# Modularized Bayesian analyses and cutting feedback in likelihood-free inference

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#### Abstract

There has been much recent interest in modifying Bayesian inference for misspecified models so that it is useful for specific purposes. One popular modified Bayesian inference method is "cutting feedback" which can be used when the model consists of a number of coupled modules, with only some of the modules being misspecified. Cutting feedback methods represent the full posterior distribution in terms of conditional and sequential components, and then modify some terms in such a representation based on the modular structure for specification or computation of a modified posterior distribution. The main goal of this is to avoid contamination of inferences for parameters of interest by misspecified modules. Computation for cut posterior distributions is challenging, and here we consider cutting feedback for likelihood-free inference based on Gaussian mixture approximations to the joint distribution of parameters and data summary statistics. We exploit the fact that marginal and conditional distributions of a Gaussian mixture are Gaussian mixtures to give explicit approximations to marginal or conditional posterior distributions so that we can easily approximate cut posterior analyses. The mixture approach allows repeated approximation of posterior distributions for different data based on a single mixture fit, which is important for model checks which aid in the decision of whether to "cut". A semi-modular approach to likelihood-free inference where feedback is partially cut is also developed. The benefits of the method are illustrated in two challenging examples, a collective cell spreading model and a continuous time model for asset returns with jumps.

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# 1 Introduction

Statisticians are increasingly using complex models which can be thought of as a collection of coupled modules. The modules represent different aspects of our knowledge of the problem, and in a Bayesian analysis each module consists of likelihood terms for different data sources and hierarchical prior terms for parameters or latent variables. There is much recent interest in ways to modify Bayesian inference so that it is fit for purpose when the model is misspecified, and for modularized Bayesian analyses so-called "cutting feedback" methods are common. The main goal of such methods is to ensure that inference about parameters of interest is not contaminated by misspecified modules. Cutting feedback approaches consider representations of the conventional Bayesian posterior distribution in terms of conditional or sequential components, but then modify certain terms for specification or computation of a modified posterior distribution. Existing cutting feedback methods have been developed in the context where the likelihood is tractable, and the purpose of this work is to develop suitable methods for intractable likelihood settings, where likelihood-free computational methods not requiring likelihood evaluations are used.

To understand the motivation for cutting feedback methods, it is helpful to consider their use in pharmacokinetic/pharmacodynamic (PK/PD) modelling (Bennett and Wakefield, 2001; Lunn et al., 2009). This is one of the areas in which cutting feedback methods were first used and formalized. In PK/PD applications, models with a two module structure are often considered. One of the modules is a pharmacokinetic (PK) model describing the evolution of a drug concentration in the blood stream, and the other is a pharmacodynamic (PD) model which describes the effects of the drug on the body. There are module-specific data sources informing corresponding module parameters, and the output of the PK module is used as an input to the PD module. Often it can be difficult to specify the PD module adequately. This can result in contamination of inferences of interest due to the misspecification, as parameters of the PK module adapt to accommodate the misspecification in the PD module. Cutting feedback methods have been used to prevent harmful effects arising from misspecification in a situation like this, while still appropriately propagating uncertainty. There are many other applications of cutting feedback methods – see Bayarri et al. (2009) and Jacob et al. (2017) for further discussion of these and modularized Bayesian analyses more generally. Pompe and Jacob (2021) and Frazier and Nott (2022) have recently studied the theoretical behaviour of the cut posterior distribution.

How are the goals of cutting feedback achieved? One way of formulating cutting feedback involves the modification of the steps of an MCMC sampling algorithm, where the cut posterior distribution is defined implicitly as the stationary distribution of the sampler. The "cut" function in the WinBUGS and OpenBUGS packages (see Lunn et al. (2009) for details) is one way to make this operational. For a Gibbs sampling approach, the modified MCMC sampler draws parameter blocks from distributions obtained by removing some terms in the conventional posterior full conditional distributions. Removing misspecified terms when forming some of the full conditionals reduces the influence of misspecified components on the final inference. The modified full conditional distributions are not in general consistent with any well-defined joint distribution (see for example Clarté et al. (2020) and Rodrigues et al. (2020) for a discussion of Gibbs sampling for inconsistent conditionals in the likelihood-free inference setting). When exact Gibbs sampling is intractable, it is natural to use Metropolis-within-Gibbs steps in detailed balance with the modified conditional distributions. In this case, the stationary distribution of the resulting Markov chain may depend on the proposal distribution used (Woodard et al., 2013; Plummer, 2015). To clarify the idea of cutting feedback, Plummer (2015) and several other authors have considered cutting feedback for a certain "two module" system discussed further in Section 2, which is general enough to cover many applications of interest. In this case, it is possible to characterize the cut posterior distribution explicitly. This cut posterior distribution is not easy to sample from in most cases, due to the need to calculate or approximate a difficult normalizing constant.

In this work we develop cutting feedback for likelihood-free inference. Likelihood-free inference methods are used with complex models where computation of the likelihood is impractical, but where it is possible to simulate data under the model for any given value of the parameter. We discuss these methods further in Section 3. To address the computational challenges of cutting feedback in the likelihood-free setting, we use Gaussian mixture approximations to the joint distribution of parameters and data summary statistics. We exploit the fact that marginal and conditional distributions of a Gaussian mixture are Gaussian mixtures to give explicit approximations to marginal or conditional posterior distributions for any summary statistic value based on a single fitted mixture model, so that we can easily approximate cut posterior analyses and perform appropriate diagnostics. To the best of our knowledge, our work is the first time that cutting feedback methods have been considered for likelihood-free inference. A semi-modular inference approach where feedback is partially cut is also developed, extending work by Carmona and Nicholls (2020) and Nicholls et al. (2022) to the case of likelihood-free inference.

In the next section we discuss cutting feedback for the two module system discussed in Plummer (2015). This is followed by a discussion of mixture approximations for likelihood-free inference. Section 4 describes how mixture approximations used in likelihood-free inference are able to address some of the computational difficulties of cutting feedback. Use of these mixture approximations for semi-modular inference methods is then discussed, and our methodology is applied to two challenging examples in Section 6, a collective cell spreading model and a continuous time model for asset returns with jumps. Section 7 gives some concluding discussion.

## 2 Cutting feedback

To describe cutting feedback methods and clarify previous "implicit" definitions of cutting feedback in terms of modified MCMC algorithms, Plummer (2015) considered the two module system represented graphically in Figure 1. Our discussion in this paper will be restricted to the case of two modules, which is general enough to cover many applications of interest. For a discussion of cutting feedback in a more general context, see Lunn et al. (2009). In Figure 1,

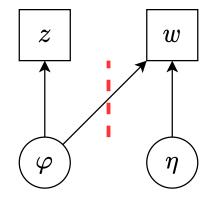


Figure 1: Graphical representation of a two module system with cutting feedback. The two modules are represented by the components on the left and right of the dashed line, which indicates the cut.

the complete data, which we will write as y, is comprised of two data sources, w and z. The distribution of z depends on parameter  $\varphi$ , and the distribution of w depends on parameters  $\eta$  and  $\varphi$ . We have two modules, the first consisting of the prior for  $\varphi$  and likelihood term for z (module 1), and the second consisting of the prior for  $\eta$  and likelihood term for w, which also depends on  $\varphi$  (module 2). Suppose that we are concerned that module 2 is misspecified, and that this might adversely affect inferences of interest. The full posterior distribution can

be written as

$$p(\varphi, \eta | y) = p(\varphi | y) p(\eta | \varphi, w)$$
  
 
$$\propto p(\varphi | z) p(w | \varphi) p(\eta | \varphi, w).$$
(1)

We have used the conditional independence of  $\eta$  and z given w and  $\varphi$  in the first line above, and the conditional independence of w and z given  $\varphi$  in the second line. In (1) the term  $p(w|\varphi)$  is the "feedback" term which modifies the marginal posterior distribution for  $\varphi$  from  $p(\varphi|z)$  to account for the information about  $\varphi$  in the second module. If the second module is suspect, we might drop this term to obtain the so-called "cut" posterior distribution

$$p_{\rm cut}(\varphi, \eta | y) = p(\varphi | z) p(\eta | \varphi, y)$$
$$= p(\varphi | z) p(\eta | \varphi, w)$$
(2)

which rejects the feedback from module 2 about  $\varphi$  while propagating uncertainty about  $\varphi$  from module 1 for making inferences about  $\eta$ . The red line in the figure indicates the cut.

