt{2F'(0)}$  and  $\phi(\cdot)$ ,  $\Phi(\cdot)$  denote the pdf and cdf of the standard normal distribution, respectively. In other words,  $V(\infty)$  is distributed as truncated normal with mean  $m$  and variance  $b^2$ , conditioned to be on  $\mathbb{R}_+$ .

In this Section, we state our main results, first that a unique stationary distribution exists for each system  $n$ , and second that the convergence in Figure 1 is valid when the limit is first taken as  $t \rightarrow \infty$  and second taken as  $n \rightarrow \infty$ . In order to do this, we first specify the relevant Markov process.

The offered waiting time process  $\{V^n(t) : t \geq 0\}$  alone is not Markovian due to the remaining arrival time. (In contrast, the offered waiting time process tracked only at

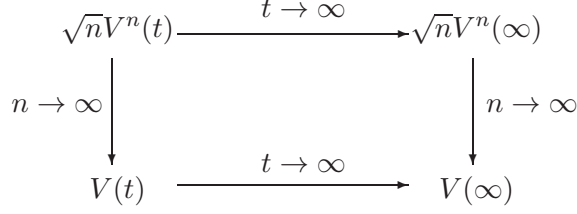


Figure 1: A graphical representation of the limit interchange.

customer arrival times is a Markov chain with state space  $\mathbb{R}_+$ ; see [2] and the recursive equations therein.) Defining the remaining arrival time (i.e., the forward recurrence time of the arrival process)

$$\tau^n(t) \equiv t_{j+1}^n - t \text{ for } t \in [t_j^n, t_{j+1}^n), j \in \mathbb{N}_0,$$

where  $\tau^n(0) = u_1$ , the vector-valued process

$$\mathbb{X}^n \equiv \{(\tau^n(t), V^n(t)) : t \geq 0\}$$

having state space  $\mathbb{S} \equiv \mathbb{R}_+ \times \mathbb{R}_+$  is strong Markov (cf. Problem 3.2 of Chapter X in [1] and also Section 2 in [12]).

Ensuring the existence of a stationary distribution requires the following technical condition on the interarrival time. More precisely, this assumption is used in the proof of Theorem 1 below to verify a petite set requirement that implies positive Harris recurrence (see Lemma 4), from which the existence of a unique stationary distribution follows immediately. Such an assumption has been frequently used in the literature; for example, Proposition 4.8 in [5], Lemma 3.7 in [21], and Theorem 3.1 in [9]. For  $x = (\tau, v) \in \mathbb{S}$ , define its norm  $|x|$  as  $|x| \equiv \tau + v$ . Define the norm  $|\mathbb{X}^n(t)|$  of  $\mathbb{X}^n(t)$  to be the sum of the offered waiting time and the remaining arrival time at  $t$ , that is,

$$|\mathbb{X}^n(t)| \equiv V^n(t) + \tau^n(t), t \geq 0.$$

(A3) The i.i.d. interarrival times  $\{u_i, i \geq 2\}$  are unbounded (that is,  $\mathbb{P}(u_2 \geq u) > 0$  for any  $u > 0$ ).

**Theorem 1.** (STATIONARY DISTRIBUTION EXISTENCE) *Assume (A1)–(A3). For each  $n \geq 1$ , there exists a unique stationary probability distribution for the Markov process  $\mathbb{X}^n$ .*

Now that we know the stationary distribution, denoted as  $\pi^n$ , exists for each fixed  $n$ , we can establish its convergence to the stationary distribution of the diffusion (3) given in (5). To state this result, we require the diffusion-scaled process:

$$\tilde{\mathbb{X}}^n \equiv (\tilde{\tau}^n, \tilde{V}^n), \text{ where } \tilde{\tau}^n(t) \equiv \sqrt{n}\tau^n(t), \tilde{V}^n(t) \equiv \sqrt{n}V^n(t).$$

Notice the time is not scaled in the process  $\tilde{V}^n$  because the arrival and service rate parameters are scaled instead (from (A2), both  $\lambda^n$  and  $\mu^n$  are order  $n$  quantities); therefore, scaling the state by  $\sqrt{n}$  produces the traditional diffusion scaling. Also, a motivation behind the scaling for the residual arrival time  $\tau^n$  comes from the way how the  $\sqrt{n}$  diffusion scaling affects the  $\tau^n$  under the arrival rate  $\lambda^n = n\lambda$  (cf. see (9) below).

**Theorem 2.** (STATIONARY CONVERGENCE) *Assume (A1)–(A3).*

(a) (DISTRIBUTION) *Denote by  $\pi_0^n$  the marginal distribution of  $\pi^n$  on the second coordinate of  $\tilde{\mathbb{X}}^n$ , i.e.,  $\pi_0^n(A) = \pi^n(\mathbb{R}_+ \times A)$  for  $A \in \mathcal{B}(\mathbb{R}_+)$ . Let  $\tilde{V}^n(\infty)$  be a random variable having distribution  $\pi_0^n$  and also  $V(\infty)$  a random variable having density (5). We have that  $\tilde{V}^n(\infty) \Rightarrow V(\infty)$  as  $n \rightarrow \infty$ .*

(b) (MOMENTS) *For any  $m \in (0, p-1)$ ,*

$$\mathbb{E}[(\tilde{V}^n(\infty))^m] \rightarrow \mathbb{E}[(V(\infty))^m] \text{ as } n \rightarrow \infty.$$

**Remark 1.** (QUEUE-LENGTH CONVERGENCE) *The queue-length process  $Q^n(t)$  represents the number of customers that are in system at time  $t > 0$ , either waiting or with the server. In contrast to  $V^n$ ,  $Q^n$  includes customers that will eventually abandon but have not yet done so. When the initial condition satisfies  $Q^n(0)/\sqrt{n} \Rightarrow Q(0)$  as  $n \rightarrow \infty$ , Theorem 3 in [31] shows that*

$$\frac{Q^n}{\sqrt{n}} \Rightarrow \lambda V \text{ in } D(\mathbb{R}) \text{ as } n \rightarrow \infty.$$

*In other words, recalling (4), a process-level version of Little’s law holds. This suggests that a version of Theorem 2 should hold for the queue-length process as well. However, proving this is more involved technically due to the need to track both customers in queue that will eventually receive service and customer in queue that will eventually abandon; see the “potential queue measure”, a measure-valued state descriptor in Section 2.2 of [15], and also see Figure 4 in [25] for a graphic depiction of that measure. This is the reason we leave that analysis as future research.*

## 4 Uniform Moment Estimates

The proofs of both Theorems 1 and 2 rely on a tightness result for the family of stationary distributions of  $\{\tilde{\mathbb{X}}^n\}_{n \geq 1}$ . The key to the desired tightness is to obtain *uniform* (in  $n$ ) bounds for the moments of the stationary distributions. Henceforth, we use the subscript  $x$  to denote the scaled Markov process  $\tilde{\mathbb{X}}^n$  has an initial state  $(\tilde{\tau}^n(0), \tilde{V}^n(0)) = (\tau, v) \equiv x \in \mathbb{S}$ . Our convention is to subscript any process that depends on the initial state  $x \in \mathbb{S}$  by  $x$ . Then,  $\tilde{V}_x^n$  has initial state  $V^n(0) = v/\sqrt{n}$  and is defined from the process  $V_x^n$  in (1) that uses the (delayed) renewal process  $A_x^n$  in (2). Recall the norm  $|x|$  of  $x = (\tau, v) \in \mathbb{S}$  is defined as  $|x| \equiv \tau + v$ .



**Proposition 2.** Assume (A1)–(A2). Let  $q \in [1, p)$ . There exists  $t_0 \in (0, \infty)$  such that for all  $t \geq t_0$ ,

$$\lim_{|x| \rightarrow \infty} \sup_n \frac{1}{|x|^q} \mathbb{E} \left[ |\tilde{\mathbb{X}}_x^n(t|x)|^q \right] = 0. \quad (6)$$

Before proving Proposition 2, we provide a roadmap about how it will be used in the proofs of the main results. Proposition 2 yields uniform (in the scaling parameter) moment bounds for the scaled Markov process of the  $GI/GI/1 + GI$  system. We use such uniform moment estimates, in conjunction with the Lyapunov function methods of Meyn and Tweedie [23] and Dai and Meyn [11], in order to obtain time uniform moment bounds (via weighted return time estimates) for the aforementioned scaled Markov process. Then, the sought-after moment bounds for the stationary distributions, uniform in the scaling parameter, readily follow (see (26) and (28) in the proof of Theorem 2) and this yields tightness of the collection of the stationary distributions and hence Theorem 2.

The crux in proving Proposition 2 lies in two versions of pathwise stability results (Lemmas 1 and 2), whose intuitive ideas are provided right after stating those results. The proof of Proposition 2 relies on a martingale representation of the offered waiting time process. We first provide this setup, and second give the proof.

