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A Comment on Work by Booth and Co-authors

Abstract. Booth and his co-authors have shown in [2], that many new approaches to theory revision (with fixed K) can be represented by two relations, $<$ and $\lhd$, where $<$ is the usual ranked relation, and $\lhd$ is a sub-relation of $<$. They have, however, left open a characterization of the infinite case, which we treat here.

Keywords: Theory Revision, Non-monotonic Logic, Completeness Proofs .

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1. Introduction

1.1. The problem we solve

Booth and his co-authors have shown in a very interesting paper, see [2], that many new approaches to theory revision (with fixed K) can be represented by two relations, $<$ and $\lhd$, where $<$ is the usual ranked relation generated by

Presented by **Heinrich Wansing**; *Received* May 27, 2009

DOI: 10.1007/s11225-010-9237-7

AGM revision for fixed K, and $\lhd$ is a sub-relation of $<$. They have, however, left open a characterization of the infinite case, which we treat here.

1.1.1. The framework

The background is AGM revision theory, see [1]. The basic problem of theory revision is to "integrate" new information into old information, preserving consistency. More precisely, both parts of information, taken separately, are supposed to be consistent, but (in the interesting case) taking the union will not do, as this union is inconsistent. AGM gave "rationality postulates" any such integration should satisfy. The revision operator is usually written as $*$, so in the usual notation we revise information K with a new formula ϕ, and the result is $K * \phi$. One of those postulates is, e.g., robustness under reformulation: If ϕ and ϕ' are (classically) equivalent, so are $K*\phi$ and $K*\phi'$.

It was shown in [5] that a distance on the set of classical models gives a semantics to AGM revision, similar to, but in one important point different from, the Stalnaker/Lewis semantics for counterfactual conditionals. When K is fixed, the AGM axioms are closely related to the axioms of rationality in preferential structures, i.e., to a preference relation. In distance terminology, we take the $\phi-$models which are closest to the set of $K-$models. In preference terminology, as K is fixed, we take the "best" or "minimal" $\phi-$models.

It was the insight of Booth and co-authors that working with two relations, instead of just one preference relation, and leaving the set of $\phi-$models, one can code many different approaches to theory revision which go beyond the AGM approach. The basic idea can perhaps best be described as follows: In the general setting, the second relation is a subrelation of the first, assume now that they are identical. But we leave the set of $\phi-$models. Consider again the preference picture. We take now not only the "best" $\phi-$models, but also all those models (forcibly $\neg\phi-$models) which are better than one of the best $\phi-$models. Different subrelations have different interpretations, but this picture should suffice here. If $\nu(\phi)$ is the function which chooses for ϕ all those models we just described, then we can split ν in two, one part $\mu^+(\phi)$, which chooses as usual the best $\phi-$models, one part $\mu^-(\phi)$, which chooses the $\neg\phi-$models below the best $\phi-$models. So $\mu^+(\phi) = \nu(\phi) \cap M(\phi)$, and $\mu^-(\phi) = \nu(\phi) - M(\phi)$.

The characterisation problem is now to give conditions for ν (or for μ^+ and μ^-) which hold for all such choice functions as described above, and conversely, if they hold, then we can construct both relations, and find again the same choice functions by above definition.

Booth et al. gave such a characterisation for the finite case. Characterising ranked structures in the finite case is, in principle, easy, as we can "grab" any single model by a formula. The infinite case is similar, if we are allowed to use full, perhaps infinite, theories, i.e., sets of formulas. Again, we can "grab" one model with a full theory. The more challenging case is the infinite version, when we are allowed to consider formulas only, as any formula will (if consistent) always have an infinity of models. So, if we want to find for x some y with $x < y$ and some additional properties for y, we have somehow to "circumscribe" y, without ever being allowed to use the full information we need. This is the problem we solve in Section 4 (page 422).