Computation for the cut posterior distribution is challenging. We can write

$$p_{\rm cut}(\varphi,\eta|y) \propto p(\varphi)p(z|\varphi)\frac{p(\eta|\varphi)p(w|\eta,\varphi)}{p(w|\varphi)},\tag{3}$$

but implementing an MCMC sampler is not easy from the likelihood and prior specification, because of the  $p(w|\varphi)$  term on the right-hand side above which is usually not tractable. A number of computationally intensive methods for dealing with the intractable normalizing factor have been suggested (Plummer, 2015; Jacob et al., 2020; Liu and Goudie, 2020). Yu et al. (2021) and Carmona and Nicholls (2022) consider variational inference methods which do not require approximation of the normalizing constant. However, these previous works were concerned with the case where the likelihood is tractable, and our main interest is in applications of cutting feedback to models with intractable likelihoods, where likelihood-free methods are used.

## **3** Mixture approximations for likelihood-free inference

The approach we use for likelihood-free inference will now be introduced, which allows us to easily approximate cut posterior analyses. Write y for the data and  $\theta$  for the parameters and we consider Bayesian inference with prior density  $p(\theta)$ . If the likelihood  $p(y|\theta)$  is impractical to compute, then conventional Bayesian computation methods are inapplicable. However, there is now a large literature on likelihood-free inference methods able to perform Bayesian inference using only model simulation, with approximate Bayesian computation (ABC) (Sisson et al., 2018b) and synthetic likelihood (Wood, 2010; Price et al., 2018) being the traditional approaches. More recently, approaches using flexible classification and regression methods from machine learning (Gutmann and Corander, 2016; Raynal et al., 2018; Hermans et al., 2020; Thomas et al., 2022; Pacchiardi and Dutta, 2022) are increasingly used.

#### 3.1 Mixture modelling of parameters and summaries

Here we will use an approach to likelihood-free inference based on mixture modelling of the joint distribution of the parameters and some summary statistics of the data. This approach is developed in Bonassi et al. (2011), where they consider induced conditional distributions from a mixture model as a form of nonlinear regression adjustment. The method has been refined within a sequential Monte Carlo framework in Bonassi and West (2015), although this extension is not helpful for the application considered here where we require analytic forms for marginal and conditional posterior densities. Other authors have considered mixture models in likelihood-free inference as well, such as Fan et al. (2013) who consider mixture of experts approximations to marginal summary statistics and copulas to approximate the likelihood, and Forbes et al. (2021) who consider mixture posterior approximations to define suitable functional discrepancy measures for ABC algorithms. Papamakarios and Murray (2016), Lueckmann et al. (2017) and Greenberg et al. (2019) have considered machine learning approaches based on mixture density networks. He et al. (2021) have recently developed mixture variational posterior approximations for likelihood-free inference using a population Monte Carlo algorithm.

The simple mixture approach of Bonassi et al. (2011) will be used here, since the ability to approximate arbitrary marginal and conditional distributions from a single mixture fit given any subset of the summary statistics, and for any values of those summary statistics, is crucial for the cutting feedback applications we describe. The more sophisticated mixture methods mentioned above either do not lead to closed form expressions for posterior approximations or require expensive additional computations for each new posterior distribution to be approximated, or both. The method of Bonassi et al. (2011) uses the following approach. First, we suppose that we have some summary statistics of the data S = S(y) available which are informative about the model parameters. The observed value of S is written as  $S_{obs} = S(y_{obs})$ , where  $y_{obs}$  is the observed value of y. We approximate  $p(\theta|y_{obs})$  by  $p(\theta|S_{obs})$ . If S is a sufficient statistic then  $p(\theta|y_{obs})$  and  $p(\theta|S_{obs})$  are the same, but a low-dimensional sufficient statistic is rarely available in likelihood-free inference applications. It is important that S is lowdimensional if we are to estimate the distribution of S from simulated data. Having chosen S, we simulate samples  $(\theta_i, S_i)$ , i = 1, ..., n from  $p(\theta)p(S|\theta)$ . Next, we fit a Gaussian mixture model to the simulated data. Writing  $U = (\theta, S)$ , this gives an estimate  $\tilde{p}(u) = \tilde{p}(\theta, S)$  for the joint density. The posterior density  $p(\theta|S_{obs})$  is just the conditional density of  $\theta$  given  $S = S_{obs}$  in  $p(\theta, S)$ , which we can approximate by the corresponding conditional density in the mixture model,  $\tilde{p}(\theta|S = S_{obs})$ .

An observation which is crucial later is that any marginal distribution of  $\tilde{p}(u)$  is a Gaussian mixture model, and any conditional distribution is also a Gaussian mixture. So from the mixture approximation  $\tilde{p}(u)$ , we can obtain a closed form approximation to any marginal or conditional posterior distribution conditional on any subset of summary statistics, and we can do this for any value of the summary statistics based on a single mixture model fit. To make this explicit, suppose our mixture approximation is

$$\widetilde{p}(u) = \sum_{j=1}^{J} w_{j,u} \phi_{j,u}(u),$$

where J is the number of mixture components,  $w_{j,u}$ ,  $j = 1, \ldots, J$  are non-negative mixing weights summing to 1, and  $\phi_{j,u}(u)$  is a multivariate normal component density with mean  $\mu_{j,u}$ and covariance matrix  $\Sigma_{j,u}$ ,  $j = 1, \ldots, J$ . Consider a subvector V = (X, W) of U where X and W are disjoint. In our later applications to cutting feedback, where we approximate marginal and conditional posterior distributions, X will be a subset of the model parameters, and W can consist of both parameters and summary statistics. Write the marginal distribution of V for the mixture component  $\phi_{j,u}(u)$  as  $\phi_{j,v}(v)$ , which is multivariate normal with mean vector  $\mu_{j,v}$ and covariance matrix  $\Sigma_{j,v}$ ,  $j = 1, \ldots, J$ . Write  $\mu_{j,v}$  in partitioned form as  $\mu_{j,v} = (\mu_{j,x}^{\top}, \mu_{j,w}^{\top})^{\top}$ , and

$$\Sigma_{j,v} = \begin{bmatrix} \Sigma_{j,x} & \Sigma_{j,xw} \\ \Sigma_{j,xw}^{\top} & \Sigma_{j,w} \end{bmatrix},$$

where the partitioning is conforming to the partition of V as (X, W). We also write the marginal density of W for component j of the mixture as  $\phi_{j,w}(w)$ ,  $j = 1, \ldots, J$ . The conditional density of X|W is a Gaussian mixture,

$$\widetilde{p}(x|w) = \sum_{j=1}^{J} w_{j,x|w} \phi_{j,x|w}(x),$$

where

$$w_{j,x|w} = \frac{w_{j,u}\phi_{j,w}(w)}{\sum_{l=1}^{J} w_{l,u}\phi_{l,w}(w)}$$

and  $\phi_{j,x|w}(x)$  is multivariate normal with mean and covariance matrix

$$\mu_{j,x|w} = \mu_{j,w} + \sum_{j,xw} \sum_{j,w}^{-1} (w - \mu_{j,w}),$$

and

$$\Sigma_{j,x|w} = \Sigma_{j,x} - \Sigma_{j,xw} \Sigma_{j,w}^{-1} \Sigma_{j,xw}^{\top}$$

respectively.