#### 4.1 Martingale Representation and Diffusion Scaling

Define the  $\sigma$ -fields  $(\hat{\mathcal{F}}_i^n)_{i \geq 1}$  where

$$\hat{\mathcal{F}}_i^n \equiv \sigma((t_1^n, v_1^n, d_1), \dots, (t_i^n, v_i^n, d_i), t_{i+1}^n) \subseteq \mathcal{F},$$

and let  $\hat{\mathcal{F}}_0^n \equiv \sigma(t_1^n)$ . Notice that  $V_x^n(t_i^n -)$  is  $\hat{\mathcal{F}}_{i-1}^n$ -measurable and the patience time  $d_i$  of the  $i$ -th customer is independent of  $\hat{\mathcal{F}}_{i-1}^n$ . Hence,

$$\mathbb{P}[V_x^n(t_i^n -) \geq d_i | \hat{\mathcal{F}}_{i-1}^n] = F(V_x^n(t_i^n -)), \quad i = 1, 2, \dots, \quad (7)$$

holds almost surely, recalling that  $F$  is the distribution function of  $d_i$ . We then have a martingale with respect to the filtration  $(\hat{\mathcal{F}}_i^n)_{i \geq 1}$  given by

$$M_x^n(i) \equiv \sum_{j=1}^i \left( \mathbf{1}_{[V_x^n(t_j^n -) \geq d_j]} - \mathbb{E}(\mathbf{1}_{[V_x^n(t_j^n -) \geq d_j]} | \hat{\mathcal{F}}_{j-1}^n) \right).$$

Using (7), we also see that for all  $i \in \mathbb{N}$

$$M_x^n(i) = \sum_{j=1}^i \left[ \mathbf{1}_{[V_x^n(t_j^n -) \geq d_j]} - F(V_x^n(t_j^n -)) \right].$$

Next, define the following centered quantities

$$S^n(i) \equiv \frac{1}{n} \sum_{j=1}^i (v_j - 1), \quad S_{d,x}^n(i) \equiv \frac{1}{n} \sum_{j=1}^i (v_j - 1) \mathbf{1}_{\{V_x^n(t_j^n-) \geq d_j\}}.$$

From (1), algebra, and the above definitions, we have for  $t \geq 0$

$$\begin{aligned} V_x^n(t) &= \frac{v}{\sqrt{n}} - t - \frac{1}{\mu^n} \sum_{j=1}^{A_x^n(t)} F(V_x^n(t_j^n-)) + \int_0^t \mathbf{1}_{\{V_x^n(s)=0\}} ds \\ &\quad + \frac{n}{\mu^n} \left( \frac{A_x^n(t)}{n} + S^n(A_x^n(t)) - S_{d,x}^n(A_x^n(t)) - \frac{1}{n} M_x^n(A_x^n(t)) \right). \end{aligned} \quad (8)$$

With the initial state  $\tilde{\mathbb{X}}^n(0) = (\tau, v) \equiv x \in \mathbb{S}$ , define fluid-scaled and diffusion-scaled quantities to carry out our analysis. For  $t \geq 0$ , let

$$\bar{A}_x^n(t) \equiv \frac{A_x^n(t)}{n}, \quad \tilde{A}_x^n(t) \equiv \sqrt{n} \left( \frac{1}{n} A_x^n(t) - \lambda \left( t - \frac{\tau}{\sqrt{n}} \wedge t \right) \right), \quad \tilde{S}^n(t) \equiv \sqrt{n} S^n([nt]), \quad (9)$$

$$\tilde{S}_{d,x}^n(t) \equiv \sqrt{n} S_{d,x}^n([nt]), \quad \tilde{M}_x^n(t) \equiv \frac{1}{\sqrt{n}} M_x^n([nt]).$$

Algebra, (8) and substitution of the above scaled quantities into the scaled offered waiting time process

$$\tilde{V}_x^n(\cdot) \equiv \sqrt{n} V_x^n(\cdot),$$

shows that for  $t \geq 0$

$$\tilde{V}_x^n(t) = v + b^n t + \tilde{N}_x^n(t) - \frac{n}{\mu^n} \int_0^t \sqrt{n} F \left( \frac{1}{\sqrt{n}} \tilde{V}_x^n(s^-) \right) d\bar{A}_x^n(s) + \tilde{I}_x^n(t), \quad (10)$$

where

$$\begin{aligned} b^n &\equiv \left( \frac{n}{\mu^n} \right) \sqrt{n} \left( \frac{\lambda^n}{n} - \frac{\mu^n}{n} \right) \\ \tilde{N}_x^n(t) &\equiv \left( \frac{n}{\mu^n} \right) \left( \tilde{S}^n(\bar{A}_x^n(t)) - \tilde{S}_{d,x}^n(\bar{A}_x^n(t)) - \tilde{M}_x^n(\bar{A}_x^n(t)) + \tilde{A}_x^n(t) \right), \\ \tilde{I}_x^n(t) &\equiv \sqrt{n} \int_0^t \mathbf{1}_{[\tilde{V}_x^n(s)=0]} ds. \end{aligned}$$

## 4.2 Proof of Proposition 2

We will establish the claim when  $q \in [2, p)$ . Then the claim with  $q \in [1, 2)$  follows from Jensen's inequality. From the inequality (cf. Lemma 2 on page 98 in [28])

$$(a + b)^r \leq (1 \vee 2^{r-1})(a^r + b^r) \quad \text{for } a, b, r \geq 0,$$

we obtain for  $q \in [2, p)$  that

$$\mathbb{E} \left[ \left| \tilde{\mathbb{X}}_x^n(t|x) \right|^q \right] \leq 2^{q-1} \left( \mathbb{E} [\tilde{\tau}_x^n(t|x)^q] + \mathbb{E} [\tilde{V}_x^n(t|x)^q] \right).$$

Therefore, it is sufficient to show that there exists  $t_0 \in \mathbb{R}_+$  such that for all  $t \geq t_0$ ,

$$\lim_{|x| \rightarrow \infty} \sup_n \frac{1}{|x|^q} \mathbb{E} [\tilde{\tau}_x^n(t|x)^q] = 0 \quad (11)$$

and

$$\lim_{|x| \rightarrow \infty} \sup_n \frac{1}{|x|^q} \mathbb{E} [\tilde{V}_x^n(t|x)^q] = 0 \quad (12)$$

We first show (11) and second show (12).

**Proof of (11).** We begin by observing that by definition  $\tilde{\tau}_x^n(t|x) \leq \sqrt{n}u_{(A_x^n(t|x)+1)/(\lambda n)}$ , for all  $t \geq 0$  and  $x \in \mathbb{S}$ . Next, define the regular (non-delayed) renewal process  $A_2^n(\cdot)$  via

$$A_2^n(t) \equiv \max \left\{ i \in \mathbb{N} : \sum_{j=1}^i \frac{u_{j+1}}{\lambda n} \leq t \right\}, \quad t \geq 0,$$

where the maximum over an empty set is 0, and observe that

$$A_x^n(t) \leq A_2^n(t) + 1.$$

Now, take  $t_0 = 1$ . Then, for  $t \geq t_0$ ,

$$\tilde{\tau}_x^n(t|x)^q \leq n^{q/2} \sum_{k=2}^{A_x^n(t|x)+1} \left( \frac{u_k}{\lambda n} \right)^q \leq n^{q/2} \sum_{k=2}^{A_2^n(t|x)+2} \left( \frac{u_k}{\lambda n} \right)^q,$$

where the first inequality above uses the fact that  $A_x^n(t|x) \geq 1$  because  $t_0|x| = |x| \geq \tau \geq u_1 = \tau/\sqrt{n}$ . From Wald's identity,

$$\mathbb{E} \left[ \sum_{k=2}^{A_2^n(t|x)+2} \left( \frac{u_k}{\lambda n} \right)^q \right] = \mathbb{E} [A_2^n(t|x) + 1] \frac{\mathbb{E}[u_2^q]}{\lambda^q n^q}.$$

Together, the above two displays imply

$$\mathbb{E} [\tilde{\tau}_x^n(t|x)^q] \leq \frac{1}{\lambda^q} \frac{1}{n^{q/2}} \mathbb{E}[u_2^q] \mathbb{E} [A_2^n(t|x) + 1]. \quad (13)$$

From the elementary renewal theorem, for any  $t > 0$  and fixed  $n$ ,

$$\frac{\mathbb{E} [A_2^n(t|x)]}{\lambda n t |x|} \rightarrow 1 \text{ as } |x| \rightarrow \infty,$$

and so

$$\mathbb{E} [A_2^n(t|x)] \leq 1 + \lambda n t |x|, \text{ for all large enough } |x|. \quad (14)$$

Substituting (14) into the right-hand-side of (13) shows that for any  $t \geq t_0$  and fixed  $n$

$$\mathbb{E} [\tilde{\tau}_x^n(t|x)^q] \leq \frac{1}{\lambda^q} \frac{1}{n^{q/2}} \mathbb{E}[u_2^q] (2 + \lambda n t |x|) \leq \frac{1}{\lambda^q} \frac{1}{n^{q/2}} 2 \mathbb{E}[u_2^q] (1 + \lambda n t |x|), \quad (15)$$

where the second inequality follows provided  $\lambda n t |x| \geq 1$ , which is true for large enough  $|x|$  and fixed  $n$ . Finally, (11) follows from (15) taking, for example,  $t_0 = 1$ .  $\square$

To complete the proof, we must show (12), which is more involved than (11), and proceeds following the approach of Ye and Yao ([33], Lemma 10 and Proposition 11). First, we establish two versions of pathwise stability results (Lemmas 1 and 2), one for any (fixed)  $n$ -th system and the other for the whole sequence. With the moment condition (A1) on the system primitives, the pathwise stability results are then turned into the moment stability in Lemma 3, which finally leads to (12).