We first give there the conditions, formulate our main result, and turn to the proof. The proof consists of two parts. In the first part, we construct the ranked relation $<$. This is quite straightforward, using usual techniques of constructing such a relation. The proof is also quite modular, e.g., we show transitivity, absence of cycles, and then use a known lemma for the final construction. The more interesting and challenging part is to construct the subrelation $\lhd$. It is there that we have to do with finite bits of information, striving for a full characterisation of one model (for which we need an infinite amount of information). The trick is to construct the infinite amount of information using inductively a finite amount of information. (A picture might illustrate this: we want to draw a 30 cm long straight line, but have only a rule of length 20 cm. We draw first 20 cm, then push the rule down, and draw the last 10 cm, holding the rule against the first part.) This part of the proof is not very modular, and the interested reader will just have to follow the line.

The, for us, main definition Booth et al. give is (in slight modification, we use the strict subrelations):

Definition 1.1. Given a deductively closed set of formulas K (as ususal in AGM style revision, see [1]), and $\leq$ and $\unlhd$, we define

$K \infty \phi := Th(\{w : w \unlhd w'$ for some $w' \in min(M(\neg\phi), \leq)\})$,

i.e. $K \infty \phi$ is given by all those worlds, which are below the $K-$closest $\neg\phi-$worlds, as seen from K.

We want to characterize $K \infty \phi$, for fixed K. Booth et al. have done the finite case by working with complete consistent formulas, i.e. single models. We want to do the infinite case without using complete consistent theories, i.e. in the usual style of completeness results in the area.

Our approach is basically semantic, though we use sometimes the language of logic, on the one hand to show how to approximate with formulas a single model, and on the other hand when we use classical compactness.

This is, however, just a matter of speaking, and we could translate it into model sets, too, but we do not think that we would win much by doing so. Moreoever, we will treat only the formula case, as this seems to be the most interesting (otherwise the problem of approximation by formulas would not exist), and restrict ourselves to the definability preserving case. The more general case is left open, for a young researcher who wants to sharpen his tools by solving it. Another open problem is to treat the same question for variable K, and for distance based revision.

1.2. Outline

After giving the framework of the article by Booth et al., we give background information about preferential logic. We then state the conditions we need, formulate our result (the extension to the infinite case of the representation result for the finite case stated in [2]), and proceed to the proof.

We introduce two functions μ^+ and μ^-, define a relation $<$ and show that it represents μ^+, and then (the more difficult part) define a relation $\lhd$, and show that it represents μ^-.

We will work in propositional logic here, where the language can be infinite, i.e. where there may be infinitely (also uncountably) many propositional variables.

2. The framework of [2]

For the reader's convenience, and to put our work a bit more into perspective, we repeat now some of the definitions and results given by Booth and his co-authors. We will be very brief, and just prepare the terrain so to say.

Consequently, all material in this section is due to Booth and his co-authors.

$\leq$ will be a total pre-order, anchored on $M(K)$, the models of K, i.e. $M(K) = min(W, \leq)$, the set of $\leq -$minimal worlds.

We have a second binary relation $\preceq$ on W, which is a reflexive subrelation of $\leq$.

Definition 2.1. (1) $(\leq, \preceq)$ is a $K-$context iff $\leq$ is a total pre-order on W, anchored on $M(K)$, and $\preceq$ is a reflexive sub-relation of $\leq$.

(2) $K \infty \phi := Th(\{w : w \preceq w' \text{ for some } w' \in min(M(\neg\phi), \leq)\})$ is called a basic removal operator.

When we translate to model sets, or even generalize to arbitrary sets, and reformulate a little, forgetting about the fixed K, we can obtain:

$\mu^+(X) := \{x \in X : x \text{ is } \leq -\text{minimal in } X\}$, and
$\mu^-(X) := \{y \notin X : y \preceq x \text{ for some } x \in \mu^+(X)\}$,
and separate the usual AGM operation, $\mu^+(X)$, from the new part: everything which is below $\mu^+(X)$.

This is, roughly, the perspective from which we see the problem.

THEOREM 2.1. (Note that Booth et al. impose that ϕ on the right of ∞ is always consistent, so this caveat for some conditions in the AGM theory is not necessary.)

