# Parameter estimation for fractional birth and fractional death processes

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Abstract The fractional birth and the fractional death processes are more desirable in practice than their classical counterparts as they naturally provide greater flexibility in modeling growing and decreasing systems. In this paper, we propose formal parameter estimation procedures for the fractional Yule, the fractional linear death, and the fractional sublinear death processes. The methods use all available data possible, are computationally simple and asymptotically unbiased. The procedures exploited the natural structure of the random inter-birth and inter-death times that are known to be independent but are not identically distributed. We also showed how these methods can be applied to certain models with more general birth and death rates. The computational tests showed favorable results for our proposed methods even with relatively small sample sizes. The proposed methods are also illustrated using the branching times of the plethodontid salamanders data of Highton and Larson (1979).

**Keywords** birth process  $\cdot$  Yule process  $\cdot$  Yule–Furry process  $\cdot$  death process  $\cdot$  Mittag–Leffler

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### 1 Introduction

Recently, generalizations of the classical birth and death processes have been developed using the techniques of fractional calculus. These are called the fractional birth (Uchaikin et al. 2008; Orsingher and Polito 2010; Cahoy and Polito 2012) and the fractional death (Orsingher et al. 2010) processes, correspondingly. A major advantage of these models over their classical counterparts is that they can capture both Markovian and non-Markovian structures of a growing or decreasing system.

When the birth and death rates are both linear, they are then called the fractional linear birth or fractional Yule or Yule–Furry process (fYp) and fractional linear death process, respectively. The classical linear birth or Yule process has been widely used to model various stochastic systems such as cosmic showers in physics and epidemics in biology to name a few (see e.g., Nee et al. 1994a; Aldous 2001; Nee 2001; Paradis 2012). Note also that the fractional linear birth process was partially investigated by Uchaikin et al. (2008) using the Riemann-Liouville derivative operator but was continued and generalized by Orsingher and Polito (2010) using the Caputo derivative. The inter-birth time distribution, which provided a way to simulate the fYp was derived in Cahoy and Polito (2012). With this, we adopt the fYp from Orsingher and Polito (2010). In addition, the definition of the fractional linear and fractional sublinear death processes are taken from Orsingher et al. (2010).

For completeness, we first enumerate some properties of the fractional Yule (with one progenitor) and the fractional linear death (with initial population size  $n_0 > 1$ ) processes, which will be used in the subsequent discussions. Table 1.1 below shows the probability  $\tilde{P}(t)$  of no event (no birth or no death) at time t, the state probability mass function  $P_i(t)$  or the probability of having (i-1) births or  $(n_0 - i)$  deaths by time t, the probability density function  $f_i(t)$  of the independent but non-identically distributed random inter-event times, the mean, and the variance of the fractional Yule and the fractional linear death processes. Note that the fractional Yule and the fractional linear death processes have the parameters  $\lambda > 0$  and  $\mu > 0$  as the birth and death intensities, correspondingly.

Note that

$$E_{\delta,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\delta j + \beta)}$$
(1.1)

is the Mittag–Leffler function.  $^{1}$ 

In this article, we propose regression-based procedures to estimate the parameters of the fractional linear birth, the fractional linear death, and the fractional sublinear death processes. The rest of the paper is organized as follows. In Section 2, the specific functional forms of the inter-death time distributions and the variances of the fractional linear and sublinear death processes are obtained. These results allowed us to apply our methods to these processes. Section 3 introduces the proposed method using the fractional Yule, the fractional linear death, and the fractional sublinear death processes as examples. The section also shows some extensions of the procedures to certain models. Section 4 contains the empirical test results and the real-data application of the proposed methods for the case of the fYp only as similar inference procedures can be applied to the fractional linear and fractional sublinear death processes. The summary and extensions of our study are given in Section 5.

# 2 More properties of the fractional linear and fractional sublinear death processes

We now derive some properties which will permit us to apply the proposed estimation procedures to the fractional linear death and the fractional sublinear death processes. More specifically, the theorems below showed that the inter-death times for both the fractional linear and sublinear death processes are Mittag-Leffler distributed. The variances of both processes are also derived. **Theorem 2.1** The inter-death time  $T_k^{\nu}$  of the fractional linear death process  $\{M^{\nu}(t), t > 0\}$  with death rate intensity  $\mu > 0$ , and  $n_0 \in \mathbb{N}$  initial individuals are independent but are non-identically distributed with probability density function

$$\Pr\{T_k^{\nu} \in dt\}/dt = \mu(n_0 - k)t^{\nu - 1}E_{\nu,\nu}(-\mu(n_0 - k)t^{\nu}),$$

where  $k = 0, 1, ..., n_0 - 1$ , and  $T_k^{\nu}$  is the random time separating the kth and (k + 1)th death.

Proof We prove the theorem by induction. When k = 0 we obtain

$$Pr\{T_0^{\nu} \le t\} = Pr\{M^{\nu}(t) < n_0\}$$

$$= 1 - Pr\{M^{\nu}(t) = n_0\}$$

$$= 1 - \tilde{P}(t) \quad (see \ Table \ 1.1)$$

$$= 1 - E_{\nu,1}(-\mu n_0 t^{\nu}).$$
(2.1)

Therefore

$$\Pr\{T_0^{\nu} \in dt\}/dt = \frac{d}{dt} \Pr\{T_0^{\nu} \le t\}$$

$$= \mu n_0 t^{\nu-1} E_{\nu,\nu} (-\mu n_0 t^{\nu}).$$
(2.2)

For k = 1 we observe

$$\Pr\{T_0^{\nu} + T_1^{\nu} \in dt\}/dt$$

$$= \frac{d}{dt} \Pr\{T_0^{\nu} + T_1^{\nu} < t\}$$

$$= \frac{d}{dt} \Pr\{M^{\nu}(t) < n_0 - 1\}$$

$$= \frac{d}{dt} \left[1 - \Pr\{M^{\nu}(t) = n_0\} - \Pr\{M^{\nu}(t) = n_0 - 1\}\right].$$
(2.3)

Using Table 1.1 we get

$$\Pr\{T_{0}^{\nu} + T_{1}^{\nu} \in dt\}/dt$$

$$= -\frac{d}{dt}E_{\nu,1}(-\mu n_{0}t^{\nu})$$

$$-\frac{d}{dt}[n_{0}E_{\nu,1}(-(n_{0}-1)\mu t^{\nu}) - n_{0}E_{\nu,1}(-n_{0}\mu t^{\nu})]$$

$$= \mu n_{0}t^{\nu-1}E_{\nu,\nu}(-\mu n_{0}t^{\nu})$$

$$+ n_{0}(n_{0}-1)\mu t^{\nu-1}E_{\nu,\nu}(-\mu(n_{0}-1)t^{\nu})$$

$$- n_{0}^{2}\mu t^{\nu-1}E_{\nu,\nu}(-\mu n_{0}t^{\nu})$$

$$= n_{0}(n_{0}-1)\mu t^{\nu-1}[E_{\nu,\nu}(-\mu(n_{0}-1)t^{\nu}) - E_{\nu,\nu}(-\mu n_{0}t^{\nu})].$$
(2.4)

To check the preceding results, we can obtain the Laplace transform as

$$\int_{0}^{\infty} e^{-wt} \Pr\{T_{0}^{\nu} + T_{1}^{\nu} \in dt\}$$

$$= \frac{n_{0}(n_{0} - 1)\mu}{w^{\nu} + \mu(n_{0} - 1)} - \frac{n_{0}(n_{0} - 1)\mu}{w^{\nu} + \mu n_{0}}$$
(2.5)