### 3.2 Mixture approximations and cutting feedback

Now we consider the issue of cutting feedback for likelihood-free inference. Suppose the summary statistics are partitioned as  $S = (S_1^{\top}, S_2^{\top})^{\top}$ , and write the corresponding partition of  $S_{\text{obs}}$  as  $S_{\text{obs}} = (S_{\text{obs},1}^{\top}, S_{\text{obs},2}^{\top})^{\top}$ . We wish to base inference about parameters  $\varphi$  only on  $S_1$ , because we are worried that the summaries  $S_2$  adversely affect inference about  $\varphi$ . The information in  $S_2$ , however, may be valuable for inference about the remaining parameters  $\eta$ . Example 5 of Sisson et al. (2018a) illustrates a simple situation where model misspecification can lead to summary statistics with conflicting information. In most likelihood-free inference applications there is no graphical structure to the model such as in Figure 1, and if we regard  $S_1$  and  $S_2$  as data sources associated with two modules, there is no conditional independence between them given the parameters. However, similar to (2) it is still useful to define a cut posterior which ignores  $S_2$  in inference about  $\varphi$ . For any value of S, we write

$$p_{\rm cut}(\theta|S) = p(\varphi|S_1)p(\eta|\varphi,S). \tag{4}$$

As discussed in the last section, given a mixture approximation  $\tilde{p}(\theta, S)$  to  $p(\theta, S)$ , an analytic form for the conditional densities for  $\varphi|S_1$  and  $\eta|\varphi, S$  can be written down. Then (4) can be approximated by

$$\widetilde{p}_{\rm cut}(\theta|S) = \widetilde{p}(\varphi|S_1)\widetilde{p}(\eta|\varphi, S),\tag{5}$$

where  $\tilde{p}(\varphi|S_1)$  and  $\tilde{p}(\eta|\varphi, S)$  are the conditional densities induced from  $\tilde{p}(\theta, S)$ . Monte Carlo summarization of the cut posterior approximation (5) is easy, since we just need to do sequential simulation from two Gaussian mixture models. In the next subsection we discuss methods for deciding whether or not to cut, where it is required to compute certain posterior distributions repeatedly for different data simulated under a reference distribution; the mixture approach can perform the required computations based on only a single mixture model fit.

An interesting case of the framework above is when S = y, and the likelihood is tractable but we wish to base inference about  $\varphi$  only on a low-dimensional summary statistic  $S_1$  for which  $p(S_1|\varphi)$  is not analytically available. The approximate Bayesian forecasting approach considered in Frazier et al. (2019) falls into this framework, where the authors consider inferring the parameter  $\varphi$  in a state space model using ABC with summary statistics, and then for ABC draws for  $\varphi$  they sample the conditional posterior distribution of latent states  $\eta$  given y using a particle filter in order to produce forecasts. Only filtering, and not smoothing, is needed for predictive inference. Here there is no need for ABC approximations in inferring the conditional posterior distribution of the states  $p(\eta|\varphi, y)$ , but ABC methods can be useful for inferring the parameters  $\varphi$ . Frazier et al. (2019) use predictive criteria for the choice of summary statistics, which might be particularly beneficial in the case of misspecification.

#### 3.3 Deciding whether or not to cut

When considering the use of cut methods we may need to decide whether to use the cut or conventional posterior. It is easy to see that the Kullback-Leibler divergence between the cut posterior distribution (4) and the full posterior distribution is the Kullback-Leibler divergence between their marginal posterior distributions for  $\varphi$ :

$$\mathrm{KL}(p(\theta|S) \| p_{\mathrm{cut}}(\theta|S)) = \mathrm{KL}(p(\varphi|S) \| p(\varphi|S_1)).$$

This follows from the fact that the conditional posterior distribution for  $\eta$  given  $\varphi$  is the same in both distributions. For a proof of this see Lemma 1 of Yu et al. (2021). Let us write

$$G(S_2|S_1) = \mathrm{KL}(p(\varphi|S)||p(\varphi|S_1)) \tag{6}$$

The statistic  $G(S_2|S_1)$  can be thought of as a prior-to-posterior divergence, in the situation where  $S_1$  is known when forming the prior but before we know  $S_2$ . Nott et al. (2020) consider prior-data conflict checks based on such prior-to-posterior divergences, and Yu et al. (2021) consider the use of these checks for deciding whether or not to cut feedback in a Bayesian analysis with tractable likelihood. Similar conflict checks were developed in likelihood-free inference in Chakraborty et al. (2022), although Chakraborty et al. (2022) do not consider cutting feedback methods.

We can approximate (6) by replacing  $p(\varphi|S)$  and  $p(\varphi|S_1)$  by  $\tilde{p}(\varphi|S)$  and  $\tilde{p}(\varphi|S_1)$  respectively. Since both of these densities are Gaussian mixtures, we can also make use of closed form approximations to Kullback-Leibler divergences between mixtures (Hershey and Olsen, 2007, Section 7) to obtain an approximation  $\tilde{G}(S_2|S_1)$  to  $G(S_2|S_1)$ . For more details, see Chakraborty et al. (2022). To decide whether or not to cut, we compare the statistic  $\tilde{G}(S_{\text{obs},2}|S_{\text{obs},1})$  to the distribution of  $\tilde{G}(S'_2|S_{\text{obs},1})$ , where  $S'_2$  is a draw from  $p(S_2|S_{\text{obs},1})$ , the conditional prior predictive for  $S_2$  given  $S_1 = S_{\text{obs},1}$ . Simulation from  $p(S_2|S_{\text{obs},1})$  can be approximated by simulation from  $\tilde{p}(S_2|S_{\text{obs},1})$  if necessary. If  $\tilde{G}(S_{\text{obs},2}|S_{\text{obs},1})$  lies out in the tails of this distribution, it says that the cut posterior distribution has changed an unusually large amount from the full posterior distribution if the model is correct for the observed  $S_2$ . Precisely, we consider the tail probability

$$p = P(\widetilde{G}(S'_2|S_{\text{obs},1}) \ge \widetilde{G}(S_{\text{obs},2}|S_{\text{obs},1})), \tag{7}$$

where  $S'_2 \sim p(S_2|S_{\text{obs},1})$ . If this tail probability is small, the change from the cut posterior distribution to the full posterior distribution is unusually large for the observed data compared to what is expected if the model is correct. In approximating (7) by Monte Carlo simulation, computation of the approximate Kullback-Leibler divergence between the full and cut marginal posterior distributions for  $\varphi$  can be done for repeated simulated summary statistics  $S'_2$  under the reference distribution based on the same single mixture fit that was used for the cut model computations for the observed summary statistics.

## 4 Semi-modular inference

As a generalization of cut posterior approaches, Carmona and Nicholls (2020) introduced semimodular inference, which gives a mechanism for partially cutting feedback. They consider the two module system of Figure 1, and suggest using some of the full module structure in making inference about  $\varphi$ . An influence parameter  $\gamma \in [0, 1]$  tempers the influence of the possibly misspecified module on inference about  $\varphi$ , whereas the conditional posterior distribution for  $\eta$  given  $\varphi$  is that of the full posterior.

The construction of Carmona and Nicholls (2020) considers a two stage approach. First, a "power posterior distribution" is constructed for inference about  $\varphi$  and an auxiliary replicate parameter  $\tilde{\eta}$  of  $\eta$ , given the data z and w. The use of similar power posterior distributions (Bissiri et al., 2016; Grünwald and van Ommen, 2017; Miller and Dunson, 2019) for robust Bayesian inference originates outside the modular inference context. Following Carmona and Nicholls (2020), we use the notation  $p_{\text{pow},\gamma}(\varphi, \tilde{\eta}|y)$  for the power posterior for  $(\varphi, \tilde{\eta})$  with influence parameter  $\gamma$ , and this is defined to be

$$p_{\text{pow},\gamma}(\varphi,\widetilde{\eta}|y) \propto p(z|\varphi)p(w|\varphi,\widetilde{\eta})^{\gamma}p(\varphi,\widetilde{\eta}).$$

The influence parameter tempers the likelihood term from the second module, reducing its influence. Next, the semi-modular posterior distribution is defined as

$$p_{\mathrm{smi},\gamma}(\varphi,\eta,\widetilde{\eta}|z,w) = p_{\mathrm{pow},\gamma}(\varphi,\widetilde{\eta}|z,w)p(\eta|w,\varphi),$$

and inference about  $\theta = (\varphi, \eta)$  is achieved by integrating out  $\tilde{\eta}$ . Setting  $\gamma = 0$ , the cut posterior for  $\theta$  is obtained, and setting  $\gamma = 1$  gives the full posterior. Hence the semi-modular

approach interpolates between the cut and full posterior distributions. Carmona and Nicholls (2020) suggest choosing  $\gamma$  using predictive methods. More recently, Nicholls et al. (2022) consider validity of semi-modular inference in a generalized Bayesian inference framework, and consider alternative forms of semi-modular inference. We consider one more alternative below for the likelihood-free setting.