Basic removal is characterized by:

$(B1)$ $K\infty\phi = Cn(K\infty\phi)$ - Cn classical consequence,
$(B2)$ $\phi \notin K\infty\phi$,
$(B3)$ If $\models \phi \leftrightarrow \phi'$, then $K\infty\phi = K\infty\phi'$,
$(B4)$ $K\infty\bot = K$,
$(B5)$ $K\infty\phi \subseteq Cn(K \cup \{\neg\phi\})$,
$(B6)$ if $\sigma \in K\infty(\sigma \wedge \phi)$, then $\sigma \in K\infty(\sigma \wedge \phi \wedge \psi)$,
$(B7)$ if $\sigma \in K\infty(\sigma \wedge \phi)$, then $K\infty\phi \subseteq K\infty(\sigma \wedge \phi)$,
$(B8)$ $(K\infty\sigma) \cap (K\infty\phi) \subseteq K\infty(\sigma \wedge \phi)$,
$(B9)$ if $\phi \notin K\infty(\sigma \wedge \phi)$, then $K\infty(\sigma \wedge \phi) \subseteq K\infty\phi$.

$(B1) - (B3)$ belong to the basic AGM contraction postulates, $(B4) - (B5)$ are weakened versions of another basic AGM postulate:

$(Vacuity)$ If $\phi \notin K$, then $K\infty\phi = K$

which does not necessarily hold for basic removal operators.

The same holds for the remaining two basic AGM contraction postulates:

$(Inclusion)$ $K\infty\phi \subseteq K$

$(Recovery)$ $K \subseteq Cn((K\infty\phi) \cup \{\phi\})$.

The main definition towards the completeness result of Booth et al. is:

DEFINITION 2.2. Given K and ∞, the structure $C(K, \infty)$ is defined by:

$(\leq)$ $w \leq w'$ iff $\neg\alpha \notin K\infty(\neg\alpha \wedge \neg\alpha')$ and
$(\preceq)$ $w \preceq w'$ iff $\neg\alpha \notin K\infty\neg\alpha'$,
where α is a formula which holds exactly in w, analogously for w' and α'.

(Thus, α and α' are equivalent to complete consistent theories, with models w and w'. This is, of course, only possible in finite languages of classical propositional languages, as we only have finite conjunctions at our disposal. Herein lies the restriction.)

As it is easy to get confused here, we add a comment — where we take liberties to make things intuitively clearer.

Looking back at Definition 2.1 (page 406), we have

$(\leq)$ $w \leq w'$ iff $w \in \{v : v \preceq v'$ for some $v' \in min(\{w, w'\}, \leq)\}$

and

$(\preceq)$ $w \preceq w'$ iff $w \in \{v : v \preceq v'$ for some $v' \in min(\{w'\}, \leq)\}$

In our perspective, in the case $(\leq)$ we stay in $X := \{w, w'\}$, so we speak about $\mu^+(X)$, in the second case, we leave $X := \{w'\}$, so we speak about $\mu^-(X)$. So we have $w \leq w'$ iff $w \in \mu^+(\{w, w'\})$, $w \preceq w'$ iff $w \in \mu^-(\{w'\})$.

Booth et al. then give a long list of Theorems showing equivalence between various postulates, and conditions on the orderings $\leq$ and $\preceq$. This, of course, shows the power of their approach.

We give three examples:

CONDITION 2.1.

(c) If (for each $i = 1, 2$) $w_i \leq w'$ for all w', then $w_1 \preceq w_2$.

(d) If $w_1 \leq w_2$ for all w_2, then $w_1 \preceq w_2$ for all w_2.

(e) If $w_1 \preceq w_2$, then $w_1 = w_2$ or $w_1 \leq w'$ for all w'.

THEOREM 2.2. Let ∞ be a basic removal operator as defined above.

(1) ∞ satisfies one half of $(Vacuity)$: If $\phi \notin K$, then $K \subseteq K \infty \phi$,

(2.1) If $(\leq, \preceq)$ satisfies (c), then ∞ satisfies $(Vacuity)$.

(2.2) If ∞ satisfies $(Vacuity)$, then $C(K, \infty)$ satisfies (c).

(3.1) If $(\leq, \preceq)$ satisfies (d), then ∞ satisfies $(Inclusion)$.

(3.2) If ∞ satisfies $(Inclusion)$, then $C(K, \infty)$ satisfies (d).

(4.1) If $(\leq, \preceq)$ satisfies (e), then ∞ satisfies $(Recovery)$.