 $<sup>^1</sup>$  Note: The entries with (\*\*) are new results and are derived in Section 2.

	fractional Yule process	fractional linear death process		
$\tilde{P}(t)$	$E_{\nu,1}(-\lambda t^{\nu})$	$E_{\nu,1}(-\mu n_0 t^{\nu})$		
$P_i(t)$	$\sum_{j=1}^{i} {\binom{i-1}{j-1}} (-1)^{j-1} E_{\nu,1}(-\lambda j t^{\nu}),  i \ge 1$	$\binom{n_0}{i} \sum_{j=0}^{n_0-i} \binom{n_0-i}{j} (-1)^j E_{\nu,1}(-(i+j)\mu t^{\nu}),  0 \le i \le n_0$		
$f_i(t)$		$\mu(n_0 - i)t^{\nu - 1}E_{\nu,\nu}(-\mu(n_0 - i)t^{\nu}),  0 \le i \le n_0 - 1 (**)$		
Mean	$E_{ u,1}\left(\lambda t^{ u} ight)$	$n_0 E_{ u,1} \left(-\mu t^ u ight)$		
Variance	$2E_{\nu,1}(2\lambda t^{\nu}) - E_{\nu,1}(\lambda t^{\nu}) - (E_{\nu,1}(\lambda t^{\nu}))^2$	$n_0(n_0-1)E_{\nu,1}(-2\mu t^{\nu}) + n_0E_{\nu,1}(-\mu t^{\nu}) - n_0^2\left(E_{\nu,1}(-\mu t^{\nu})\right)^2 (**)$		

**Table 1.1** Known properties of fractional Yule  $(N^{\nu}(t))$  and linear death  $(M^{\nu}(t))$  processes.

$$= \frac{\mu n_0}{w^{\nu} + \mu n_0} \cdot \frac{\mu (n_0 - 1)}{w^{\nu} + \mu (n_0 - 1)}$$
  
=  $\int_0^{\infty} e^{-ws} \Pr\{T_0^{\nu} \in ds\} \int_0^{\infty} e^{-wy} \Pr\{T_1^{\nu} \in ds\}$   
=  $\int_0^{\infty} e^{-wt} \int_0^t \Pr\{T_1^{\nu} \in d(t - s)\} \Pr\{T_0^{\nu} \in ds\}$   
=  $\int_0^{\infty} \Pr\{T_0^{\nu} \in ds\} \int_s^{\infty} e^{-zt} \Pr\{T_1^{\nu} \in d(t - s)\},$ 

which is just a convolution of two independent variables  $T_0^{\nu}$  and  $T_1^{\nu}$ . For a general k it is sufficient to note that

$$\Pr\{T_0^{\nu} + \dots + T_k^{\nu} \in dt\}$$

$$= \int_0^t \Pr\{T_k^{\nu} \in d(t-s)\} \Pr\{T_0^{\nu} + \dots + T_{k-1}^{\nu} \in ds\}.$$
(2.6)

By exploiting again the Laplace transform and writing  $\mathfrak{D}_k^{\nu} = T_0^{\nu} + \cdots + T_k^{\nu}$ , we have

$$\int_{0}^{\infty} e^{-wt} \Pr\{\mathfrak{D}_{k}^{\nu} \in dt\}$$
(2.7)  
=  $\int_{0}^{\infty} e^{-wt} \int_{0}^{t} \Pr\{T_{k}^{\nu} \in d(t-s)\} \Pr\{\mathfrak{D}_{k-1}^{\nu} \in ds\}$   
=  $\int_{0}^{\infty} \Pr\{\mathfrak{D}_{k-1}^{\nu} \in ds\} \int_{s}^{\infty} e^{-zt} \Pr\{T_{k}^{\nu} \in d(t-s)\}$   
=  $\int_{0}^{\infty} e^{-ws} \Pr\{\mathfrak{D}_{k-1}^{\nu} \in ds\} \int_{0}^{\infty} e^{-wy} \Pr\{T_{k}^{\nu} \in dy\}$   
=  $\prod_{j=0}^{k} \int_{0}^{\infty} e^{-ws} \Pr\{T_{j}^{\nu} \in ds\}$   
=  $\prod_{j=0}^{k} \frac{\mu(n_{0}-j)}{w^{\nu} + \mu(n_{0}-j)}.$ 

We now determine the variance of the fractional linear death process  $\{M^{\nu}(t), t > 0\}$ . Consider equation (1.6) of Orsingher et al. (2010). That is,

$$\begin{cases} \frac{d^{\nu}}{dt^{\nu}} p_k^{\nu}(t) = \mu(k+1) p_{k+1}^{\nu}(t) - \mu k p_k^{\nu}(t), & 0 \le k \le n_0, \\ p_k^{\nu}(0) = \begin{cases} 1, & k = n_0, \\ 0, & 0 \le k < n_0. \end{cases} \end{cases}$$
(2.8)

It is then straightforward to arrive at

$$\begin{cases} \frac{\partial^{\nu}}{\partial t^{\nu}} G^{\nu}(u,t) = -\mu(u-1) \frac{\partial}{\partial u} G^{\nu}(u,t), \\ G^{\nu}(u,0) = u^{n_0}, \end{cases}$$
(2.9)

where  $G^{\nu}(u,t) = \sum_{k=0}^{n_0} u^k p_k^{\nu}(t)$  is the probability generating function of the fractional linear death process. This in turn leads to

$$\begin{cases} \frac{\partial^{\nu}}{\partial t^{\nu}} H(t) = -2\mu H(t), \\ H(0) = n_0(n_0 - 1), \end{cases}$$
(2.10)

where  $H(t) = \mathbb{E}(M^{\nu}(t)(M^{\nu}(t)-1))$  is the second factorial moment. The solution to (2.10) reads

$$H(t) = n_0(n_0 - 1)E_{\nu,1}(-2\mu t^{\nu}), \qquad (2.11)$$

and the variance can be immediately obtained as

$$\begin{aligned} \mathbb{V}arM^{\nu}(t) &= H(t) + \mathbb{E}M^{\nu}(t) - (\mathbb{E}M^{\nu}(t))^{2} \\ &= n_{0}(n_{0} - 1)E_{\nu,1}(-2\mu t^{\nu}) \\ &+ n_{0}E_{\nu,1}(-\mu t^{\nu}) - n_{0}^{2}(E_{\nu,1}(-\mu t^{\nu}))^{2}. \end{aligned} \tag{2.12}$$

Note that the above expression, when  $\nu = 1$ , simplifies to the variance of the classical linear death process, i.e.

$$\mathbb{V}arM^{1}(t) = n_{0}e^{-\mu t}(1 - e^{-\mu t}).$$
 (2.13)

Below is the algorithm to generate a typical sample path of a fractional linear death process in Figure 2.1. Note that there are several sub-algorithms to generate the inter-death times  $T_j^{\nu}$ 's that are available in the literature (see e.g., Cahoy and Polito 2012).

# Algorithm:

Step 1. Let k = 0 and the population size equal  $n_0$ . Step 2. Simulate  $T_k^{\nu}$ , and let the *k*th death time be  $\mathfrak{D}_k^{\nu} = T_0^{\nu} + T_1^{\nu} + T_2^{\nu} + \cdots + T_k^{\nu}$ .

Step 3. Set the population size  $n_0 - k$ , and k = k + 1. Step 4. Repeat Steps 2–3 for  $k = 1, \ldots, n_0 - 1$ . 4

Fig. 2.1 Sample paths of the classical linear death process (top) and the fractional linear death process (bottom) in the interval with parameters  $(\nu, \lambda) = (0.75, 1)$  and initial population size  $n_0 = 40$ .