**Lemma 1.** (STABILITY OF  $\tilde{V}^n(\cdot)$  FOR ANY (FIXED)  $n$ ) *Let  $\{r_i\}_{i \geq 1}$  be a sequence of numbers such that  $r_i \rightarrow \infty$  as  $i \rightarrow \infty$  and assume the sequence of initial states  $\{x^i \in \mathbb{S}\}_{i \geq 1}$  satisfies  $|x^i| \leq r_i$  for all  $i$ . Pick any constant  $c > 1$ . Then, for any fixed  $n$ , the following holds (with probability one),*

$$\lim_{i \rightarrow \infty} \frac{1}{r_i} \tilde{V}_{x^i}^n(r_i t) = 0, \text{ u.o.c. for } t \geq \frac{c}{\sqrt{n}}. \quad (16)$$

**Lemma 2.** (STABILITY OF  $\tilde{V}^n(\cdot)$ ) *Let  $\{r_n\}$  be a sequence of numbers such that  $r_n \rightarrow \infty$  as  $n \rightarrow \infty$  and assume that the sequence of initial states  $\{x^n \in \mathbb{S}\}_{n \geq 1}$  satisfies  $|x^n| \leq r_n$ . Then, for any  $\epsilon > 0$ , the following holds (with probability one),*

$$\lim_{n \rightarrow \infty} \frac{1}{r_n} \tilde{V}_{x^n}^n(r_n t) = 0, \text{ u.o.c. for } t \geq \epsilon. \quad (17)$$

**Lemma 3.** (MOMENT STABILITY) *Assume (A1) and (A2).*

(a) *Letting  $\{r_i\}$  and  $\{x^i\}$  as in Lemma 1,*

$$\lim_{i \rightarrow \infty} \mathbb{E} \frac{1}{r_i^q} \tilde{V}_{x^i}^n(r_i t)^q = 0, \text{ for } t \geq \frac{1}{\sqrt{n}}. \quad (18)$$

(b) Letting  $\{r_n\}$  and  $\{x^n\}$  as in Lemma 2,

$$\lim_{n \rightarrow \infty} \mathbb{E} \frac{1}{r_n^q} \tilde{V}_{x^n}^n(r_n t)^q = 0, \quad \text{for } t > 0. \quad (19)$$

While the proofs of the above three lemmas are provided in Section 6, we provide some intuitions here, for Lemmas 1 and 2 in particular. Consider the (fixed)  $n$ -th system in Lemma 1. During the initial period, if it starts with a large initial state, say,  $\tilde{V}_{x^i}^n(0) = r_i$ , new arrivals will abandon the service with nearly probability one. On the other hand, the existing workload, under fluid scaling as in (16), drains (i.e., is processed) at the rate  $\sqrt{n}$  approximately. Therefore, the workload  $\tilde{V}_{x^i}^n(t)$  will reach the “normal” operating state after the initial period with an order of  $r_i/\sqrt{n}$ . The normal operating state, scaled by  $1/r_i$  (where  $r_i \rightarrow \infty$ ), will be approximately zero, and this is characterized by the convergence in (16). In Lemma 2, the index  $n$  approaches infinity. For large  $n$  and hence large  $r_n$ , consider the  $n$ -th (scaled) system and suppose it restarts at a time  $t$ , i.e.,  $\frac{1}{r_n} \tilde{V}_{x^n}^n(r_n(t + \cdot))$ . Then, the key observation is similar to the above case; that is, if the initial state (starting at  $t$ ) is bounded by a constant (say,  $\frac{1}{r_n} \tilde{V}_{x^n}^n(r_n t) \leq 1$ ), it should be approximately zero after a time of  $O(1/\sqrt{n})$ . Indeed by applying Bramson’s hydrodynamic approach, we are able to bound  $\frac{1}{r_n} \tilde{V}_{x^n}^n(r_n t)$  for any time  $t$  during the (arbitrarily given) period  $[0, T]$ . Therefore, our key observation applies for any time  $t$ , which will establish (17). Lastly, Lemma 3 plays a pivotal role in establishing the key moment estimate in (12). Given Lemma 3, the proof of (12) repeats the one for Proposition 11 of [33] and is provided below for completeness.

**Proof of (12).** Pick any time  $t > 0$ . Suppose (12) does not hold; then, there exists an  $\epsilon_0 > 0$  and a sequence of initial states  $\{x^i \in \mathbb{S} : i = 1, 2, \dots\}$  satisfying  $\lim_{i \rightarrow \infty} |x^i| = \infty$  such that

$$\sup_n \frac{1}{|x^i|^q} \mathbb{E} \left| \tilde{V}_{x^i}^n(t|x^i|) \right|^q > 2\epsilon_0. \quad (20)$$

Corresponding to each  $x^i$ , choose an index in the sequence  $\{n\}_{n \geq 1}$ , denoted by  $n_i$ , such that

$$\frac{1}{|x^i|^q} \mathbb{E} \left| \tilde{V}_{x^i}^{n_i}(t|x^i|) \right|^q > \epsilon_0. \quad (21)$$

We claim that  $\{n_i\}_{i \geq 1}$  cannot be bounded. Otherwise, at least an index, say  $n'$ , repeats in the sequence for infinitely many times; this contradicts to Lemma 3(a). Otherwise, without loss of generality, assume  $n_i \rightarrow \infty$  as  $i \rightarrow \infty$ . Then, the bound in (21) contradicts to Lemma 3(b). We conclude that the aforementioned sequence of initial states  $\{x^i \in \mathbb{S} : i = 1, 2, \dots\}$  satisfying (20) cannot exist, which implies (12) holds.  $\square$

## 5 Proofs of the Main Results (Theorems 1 and 2)

**Proof of Theorem 1.** From Proposition 2, there exists  $\delta \equiv t_0 > 0$  such that

$$\lim_{|x| \rightarrow \infty} \frac{1}{|x|} \mathbb{E} |\tilde{\mathbb{X}}_x^n(|x|\delta)| = 0. \quad (22)$$

We require the following lemma, whose proof follows along the same lines of Proposition 4.8 in Bramson [5]. We provide its proof and the notion of petite set in the Appendix for the sake of completeness.

**Lemma 4.** *Assume (A3). The set  $C = \{x \in \mathbb{S} : |x| \leq \kappa\}$  is closed petite for every  $\kappa > 0$ .*

Given Lemma 4, Theorem 3.1 of [9] implies the Markov process  $\mathbb{X}^n(\cdot)$  and the scaled process  $\tilde{\mathbb{X}}^n(\cdot)$  are positive Harris recurrent and hence the existence of a unique stationary distribution follows.  $\square$

Having Proposition 2 and Theorem 1 at hand, the proof of Theorem 2 follows a similar outline to that of Theorem 3.1 in [6]. (Recall [6] establishes the validity of the heavy traffic stationary approximation for a generalized Jackson network without customer abandonment, assuming the inter-arrival and service time distributions have finite polynomial moments, as in (A1).) A global strategy of the proof is as follows. First, Proposition 3 below establishes uniform (in  $n$ ) estimates on the expected return time of the general Markov process to a compact set. Second, from such return time estimates, moment bounds for the stationary distributions of  $\{\tilde{\mathbb{X}}^n\}$ , uniform in the scaling parameter  $n$ , follow readily, yielding tightness of these distributions. Third, the distributional convergence in Theorem 2(a) follows by combining this tightness property with the known weak convergence results of  $\tilde{V}^n$  in (4) and [31]. Next, we obtain convergence of moments of stationary distributions, i.e., Theorem 2(b). We begin by providing a general statement concerning strong Markov processes.