(4.2) If ∞ satisfies $(Recovery)$, then $C(K, \infty)$ satisfies (e).

(5) The following are equivalent:

(5.1) ∞ is a full AGM contraction operator,

(5.2) ∞ satisfies $(B1) - (B9)$, $(Inclusion)$, and $(Recovery)$

(5.3) ∞ is generated by some $(\leq, \preceq)$ satisfying (d) and (e).

3. Background material

We give here some background material showing without proof connections between conditions for preferential structures and an abstract result on orderings. Moreover, we give an introduction to AGM revision.

The material presented here goes beyond what is strictly needed, but helps to put the question more into perspective.

DEFINITION 3.1.

(1) We use $\mathcal{P}$ to denote the power set operator, $\Pi\{X_i : i \in I\} := \{g : g : I \to \bigcup\{X_i : i \in I\}, \forall i \in I.g(i) \in X_i\}$ is the general cartesian product, $card(X)$ shall denote the cardinality of X, and V the set-theoretic universe we work in - the class of all sets. Given a set of pairs $\mathcal{X}$, and a set X, we denote by $\mathcal{X} \upharpoonright X := \{\langle x, i\rangle \in \mathcal{X} : x \in X\}$. When the context is clear, we will sometime simply write X for $\mathcal{X} \upharpoonright X$. (The intended use is for preferential structures, where x will be a point (intention: a classical propositional model), and i an index, permitting copies of logically identical points.)

(2) $A \subseteq B$ will denote that A is a subset of B or equal to B, and $A \subset B$ that A is a proper subset of B, likewise for $A \supseteq B$ and $A \supset B$.
Given some fixed set U we work in, and $X \subseteq U$, then $\boldsymbol{C}(X) := U - X$.

(3) If $\mathcal{Y} \subseteq \mathcal{P}(X)$ for some X, we say that $\mathcal{Y}$ satisfies
$(\cap)$ iff it is closed under finite intersections,
$(\bigcap)$ iff it is closed under arbitrary intersections,
$(\cup)$ iff it is closed under finite unions,
$(\bigcup)$ iff it is closed under arbitrary unions,
$(\boldsymbol{C})$ iff it is closed under complementation,
$(-)$ iff it is closed under set difference.

(4) We will sometimes write $A = B \parallel C$ for: $A = B$, or $A = C$, or $A = B \cup C$.

We make ample and tacit use of the Axiom of Choice.

DEFINITION 3.2.

(1) We work here in a classical propositional language $\mathcal{L}$, a theory T will be an arbitrary set of formulas. Formulas will often be named ϕ, ψ, etc., theories T, S, etc.
$v(\mathcal{L})$ will be the set of propositional variables of $\mathcal{L}$.
$F(\mathcal{L})$ will be the set of formulas of $\mathcal{L}$.
$M_{\mathcal{L}}$ will be the set of (classical) models for $\mathcal{L}$, $M(T)$ or M_T is the set of models of T, likewise $M(\phi)$ for a formula ϕ.

(2) $\boldsymbol{D}_{\mathcal{L}} := \{M(T) : T \text{ a theory in } \mathcal{L}\}$, the set of *definable* model sets.
Note that, in classical propositional logic, $\emptyset, M_{\mathcal{L}} \in \boldsymbol{D}_{\mathcal{L}}$, $\boldsymbol{D}_{\mathcal{L}}$ contains singletons, is closed under arbitrary intersections and finite unions.
An operation $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ for $\mathcal{Y} \subseteq \mathcal{P}(M_{\mathcal{L}})$ is called *definability preserving* , (dp) or (μdp) in short, iff for all $X \in \boldsymbol{D}_{\mathcal{L}} \cap \mathcal{Y}$ $f(X) \in \boldsymbol{D}_{\mathcal{L}}$.
We will also use (μdp) for binary functions $f : \mathcal{Y} \times \mathcal{Y} \to\to \mathcal{P}(M_{\mathcal{L}})$ — as needed for theory revision — with the the obvious meaning.

(3) $\vdash$ will be classical derivability, and
$\overline{T} := \{\phi : T \vdash \phi\}$, the closure of T under $\vdash$.