It can be gleaned from Figure 2.1 that the sample path of the fractional linear death process (bottom) seems to decay faster at small times but is slower for large times than its classical counterpart. The figure also indicates that it is capable of producing death bursts especially at early stages (corresponding to small times).

The inter-death time distribution for the fractional sublinear death process can be easily deduced (whose proof follows from the previous result and is omitted) from the preceding theorem as follows.

**Theorem 2.2** The fractional sublinear death process  $\{\mathfrak{M}^{\nu}(t), t > 0\}$ , with death intensity rate  $\mu > 0$ , and  $n_0 \in \mathbb{N}$  initial individuals has the following probability density function of the inter-death times  $\mathfrak{T}_k^{\nu}$ 's

$$\Pr\{\mathfrak{T}_k^{\nu} \in dt\}/dt = \mu(k+1)t^{\nu-1}E_{\nu,\nu}(-\mu(k+1)t^{\nu}),$$

with  $k = 0, 1, ..., n_0 - 1$ , where  $\mathfrak{T}_k^{\nu}$  is the random time separating the kth and (k + 1)th death.

The variance of the fractional sublinear death process can be determined by considering equation (3.45) of Orsingher et al. (2010). Recall that

$$\frac{\partial^2}{\partial u^2} \mathfrak{G}^{\nu}(u,t) \bigg|_{u=1} = \mathbb{E} \left[ \mathfrak{M}^{\nu}(t) \left( \mathfrak{M}^{\nu}(t) - 1 \right) \right]$$
(2.14)  
=  $H(t).$ 

Then  

$$\frac{d^{\nu}}{dt^{\nu}}H(t) = -2\mu(n_0+1) \left(\mathbb{E}\mathfrak{M}^{\nu}(t) + \Pr\{\mathfrak{M}^{\nu}(t) = 0\} - 1\right)$$

$$+ 2\mu H(t)$$
(2.15)  
=  $-2\mu(n_0+1)\left(\sum_{k=1}^{n_0} \binom{n_0}{k}(-1)^k E_{\nu,1}(-k\mu t^{\nu}) + \sum_{k=1}^{n_0} \binom{n_0+1}{k+1}(-1)^{k+1} E_{\nu,1}(-\mu k t^{\nu})\right) + 2\mu H(t)$   
=  $-2\mu(n_0+1)\left[\sum_{k=1}^{n_0} \left[\binom{n_0}{k} - \binom{n_0+1}{k+1}\right](-1)^k E_{\nu,1}(-k\mu t^{\nu})\right] + 2\mu H(t)$ 

$$= 2\mu(n_0+1)\sum_{k=1}^{n_0} \binom{n_0}{k+1} (-1)^k E_{\nu,1}(-k\mu t^{\nu}) + 2\mu H(t).$$

Using the initial condition  $H(0) = n_0(n_0 - 1)$  and letting  $\tilde{H}(w)$  be the Laplace transform of H(t), we write

$$w^{\nu}\tilde{H}(w) - w^{\nu-1}n_0(n_0 - 1)$$

$$= 2\mu(n_0 + 1)\sum_{k=1}^{n_0} \binom{n_0}{k+1} (-1)^k \frac{w^{\nu-1}}{w^{\nu} + k\mu} + 2\mu\tilde{H}(w).$$
(2.16)

Hence,

$$\tilde{H}(w) \tag{2.17}$$

$$= n_0(n_0 - 1)\frac{w^{\nu - 1}}{w^{\nu} - 2\mu} + 2\mu(n_0 + 1)\sum_{k=1}^{n_0} \binom{n_0}{k+1} \times (-1)^k w^{\nu - 1} \frac{1}{(w^{\nu} + k\mu)(w^{\nu} - 2\mu)}$$
  
$$= n_0(n_0 - 1)\frac{w^{\nu - 1}}{w^{\nu} - 2\mu} + 2\mu(n_0 + 1) \times \sum_{k=1}^{n_0} \binom{n_0}{k+1}(-1)^k w^{\nu - 1} \left[\frac{1}{w^{\nu} + k\mu} - \frac{1}{w^{\nu} - 2\mu}\right] \frac{1}{(-2\mu)}$$
  
$$= n_0(n_0 - 1)\frac{w^{\nu - 1}}{w^{\nu} - 2\mu} + \frac{w^{\nu - 1}}{w^{\nu} - 2\mu}(n_0 + 1)\sum_{k=1}^{n_0} \binom{n_0}{k+1}(-1)^k$$

$$-(n_0+1)\sum_{k=1}^{n_0} \binom{n_0}{k+1} (-1)^k \frac{w^{\nu-1}}{w^{\nu}+k\mu}$$
$$= \frac{w^{\nu-1}}{w^{\nu}-2\mu} (1-n_0) - (n_0+1)\sum_{k=1}^{n_0} \binom{n_0}{k+1} (-1)^k \frac{w^{\nu-1}}{w^{\nu}+k\mu}$$

The second factorial moment can be easily shown as

$$H(t) = -(n_0 - 1)E_{\nu,1}(2\mu t^{\nu})$$
(2.18)

$$+(n_0+1)\sum_{k=1}^{n_0} \binom{n_0}{k+1} (-1)^{k+1} E_{\nu,1}(-k\mu t^{\nu})$$

Thus, the variance simply follows as

$$\begin{aligned} \mathbb{V}ar \ \mathfrak{M}^{\nu}(t) &= H(t) + \mathbb{E}\mathfrak{M}^{\nu}(t) - \left[\mathfrak{M}^{\nu}(t)\right]^{2} \\ &= -(n_{0} - 1)E_{\nu,1}(2\mu t^{\nu}) + (n_{0} + 1)\sum_{k=1}^{n_{0}} \binom{n_{0}}{k+1} \\ &\times (-1)^{k+1}E_{\nu,1}(-k\mu t^{\nu}) \\ &+ \sum_{k=1}^{n_{0}} \binom{n_{0} + 1}{k+1}(-1)^{k+1}E_{\nu,1}(-\mu kt^{\nu}) \\ &- \left[\sum_{k=1}^{n_{0}} \binom{n_{0} + 1}{k+1}(-1)^{k+1}E_{\nu,1}(-k\mu t^{\nu})\right]^{2}. \end{aligned}$$

Note that the algorithm above could be easily adopted to simulate sample trajectories of the fractional sublinear death process.

#### **3** Parameter estimation

3.1 Estimation for the fractional Yule or linear birth process

We now illustrate our estimation approach for the fYp with birth rate  $\lambda i, i \geq 1$ . Furthermore, assume that a sample trajectory of n births corresponding to n random inter-birth times  $T_i$ 's of the fractional linear birth process is observed. That is, n independent but are not identically distributed random inter-birth times of the fractional linear birth process are given. This also insinuates that only a single datum is obtained from each of the n different Mittag-Leffler distributions. This observation and the Mittag-Leffler's seemingly complex structure pose a computational challenge on how to estimate the model parameters more efficiently especially for small population sizes. Recall the structural representation of the Mittag-Leffler distributed random inter-birth time  $T_i \stackrel{d}{=} E^{1/\nu} S_{\nu}$  (see Cahoy and Polito 2012), where  $E \stackrel{d}{=} \exp(\lambda i)$  is independent of  $S_{\nu}$  which is a one-sided  $\alpha^+$ -stable distributed random variable. Applying the logarithmic transformation and taking the

expectation on both sides, it can be easily shown that the mean and variance (see details in Cahoy et al. 2010) of the log-transformed *i*-th random sojourn time  $T'_i = \ln(T_i)$  of the fYp are