**Proposition 3.** *(Theorem 3.5 of [6], cf. Proposition 5.4 of [11]) For  $n \geq 1$ , consider a strong Markov process  $\{\mathbb{Y}_x^n(t) : t \geq 0\}$  with initial condition  $x$  on a state space  $\mathbb{T}$ . For  $\bar{\delta} \in (0, \infty)$ , define the return time to a compact set  $C \subset \mathbb{T}$  by  $\tau_C^n(\bar{\delta}) \equiv \inf\{t \geq \bar{\delta} : \mathbb{Y}_x^n(t) \in C\}$ . Let  $f : \mathbb{T} \rightarrow [0, \infty)$  be a measurable map. For  $\bar{\delta} \in (0, \infty)$  and a compact set  $C \subset \mathbb{T}$ , define*

$$G_n(x) \equiv \mathbb{E} \left[ \int_0^{\tau_C^n(\bar{\delta})} f(\mathbb{Y}_x^n(t)) dt \right], \quad x \in \mathbb{T}.$$

*If  $\sup_n G_n$  is everywhere finite and uniformly bounded on  $C$ , then there exists a constant  $\eta \in (0, \infty)$ , that is independent of  $n$ , such that for all  $n \in \mathbb{N}$ ,  $t \in (0, \infty)$ ,  $x \in \mathbb{T}$ ,*

$$\frac{1}{t} \mathbb{E}[G_n(\mathbb{Y}_x^n(t))] + \frac{1}{t} \int_0^t \mathbb{E}[f(\mathbb{Y}_x^n(s))] ds \leq \frac{1}{t} G_n(x) + \eta. \quad (23)$$

**Proof of Theorem 2.** We first prove (a) and then (b).

**Part (a):** Using standard arguments (cf. [13]), it suffices to establish the tightness of the family of stationary distributions  $\{\pi^n : n \geq 1\}$ . Indeed, the tightness implies every subsequence of  $\{\pi^n : n \geq 1\}$  admits a convergent subsequence. Denote a typical limit point by  $\tilde{\pi}$  and also define the marginal distribution (corresponding to the limiting stationary distribution of  $\tilde{V}^n$ )  $\tilde{\pi}_0$  as  $\tilde{\pi}_0(A) \equiv \tilde{\pi}(\mathbb{R}_+ \times A)$ ,  $A \in \mathcal{B}(\mathbb{R}_+)$ . Then, as in (4), we see that the process  $\tilde{V}^n$ , with  $\tilde{X}^n(0)$  distributed as  $\pi^n$ , converges in distribution to  $V$  defined in (3) with  $V(0) \sim \tilde{\pi}_0$ . The stationarity of  $\tilde{V}^n$  implies that  $\tilde{\pi}_0$  is a stationary distribution for  $V$ . Since  $V$  has a unique stationary distribution, say  $\pi$ , it must be  $\tilde{\pi}_0 = \pi$ .

To prove the desired tightness, it suffices to show that there exists a positive integer  $N$  such that for all  $n \geq N$

$$\int_{\mathbb{S}} |y| \pi^n(dy) \leq \tilde{c}, \quad (24)$$

where  $\tilde{c} \in (0, \infty)$  is a constant independent of  $n$ . The following arguments proceed according to the same outline as the proof of Theorem 3.2 of [6], but with some details that differ. A key observation from Proposition 2 is that there exists  $\gamma_0 \in (0, \infty)$  such that, for  $t_0$  as in that same proposition,

$$\sup_n \mathbb{E} |\tilde{X}_x^n(t_0|x)|^q \leq \frac{1}{2} |x|^q, \text{ for all } x \in C^c, \ C \equiv \{x \in \mathbb{S} : |x| \leq \gamma_0\}. \quad (25)$$

Next, we apply Proposition 3 above with

$$\mathbb{Y}_x^n = \tilde{X}_x^n, \ \mathbb{T} = \mathbb{S}, \ \bar{\delta} \equiv t_0 \gamma_0, \ f(x) \equiv 1 + |x|^{q-1} \text{ for } x \in \mathbb{S}, \ q \in [2, p), \ C \equiv \{x \in \mathbb{S} : |x| \leq \gamma_0\}.$$

Suppose we can show that there exist  $N \in \mathbb{N}$  and  $\bar{c} \in (0, \infty)$  such that

$$\sup_{n \geq N} G_n(x) = \sup_{n \geq N} \mathbb{E} \left[ \int_0^{\tau_C^n(\bar{\delta})} (1 + |\tilde{X}_x^n(t)|^{q-1}) dt \right] \leq \bar{c}(1 + |x|^q), \quad x \in \mathbb{S}, \quad (26)$$

so that the conditions of Proposition 3 are satisfied for the family  $\{\tilde{X}_x^n : n \geq N\}$ . Then, for  $x \in \mathbb{S}$  and  $\eta \in (0, \infty)$  as in Proposition 3,

$$\Phi_n(x) \equiv \frac{1}{t} G_n(x) - \frac{1}{t} \mathbb{E}[G_n(\tilde{X}_x^n(t))] \geq \frac{1}{t} \int_0^t \mathbb{E}(f(\tilde{X}_x^n(s))) ds - \eta,$$

and thus an expectation with respect to the stationary distribution  $\pi^n$  has a lower bound,

$$\begin{aligned} \int_{\mathbb{S}} \Phi_n(x) \pi^n(dx) &\geq \int_{\mathbb{S}} \left( \frac{1}{t} \int_0^t \mathbb{E}(f(\tilde{\mathbb{X}}_x^n(s))) ds - \eta \right) \pi^n(dx) \\ &= \int_{\mathbb{S}} \left( \frac{1}{t} \int_0^t \mathbb{E}(f(\tilde{\mathbb{X}}_x^n(s))) ds \right) \pi^n(dx) - \eta \\ &= \frac{1}{t} \int_0^t \left( \int_{\mathbb{S}} \mathbb{E}(f(\tilde{\mathbb{X}}_x^n(s))) \pi^n(dx) \right) ds - \eta \end{aligned} \quad (27)$$

$$= \int_{\mathbb{S}} f(x) \pi^n(dx) - \eta, \quad (28)$$

where (27) is from Fubini's theorem and (28) follows from the fact that  $\pi^n$  is a stationary distribution. Furthermore, if  $\Phi_n(x)$  is a bounded function in  $x$ , then from the definitions of the stationary distribution  $\pi^n$  and the function  $\Phi_n(x)$ , it is seen that  $0 = \int_{\mathbb{S}} \Phi_n(x) \pi^n(dx)$ . Otherwise, if  $\Phi_n(x)$  is unbounded, then Fatou's lemma implies that (cf. proof of Theorem 3.2 in [6] on page 55, also proof of Theorem 5 in [13])

$$0 \geq \int_{\mathbb{S}} \Phi_n(x) \pi^n(dx). \quad (29)$$

Finally, it follows from (28)–(29) that

$$0 \geq \int_{\mathbb{S}} \Phi_n(x) \pi^n(dx) \geq \int_{\mathbb{S}} f(x) \pi^n(dx) - \eta, \quad (30)$$

which establishes the desired uniform moment bound in (24).

To complete the proof of part (a), it only remains to show (26). The following arguments are similar to those leading to Theorem 3.4 in [6]. We note that the terms  $\bar{\delta}$ ,  $f(x)$ , and the compact set  $C$  are chosen exactly the same way as in the cited theorem. However, minor modifications are necessary because the Markov process  $\tilde{\mathbb{X}}_x^n$  in this paper is defined differently from the Markov process  $\hat{X}_x^n$  in [6]. This entails to checking the following bound, which corresponds to (38) in [6]: there exist  $N \in \mathbb{N}$  and  $c_0 \in (0, \infty)$  such that

$$\sup_{n \geq N} \mathbb{E} \left[ \int_0^{\sigma_1} \left( 1 + \tilde{\mathbb{X}}_x^n(t) \right)^{q-1} dt \right] \leq c_0 (1 + |x|^q), \quad x \in \mathbb{S}, \quad (31)$$

where  $\sigma_1 \equiv t_0(|x| \vee \gamma_0)$  is defined exactly as in the proof of Theorem 3.4 in [6]. Notice that  $\sigma_1 \leq c_1(1 + |x|)$  for some  $c_1 \in (0, \infty)$ , which is the same estimate as in (39) of [6]. For  $t \geq 0$  and a real-valued function  $f$  on  $\mathbb{R}_+$ , define  $\|f\|_t \equiv \sup_{0 \leq s \leq t} |f(s)|$ . To show (31), it is sufficient to show there exist constants  $c_2, c_3 \in (0, \infty)$ , and  $N \in \mathbb{N}$  such that, for all  $n \geq N$ ,

$$\mathbb{E} \left\| \tilde{V}_x^n \right\|_{c_1(1+|x|)}^{q-1} \leq c_2 (1 + |x|)^{q-1} \quad (32)$$



and

$$\mathbb{E} \|\tilde{\tau}_x^n\|_{c_1(1+|x|)}^{q-1} \leq c_3(1+|x|)^{q-1}. \quad (33)$$

*The argument to show (32).* It follows from the proof of Lemma 3 (more precisely, (61) in Section 6 and the fact that  $\|\tilde{V}_x^n\|_t \leq \|\tilde{W}_x^n\|_t$ , where  $\{W_x^n(t) : t \geq 0\}$  denotes the offered waiting time process of the  $GI/GI/1$  queue defined as in the proof of Lemma 3).