(4) $Con(.)$ will stand for classical consistency, so $Con(\phi)$ will mean that ϕ is classical consistent, likewise for $Con(T)$. $Con(T, T')$ will stand for $Con(T \cup T')$, etc.

(5) Given a consequence relation $\mid\!\sim$, we define
$\overline{\overline{T}} := \{\phi : T \mid\!\sim \phi\}$.
(There is no fear of confusion with $\overline{T}$, as it just is not useful to close twice under classical logic.)

(6) $T \vee T' := \{\phi \vee \phi' : \phi \in T, \phi' \in T'\}$.

(7) If $X \subseteq M_{\mathcal{L}}$, then $Th(X) := \{\phi : X \models \phi\}$, likewise for $Th(m)$, $m \in M_{\mathcal{L}}$. ($\models$ will usually be classical validity.)

Explanation of Table 1 (page 417), "Logical rules, definitions and connections Part I" and Table 2 (page 418), "Logical rules, definitions and connections Part II":

The tables are split in two, as they would not fit onto a page otherwise. The difference between the first two columns is that the first column treats the formula version of the rule, the second the more general theory (i.e., set of formulas) version.

The numbers in the first column "Corr.", meaning "Correspondence", refer to Proposition 21 in [3]. The first column "Corr." is to be understood as follows:

Let a logic $\mid\!\sim$ satisfy (LLE) and (CCL), and define a function $f : \boldsymbol{D}_{\mathcal{L}} \to \boldsymbol{D}_{\mathcal{L}}$ by $f(M(T)) := M(\overline{\overline{T}})$. Then f is well defined, satisfies (μdp), and $\overline{\overline{T}} = Th(f(M(T)))$.

If $\mid\!\sim$ satisfies a rule in the left hand side, then - provided the additional properties noted in the middle for $\Rightarrow$ hold, too - f will satisfy the property in the right hand side.

Conversely, if $f : \mathcal{Y} \to \mathcal{P}(M_{\mathcal{L}})$ is a function, with $\boldsymbol{D}_{\mathcal{L}} \subseteq \mathcal{Y}$, and we define a logic $\mid\!\sim$ by $\overline{\overline{T}} := Th(f(M(T)))$, then $\mid\!\sim$ satisfies (LLE) and (CCL). If f satisfies (μdp), then $f(M(T)) = M(\overline{\overline{T}})$.

If f satisfies a property in the right hand side, then - provided the additional properties noted in the middle for $\Leftarrow$ hold, too - $\mid\!\sim$ will satisfy the property in the left hand side.

We use the following abbreviations for those supplementary conditions in the "Correspondence" columns: "$T = \phi$" means that, if one of the theories (the one named the same way) is equivalent to a formula, we do not need (μdp). $-(\mu dp)$ stands for "without (μdp)".

$A = B \parallel C$ will abbreviate $A = B$, or $A = C$, or $A = B \cup C$.

The numbers in the left hand column of the following Fact 3.1 (page 411) refer to the proof of Fact 3.6 in [3].

FACT 3.1. Table 3 (page 419), "Interdependencies of algebraic rules", is to be read as follows: If the left hand side holds for some function $f : \mathcal{Y} \to \mathcal{P}(U)$, and the auxiliary properties noted in the middle also hold for f or $\mathcal{Y}$, then the right hand side will hold, too — and conversely.

"sing." will stand for: "$\mathcal{Y}$ contains singletons"

DEFINITION 3.3. Fix $U \neq \emptyset$, and consider arbitrary X. Note that this X has not necessarily anything to do with U, or $\mathcal{U}$ below. Thus, the functions $\mu_{\mathcal{M}}$ below are in principle functions from V to V — where V is the set theoretical universe we work in.

Note that we work here often with copies of elements (or models). In other areas of logic, most authors work with valuation functions. Both definitions — copies or valuation functions — are equivalent, a copy $\langle x, i\rangle$ can be seen as a state $\langle x, i\rangle$ with valuation x. In the beginning of research on preferential structures, the notion of copies was widely used, whereas e.g., [4] used that of valuation functions. There is perhaps a weak justification of the former terminology. In modal logic, even if two states have the same valid classical formulas, they might still be distinguishable by their valid modal formulas. But this depends on the fact that modality is in the object language. In most work on preferential structures, the consequence relation is outside the object language, so different states with same valuation are in a stronger sense copies of each other.