### 4.1 Likelihood-free semi-modular inference

We now develop an alternative semi-modular posterior construction for the likelihood-free setting, using a method for constructing marginal inferences for  $\varphi$  inspired by linear opinion pooling (Stone, 1961). Linear opinion pooling (Stone, 1961) combines distributions representing opinions of different experts using a mixture model. We define a semi-modular marginal posterior distribution for  $\varphi$  as a mixture between the cut and full marginal posterior distributions with mixing weight  $\gamma$ . Such an approach is natural when mixture approximations are used for computation. In that case, the cut marginal posterior approximation for  $\varphi$  and the full marginal posterior approximation for  $\varphi$  are Gaussian mixtures, and a mixture of them is also a Gaussian mixture. This makes it easy to approximate our proposed semi-modular posterior distribution with an explicit form, and we suggest a convenient way to choose the influence parameter  $\gamma$  based on calculations similar to those used for the prior-data conflict checks in Section 3.3.

Define

$$p_{\gamma}(\varphi|S) = \gamma p(\varphi|S) + (1 - \gamma)p(\varphi|S_1), \tag{8}$$

where  $\gamma \in [0, 1]$  is the influence parameter. We define a semi-modular posterior distribution for  $\theta = (\varphi, \eta)$  by

$$p_{\mathrm{smi},\gamma}(\theta|S) = p_{\gamma}(\varphi|S)p(\eta|\varphi,S).$$
(9)

Using the mixture posterior approximations of Section 3 for computation, the approximations to the posterior densities of  $\varphi$  given  $S_1$  and  $\varphi$  given S respectively are written

$$\widetilde{p}(\varphi|S_1) = \sum_{j=1}^J w_{j,\varphi|S_1} \phi_{j,\varphi|S_1}(\varphi), \tag{10}$$

and

$$\widetilde{p}(\varphi|S) = \sum_{j=1}^{J} w_{j,\varphi|S} \phi_{j,\varphi|S}(\varphi).$$
(11)

Similarly, write the mixture approximation to the density of  $\eta | \varphi, S$  as

$$\widetilde{p}(\eta|\varphi,S) = \sum_{j=1}^{J} w_{j,\eta|\varphi,S} \phi_{j,\eta|\varphi,S}(\varphi).$$

We approximate (8) by

$$\widetilde{p}_{\gamma}(\varphi|S) = \gamma \widetilde{p}(\varphi|S) + (1-\gamma)\widetilde{p}(\varphi|S_1),$$
(12)

and the semi-modular posterior (9) by

$$\widetilde{p}_{\mathrm{smi},\gamma}(\varphi,\eta|S) = \widetilde{p}_{\gamma}(\varphi|S)\widetilde{p}(\eta|S).$$
(13)

 $\widetilde{p}_{\gamma}(\varphi|S)$  is itself a Gaussian mixture, with 2J components, and component densities

$$\{\phi_{j,\varphi|S_1}(\varphi), \phi_{j,\varphi|S}(\varphi) : j = 1, \dots, J\}$$

and corresponding mixing weights

$$\{\gamma w_{j,\varphi|S}, (1-\gamma)w_{j,\varphi|S_1} : j = 1, \dots, J\}.$$

Hence both terms on the right-hand side of (13) are Gaussian mixtures. Similar to the cutting feedback case of Section 3.2, Monte Carlo summarization of the semi-modular approximation is easy, involving sequential simulation from two mixture models.

## 4.2 Choosing the influence parameter

We now outline a convenient approach to the choice of the influence parameter  $\gamma$ . We write

$$G_{\gamma}(S) = \mathrm{KL}(p_{\gamma}(\varphi|S) \| p(\varphi|S_1))$$

for the Kullback-Leibler divergence between the semi-modular marginal posterior distribution for  $\varphi$  and  $p(\varphi|S_1)$ . If  $\gamma = 1$ , then this is the conflict checking statistic  $G(S_2|S_1)$  considered in Section 3.3. Similar to the discussion of Section 3.3, we can replace  $p_{\gamma}(\varphi|S)$  and  $p(\varphi|S_1)$ by their mixture approximations  $\tilde{p}_{\gamma}(\varphi|S)$  and  $\tilde{p}(\varphi|S_1)$ , and use a closed-form approximation to Kullback-Leibler divergences between mixtures (Hershey and Olsen, 2007, Section 7) to obtain an approximate statistic  $\tilde{G}_{\gamma}(S)$  that is easy to compute.

Define a tail probability

$$p(\gamma) = P(\widetilde{G}(S'_2|S_{\text{obs},1}) \ge \widetilde{G}_{\gamma}(S_{\text{obs}})), \tag{14}$$

where  $S'_2 \sim p(S_2|S_{\text{obs},1})$ . Simulation of  $S'_2$  can be approximated by simulation from  $\tilde{p}(S_2|S_{\text{obs},1})$ if needed. We propose to choose  $\gamma$  as the largest value  $\gamma'$  such that  $p(\gamma') > \alpha$ , where  $\alpha$  is a cutoff for a measure of surprise such as 0.05. Finding  $\gamma'$  can be done by computing  $p(\gamma)$  for  $\gamma$  on a grid. The intuitive meaning of choosing  $\gamma$  in this way is the following. If there is no conflict at level  $\alpha$  according to the check of Section 3.3, then we choose  $\gamma = 1$  and we use the full posterior. If there is a conflict, then we back off from  $\gamma = 1$  to a smaller value such that the conflict would be avoided if  $\tilde{p}_{\gamma}(\varphi|S)$  had been the full posterior marginal for  $\varphi$ . The idea is to use as much of the full posterior information as possible, subject to retaining an interpretation for the inference that is not in conflict with that based only on the summary  $S_1$ .

## 5 Examples

We consider two examples. The first concerns a collective cell spreading model. A common use for cutting feedback methods is to explore whether misspecification of one module impacts inference about certain parameters. If cut and full posterior inferences have a similar interpretation, this might be reassuring that inferences of interest are not sensitive to the misspecification. In the cell spreading model it has been noted in past work by Frazier and Drovandi (2021) that the cell interaction component fails to capture some aspects of the observed data, and for this example we use cutting feedback to demonstrate that this inadequacy does not affect inference about cell proliferation. The example also illustrates the usefulness of the mixture modelling approach for exploring the informativeness of summary statistics for inference about different parameters in a computationally thrifty way. Our second example considers time series models for asset returns with jumps. We start by considering a continuous time model and explore cutting feedback so that jump parameters are estimated using only summary statistics on high frequency intra-day returns. This results in different inferences about the jump parameters compared to the ordinary posterior where summary statistics incorporate both information from both daily and intra-day returns. A similar discrete-time model is then discussed, where the performance of full, cut and semi-modular posterior distributions are explored for forecasting purposes, and where cut and semi-modular approaches improve on the conventional posterior distribution for predictive purposes.

## 5.1 Collective cell spreading

Our first example considers a model developed in Browning et al. (2018) for collective cellspreading. Their model is useful in applications to understanding skin cancer growth and wound healing. Misspecification for this model was discussed in the supplementary material of Frazier and Drovandi (2021), and we apply cutting feedback methods to understand the effect of this misspecification on inference. The model has three unknown parameters,  $\theta = (m, \rho, \gamma_b)^{\top}$ . The rates of motility (cell movement) and proliferation (cell birth) are given by m and  $\rho$ , respectively. The parameter  $\gamma_b$  is part of a Gaussian kernel used to measure the closeness of cells. The prior distribution for  $\theta$  has independent uniform components,  $\rho \sim U(0, 10), m \sim U(0, 0.1)$  and  $\gamma_b \sim U(0, 20)$ . The reader is referred to Browning et al. (2018) for a detailed discussion of the model.