$$\mu_{T_i'} = \frac{-\ln\left(\lambda i\right)}{\nu} - \gamma,\tag{3.1}$$

and

$$\sigma_{T_i}^2 = \pi^2 \left( \frac{1}{3\nu^2} - \frac{1}{6} \right), \tag{3.2}$$

respectively, where  $\gamma \approx 0.5772156649$  is the Euler - Mascheroni's constant. The first two moments above therefore suggest that the following simple linear regression model can be fitted/formulated:

$$T'_{i} = a_0 + a_1 \ln i + \varepsilon_i, \qquad i = 1, \dots, n,$$
(3.3)

where

$$a_0 = \frac{-\ln(\lambda)}{\nu} - \gamma, \qquad a_1 = \frac{-1}{\nu},$$
 (3.4)

and  $\varepsilon_i \stackrel{iid}{=} N\left(\mu_{\varepsilon} = 0, \sigma_{\varepsilon}^2 = \sigma_{T'_i}^2\right)$ . The trick used here was to factor out the non-identical means of the log-transformed random inter-birth or sojourn times, which are linear functions of the logarithm of the known fixed *i*. Thus, this leads to studying the widely used simple linear regression model (see Montgomery et al. 2006).

# 3.1.1 Point estimation

Inverting the least squares (LS) estimators

$$\hat{a}_{1} = \frac{\sum_{j=1}^{n} T'_{j} \left( \ln j - \overline{\ln i} \right)}{\sum_{j=1}^{n} \left( \ln j - \overline{\ln i} \right)^{2}}$$
(3.5)

and  $\hat{a}_0 = \overline{T'_i} - \hat{a}_1 \cdot \overline{\ln i}$  gives the LS-based point estimators of  $\nu$  and  $\lambda$  as

$$\widehat{\nu}_{ls} = \frac{-1}{\widehat{a}_1} \tag{3.6}$$

and

$$\widehat{\lambda}_{ls} = \exp\left(\left(\widehat{a}_0 + \gamma\right) / \widehat{a}_1\right), \qquad (3.7)$$

respectively, where  $\overline{\ln i} = \sum_{j=1}^{n} \ln j/n$ , and  $\overline{T'} = \sum_{j=1}^{n} T'_j/n$ . Equating  $\sigma_{\varepsilon}^2$  or  $\sigma_{T'_i}^2$  in (3.1) with its unbiased estimator

$$\widehat{\sigma}_u^2 = \sum_{j=1}^n \widehat{\varepsilon}_j^2 / (n-2), \qquad (3.8)$$

we get the residual-based point estimators

$$\hat{\nu}_{res} = \frac{1}{\sqrt{3\left(\hat{\sigma}_u^2/\pi^2 + \frac{1}{6}\right)}}$$
 (see Cahoy et al. 2010)

and

$$\widehat{\lambda}_{res} = \exp\left(-\widehat{\nu}_{res}\left(\widehat{a}_0 + \gamma\right)\right) \tag{3.10}$$

of the model parameters  $\nu$  and  $\lambda$ , correspondingly where  $\widehat{\varepsilon}_i = T'_i - \widehat{T'_i}$ , and  $\widehat{T'_i} = \widehat{a}_0 + \widehat{a}_1 \ln i$ . Note that the residual-based estimators exploit the residuals to estimate  $\nu$  rather than the negative inverse of the LS estimate of the slope  $a_1$ .

### 3.1.2 Interval estimation

We now develop interval estimators using the largesample properties of the least squares estimators  $\hat{b}_0$  and  $\hat{b}_1$  above. The following result shows the joint asymptotic behavior of the proposed point estimators of  $\nu$  and  $\lambda$  for the fYp.

**Theorem 3.1** Let  $0 < \nu \leq 1$  and  $\lambda > 0$ . Then

$$\sqrt{n} \left( \frac{\widehat{\nu}_{ls} - \nu}{\widehat{\lambda}_{ls} - \lambda} \right) \stackrel{d}{\longrightarrow} N\left[ \mathbf{0}, n\sigma_{\varepsilon}^{2} \mathbf{C} \right]$$

where " $\stackrel{d}{\longrightarrow}$ " denotes convergence in distribution,

$$\mathbf{C} = \begin{pmatrix} C_1 & C_{12} \\ C_{21} & C_2 \end{pmatrix},$$

$$C_1 = \nu^4 s^{-1},$$

$$C_{12} = C_{21} = \lambda \nu^3 \left( \left( \overline{\ln i} + \ln(\lambda) \right) / s \right),$$

$$C_2 = (\nu \lambda)^2 \left( 1/n + \left( \overline{\ln i^2} + 2\ln(\lambda) \overline{\ln i} + (\ln(\lambda))^2 \right) / s \right),$$
and  $s = \sum_{j=1}^n \left( \ln j - \overline{\ln i} \right)^2.$ 

Proof Recall the large-sample normality of the least squares estimators  $\hat{a}_0$  and  $\hat{a}_1$ , i.e.,

$$\sqrt{n} \left( \begin{array}{c} \widehat{a}_0 - a_0 \\ \widehat{a}_1 - a_1 \end{array} \right) \stackrel{d}{\longrightarrow} N[\mathbf{0}, \mathbf{\Sigma}]$$

where the covariance matrix  $\Sigma$  is defined as

$$\boldsymbol{\Sigma} = n\sigma_{\varepsilon}^{\mathbf{2}} \begin{pmatrix} \left(1/n + \overline{\ln i}^2/s\right) & -\overline{\ln i}/s \\ & & \\ & -\overline{\ln i}/s & s^{-1} \end{pmatrix}.$$

Recall the multivariate delta method (Ferguson 1996): If  $\sqrt{n}(\widehat{\beta}_n - \beta) \longrightarrow N[\mathbf{0}, \Sigma]$  then

$$\sqrt{n} \left( \boldsymbol{g}(\widehat{\boldsymbol{\beta}}_n) - \boldsymbol{g}(\boldsymbol{\beta}) \right) \stackrel{d}{\to} N \left[ \boldsymbol{0}, \ \dot{\boldsymbol{g}}(\boldsymbol{\beta})^T \boldsymbol{\Sigma} \dot{\boldsymbol{g}}(\boldsymbol{\beta}) \right].$$

Hence, using the delta method above,  $\hat{\boldsymbol{\beta}}_n = (\hat{a}_0, \hat{a}_1)^T$ ,  $\boldsymbol{g}(\hat{\boldsymbol{\beta}}_n) = (\hat{\nu}_{ls}, \hat{\lambda}_{ls})^T$ ,  $\boldsymbol{g}(\boldsymbol{\beta}) = (\nu, \lambda)^T$ , and the Jacobian matrix

$$\dot{\boldsymbol{g}}(\boldsymbol{\beta}) = \begin{pmatrix} 0 & \exp\left((a_0 + \gamma)/a_1\right)/a_1\\ \\ 1/a_1^2 & -\exp\left((a_0 + \gamma)/a_1\right)(a_0 + \gamma)/a_1^2 \end{pmatrix}$$

we obtain the final expression of the covariance matrix by simply substituting back  $a_0 = -\ln(\lambda)/\nu - \gamma$  and  $a_1 = -1/\nu$ .