*The argument to show (33).* The same logic used to show (13) also shows

$$\mathbb{E} [\tilde{\tau}_x^n(t)^q] \leq \frac{1}{\lambda^q} \frac{1}{n^{q/2}} \mathbb{E} [u_2^q] (\mathbb{E} [A_2^n(t)] + 1), \text{ for } t \geq 0,$$

where  $A_2^n$  is the regular (non-delayed) renewal process defined in the second paragraph of the proof of Proposition 2. Since  $A_2^n$  is a non-decreasing process,

$$\mathbb{E} [\tilde{\tau}_x^n(t)^q] \leq \frac{1}{\lambda^q} \frac{1}{n^{q/2}} \mathbb{E} [u_2^q] (\mathbb{E} [A_2^n(c_1 + (1 + c_1)|x|)] + 1), \text{ for } 0 \leq t \leq c_1(1 + |x|).$$

From the elementary renewal theorem, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$\mathbb{E} [A_2^n(c_1 + (1 + c_1)|x|)] \leq \lambda n (c_1 + (1 + c_1)|x| + 1) = \lambda n(1 + c_1)(1 + |x|),$$

and so

$$\mathbb{E} [\tilde{\tau}_x^n(t)^q] \leq \frac{1}{\lambda^q} n^{1-q/2} \mathbb{E} [u_2^q] (\lambda(1 + c_1)(1 + |x|) + 1), \text{ for } n \geq N \text{ and } 0 \leq t \leq c_1(1 + |x|).$$

Recalling that  $q \geq 2$  and  $n \geq 1$  ensures  $n^{1-q/2} \leq 1$  and so

$$\mathbb{E} [\tilde{\tau}_x^n(t)^q] \leq \frac{1}{\lambda^q} \mathbb{E} [u_2^q] (\lambda(1 + c_1) + 1) (1 + |x|), \text{ for } n \geq N \text{ and } 0 \leq t \leq c_1(1 + |x|).$$

Since  $0 < q - 1 < q$ , Hölder's inequality implies

$$\mathbb{E} [\tilde{\tau}_x^n(t)^{q-1}] \leq (\mathbb{E} [\tilde{\tau}_x^n(t)^q])^{(q-1)/q}, \text{ for } t \geq 0.$$

Hence for  $c_3 \equiv (\mathbb{E} [u_2^q] (\lambda(1 + c_1) + 1) / \lambda^q)^{(q-1)/q}$ ,

$$\mathbb{E} [\tilde{\tau}_x^n(t)^{q-1}] \leq c_3 (1 + |x|)^{(q-1)/q} \leq c_3 (1 + |x|)^{q-1}, \text{ for } n \geq N \text{ and } 0 \leq t \leq c_1(1 + |x|),$$

from which (33) follows.

**Part (b):** Proposition 2 implies the moment estimate (25) with  $q \in [1, p)$ . However, when applying Proposition 3 with  $f(x) \equiv 1 + |x|^{q-1}$ , we require  $q \in [2, p)$  in order to obtain (33) and (30). Therefore, we have a uniform (in  $n$ ) moment bound on the  $(q - 1)$ -th moment of the family of stationary distributions  $\{\pi^n\}_{n \geq 1}$ . If we pick  $q$  such that  $m < q - 1$ , this uniform moment bound, in turn, implies the uniform integrability of  $\{\|\tilde{X}^n\|^m\}_{n \geq 1}$  with  $m \in (0, q - 1)$  in stationarity (i.e.,  $\tilde{X}^n(0) \sim \pi^n$ ). Combining the weak convergence result established in part (a) and the aforementioned uniform integrability, we conclude the desired moment convergence in stationarity as in part (b). This completes the proof.  $\square$

## 6 Proofs of Lemmas 1–3

**Proof of Lemma 1.** Without loss of generality, assume that as  $i \rightarrow \infty$ ,  $x^i/r_i \rightarrow \bar{x} \equiv (\bar{\tau}, \bar{v}(0))$  with  $|\bar{x}| = \bar{\tau} + \bar{v}(0) \leq 1$ ; otherwise, it suffices to consider any convergent subsequence. Fix the index  $n$  throughout the proof. We also omit the index  $n$  and the subscript  $x^i$  whenever it does not cause any confusion.

For the  $i$ -th copy of the  $n$ -th system, write the offered waiting time as:

$$\frac{1}{r_i} \tilde{V}_{x^i}^n(r_i t) \equiv \tilde{v}_i(t) = \phi_i(t) + \eta_i(t), \quad \text{with} \quad (34)$$

$$\begin{aligned} \phi_i(t) \equiv & \frac{v^i}{r_i} + \frac{n\sqrt{n}}{\mu^n} \cdot \frac{1}{r_i n} \sum_{j=1}^{A^n(r_i t)} (1 - F(V^n(t_j^n -))) - \sqrt{n}t \\ & + \frac{n\sqrt{n}}{\mu^n} \cdot \frac{1}{r_i} \left( S^n(A^n(r_i t)) - S_d^n(A^n(r_i t)) - \frac{1}{n} M^n(A^n(r_i t)) \right), \end{aligned} \quad (35)$$

$$\eta_i(t) \equiv \frac{\sqrt{n}}{r_i} \int_0^{r_i t} \mathbf{1}_{\{V^n(s)=0\}} ds. \quad (36)$$

First, estimate the item associated with the arrival in the above (in the first summation):

$$\begin{aligned} \frac{A^n(r_i t)}{r_i n} &= \frac{1}{r_i n} \left( A^n(r_i t) - \lambda n \left( r_i t - \frac{\tau_i}{\sqrt{n}} \wedge r_i t \right) \right) + \lambda \left( t - \frac{\tau_i}{r_i \sqrt{n}} \wedge t \right) \\ &\rightarrow \lambda \left( t - \frac{\bar{\tau}}{\sqrt{n}} \wedge t \right), \quad \text{as } i \rightarrow \infty \text{ a.s.} \end{aligned} \quad (37)$$

Second, denote the term associated with the arrival and abandonment as

$$\xi_i(t) \equiv \frac{1}{r_i n} \sum_{j=1}^{A^n(r_i t)} (1 - F(V^n(t_j^n -))). \quad (38)$$

Observe that for any  $0 \leq t_1 < t_2$ , we have

$$0 \leq \xi_i(t_2) - \xi_i(t_1) \leq \frac{1}{r_i n} (A^n(r_i t_2) - A^n(r_i t_1)). \quad (39)$$

From (37), we note that the right-hand side in the above converges uniformly to  $\lambda(t_2 - t_1 - \frac{\bar{\tau}}{\sqrt{n}} \wedge t_2 + \frac{\bar{\tau}}{\sqrt{n}} \wedge t_1)$ . Therefore, any subsequence of  $i$  contains a further subsequence such that as  $i \rightarrow \infty$  along the further subsequence, we have the weak convergence

$$\xi_i(\cdot) \Rightarrow \bar{\xi}(\cdot), \quad \text{in } D(\mathbb{R}) \text{ as } i \rightarrow \infty,$$

where the limit  $\bar{\xi}(\cdot)$  is Lipschitz continuous (recall (39)) with a Lipschitz constant  $\lambda$  (with probability one). Without loss of generality, we can assume the above convergence is along the full sequence, and furthermore, by using the coupling technique, we can further assume the convergence is almost surely:

$$\xi_i(t) \rightarrow \bar{\xi}(t), \quad \text{as } i \rightarrow \infty \quad \text{a.s.}$$

Third, for the martingale terms, we have as  $i \rightarrow \infty$  with probability one,

$$\frac{1}{r_i} \left( S^n(A^n(r_i t)) - S_d^n(A^n(r_i t)) - \frac{1}{n} M^n(A^n(r_i t)) \right) \rightarrow 0. \quad (40)$$

Putting the above convergences together yields, as  $i \rightarrow \infty$ ,

$$\phi_i(t) \rightarrow \bar{\phi}(t) \equiv \bar{v}(0) + \frac{n\sqrt{n}}{\mu^n} \bar{\xi}(t) - \sqrt{nt}, \quad \text{u.o.c. of } t \geq 0. \quad (41)$$

Note from (34)–(36) that the tuple  $(\tilde{v}_i(t), \phi_i(t), \eta_i(t))_{t \geq 0}$  satisfies the one-dimensional linear Skorokhod problem (cf. §6.2 of [7]):

$$\tilde{v}_i(t) = \phi_i(t) + \eta_i(t) \geq 0, \quad d\eta_i(t) \geq 0 \text{ with } \eta_i(0) = 0, \quad \tilde{v}_i(t)d\eta_i(t) = 0.$$