The summary statistics are as follows:

- 1. The number of cells at 12, 24 and 36 hours  $(S_1)$ .
- 2. The pair correlation computed at 12, 24 and 36 hrs  $(S_2)$ .

The summary statistic vector  $S_1$  is intended to be informative about  $\rho$ , and  $S_2$  is intended to be informative about m and  $\gamma_b$ . It is reasonable to suspect that the observed number of cells can be recovered for appropriate values of the model parameter. However, it is more challenging to capture the spatial dependence of the cell population, i.e. how the cells interact with each other. Accurate estimation of  $\rho$  is important as cell proliferation drives cancer growth, and cancer treatments would aim to reduce this parameter. Therefore, there is an interest in inferring  $\rho$  in a way that is robust to potential misspecification of the cell interaction component of the model. We consider a simulated dataset first where the model is correctly specified, using  $\theta = (1, 0.04, 6)$ . For the real data, Frazier and Drovandi (2021) show that the model does not capture the way that the observed pair correlation changes over time.

We generate  $N = 10^5$  simulations from the prior predictive distribution. In an effort to improve the Gaussian mixture model (GMM) fit, we first transform each marginal summary statistic and parameter distribution using the probability integral transform to uniform, and then push the transformed samples through the standard normal quantile function. For the prior distribution of the parameters, the distribution function has an analytic form. For the summary statistics, the distribution function is estimated by kernel density estimation, and so the marginal distribution of the transformed summaries is only approximately standard normal. Based on Figure 2, even after transforming the marginal distributions, the dependence structure is complex, both between parameters and summary statistics, and between the summary statistics themselves. Hence this represents a challenging application for the GMM approach.

A GMM with 10 components was fitted to the transformed prior predictive samples. In the GMM we assumed unrestricted component covariance matrices, and estimation was performed

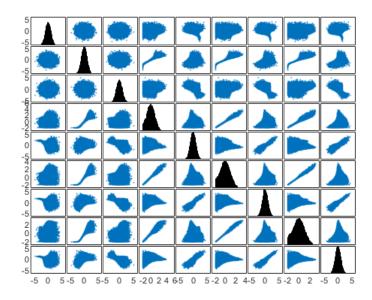


Figure 2: Marginal distributions and bivariate scatterplots based on the joint distribution of parameters and summary statistics generated from prior predictive distribution (after a marginal transformation of the parameters and summary statistics).

using an EM algorithm. The fitted GMM was then used to approximately sample the posterior density  $p(\theta|S_1, S_2)$  and the cut posterior density  $p_{\text{cut}}(\theta|S_1, S_2) = p(\rho|S_1)p(m, \gamma_b|\rho, S_1, S_2)$ . The parameter samples generated from the fitted GMM were passed through the inverse transform to generate samples from the approximate posterior distributions on the original parameter space.

The marginal posterior estimates for the simulated and real datasets are shown in Figures 3 and 4, respectively. We also include the estimates from Frazier and Drovandi (2021) using their robust BSL method with variance inflation, which should be robust to potential misspecification of the spatial dependence of cells, in the sense that it can reduce the influence of summaries that the model is not compatible with. It can be seen that the cutting feedback GMM posterior approximation does not differ greatly from the GMM posterior approximation, even for the real data. This is consistent with the results obtained in Frazier and Drovandi (2021). This indicates that the inability to recover  $S_2$  is not adversely affecting inference about  $\rho$ . The GMM posterior approximations are surprisingly accurate given the greatly reduced number of model simulations used compared to BSL.

Another potential advantage of the GMM approach is that we can thriftily explore the sensitivity of different summary statistic choices on each parameter. Figure 5 illustrates

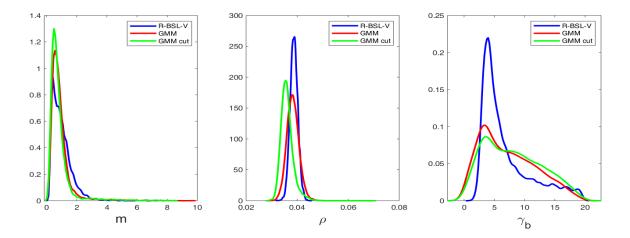


Figure 3: Estimates of marginal posterior densities for m,  $\rho$  and  $\gamma_b$  obtained using robust Bayesian synthetic likelihood with variance inflation (R-BSL-V), GMM approximation to the full posterior (GMM) and GMM approximation to the cut posterior (GMM cut) for the simulated data for collective cell spreading model.

this using the simulated data, where marginal posterior densities for different parameters are estimated using each summary statistic individually. It can be seen that  $\rho$  is informed by the number of cells at each time point, and the pair correlation statistics provide no information. From this perspective, it is not surprising that  $\rho$  is not adversely influenced by potential incompatibility of  $S_2$ , since the parameter is not sensitive to these statistics. The estimated posterior of  $\rho$  conditional on only the number of cells at 36 hrs is similar to the posterior approximation conditional on all the statistics. The results suggest that it is only necessary to include the number of cells at the final time point in the summary statistic vector. It is evident that m and  $\gamma_b$  are informed by  $S_2$  and not  $S_1$ . In the case of m, there does seem to be a benefit in including all time points, as the posterior approximation with the three pair correlation statistics is more concentrated than with any individual time point.

## 5.2 Continuous time model for asset returns with jumps

Our next example considers a continuous time model for asset returns with jumps and illustrates the use of our conflict checking approach for deciding whether or not to cut feedback, as well as our proposed semi-modular method. Here the use of a cut posterior distribution results in different inferences to those of the full posterior distribution. Let  $P_t$  denote the instantaneous price of an asset at time  $t \ge 0$ . For  $p_t = \ln P_t$ , suppose that  $p_t$  evolves according

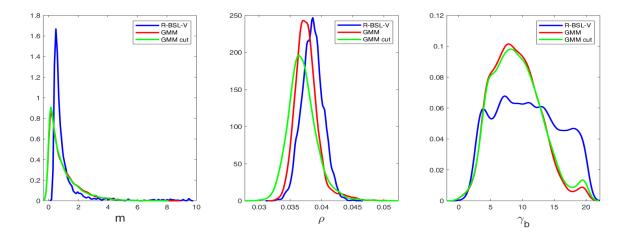


Figure 4: Estimates of marginal posterior densities for m,  $\rho$  and  $\gamma_b$  obtained using robust Bayesian synthetic likelihood with variance inflation (R-BSL-V), GMM approximation to the full posterior (GMM) and GMM approximation to the cut posterior (GMM cut) for the real data for collective cell spreading model.

to the bivariate jump-diffusion process

$$dp_t = \mu_p dt + \exp\left(V_t/2\right) dW_t^p + dJ_t^p$$
(15)

$$dV_t = \kappa \left(\alpha - V_t\right) dt + \sigma_v dW_t^v \tag{16}$$

$$dJ_t^p = Z_t dN_t, \quad Z_t \sim N\left(\mu_z, \sigma_z^2\right), \tag{17}$$

where  $W_t^j$ ,  $j \in \{v, p\}$ , are correlated Brownian motion processes, with instantaneous correlation  $\rho$ ,  $N_t$  is a counting process,  $V_t$  is a latent volatility process  $J_t^p$  is a process of unobservable jumps, and  $\mu_p$ ,  $\kappa$ ,  $\alpha$ ,  $\sigma_v$ ,  $\mu_z$  and  $\sigma_z$  are unknown parameters.

The above model is similar to the model of Creel and Kristensen (2015), however, we model the jump process via a conditionally deterministic Hawkes process (Aït-Sahalia et al., 2015; Maneesoonthorn et al., 2017) for which

$$\Pr\left(\mathrm{d}N_t = 1\right) = \delta_t \mathrm{d}t + o(1), \quad \mathrm{d}\delta_t = (d + \beta\delta_t)\mathrm{d}t + \tau \mathrm{d}N_t, \tag{18}$$

where d,  $\beta$  and  $\tau$  are positive unknown parameters. We refer to (15) as the returns model, (16) as the volatility model, and (17) and (18) as the jump model. The unknown parameters are collected as  $\theta = (\varphi^{\top}, \eta^{\top})^{\top}$ , where

$$\varphi = (\mu_z, \sigma_z, d, \beta, \tau)^\top, \quad \eta = (\mu_p, \kappa, \alpha, \sigma_v, \rho)^\top.$$

The parameters  $\varphi$  control the jump dynamics, while the parameters  $\eta$  are those appearing in the returns and volatility models.