**Corollary 3.1** Approximate  $(1 - \alpha)100\%$  confidence intervals for  $\nu$  and  $\lambda$  can be deduced as

$$\widehat{\nu}_{ls} \pm z_{\alpha/2} \widehat{\sigma}_{\varepsilon} \widehat{\nu}_{ls}^2 \sqrt{s^{-1}}, \qquad (3.11)$$

and

(3.9)

$$\begin{aligned} \widehat{\lambda}_{ls} &\pm z_{\alpha/2} \widehat{\sigma}_{\varepsilon} \widehat{\nu}_{ls} \widehat{\lambda}_{ls} \Big( 1/n \\ &+ \left( \overline{\ln i}^2 + 2 \ln(\widehat{\lambda}_{ls}) \overline{\ln i} + (\ln(\widehat{\lambda}_{ls}))^2 \right) /s \Big)^{1/2}, \quad (3.12) \end{aligned}$$

respectively, where  $z_{\alpha/2}$  is the  $(1 - \alpha/2)$ th quantile of the standard normal distribution and  $0 < \alpha < 1$ .

We now propose another interval estimators which utilize the residual-based estimate of  $\nu$ , and a bootstrap technique. It can be inferred from Cahoy et al. (2010) that a residual-based  $(1 - \alpha)100\%$  confidence interval for  $\nu$  can be

$$\hat{\nu}_{res} \pm z_{\alpha/2} \sqrt{\frac{\hat{\nu}_{res}^2 \left(32 - 20\hat{\nu}_{res}^2 - \hat{\nu}_{res}^4\right)}{40n}}, \qquad (3.13)$$

where  $z_{\alpha/2}$  is defined above. A residual-based  $(1-\alpha)100\%$ interval estimate for  $\lambda$  can also be

$$\widehat{\lambda}_{res} \pm z_{\alpha/2} \left[ \frac{e^{-2\widehat{\nu}_{res}(\widehat{a}_0 + \gamma)} \left( \frac{\widehat{\nu}_{res}^2 (32 - 20\widehat{\nu}_{res}^2 - \widehat{\nu}_{res}^4)}{40} \right)}{n} + \widehat{\nu}_{res}^2 \widehat{\sigma}_u^2 \left( 1/n + \overline{\ln i}^2/s \right) \right]^{1/2}.$$
(3.14)

Since the small-sample performance of  $\hat{\nu}_{res}$  and the residual-based interval estimator in (3.13) have been shown to perform well already (see, e.g., Cahoy et al. 2010), we apply a non-parametric percentile bootstrap technique to  $\hat{\lambda}_{res}$  using the fixed-regressor approach to obtain a small-sample interval estimator of  $\lambda$ . This well-known procedure is slightly modified by first dividing each residual  $\hat{\varepsilon}$  by  $\sqrt{1-h_i}$ , where  $h_i$  is the *i*th leverage or the *i*th diagonal entry in the hat matrix before sampling from the transformed residuals. Note that the division of  $\sqrt{1-h_i}$  is simply for correction as the true variance of the residual  $\hat{\varepsilon}_i$  is  $\mathbb{Var} \ \hat{\varepsilon}_i =$ 

 $\sigma_{\varepsilon}^2(1-h_i)$  (see Montgomery et al. 2006). Hence, the bootstrap counterpart of  $\hat{\lambda}_{res}$  is calculated as  $\hat{\lambda}_{res}^* = \exp(-\hat{\nu}_{res}^*(\hat{a}_0^*+\gamma))$  where  $\hat{\nu}_{res}^*$  used the bootstrapped transformed or weighted residuals. A clear advantage of the asymptotic-based procedures over the re-samplingbased ones is that they are faster to calculate especially for large sample sizes.

# 3.2 Estimation for the fractional linear and the fractional sublinear death processes

Assuming that a sample trajectory of  $n_0$  deaths corresponding to  $n_0$  random inter-death times  $T_k^{\nu}$ 's of a fractional death process is observed. Following the procedure for the fractional linear birth process in the preceding subsection, we can estimate the parameters  $\nu$  and  $\mu$  by regressing  $\ln(T_k^{\nu})$  with  $\ln(n_0 - k)$ . That is, we fit the following simple linear regression model:

$$\ln (T_k^{\nu}) = b_0 + b_1 \ln(n_0 - k) + \varepsilon_k, \qquad (3.15)$$

where  $k = 0, ..., n_0 - 1, b_0 = -\ln(\mu)/\nu - \gamma, b_1$  is given in (3.4) of subsection 3.1, and  $\varepsilon_k \stackrel{iid}{=} N\left(\mu_{\varepsilon} = 0, \sigma_{\ln(T_k^{\nu})}^2\right)$ . Following the methodology in the preceding subsection, we can straightforwardly obtain the corresponding LSbased point estimates of  $\nu$  and  $\mu$  from (3.6) and (3.7) as

$$\widehat{\nu}_{ls} = \frac{-1}{\widehat{b}_1} \tag{3.16}$$

and

$$\widehat{\mu}_{ls} = \exp\left(\left(\widehat{b}_0 + \gamma\right) / \widehat{b}_1\right), \qquad (3.17)$$

respectively, where  $\overline{\ln(n_0 - k)} = \sum_{j=0}^{n_0 - 1} \frac{\ln(n_0 - j)}{n_0}, \overline{\ln(T^{\nu})} = \sum_{j=0}^{n_0 - 1} \frac{\ln(T_j^{\nu})}{n_0}, \hat{b}_0 = \overline{\ln(T^{\nu})} - \hat{b}_1 \cdot \overline{\ln(n_0 - k)}, \text{ and}$  $\hat{b}_1 = \frac{\sum_{j=0}^{n_0 - 1} \ln(T_j^{\nu}) \left(\ln(n_0 - j) - \overline{\ln(n_0 - k)}\right)}{\sum_{j=0}^{n_0 - 1} \left(\ln(n_0 - j) - \overline{\ln(n_0 - k)}\right)^2}.$  (3.18)

Furthermore, the LS-based interval estimates for  $\nu$  and  $\mu$  directly follow from (3.11) and (3.12) of Corollary 3.1 in subsection 3.1.2, correspondingly. Hence, the approximate  $(1-\alpha)100\%$  for  $\nu$  and  $\mu$  can be explicitly written as

$$\widehat{\nu}_{ls} \pm z_{\alpha/2} \widehat{\sigma}_{\varepsilon} \widehat{\nu}_{ls}^2 \sqrt{\left(\sum_{j=0}^{n_0-1} \left(\ln(n_0-j) - \overline{\ln(n_0-k)}\right)^2\right)^{-1}}$$

and

$$\begin{aligned} \widehat{\lambda}_{ls} &\pm z_{\alpha/2} \widehat{\sigma}_{\varepsilon} \widehat{\nu}_{ls} \widehat{\lambda}_{ls} \Big( 1/n \\ &+ \left( \overline{\ln(n_0 - k)}^2 + 2\ln(\widehat{\lambda}_{ls}) \overline{\ln(n_0 - k)} + (\ln(\widehat{\lambda}_{ls}))^2 \right) / s \Big)^{1/2}, \end{aligned}$$

correspondingly. On the other hand, the residual-based point and interval estimators of  $\nu$  and  $\mu$  immediately follow from subsection 3.1 as well, where  $\hat{a}_0$  is replaced by  $\hat{b}_0$  in (3.10),  $\hat{\varepsilon}_k = \ln(T_k^{\nu}) - \ln(T_k^{\nu})$ , and  $\ln(T_k^{\nu}) = \hat{b}_0 + \hat{b}_1 \ln(n_0 - k)$ .

A similar approach can be done to obtain estimates for the fractional sublinear death process. That is, we regress  $\ln(\mathfrak{T}_k^{\nu})$  with  $\ln(k+1), k = 0, 1, \ldots, n_0 - 1$ , or fit the model

$$\ln\left(\mathfrak{T}_{k}^{\nu}\right) = c_{0} + c_{1}\ln(k+1) + \varepsilon_{k}, \qquad (3.19)$$

and follow the procedures used for fractional Yule and the fractional linear death processes. In general, we simply replace  $\lambda, \ln i, \overline{\ln i}$  by  $\mu, \ln(n_0 - k)$  or  $\ln(k + 1)$ , and  $\overline{\ln(n_0 - k)}$  or  $\overline{\ln(k + 1)}$ , accordingly in the methods of subsection 3.1 to obtain the parameter estimators for the fractional linear death and the fractional sublinear death processes.