Hence, by invoking the Lipschitz continuity of the Skorokhod mapping (cf. Theorem 6.1 of [7]), the convergence in (41) implies

$$\frac{1}{r_i} \tilde{V}_{x_i}^n(r_i t) \rightarrow \bar{v}(t) \quad \text{and} \quad \eta_i(t) \rightarrow \bar{\eta}(t) \quad \text{u.o.c. of } t \geq 0, \quad (42)$$

with the limit satisfying the Skorokhod problem as well:

$$\bar{v}(t) = \bar{\phi}(t) + \bar{\eta}(t) \geq 0, \quad d\bar{\eta}(t) \geq 0 \text{ with } \bar{\eta}(0) = 0, \quad \bar{v}(t)d\bar{\eta}(t) = 0. \quad (43)$$

Next, we further examine the limit  $\bar{\xi}(\cdot)$  following the approach of Chen and Ye ([8], Proposition 3(b)). From (37) and (38), and noting that  $\xi_i(t) \leq \frac{A^n(r_i t)}{r_i n}$ , we have

$$\bar{\xi}(t) = 0, \quad 0 \leq t \leq \frac{\bar{\tau}}{\sqrt{n}}. \quad (44)$$

Now, consider any regular time  $t_1 > \bar{\tau}/\sqrt{n}$ , at which all processes concerned, i.e.,  $\bar{v}(\cdot)$ ,  $\bar{\phi}(\cdot)$ , and  $\bar{\eta}(\cdot)$  are differentiable, and  $\bar{v}(t_1) > 0$ . Note that the Lipschitz continuity of  $\bar{\xi}_i(\cdot)$  implies that  $(\bar{v}(\cdot), \bar{\phi}(\cdot), \bar{\eta}(\cdot))$  are also Lipschitz continuous. Therefore, we can find (small) constants  $\epsilon > 0$  and  $\delta > 0$  such that the following inequality holds for all sufficiently large  $i$ :

$$\tilde{v}_i(t_2) > \epsilon \quad \text{i.e.,} \quad \tilde{V}^n(r_i t_2) > r_i \epsilon, \quad t_2 \in [t_1, t_1 + \delta). \quad (45)$$

Observe that if the  $j$ -th arrival falls between  $A^n(r_it_1)+1$  and  $A^n(r_it_2)$ , then its arrival time,  $t_j^n$ , shall also falls between the corresponding time epochs, i.e.,  $r_it_1 < t_j^n \leq r_it_2$ . Given the estimate in (45), this implies the following estimate holds:

$$\tilde{V}^n(t_j^n) > r_i\epsilon.$$

Consequently, we have for all sufficiently large  $i$  that

$$\begin{aligned} \xi_i(t_2) - \xi_i(t_1) &= \frac{1}{r_in} \sum_{j=A^n(r_it_1)+1}^{A^n(r_it_2)} (1 - F(V^n(t_j^n))) \\ &\leq \frac{1}{r_in} (A^n(r_it_2) - A^n(r_it_1))(1 - F(r_i\epsilon)), \quad t_2 \in [t_1, t_1 + \delta]. \end{aligned}$$

Taking  $i \rightarrow \infty$ , the above yields  $\bar{\xi}(t_2) - \bar{\xi}(t_1) \leq 0$ , which is effectively  $\bar{\xi}(t_2) - \bar{\xi}(t_1) = 0$ . In summary, the above implies for any regular time  $t > \bar{\tau}/\sqrt{n}$  with  $\bar{v}(t) > 0$ ,

$$\frac{d\bar{\xi}(t)}{dt} = 0. \tag{46}$$

Finally, using the properties in (41), (43), (44) and (46), it is direct to show that  $\bar{v}(t) = \bar{v}(0) - \sqrt{n}t$  for  $t \leq \bar{v}(0)/\sqrt{n} (\leq 1/\sqrt{n})$  and  $\bar{v}(t) = 0$  for  $t \geq \bar{v}(0)/\sqrt{n}$ . This property, along with the convergence in (42) yields the desired convergence in (16).  $\square$

**Proof of Lemma 2.** We adopt Bramson's hydrodynamics approach (cf. [4, 20, 32]), and its variation (in Appendix B.2 of [33]) in particular, to examine the processes involved in (17). Define for each  $n$  and for  $j = 0, 1, \dots$ ,

$$\bar{V}^{n,j}(u) = \frac{1}{r_n} \tilde{V}_{x^n}^n(r_n \frac{j+u}{\sqrt{n}}), \quad u \geq 0. \tag{47}$$

That is, the  $n$ -th diffusion-scaled process (say,  $\{\frac{1}{r_n} \tilde{V}_{x^n}^n(r_nt)\}_{t \geq 0}$ ) breaks into many pieces of fluid-scaled process (i.e.,  $\{\bar{V}^{n,j}(u) : u \in [0, 1]\}$  and  $j = 0, 1, \dots$ ), with each piece covering a period of  $1/\sqrt{n}$  in the diffusion-scaled process. (As in the proof of the previous Lemma 1, we omit the index  $n$  and the subscript  $x^n$  whenever it does not cause any confusion.)

Let  $j_n$  be any nonnegative integer for each  $n$  and consider any subsequence of positive integer, denoted  $\mathcal{N}$ . If  $\limsup_{n \rightarrow \infty, n \in \mathcal{N}} [\bar{V}^{n,j_n}(0)] \leq 1$ , then it can be seen that: as  $n \rightarrow \infty$  along  $\mathcal{N}$ ,

$$\bar{V}^{n,j_n}(u) \rightarrow 0, \quad \text{u.o.c. of } u \geq 1. \tag{48}$$

This can be proved in the same manner as Lemma 1, with extra modification on the shifted initial residual arrival time and the sequence of scaling factors ( $\{r_n/\sqrt{n}\}$  here versus  $\{r_i\}$

of Lemma 1). We defer the proof of (48) for now, and proceed to show the main claim in (17).

To prove (17), given the hydrodynamic representation of the waiting time in (47), it suffices to show that for any  $\Delta > 0$  and  $\epsilon > 0$ , the following holds for sufficiently large  $n$  and for  $j = 1, \dots, \lfloor \sqrt{n}\Delta \rfloor$  (excluding  $j = 0$ ),

$$\bar{V}^{n,j}(u) \leq \epsilon, \quad \text{for } u \in [0, 1]. \quad (49)$$

We prove this by contradiction. Suppose to the contrary, there exists a subsequence  $\mathcal{N}$ , and for all  $n \in \mathcal{N}$ , we can find an integer index  $j_n \in [1, \sqrt{n}\Delta]$  and a time  $u_n \in [0, 1]$  such that

$$\bar{V}^{n,j_n}(u_n) > \epsilon. \quad (50)$$

Furthermore, we can require that for each  $n \in \mathcal{N}$ ,  $j_n$  is the smallest integer (in  $[1, \sqrt{n}\Delta]$ ) for the above inequality to hold.

Observe that the following initial condition must hold for sufficiently large  $n \in \mathcal{N}$ :

$$\bar{V}^{n,j_n-1}(0) \leq 1. \quad (51)$$

First, if  $j_n = 1$ , then from the definition in (47), we evaluate the left-hand side of the above as:

$$\bar{V}^{n,0}(0) = \frac{1}{r_n} \tilde{V}_{x^n}^n(0) \leq \frac{|x^n|}{r_n} \leq 1.$$

On the other hand, we have  $j_n \geq 2$ ; and according to the definition of  $j_n$ , it means that the inequality (49) applies for  $j = j_n - 1$ . As a result, the inequality (51) also holds.

Given the above initial condition, applying (48) yields the following convergence as  $n \rightarrow \infty$  in  $\mathcal{N}$ ,

$$\bar{V}^{n,j_n-1}(u) \rightarrow 0, \quad \text{u.o.c. of } u \geq 1.$$

Therefore, for sufficiently large  $n \in \mathcal{N}$ , we have

$$\bar{V}^{n,j_n}(u) = \bar{V}^{n,j_n-1}(1+u) \leq \epsilon, \quad u \in [0, 1],$$

which contradicts to (50), and this establishes the desired result in (17).