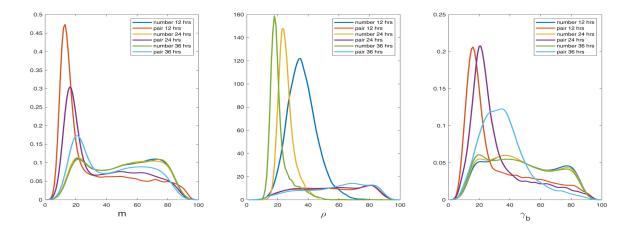


Figure 5: Sensitivity of posterior approximations to different summary statistics based on simulated data for the collective cell spreading example. Estimated marginal posterior densities are shown for m,  $\rho$  and  $\gamma_b$ . The different lines in each plot are estimates obtained by conditioning on different scalar summary statistics.

For a general value of  $\theta$ , and a given sequence of observed log-returns  $\{r_t = p_t - p_{t-1} : t \geq 1\}$ , the likelihood associated with the model in (15)-(18) is intractable. This intractability is due to the presence of the unobservable state variables  $V_t$ , the latent volatilities, and  $J_t$ , the unobservable jumps. These variables must be integrated out of the measurement equation for the observables to obtain a likelihood that depends only on the observable data. This integration is made even more difficult by the fact that the transition equations for the latent states do not admit closed-form densities, due to their continuous-time evolution, and must generally be approximated. In contrast, simulation-based methods bypass calculation of the likelihood function by simulating data directly from the model.

We consider the case where the researcher is uncertain of the error specification in the volatility/returns equations, and/or the specification of the jump dynamics. Let us focus on the jump dynamics as a single module, and the volatility/returns specification as a separate module. If intra-day returns are available, inference on the parameters  $\varphi$  governing the jump dynamics can proceed by "cutting" the link with the returns and volatility equations. We observe daily log returns  $r_t = p_t - p_{t-1}$  at integer times  $t = 1, \ldots, T$ , and, for each t, we observe M equally spaced intra-day returns  $r_{t,i}$ , with  $i = 1, \ldots, M$ . Define the bipower variation,  $BV_t$ , and jump-variation,  $JV_t$ , as

$$BV_t := \frac{\pi}{2} \left( \frac{M}{M-1} \right) \sum_{i=2}^{M} \left| r_{t,i} r_{t,(i-1)} \right|, \quad JV_t := \max \left\{ RV_t - BV_t, 0 \right\},$$

where  $\text{RV}_t$  denotes realized volatility  $\text{RV}_t = \sum_{i=1}^M r_{t,i}^2$ . ABC inference on the jump-dynamics

can be carried out using the following summary statistics (Frazier, et al., 2019):

$$S_{1,1} := \frac{1}{T} \sum_{t=1}^{T} \operatorname{sgn}\left(r_{t}\right) \sqrt{\operatorname{JV}_{t}}, \quad S_{1,2} := \frac{1}{T} \sum_{t=1}^{T} \left(\operatorname{JV}_{t} - \overline{\operatorname{JV}}_{t}\right)^{2},$$
$$S_{1,3} := \frac{1}{T} \sum_{t=2}^{T} \left(\operatorname{JV}_{t} - \overline{\operatorname{JV}}_{t}\right) \left(\operatorname{JV}_{t-1} - \overline{\operatorname{JV}}_{t}\right),$$

where  $\overline{JV}_t = T^{-1} \sum_{t=1}^T JV_t$ . We also define  $S_{1,4}$  and  $S_{1,5}$  as the sample skewness and kurtosis respectively of log BV<sub>t</sub>. The summary statistic vector  $S_1$  used for inference on  $\varphi$  in the "cut" posterior is  $S_1 = (S_{1,1}, S_{1,2}, S_{1,3}, S_{1,4}, S_{1,5})^{\top}$ .

The unknown parameters in the return and volatility equations can be identified using several possible auxiliary models, including those that explicitly capture the relationship between volatility, realized variance, and bipower variation, or various combinations of these components. However, for reasons of parsimony, we only specify an auxiliary model for returns and volatility. In particular, we follow Frazier et al. (2019) and use a TARCH-T auxiliary model (threshold GARCH auxiliary model with student-t errors).

$$r_{t} = \sigma_{t}\varepsilon_{t}, \quad \varepsilon_{t} \stackrel{iid}{\sim} t(\nu)$$
  
$$\sigma_{t}^{2} = \gamma_{1} + \gamma_{2}(r_{t-1} - \gamma_{0})^{2} + \gamma_{3} \mathbb{1}[r_{t-1} < 0] (r_{t-1} - \gamma_{0})^{2} + \gamma_{4}\sigma_{t-1}^{2}$$

with  $\gamma = (\gamma_1, \ldots, \gamma_4, \nu)'$ , and where  $\mathbb{1}[A]$  denotes the indicator function on the set A. The summary statistics  $S_2$  are obtained by evaluating the score vector for the auxiliary model, evaluated at the quasi maximum likelihood estimate (QMLE) for the observed data.

#### 5.2.1 Simulation of the Model

Simulating data from the model can be done using an Euler discretization scheme with step size 1/I and I large. Write  $p_{t,i/I} = p_{t+i/I}$ , and define  $V_{t,i/I}$  and  $\delta_{t,i/I}$  similarly. Define  $\Delta N_{t,i/I} = N_{t,(i+1)/I} - N_{t,i/I}$ . We discretize the system and generate data recursively: for each day t, and each intra-day time  $i = 1, \ldots, I$ , simulate data according to

$$p_{t,(i+1)/I} = p_{t,i/I} + \mu \frac{1}{I} + \exp\left(V_{t,i/I}/2\right) \epsilon_{t,i}^p \frac{1}{\sqrt{I}} + Z_{t,i} \Delta N_{t,i/I}$$
$$V_{t,(i+1)/I} = V_{t,i/I} + \kappa \left(\alpha - V_{t,i/I}\right) \frac{1}{I} + \frac{\sigma_v}{\sqrt{I}} \left(\rho \epsilon_{t,i}^p + \sqrt{1 - \rho^2} \epsilon_{t,i}^v\right)$$

where  $\epsilon^p, \epsilon^v$  are bivariate standard normal random variables,  $Z_{t,i} \stackrel{iid}{\sim} N(\mu_z, \sigma_z^2)$  and  $\Delta N_{t,(i+1)/I}$  is 1 with probability  $I^{-1}\delta_{t,(i+1)/I}$ ,

$$\delta_{t,(i+1)/I} = \delta_{t,i/I} + (d + \beta \delta_{t,i/I})I^{-1} + \tau \Delta N_{t,i/I},$$

and zero otherwise. Given a trajectory  $p_{t,i/I}$  define  $r_{t,i/I} = p_{t,i/I} - p_{t,(i-1)/I}$ , i = 1, ..., I. Then by downsampling these values to a sequence of M equally spaced values (assuming I is an integer multiple of M) we obtain  $r_{t,i}$ , i = 1, ..., M, corresponding to the M intraday return on day t for the observed data.

#### 5.3 Real data analysis

We consider daily returns data on the S&P500 index from 26 February 2010 to 7 February 2017. There are 1750 daily observations, and the most recent 250 observations are reserved for out-of-sample predictive assessments using a related, but more parsimonious, model described in the next subsection. These data were also used in Frazier et al. (2019). Uniform priors are used for each parameter with lower and upper bounds for each parameter given in Table 1.

Table 1: Lower and upper bounds for uniform priors.