### 3.3 Some Extensions

Assume that a fractional birth or death process exists with rates  $\theta_j$ , j = 1, 2, ..., n, where the *j*th inter-event time  $X_j$  is Mittag-Leffler distributed with parameter  $\theta_j$ . Then the mean of  $X'_j = \ln(X_j)$  is

$$\mu_{X_j'} = \frac{-\ln\left(\theta_j\right)}{\nu} - \gamma. \tag{3.20}$$

Based on the above mean formulation, we use the model

$$X'_{j} = d_0 + d_1 \cdot q(j) + \varepsilon_j \tag{3.21}$$

to estimate more forms of the parameters or rates under the two cases below.

Case 1: When  $\ln(\theta_j) = m(\theta) + q(j)$  for some appropriate known functions  $m(\theta)$  and q(j) of the parameter  $\theta$  and  $j \in \mathbb{N}$ , correspondingly.

In this case, the general form of the regression model that could be used for estimation is

$$X'_{j} = -\left(\gamma + \frac{m(\theta)}{\nu}\right) - \frac{1}{\nu} \cdot q(j) + \varepsilon_{j}.$$
(3.22)

Clearly,  $d_0 = -(\gamma + m(\theta)/\nu)$ ,  $d_1 = -1/\nu$ , and q(j) is the regressor variable. Using  $\hat{\nu}_{res}$  or  $-1/\hat{d}_1$  and inverting the least squares estimate  $\hat{b}_0$ , we can compute  $\widehat{m(\theta)}$  and

 $\hat{\theta}$  sequentially. Note that the explicitness of  $\hat{\theta}$  depends on the form of m.

Example 1: When  $\theta_j$  is linear, i.e.,  $\theta_j = \theta_j$  then  $\ln(\theta_j) = \ln(\theta) + \ln(j)$ , where  $m(\theta) = \ln(\theta)$  and  $q(j) = \ln(j)$ , respectively. Note that this parametrization corresponds to the fractional Yule, the fractional linear death, and the fractional sublinear death processes.

Example 2: If  $\theta_j = e^{\theta+j}$  then  $\ln(\theta_j) = \theta + j$ , where  $m(\theta) = \theta$  and q(j) = j, correspondingly. This suggests that  $d_0 = -(\gamma + \theta/\nu)$ .

Case 2: When  $\ln(\theta_j) = m(\theta) \cdot q(j)$  for some appropriate known functions  $m(\theta)$  and q(j) of the parameter  $\theta$  and  $j \in \mathbb{N}$ , correspondingly.

The general form of the regression model in this case is

$$X_{j}^{'} = -\gamma - \frac{m(\theta)}{\nu} \cdot q(j) + \varepsilon_{j}.$$
(3.23)

Apparently,  $d_0 = -\gamma, d_1 = -m(\theta)/\nu$ , and q(j) is the predictor variable. Using  $\hat{\nu}_{res}$  and inverting the least squares estimate  $\hat{d}_1$ , we can calculate  $\widehat{m(\theta)}$  and  $\hat{\theta}$  successively.

Example 1: If  $\theta_j = \theta^j$  then  $\ln(\theta_j) = \ln(\theta) \cdot j$ , where  $m(\theta) = \ln(\theta)$  and q(j) = j, correspondingly. This indicates that  $d_1 = -\ln(\theta)/\nu$ .

Example 2: When  $\theta_j = e^{\theta \cdot j}$  then  $\ln(\theta_j) = \theta \cdot j$ , where  $m(\theta) = \theta$  and q(j) = j, respectively. This shows that  $d_1 = -\theta/\nu$ .

### 4 Method testing and application

### 4.1 Empirical test

For the sake of reproducibility, we now test our procedures using the fYp as a particular example as similar approach can be carried out for both the fractional linear death and the fractional sublinear death processes. In point estimation testing, we evaluated the finite-sample properties (unbiasedness and homogeneity) by computing the average and the median absolute deviation (MAD) of the estimates using 1000 simulations for sample sizes n = 100,500, and 1000. These values are shown in Table 4.1 below. The relative fluctuation (RF=100% ×MAD/mean) of  $\hat{\nu}_{ls}$  decreases from 19.23% (corresponds to  $\nu = 0.1$ , n = 100) to as little as 4.41% (with  $\nu = 0.95$  and n = 1000). On the other hand, the residual-based  $\hat{\nu}_{res}$ 's RF ranges from 4.41% (with  $\nu = 0.95$  and n = 1000) to 1.89% (corresponds to  $\nu = 0.95$ , n = 1000). While  $\hat{\lambda}_{ls}$ 's RF improves from

33.94% (corresponds to  $\lambda = 0.5$ , n = 100) to 30.48% (with  $\lambda = 5$  and n = 1000),  $\hat{\lambda}_{res}$ 's RF decays faster from 55% ( $\lambda = 1$ , n = 100) to 25.29% ( $\lambda = 5$  and n = 1000). In general, the relative fluctuations of the residual-based estimators tend to decay faster than the LS-based estimators especially for  $n \leq 100$ . Nonetheless, both the residual- and LS-based point estimators are asymptotically unbiased as expected.

Table 4.2 below shows the averaged lower and upper 95% confidence bounds using the formulae in Section 3. These bounds used 1000 simulation runs for each of the sample sizes n = 15, 30, 100, and 500. Note that the residual-based interval estimator  $\lambda^*_{res}$  utilized 500 bootstrap samples and  $\hat{\sigma}_{\varepsilon}^2 = \pi^2 \left( 1/(3\hat{\nu}_{res}^2) - 1/6 \right)$  is used to estimate the error variance in our LS-based procedures. Observe that some of the interval estimates for sample sizes n = 15, and n = 30 are omitted as they are unreliable due to the multiple error warnings that showed up during the computation process. Moreover, the convergence of the coverage probabilities to their true levels for the LS-based method is made faster by using the error variance estimate  $\hat{\sigma}_{\varepsilon}^2 = \pi^2 \left( 1/(3\hat{\nu}_{ls}^2) - 1/6 \right).$ From Table 4.2, it is apparent that the residual-based interval estimates of  $\nu$  are narrower and are better centered around the true parameter values than the leastsquares' even when the sample size is as large as 500. In addition, our simulations showed that the asymptotic or non-bootstrapped residual-based interval estimator of  $\lambda$  gives more sensible results than the LS-based procedure for small samples. Nevertheless, the LS-based interval estimates for  $\lambda$  are more accurately centered than the bootstrapped residual-based estimates especially for large samples.