Finally, it remains to show the claim (48) holds, which resembles the proof of Lemma 1 and hence we provide the outline only. Without loss of generality, assume that as  $n \rightarrow \infty$ ,  $\bar{V}^{n,j_n}(0) \rightarrow \bar{v}(0) \leq 1$ . Similar to (34), the offered waiting time for the  $n$ -system can be expressed as

$$\bar{V}^{n,j_n}(u) = \frac{1}{r_n} \tilde{V}^n(r_n \frac{j_n + u}{\sqrt{n}}) \equiv \tilde{v}_n(u) \equiv \phi_n(u) + \eta_n(u), \quad (52)$$

where

$$\begin{aligned}
\phi_n(u) &\equiv \bar{V}^{n,j_n}(0) + \frac{n}{\mu^n} \cdot \frac{1}{r_n \sqrt{n}} \sum_{j=A^n(r_n j_n / \sqrt{n})+1}^{A^n(r_n(j_n+u)/\sqrt{n})} (1 - F(V^n(t_j^n))) - u \\
&\quad + \frac{n}{\mu^n} \cdot \frac{\sqrt{n}}{r_n} \left( S^n(A^n(r_n \frac{j_n+u}{\sqrt{n}})) - S^n(A^n(r_n \frac{j_n}{\sqrt{n}})) \right) \\
&\quad - \frac{n}{\mu^n} \cdot \frac{\sqrt{n}}{r_n} \left( S_d^n(A^n(r_n \frac{j_n+u}{\sqrt{n}})) - S_d^n(A^n(r_n \frac{j_n}{\sqrt{n}})) \right) \\
&\quad - \frac{n}{\mu^n} \cdot \frac{1}{r_n \sqrt{n}} \left( M^n(A^n(r_n \frac{j_n+u}{\sqrt{n}})) - M^n(A^n(r_n \frac{j_n}{\sqrt{n}})) \right), \\
\eta_n(u) &\equiv \frac{\sqrt{n}}{r_n} \int_{\frac{j_n}{\sqrt{n}}}^{\frac{j_n+u}{\sqrt{n}}} \mathbf{1}_{\{V^n(s)=0\}} ds.
\end{aligned}$$

In parallel to the convergence in (37), we have:

$$\begin{aligned}
&\frac{1}{r_n \sqrt{n}} \left( A^n(r_n \frac{j_n+u}{\sqrt{n}}) - A^n(r_n \frac{j_n}{\sqrt{n}}) \right) \\
= &\frac{1}{r_n \sqrt{n}} \left( A^n(r_n \frac{j_n+u}{\sqrt{n}}) - A^n(r_n \frac{j_n}{\sqrt{n}}) - \lambda r_n \sqrt{n} \left( u - u \wedge \left( \frac{1}{r_n} \tilde{\tau}_n(r_n \frac{j_n}{\sqrt{n}}) \right) \right) \right) \\
&+ \lambda \left( u - u \wedge \left( \frac{1}{r_n} \tilde{\tau}_n(r_n \frac{j_n}{\sqrt{n}}) \right) \right) \\
\rightarrow &\lambda u, \quad \text{as } n \rightarrow \infty \quad \text{a.s.}
\end{aligned} \tag{53}$$

While the convergence in (37) is justified by the conventional functional strong law of large numbers for a renewal process (e.g., [7]), we have applied here a version established by Bramson ([4], Proposition 4.2) and Stolyar ([29], Appendix A.2), which accompanies the hydrodynamic scaling approach.

Then, following the arguments from (38)–(43), it is seen that, as  $n \rightarrow \infty$ ,

$$(\bar{V}^{n,j_n}(u), \phi_n(u), \eta_n(u)) \rightarrow (\bar{v}(u), \bar{\phi}(u), \bar{\eta}(u)) \quad \text{u.o.c. of } u \geq 0, \tag{54}$$

where

$$\bar{\phi}(u) = \bar{v}(0) + \frac{1}{\lambda} \bar{\xi}(u) - u. \tag{55}$$

Moreover, the limits,  $\bar{\xi}(\cdot)$ ,  $\bar{v}(\cdot)$ ,  $\bar{\phi}(\cdot)$  and  $\bar{\eta}(\cdot)$ , are Lipschitz continuous and satisfy the Skorokhod problem:

$$\bar{v}(u) = \bar{\phi}(u) + \bar{\eta}(u) \geq 0, \quad d\bar{\eta}(u) \geq 0 \text{ with } \bar{\eta}(0) = 0, \quad \bar{v}(u) d\bar{\eta}(u) = 0. \tag{56}$$

Next, a similar analysis on the limit  $\bar{\xi}(\cdot)$  yields that, for any regular time  $u > \bar{\tau}$  with  $\bar{v}(u) > 0$ ,

$$\frac{d\bar{\xi}(u)}{du} = 0. \quad (57)$$

Finally, using the properties in (55), (56) and (57), it is direct to show that  $\bar{v}(u) = \bar{v}(0) - u$  for  $u \leq \bar{v}(0)$  ( $\leq 1$ ) and  $\bar{v}(u) = 0$  for  $u \geq \bar{v}(0)$ . This property, along with the convergence in (54) yields the desired convergence in the claim (48).  $\square$

**Proof of Lemma 3.** We first show that for any (fixed)  $p' \in (q, p)$ , there exists a constant  $\kappa' \in (0, \infty)$  such that the following bound holds for any initial state  $x$  and any time  $t \geq 0$ ,

$$\mathbb{E}[\tilde{V}_x^n(t)]^{p'} \leq \kappa'(1 + |x|^{p'} + t^{p'}). \quad (58)$$

Indeed, if we discard the abandonment component, the systems will be reduced to the more basic  $GI/GI/1$  queues. Dropping the abandonment component in the original state equation (1), one gets the offered waiting time process  $\{W^n(t) : t \geq 0\}$ , for the  $n$ -th system, as

$$W_x^n(t) = V_x^n(0) + \sum_{j=1}^{A_x^n(t)} v_j^n - \int_0^t \mathbf{1}_{[W_x^n(s) > 0]} ds.$$

Then, because the system without abandonment dominates the system with abandonment at all times (as can be seen using the one-dimensional Skorokhod mapping),  $V_x^n(t) \leq W_x^n(t)$  for all  $t \geq 0$ . Rewrite the above, with diffusion scaling, as,

$$\widetilde{W}_x^n(t) [\equiv \sqrt{n}W_x^n(t)] = \widetilde{V}_x^n(0) + \xi_n(t) + \sqrt{n} \int_0^t \mathbf{1}_{[\widetilde{W}_x^n(s) = 0]} ds,$$

where

$$\begin{aligned} \xi_n(t) &\equiv \sqrt{n} \sum_{j=1}^{A_x^n(t)} v_j^n - \sqrt{n}t \\ &= \frac{\sqrt{n}}{\mu^n} \sum_{j=1}^{A_x^n(t)} (v_j - 1) + \frac{\sqrt{n}}{\mu^n} (A_x^n(t) - n\lambda t) + \sqrt{n} \left( \lambda - \frac{\mu^n}{n} \right) \frac{n}{\mu^n} t. \\ &= \frac{n}{\mu^n} (\tilde{S}^n(\bar{A}_x^n(t)) + \tilde{A}_x^n(t) + \theta^n t). \end{aligned} \quad (59)$$

Observe that  $\widetilde{W}_x^n(\cdot) = \Phi(\widetilde{V}_x^n(0) + \xi_n(\cdot))$ , where  $\Phi$  is the standard one-dimensional Skorokhod mapping (cf. [7]), Therefore, it is bounded by the “free process” as,

$$\sup_{0 \leq s \leq t} |\widetilde{W}_x^n(s)| \leq |\widetilde{V}_x^n(0)| + 2 \sup_{0 \leq s \leq t} |\xi_n(s)|, \quad (60)$$

which, under the assumed moment condition, implies the following bound (whose proof is presented after the next paragraph) for some constant  $\kappa' \in (0, \infty)$ ,

$$\mathbb{E}[\widetilde{V}_x^n(t)]^{p'} \leq \mathbb{E}[\widetilde{W}_x^n(t)]^{p'} \leq \mathbb{E}[\sup_{0 \leq s \leq t} |\widetilde{W}_x^n(s)|^{p'}] \leq \kappa'(1 + |x|^{p'} + t^{p'}), \quad (61)$$

which implies (58).

For the conclusion in (a), the above implies that the following bound holds uniformly over all (large)  $i$ ,

$$\mathbb{E} \frac{1}{r_i^{p'}} \left| \widetilde{V}_{x^i}^n(r_i t) \right|^{p'} \leq \kappa'(2 + t^{p'}).$$

This implies that the sequence  $\{\frac{1}{r_i^q} |\widetilde{V}_{x^i}^n(r_i t)|^q, i \in \mathbb{N}\}$  (where  $q < p'$ ) is uniformly integrable. Thus, the limit and the expectation in (18) can be interchanged, and then the conclusion (a) follows from Lemma 1. The conclusion (b) is proved in the same way by using Lemma 2.