Parameter	$\mu_p$	$\kappa$	$\alpha$	$\sigma_v$	$\rho$	$\mu_z$	$\sigma_z$	d	au	$\beta$
Lower	-0.1	0.05	-1.0	0.001	-0.70	-1.0	.0.0	0.01	0.001	0.5
Upper	0.1	0.50	3.0	1.99	0.0	1.0	3.0	0.2	0.2	$1\text{-}\tau$

Figure 6 shows the marginal posterior distributions for the parameters  $\varphi$  in the jump process, and the corresponding "cut" and semi-modular marginal posterior densities. The influence parameter  $\gamma$  in the semi-modular approach is chosen as described in Section 4.1, which results in the value  $\gamma = 0.41$ . The parameters  $\mu_z$  and  $\sigma_z$  in the jump model, which represent the (average) magnitude and variability of the jumps respectively, are estimated quite differently in the full and cut posterior distributions; the interpretation is that cutting feedback suggests that the magnitude of the jumps in daily returns are smaller (i.e., closer to zero) than under the full posterior, and that the variability of the jump size is smaller than under the full posteriors.

The marginal posterior distributions were estimated by fitting a Gaussian mixture model to 50,000 simulations from the prior for parameters and summary statistics. Similar to the first example, variables are transformed to be marginally univariate normal before fitting the mixture, with transformation back to the original scale for estimation of the posterior densities. The mclust package (Scrucca et al., 2016) was used to choose the number of mixture components up to a maximum of 10 by BIC, considering different covariance structures for the components. The final model had 8 mixture components, with distinct and unrestricted component covariance matrices.

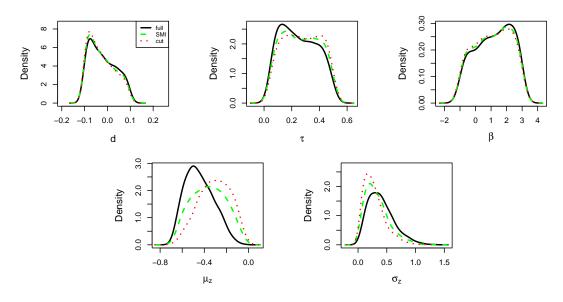


Figure 6: Marginal densities for jump parameters in asset pricing model with jumps for the full, cut and semi-modular posterior densities

Figure 7 (left) shows the location of the observed checking statistic  $\tilde{G}(S_{\text{obs},2}|S_{\text{obs},1})$  (shown by the red line) within its corresponding reference distribution, demonstrating that the tail probability (7) is very small, and hence that the cut and full posterior distributions are surprisingly different under the reference distribution for the check, supporting the decision to cut. Figure 7 (right) shows how the tail probability (14) varies with  $\gamma$  in the semi-modular approach.

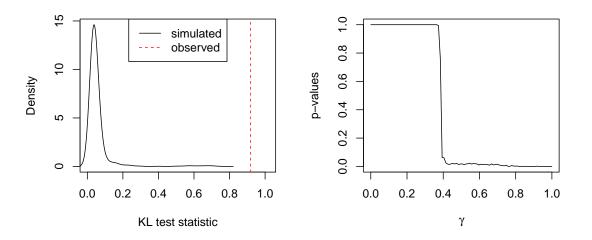


Figure 7: Observed statistic for conflict check  $\widetilde{G}(S_{\text{obs},2}|S_{\text{obs},1})$  (left), shown by the red line, within the reference distribution for the check and the tail probability  $p(\gamma)$  as  $\gamma$  varies for the semi-modular approach (right), for the asset pricing example.

Figure 8 shows how the semi-modular inference for the various parameters in the jump process change with  $\gamma$ . Our chosen value of  $\gamma$  was 0.41 here, and the Figure confirms that for larger values of  $\gamma$  the posterior inference is very different for the corresponding semi-modular and full posterior, particularly for  $\mu_z$  and  $\sigma_z$ .

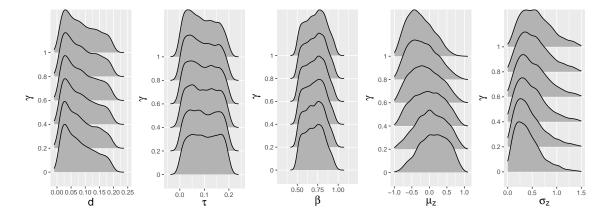


Figure 8: Marginal densities for jump parameters in asset pricing model for the semi-modular posterior densities as  $\gamma$  changes.

While the continuous-time model in equations (15)-(17) yields parameters that have meaningful structural interpretations, it is well-known that forecasts for returns obtained from continuous-time models are often outperformed by more parsimonious discrete-time models. Furthermore, the use of continuous-time models in forecasting is hindered by their computational complexity. Producing forecast densities from continuous-time models requires sequentially approximating the transition density for the states in continuous time, and the conditional density for the observable variables given the states. This approximation is carried out by discretization of the process over a very fine grid, and then simulating forward the process to accurately capture the dynamics. This procedure must then be repeated each time one wishes to produce a forecast. We also note that such an approximation yields a discretization error at each step, which can add additional noise to the resulting forecasts.

Consequently, in order to produce accurate forecasts, we follow Frazier et al. (2019) and consider a more parsimonious discrete-time analogue of the continuous-time model, with likelihood-free inference using summary statistics employed to produce the posterior for  $\theta$ , and a particle filter being used for estimation of the latent states in the production of the forecasts. The resulting model has a similar structural interpretation to the continuous-time model in (15)-(18) but is computationally much simpler to simulate, and does not require any discretization to produce predictive densities.

#### 5.4 Discrete time model

With similar notations to the previous subsection, and following Frazier et al. (2019), we consider the following discrete time model for daily logarithmic returns and bipower variation:

$$r_t = \exp\left(\frac{h_t}{2}\right)\epsilon_t + \Delta N_t Z_t, \ \epsilon_t \sim N(0, 1), \ Z_t \sim N(\mu_z, \sigma_z^2),$$
(19)

$$\log BV_t = \psi_0 + \psi_1 h_t + \sigma_{BV} \zeta_t, \quad \zeta_t \sim N(0, 1), \tag{20}$$

$$h_t = \omega + \rho h_{t-1} + \sigma_h \eta_t, \quad \eta_t \sim S(\alpha, -1, 0, 1), \tag{21}$$

where  $S(\alpha, \beta, \mu, \sigma)$  denotes the  $\alpha$ -stable distribution with stability parameter  $\alpha$ , skewness  $\beta$ , location  $\mu$  and scale  $\sigma$ , and

$$P\left(\Delta N_t = 1 \mid \mathcal{F}_{t-1}\right) = \delta_t = d + \beta \delta_{t-1} + \tau \Delta N_{t-1}, \qquad (22)$$

where  $\mathcal{F}_t$  denotes the  $\sigma$ -field generated by the observations up to time t and  $\Delta N_t$  plays a similar role to the continuous time jump process  $dN_t$  in (17). The above model has a similar motivation and structure to the continuous-time model, but as discussed earlier it is more convenient for forecasting. It includes an additional measurement equation (20) which depends on intra-day returns, through  $\ln BV_t$ . We fix the parameter  $\psi_0$  to 0, since this parameter is hard to identify. The set of unknown parameters is  $\theta = (\varphi^{\top}, \eta^{\top})^{\top}$ , where  $\varphi = (\mu_z, \sigma_z, d, \beta, \tau)^{\top}$  are the parameters in the jump module and  $\eta = (\psi_0, \psi_1, \sigma_{BV}, \omega, \rho, \sigma_h)^{\top}$  are the parameters in the return/volatility module. Our prior distributions are the same as those in Frazier et al. (2019), except for the fixed parameter  $\psi_0$ .