The corresponding coverage probabilities and the widths of the interval estimates above with a confidence level of 95% are displayed in Table 4.3. When the sample size n = 15, the residual-based interval estimators have minimum coverage of 90.1% and 91.1% for  $\nu = 0.95$  and  $\lambda = 0.1$ , respectively. When n = 500, the bootstrap interval estimator of  $\lambda$  has coverage probabilities which are closer to the true confidence level than the LS-based procedure for large values of  $\lambda$ . However, the LS-based estimator of  $\lambda$  has a better coverage than the bootstrapped residual-based interval estimator for small  $\lambda$  values. The residual-based interval estimator for  $\lambda$  seemed to have slower convergence than the LS-based method. Furthermore, the residual-based estimator of  $\nu$  outperformed the LS-based method as its coverage

( ))	E	n = 100		n = 500		n = 1000	
$( u, \lambda)$	Estimator	Mean	MAD	Mean	MAD	Mean	MAD
	$\widehat{\nu}_{ls}$	0.104	0.020	0.101	0.008	0.100	0.006
(0.1, 1)	$\widehat{\nu}_{res}$	0.103	0.008	0.100	0.004	0.100	0.003
(0.1, 1)	$\widehat{\lambda}_{ls}$	3.190	0.665	1.151	0.407	1.077	0.318
	$\widehat{\lambda}_{res}$	1.318	0.725	1.091	0.408	1.051	0.322
	$\widehat{\nu}_{ls}$	0.261	0.048	0.252	0.021	0.251	0.014
(0.25, 0.1)	$\hat{\nu}_{res}$	0.252	0.022	0.251	0.010	0.250	0.007
(0.20, 0.1)	$\widehat{\lambda}_{ls}$	0.119	0.031	0.106	0.025	0.103	0.021
	$\widehat{\lambda}_{res}$	0.131	0.071	0.109	0.044	0.106	0.033
	$\widehat{ u}_{ls}$	0.521	0.100	0.505	0.040	0.501	0.028
(0.5, 0.5)	$\widehat{\nu}_{res}$	0.506	0.041	0.501	0.018	0.500	0.014
(0.0, 0.0)	$\lambda_{ls}$	0.825	0.280	0.565	0.190	0.532	0.142
	$\widehat{\lambda}_{res}$	0.640	0.350	0.555	0.216	0.528	0.164
	$\widehat{ u}_{ls}$	0.774	0.121	0.755	0.052	0.751	0.036
(0.75, 0.25)	$\widehat{\nu}_{res}$	0.755	0.056	0.752	0.023	0.750	0.016
(0.10, 0.20)	$\lambda_{ls}$	0.313	0.093	0.266	0.069	0.259	0.056
	$\widehat{\lambda}_{res}$	0.300	0.146	0.268	0.094	0.260	0.072
	$\widehat{\nu}_{ls}$	0.969	0.131	0.953	0.058	0.952	0.042
(0.95, 5)	$\widehat{\nu}_{res}$	0.955	0.055	0.950	0.024	0.950	0.018
(0.00, 0)	$\widehat{\lambda}_{ls}$	11.251	3.492	5.836	2.104	5.397	1.645
	$\widehat{\lambda}_{res}$	5.978	2.544	5.375	1.635	5.206	1.317

**Table 4.1** Mean point estimates of and dispersions from the true parameters  $\nu$  and  $\lambda$ .

**Table 4.2** Average 95% confidence intervals for different values of  $\nu$  and  $\lambda$ .

$( u, \lambda)$	Estimator	n = 15	n = 30	n = 100	n = 500
	$\hat{\nu}_{ls}$			(0.063, 0.142)	(0.084, 0.117)
(0.1, 1)	$\widehat{\nu}_{res}$	(0.060, 0.158)	(0.071 , 0.137)	(0.083, 0.118)	(0.092, 0.108)
(0.1, 1)	$\widehat{\lambda}_{ls}$			(-35.8067, 58.864)	(0.108, 2.228)
	$\widehat{\lambda}_{res}$			(-0.362, 2.885)	(0.183, 2.038)
	$\widehat{\lambda}^*_{res}$	(0.214, 53.963)	(0.243, 16.667)	(0.297 , 5.560)	(0.464, 2.652)
	$\widehat{\nu}_{ls}$			(0.162, 0.359)	(0.211, 0.292)
(0.25, 0.1)	$\widehat{\nu}_{res}$	(0.151, 0.389)	(0.211, 0.298)	(0.209, 0.296)	(0.231, 0.269)
(0.20, 0.1)	$\widehat{\lambda}_{ls}$			(0.034, 0.201)	(0.052, 0.159)
	$\widehat{\lambda}_{res}$			(-0.031, 0.291)	(0.018, 0.202)
	$\widehat{\lambda}^*_{res}$	(0.015, 2.183)	(0.020 , 1.271)	$(0.028 \ , \ 0.551)$	(0.044, 0.265)
	$\widehat{\nu}_{ls}$			(0.336, 0.704)	(0.427, 0.580)
(0.5, 0.5)	$\hat{\nu}_{res}$	(0.319, 0.749)	(0.366, 0.664)	(0.424, 0.586)	(0.464, 0.536)
(0.0, 0.0)	$\widehat{\lambda}_{ls}$			(-0.261, 1.881)	(0.151, 0.968)
	$\widehat{\lambda}_{res}$			(-0.103, 1.389)	(0.121, 0.978)
	$\widehat{\lambda}^*_{res}$	(0.099, 9.154)	(0.113, 5.261)	(0.164, 2.542)	(0.241, 1.246)
	$\widehat{\nu}_{ls}$			(0.534, 1.019)	(0.648, 0.855)
(0.75, 0.25)	$\hat{\nu}_{res}$	(0.527, 1.057)	(0.573, 0.953)	(0.648, 0.857)	(0.704, 0.798)
(0.10, 0.20)	$\widehat{\lambda}_{ls}$			(0.062, 0.543)	(0.117, 0.409)
	$\widehat{\lambda}_{res}$			(-0.008, 0.618)	(0.076, 0.453)
	$\widehat{\lambda}^*_{res}$	(0.054, 3.070)	(0.067 , 2.206)	(0.087, 1.055)	(0.127, 0.573)
	$\widehat{ u}_{ls}$			(0.700, 1.219)	(0.839, 1.067)
(0.95, 5)	$\hat{\nu}_{res}$	(0.719, 1.221)	(0.779, 1.150)	(0.848, 1.059)	(0.904, 0.999)
(0.00,0)	$\widehat{\lambda}_{ls}$			(-2.5731, 17.648)	(0.963, 10.334)
	$\widehat{\lambda}_{res}$			(0.285, 10.951)	(1.976, 8.557)
	$\widehat{\lambda}^*_{res}$	(1.573, 49.872)	(1.602, 30.456)	(2.006, 16.841)	(2.714, 10.067)

probabilities are closer to 95%, and has narrower intervals. Overall, the coverage probabilities and interval widths still provide good merits for our estimators even when the sample size is as small as n = 15.

Collectively, Tables 4.1–4.3 strongly indicate that the proposed point and interval estimators performed well in our computational tests. We emphasize that the point estimates could also be regarded as reasonable starting values for better iterative estimation algorithms.

### **5** Application

We now apply our proposed methods to a real dataset. In particular, we estimate the parameters of the fractional Yule model using the branching times for plethodontid salamander dataset from Highton and Larson (1979) (see also Nee et al. 1994a; Nee 2001). The 25 data points are the times measured from each node to the present of a phylogenetic tree, and can be downloaded from the package laser of the R software. The summary statistics of the inter-branching times of the plethodontid dataset are given in Table 5.1 below.

The point and the 95% confidence interval estimates are given in Table 5.2. The LS-based point estimate (0.749) of the fractional parameter  $\nu$  seemed to suggest that the plethodontid salamandar branching process is not a standard Yule process while the residualbased point estimate (1.119) appeared to suggest otherwise. Moreover, both the LS- and residual-based interval estimates of  $\nu$  indicated that  $\nu$  could be strictly less than one, which implies that a non-standard Yule process could model the plethodontid salamandar dataset with a confidence level of 95%. The residual-based point estimate (0.011) of  $\lambda$  is more conservative than the bootstrap- and LS-based estimate (0.049). A similar observation can be gleaned from the 95% interval estimates, i.e., the residual-based 95% interval estimate is narrower than the bootstrap- and LS-based interval estimates.