Finally, it remains to show the third inequality in (61). For  $t \geq 0$  and a real-valued function  $f$  on  $\mathbb{R}_+$ , define  $\|f\|_t \equiv \sup_{0 \leq s \leq t} |f(s)|$ . Recalling the definition in (59), it suffices to prove the following bounds (62) and (63):

$$\mathbb{E} \left[ \|\widetilde{A}_x^n\|_t^{p'} \right] \leq \mathbb{E} \left[ \|\widetilde{A}_2^n\|_{t+|x|}^{p'} \right] \leq c_0 \left( 1 + \sqrt{t + |x|} \right)^{p'} \quad (62)$$

and

$$\mathbb{E} \left[ \|\widetilde{S}^n \circ \bar{A}_x^n\|_t^{p'} \right] \leq \mathbb{E} \left[ \|\widetilde{S}^n \circ \bar{A}_2^n\|_{t+|x|}^{p'} \right] \leq c_1 \left( 1 + \sqrt{(t + |x|)} \right)^{p'} \quad (63)$$

for some constants  $c_0, c_1 \in (0, \infty)$ , independent of  $n$ . The first inequalities of (62) and (63) follow from recalling the regular (non-delayed) renewal process  $A_2^n(\cdot)$  defined in the second paragraph of the proof of Proposition 2, for which  $A_x^n(t) \leq A_2^n(t) + 1$ , and hence,

$$\bar{A}_x^n(t) \leq \bar{A}_2^n(t) + \frac{1}{n} \text{ and } \widetilde{A}_x^n(t) \leq \widetilde{A}_2^n(t) + \frac{1}{\sqrt{n}}.$$

Then the second inequality in (62) follows from Theorem 4 (A1.3) in [18]. For the second inequality in (63), from (A1.16) in [18],

$$\mathbb{E} \left[ \|\widetilde{S}^n \circ \bar{A}_0^n\|_t^{p'} \right] \leq c_2 \mathbb{E}[\bar{A}_0^n(t)^{p'/2}], \quad (64)$$

where  $c_2 \in (0, \infty)$  is a constant independent of  $n$ . Since  $p'/2 < p'$ , from Lyapunov's inequality,

$$\mathbb{E}[\bar{A}_0^n(t)^{p'/2}]^{2/p'} \leq \mathbb{E}[\bar{A}_0^n(t)^{p'}]^{1/p'} \leq c_3(1 + t), \quad (65)$$

for some  $c_3 \in (0, \infty)$ , where the second inequality follows from Theorem 4 (A1.1) in [18]. Thus, we conclude  $\mathbb{E} \left[ \|\widetilde{S}^n \circ \bar{A}_0^n\|_t^{p'} \right]^{2/p'} \leq c_2^{2/p'} c_3(1 + t)$ , which implies (63).  $\square$



## Appendix: Proofs of Proposition 1 and Lemma 4

**Proof of Proposition 1.** The weak convergence in (4) can be seen from the expression (10) and Theorem 1(a) of [31] with minor modifications as follows. Observe that

$$\tilde{N}_x^n \Rightarrow \sqrt{\frac{\text{var}(u_2) + \text{var}(v_2)}{\lambda}} W, \text{ in } D(\mathbb{R}) \text{ as } n \rightarrow \infty, \quad (66)$$

where  $W$  is a one-dimensional standard Brownian motion. This is because standard results on renewal processes show that the delayed renewal process defined in (2) has the limiting behavior

$$\bar{A}_x^n \rightarrow \lambda e \text{ and } \tilde{A}_x^n \Rightarrow \sqrt{\lambda} \sqrt{\text{var}(u_2)} W_1, \text{ in } D(\mathbb{R}) \text{ as } n \rightarrow \infty,$$

where  $W_1$  is a standard Brownian motion and  $e(\cdot)$  represents the identity map, i.e.,  $e(t) = t$  for all  $t \in \mathbb{R}_+$ . Donsker's theorem and a random time change show

$$\tilde{S}^n \circ \bar{A}_x^n \Rightarrow \sqrt{\lambda} \sqrt{\text{var}(v_2)} W_2, \text{ in } D(\mathbb{R}) \text{ as } n \rightarrow \infty,$$

where  $W_2$  is a standard Brownian motion independent of  $W_1$ . Arguments similar to those in Theorem 5.1(a) in [31] show

$$\tilde{S}_{d,x} \circ \bar{A}_x^n \Rightarrow 0 \text{ and } \tilde{M}_d^n \circ \bar{A}_x^n \Rightarrow 0, \text{ in } D(\mathbb{R}) \text{ as } n \rightarrow \infty.$$

Also, (A2) implies  $n/\mu^n \rightarrow 1/\lambda$  as  $n \rightarrow \infty$ , which leads to the coefficient multiplying  $W$  in (66), and  $b^n \rightarrow \theta/\lambda$  as  $n \rightarrow \infty$  by (A2). Finally, the linearly generalized regulator mapping in Section A.1 of [31] and an application of the continuous mapping theorem establish the weak convergence in (4).  $\square$

Before we prove Lemma 4 below, we recall the definition of a petite set. Given a Markov process  $\Phi = \{\Phi_t : t \geq 0\}$ , with the semigroup operator  $P^t$ , on a state space  $\mathcal{X}$ , we say that a non-empty set  $A \in \mathcal{B}(\mathcal{X})$  is  $\nu_a$ -petite if  $\nu_a$  is a non-trivial measure on  $\mathcal{B}(\mathcal{X})$ ,  $a(\cdot)$  is a probability measure on  $(0, \infty)$  (referred to as a sampling distribution), and  $K_a(x, \cdot) \geq \nu_a(\cdot)$  for all  $x \in A$ . See [22]. Here, the Markov transition function  $K_a := \int P^t a(dt)$ . The set  $A$  will be called simply petite when the specific measure  $\nu_a$  is unimportant.

**Proof of Lemma 4.** First, note that the set  $C$  is closed by definition. To establish that it is petite, we need to find a non-trivial measure  $\nu_a$  on  $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$  and a sampling distribution  $a(\cdot)$  on  $(0, \infty)$  such that  $K_a(x, \cdot) \geq \nu_a(\cdot)$  for all  $x \in C$ . We fix the scaling parameter  $n \geq 1$  and omit the associated superscript  $n$  in this proof. The following arguments are directly transferred from the proof of Proposition 4.8 in Bramson [5]. Fix  $x = (\tau, v) \in C$  and consider the transition probability  $P^t(x, [0, 1] \times \{0\}) = \mathbb{P}(\tau(t) \in [0, 1], V_x(t) = 0)$  for all  $|x| \leq \kappa$ , with  $\kappa > 0$  fixed. From (A3), there exists a  $b \geq v + 2$  so that

$$\begin{aligned} \mathbb{P}(b + 1/2 \leq u_2 \leq b + 1) &\geq \epsilon_1, \\ \mathbb{P}(v_1 \leq 2) &\geq \epsilon_2, \end{aligned}$$

for some  $\epsilon_i \in (0, 1), i = 1, 2$ . Denote the two events considered above by  $G_1$  and  $G_2$  respectively, and also define  $t_1(x) \equiv \tau + b$  and  $t_2(x) \equiv \tau + b + 1/2$ . Suppose  $\omega \in G_1 \cap G_2$ . The first arrival sees workload  $(v - \tau)^+$ , enters service if  $d_1 > (v - \tau)^+$ , and abandons the system without receiving service if  $d_1 \leq (v - \tau)^+$ . Consequently, the departure time of the first arrival is

$$t' \equiv \begin{cases} \max(\tau, v) + v_1 & \text{if } d_1 > (v - \tau)^+, \\ \tau + d_1 & \text{if } d_1 \leq (v - \tau)^+. \end{cases}$$

Algebra shows  $t' \leq \max(\tau, v) + v_1 \leq |x| + v_1$ , and so  $t' \leq \tau + v + 2 \leq t_1(x)$  for  $\omega \in G_2$ . Similarly, letting  $t''$  be the arrival time of the second job,  $t'' = \tau + u_2 \geq t_2(x)$  from the definition of  $t_2(x)$  and  $\omega \in G_1$ . Hence, for all  $t \in [t_1(x), t_2(x)]$ ,  $V(t) = 0$  and also  $\tau(t) = t'' - t \in [0, 1]$  when  $\omega \in G_1 \cap G_2$ . This implies that for all  $t \in [t_1(x), t_2(x)]$ ,

$$\mathbb{P}(\tau(t) \in [0, 1], V_x(t) = 0) \geq \mathbb{P}(G_1 \cap G_2) = \mathbb{P}(G_1)\mathbb{P}(G_2) \geq \epsilon_1\epsilon_2 \equiv \epsilon.$$

Next, let  $\bar{t} \equiv \kappa + b + 1$  and choose the probability measure  $a(\cdot)$  to be uniform over  $(0, \bar{t})$  so that for  $B = B_1 \times B_2 \in \mathcal{B}(\mathbb{S})$ ,

$$K_a(x, B_1 \times B_2) := \int_0^\infty P^t(x, B_1 \times B_2) a(dt) = \int_0^{\bar{t}} \frac{1}{\bar{t}} P^t(x, B_1 \times B_2) dt.$$

Set the measure  $\nu(\cdot)$  on  $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$  to be uniform over the empty states with residual inter-arrival time in  $(0, 1)$ , that is,

$$K_a(x, B_1 \times B_2) \geq \epsilon 1_{\{0 \in B_1\}} m((0, 1) \cap B_2) \equiv \nu(B_1 \times B_2),$$

where  $m(\cdot)$  is the Lebesgue measure on the positive real line. Since  $\nu(\cdot)$  is a nontrivial measure on  $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$  (i.e.,  $\nu(\mathbb{S}) = \nu(\mathbb{R}_+ \times \mathbb{R}_+) > 0$ ), this completes the proof.  $\square$

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