Once again write the summary statistics as  $S = (S_1^{\top}, S_2^{\top})^{\top}$ , where  $S_1$  is informative about  $\varphi$ . Our summary statistics are related to those used in Frazier et al. (2019) for their TARCH-T auxiliary model, which performed best for forecasting in their work. These summary statistics were also the motivation for those used in our continuous-time model, but must be modified here, since the jumps-variation JV<sub>t</sub> cannot be computed. The reason is that the discrete-time model above only generates the summary of intra-day returns log BV<sub>t</sub> directly, without generating the intra-day returns themselves which would be needed to compute the realized volatility. We consider for  $S_1$  the summary statistics. The jump process appears only in the model for the daily returns (19) in the discrete model, and so it is sensible to use the TARCH-T auxiliary model fitted to the daily returns data to summarize the information about the jump process. For  $S_2$ , we consider summaries based on log BV<sub>t</sub>. We consider the mean, variance and skewness of both log BV<sub>t</sub>, and log BV<sub>t</sub> - log BV<sub>t-1</sub>, as well as the correlation of BV<sub>t</sub> and BV<sub>t-1</sub>, so that  $S_2$  contains 7 summary statistics.

Similar to Figure 9 for the continuous-time model, Figure 9 shows the marginal posterior distributions for the parameters  $\varphi$  in the jump process for the discrete time model, and the corresponding "cut" and semi-modular marginal posterior densities. Once again, the influence parameter  $\gamma$  in the semi-modular approach is chosen as described in Section 4.1, resulting in  $\gamma = 0.01$  here so that the SMI posterior is nearly identical to the cut posterior. The marginal posterior distributions for d and  $\sigma_z$  in the cut posterior suggest the presence of fewer jumps with less variation around the average jump magnitude. The marginal posterior distributions from the prior for parameters and summary statistics. Once again, variables are transformed to be marginally univariate normal before fitting the mixture, and the mclust package (Scrucca et al., 2016) was used to choose the number of mixture components up to a maximum of 10 by BIC, considering different covariance structures for the components. The final model had 9 mixture components, with distinct and unrestricted component covariance matrices.

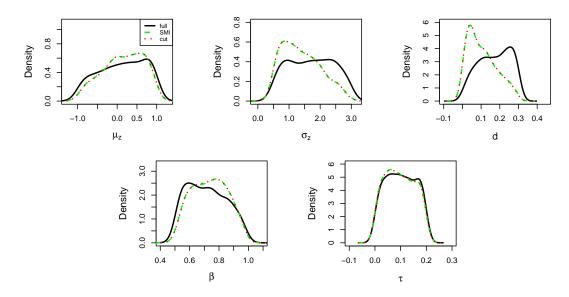


Figure 9: Marginal densities for jump parameters in discrete time asset pricing model with jumps for the full, cut and semi-modular posterior densities. The cut and semi-modular posterior densities are similar in all panels.

To assess the forecast performance, we first estimate the posterior distribution of  $\theta$  based on the training set observations. This estimate is then kept fixed throughout the forecast period, and a bootstrap filter (Gordon et al., 1993) is used to estimate the latent states  $h_t, \Delta N_t$ , conditional on the data up to time t. We write this data as  $r_{1:t} = (r_1, \ldots, r_t)^{\top}$ ,  $BV_{1:t} = (BV_1, \ldots, BV_t)^{\top}$ . Write  $y_t = (r_{1:t}^{\top}, BV_{1:t}^{\top})^{\top}$ . We use 5,000 particles in the particle filter. To obtain a one-step ahead forecast at time T, write  $h_{T,p}^s$ ,  $\Delta N_{T,p}^s$  for the pth particle values respectively for  $h_T$ ,  $\Delta N_T$  for posterior parameter sample  $\theta^s$ ,  $s = 1, \ldots, S$ ,  $p = 1, \ldots, P$ . Write  $\delta_{T,p}^s$  for the corresponding values of  $\delta_t$ . We use S = 1,000. The one-step ahead forecast density is approximated by

$$\widehat{p}(y_{T+1}|y_{1:T}) \approx \frac{1}{\mathcal{S} \times P} \sum_{s=1}^{\mathcal{S}} \sum_{p=1}^{P} p(y_{T+1}|h_{T+1} = \widetilde{h_{T+1,p}^s}, \Delta N_{T+1} = \widetilde{\Delta N_{T+1,p}^s}, \theta),$$

where  $h_{T+1,p}^s$  and  $\Delta N_{T+1,p}^s$  are obtained by simulating from (21)-(22) with  $h_T = h_{T,p}^s$  and  $\delta_T = \delta_{T,p}^s$ , and  $p(y_t|h_t, \Delta N_t, \theta)$  is defined from equation (19). Similar to Frazier et al. (2019), out-of-sample predictive performance for the one-step ahead forecasts are assessed by average predictive log score, quadratic score, and continuous ranked probability score, with results shown in Table 2. Although the largest logarithmic and CRPS score values occur for small values for  $\gamma$  for both outcomes, the differences in forecasting performance between methods are minor in any practical sense.

Table 2: Logarithmic (LS), quadratic (QS) and continuous rank probability score (CRPS) for out of sample forecast assessment for outcomes  $r_t$  and  $\log BV_t$  for the discrete time model for semi-modular inference with  $\gamma = 0.2, 0.4, 0.6$  and 0.8.  $\gamma = 0.01$  is the value of the SMI influence parameter chosen by the conflict checking method, giving similar predictive scores to the cut posterior with  $\gamma = 0$ .  $\gamma = 1$  is the full posterior. Scores are oriented so that larger values represent better forecasting performance.

	$\gamma = 0$	$\gamma = 0.2$	$\gamma = 0.4$	$\gamma = 0.6$	$\gamma = 0.8$	$\gamma = 1.0$						
	Outcome $r_t$											
LS	-0.670	-0.673	-0.677	-0.682	-0.686	-0.690						
QS	0.612	0.615	0.615	0.616	0.616	0.617						
CRPS	-0.259	-0.259	-0.260	-0.260	-0.260	-0.260						
	Outcome $\log BV_t$											
LS	-3.894	-3.877	-3.914	-3.909	-3.926	-3.924						
QS	-0.231	-0.231	-0.232	-0.228	-0.228	-0.228						
CRPS	-1.801	-1.806	-1.819	-1.823	-1.833	-1.840						

# 6 Discussion

Cutting feedback methods are useful in applications involving multi-modular models, where they can be used both as a diagnostic for understanding misspecification and posterior sensitivity as well as an alternative to using the full posterior for predictive inference when the development of an alternative model is infeasible. As far as we are aware, the use of cutting feedback methods has so far been restricted in the literature to models with tractable likelihood. The extension to the intractable likelihood setting discussed here can be useful when it is desired to restrict the information used for inference about a subset of the parameters to that obtained from a subset of the summary statistics only. Our proposed Gaussian mixture model approach to estimation of the posterior distribution makes the cutting feedback computations easy to perform, and facilitates model checks which can help guide the decision of whether or not to cut, as well as a semi-modular inference extension where feedback is partially cut.

The use of cutting feedback methods in likelihood-free inference is particularly helpful since model misspecification is known to negatively impact common likelihood-free inference procedures. In the case of ABC, Frazier et al. (2020) demonstrate that if the model is misspecified, then the ABC posterior does not produce valid inferences and can be ill-behaved. Frazier et al. (2021) show that similar problems occur for Bayesian synthetic likelihood approaches. Hence a benefit of cutting feedback is that it can be used to hedge against the potential consequences of using misspecified models in likelihood-free inference. Recently, Pacchiardi and Dutta (2021) consider a generalized Bayesian approach to likelihood-free inference based on scoring rules which can deal with misspecification.

There has been renewed interest recently in the idea of robustifying Bayesian inference by conditioning on an insufficient data summary to exclude information (Li et al., 2017; Lewis et al., 2021). This is interesting regardless of whether the likelihood is tractable or not, and the consideration of complex data summaries for conditioning leads to possible applications of likelihood-free inference methods in models with tractable likelihood when misspecification is a concern. Another recent work which is relevant to the likelihood-free inference literature is Miller and Dunson (2019), where the authors make a connection between inference for a "coarsened" version of the data, which is reminiscent of ABC methods, and the use of power posterior distributions. In traditional ABC methods the ABC kernel can be interpreted in terms of an allowance for model misspecification (Wilkinson, 2013), and an interesting avenue for future research might be to pursue ABC approaches to the semi-modular inference framework making use of this interpretation as a way of partially cutting feedback. This seems

related to the kernel-smoothing  $\delta$ -SMI approach discussed in Nicholls et al. (2022).

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