We also tested the residuals for normality using the Shapiro-Wilk, Anderson-Darling, Cramer-von Mises, Lilliefors, Pearson chi-square, and the Shapiro-Francia tests, which gave the p-values 0.811, 0.651, 0.619, 0.609, 0.849, and 0.461, correspondingly. Hence, these p-values indicated good fit of the fractional Yule process to the plethodontid salamandar data.

### 6 Concluding remarks

We have proposed closed-form expressions of the estimators of the parameters  $\nu$  and  $\lambda$  for the fractional linear birth or Yule, the fractional linear death, and the fractional sublinear death processes. The estimators were derived by taking advantage of the known structural form of the logarithm of the random inter-event times and the well-studied least squares regression procedure. The explicit formulas led to computationally simple and fast parameter estimation procedures. The inter-death time distributions and variances of the fractional linear and sublinear death processes were also obtained. These statistical properties were necessary for generating sample trajectories and for our estimation procedures to be applicable in these processes. It has also been shown that the proposed procedure can be easily extended to certain models that have different model parameterizations than the linear ones. The proposed methods were used to model a real physical process. Generally, the extensive computational tests showed favorable results for the proposed estimators.

We cite some extensions which would be worth pursuing in the future. For instance, improving the smallsample performance of the least squares-based estimators and developing other estimators using the likelihood approach or a re-sampling technique would be valuable pursuits. The application of these methods in practice, and the characterization of the appropriate functions  $m(\theta)$  and q(j) would also be of interest.

# References

- Aldous, D.J.: Stochastic models and descriptive statistics for phylogenetic trees, from Yule to today, Statistical Science 16(1), 23–34 (2001)
- Cahoy D.O., Polito, F.: Simulation and estimation for the fractional Yule process. Methodology and Computing in App. Prob. 14(2), 383–403 (2012)
- Cahoy, D.O., Uchaikin, V.V., Woyczynski, W.A.: Parameter estimation for fractional Poisson processes. J. Stat. Plan. Inf. 140, 3106–3120 (2010)
- Efron, B., Tibshirani, R.: An Introduction to the Bootstrap. CRC Press, Boca Raton, FL (1994)
- Ferguson, T.: A Course in Large Sample Theory, Chapman & Hall, Great Britain (1996)
- Highton, R., Larson, L. A.: The genetic relationships of the salamanders of the genus Plethodon. Syst. Zool. 28, 579–599 (1979)
- Montgomery, D.C., Peck, E.A., Vining, G.G.: Introduction to linear regression analysis 4 ed, John Wiley & Sons, Inc., Great Britain, United States of America (2006)

$( u, \lambda)$	Estimator	n = 15		n = 30		n = 100		n = 500	
( <i>ν</i> , <i>x</i> )	Estimator	Coverage	Width	Coverage	Width	Coverage	Width	Coverage	Width
	$\widehat{\nu}_{ls}$					0.949	0.078	0.952	0.033
(0.1, 1)	$\widehat{\nu}_{res}$	0.956	0.098	0.958	0.066	0.956	0.035	0.957	0.016
(0.1, 1)	$\widehat{\lambda}_{ls}$					0.893	94.671	0.915	2.120
	$\widehat{\lambda}_{res}$					0.900	3.248	0.925	1.854
	$\widehat{\lambda}^*_{res}$	0.919	65.411	0.928	16.424	0.946	5.131	0.946	2.161
	$\widehat{\nu}_{ls}$					0.937	0.197	0.958	0.081
(0.25, 0.1)	$\widehat{\nu}_{res}$	0.947	0.238	0.953	0.162	0.950	0.087	0.956	0.038
(0.20, 0.1)	$\widehat{\lambda}_{ls}$					0.940	0.167	0.949	0.107
	$\widehat{\lambda}_{res}$					0.895	0.322	0.925	0.184
	$\widehat{\lambda}^*_{res}$	0.911	2.168	0.938	1.251	0.952	0.521	0.951	0.219
	$\widehat{\nu}_{ls}$					0.946	0.367	0.933	0.153
(0.5, 0.5)	$\widehat{\nu}_{res}$	0.952	0.430	0.953	0.298	0.954	0.161	0.949	0.072
(0.0, 0.0)	$\hat{\lambda}_{ls}$					0.931	2.141	0.923	0.817
	$\widehat{\lambda}_{res}$					0.920	1.492	0.921	0.857
	$\widehat{\lambda}^*_{res}$	0.946	9.055	0.950	5.148	0.947	2.379	0.948	1.004
	$\widehat{\nu}_{ls}$					0.931	0.485	0.942	0.207
(0.75, 0.25)	$\widehat{\nu}_{res}$	0.921	0.529	0.945	0.379	0.934	0.209	0.956	0.094
(0110,0120)	$\lambda_{ls}$					0.941	0.481	0.950	0.291
	$\widehat{\lambda}_{res}$					0.903	0.627	0.934	0.377
	$\widehat{\lambda}^*_{res}$	0.916	3.015	0.931	2.139	0.951	0.956	0.952	0.442
	$\widehat{\nu}_{ls}$					0.923	0.519	0.945	0.228
(0.95, 5)	$\widehat{\nu}_{res}$	0.901	0.502	0.927	0.317	0.941	0.210	0.947	0.095
(0.00,0)	$\widehat{\lambda}_{ls}$					0.899	20.221	0.939	9.371
	$\widehat{\lambda}_{res}$					0.943	10.666	0.944	6.581
	$\widehat{\lambda}^*_{res}$	0.919	48.299	0.945	28.854	0.951	14.748	0.948	7.356

**Table 4.3** Coverage probabilities and mean widths of 95% interval estimates for different values of  $\nu$  and  $\lambda$ .

**Table 5.1** Summary statistics for the plethodontid dataset.

Minimum	First quartile	Median	Mean	Third quartile	Maximum	Standard deviation
0.090	0.607	1.315	3.981	4.405	22.120	5.966

**Table 5.2** Point and 95% interval estimates for  $\nu$  and  $\lambda$  of the plethodontid salamander data.

Estimator	Point estimate	Interval estimate
$\widehat{ u}_{ls}$	0.749	( 0.182 , 1.317 )
$\widehat{ u}_{res}$	1.119	$(0.955 \ , \ 1.283 \ )$
$\widehat{\lambda}_{ls}$	0.049	$( \ 0.008 \ , \ 0.089 \ )$
$\widehat{\lambda}_{res}$	0.011	(-0.005,0.027)
$\widehat{\lambda}_{res}^{*}$		$( \ 0.003 \ , \ 0.051 \ )$

Nee, S., Holmes, E.C., May, R.M., Harvey, P.H.: Extinction rates can be estimated from molecular phylogenies. Philos. Trans. R. So. Lond. B 344, 77-82 (1994a)

- Nee, S.: Inferring speciation rates from phylogenies. Evolution 55, 661-668. (2001)
- Orsingher, E., Polito, F.: Fractional pure birth processes. Bernoulli 16, 858–881 (2010)
- Orsingher, E., Polito, F.: Randomly stopped nonlinear fractional birth processes. Submitted (2011)
- Orsingher, E., Polito, F., Sakhno, L.: Fractional non-Linear, linear and sublinear death processes. J. Stat. Phys. 141, 68–93. (2010).

- Paradis, E.: Analysis of Phylogenetics and Evolution with R, 2 ed, Springer, New York, USA (2012)
- Uchaikin, V.V., Cahoy, D.O., Sibatov, R.T.: Fractional processes: from Poisson to branching one. International J. Bifurcation 18, 2717–2725 (2008)