(1) *Preferential models* or *structures.*

- (1.1) The version without copies:

  A pair $\mathcal{M} := \langle U, \prec\rangle$ with U an arbitrary set, and $\prec$ an arbitrary binary relation on U is called a *preferential model* or *structure.*

- (1.2) The version with copies :

  A pair $\mathcal{M} := \langle \mathcal{U}, \prec\rangle$ with $\mathcal{U}$ an arbitrary set of pairs, and $\prec$ an arbitrary binary relation on $\mathcal{U}$ is called a *preferential model* or *structure.*

  If $\langle x, i\rangle \in \mathcal{U}$, then x is intended to be an element of U, and i the index of the copy.

We sometimes also need copies of the relation $\prec$. We will then replace $\prec$ by one or several arrows α attacking non-minimal elements, e.g., $x \prec y$ will be written $\alpha : x \to y$, $\langle x,i\rangle \prec \langle y,i\rangle$ will be written $\alpha : \langle x,i\rangle \to \langle y,i\rangle$, and finally we might have $\langle \alpha,k\rangle : x \to y$ and $\langle \alpha,k\rangle : \langle x,i\rangle \to \langle y,i\rangle$, etc.

(2) *Minimal elements*, the functions $\mu_{\mathcal{M}}$

(2.1) The version without copies:
Let $\mathcal{M} := \langle U, \prec\rangle$, and define
$\mu_{\mathcal{M}}(X) := \{x \in X : x \in U \wedge \neg\exists x' \in X \cap U.x' \prec x\}$.
$\mu_{\mathcal{M}}(X)$ is called the set of *minimal elements* of X (in $\mathcal{M}$).
Thus, $\mu_{\mathcal{M}}(X)$ is the set of elements such that there is no smaller one in X.

(2.2) The version with copies:
Let $\mathcal{M} := \langle \mathcal{U}, \prec\rangle$ be as above. Define
$\mu_{\mathcal{M}}(X) := \{x \in X : \exists\langle x,i\rangle \in \mathcal{U}.\neg\exists\langle x',i'\rangle \in \mathcal{U}(x' \in X \wedge \langle x',i'\rangle' \prec \langle x,i\rangle)\}$.
Thus, $\mu_{\mathcal{M}}(X)$ is the projection on the first coordinate of the set of elements such that there is no smaller one in X.
Again, by abuse of language, we say that $\mu_{\mathcal{M}}(X)$ is the set of *minimal elements* of X in the structure. If the context is clear, we will also write just μ.
We sometimes say that $\langle x,i\rangle$ "*kills*" or "*minimizes*" $\langle y,j\rangle$ if $\langle x,i\rangle \prec \langle y,j\rangle$. By abuse of language we also say a set X *kills* or *minimizes* a set Y if for all $\langle y,j\rangle \in \mathcal{U}$, $y \in Y$ there is $\langle x,i\rangle \in \mathcal{U}$, $x \in X$ s.t. $\langle x,i\rangle \prec \langle y,j\rangle$.
$\mathcal{M}$ is also called *injective* or 1-copy , iff there is always at most one copy $\langle x,i\rangle$ for each x. Note that the existence of copies corresponds to a non-injective labelling function — as is often used in nonclassical logic, e.g., modal logic.

We say that $\mathcal{M}$ is *transitive*, *irreflexive*, etc., iff $\prec$ is.
Note that $\mu(X)$ might well be empty, even if X is not.

Usually, preferential structures work with copies of models (or, equivalently, with function which attribute values to points, as, e.g., in Kripke structures for modal logics). Ranked structures (usually) do not need copies. For completeness' sake, we gave the full definition, though it will not be used here. For more details, the reader is referred to [7] or [3].

Definition 3.4. We define the consequence relation of a preferential structure for a given propositional language $\mathcal{L}$.

(1)(1.1) If m is a classical model of a language $\mathcal{L}$, we say by abuse of language
$\langle m, i\rangle \models \phi$ iff $m \models \phi$,
and if X is any set of such pairs, that
$X \models \phi$ iff for all $\langle m, i\rangle \in X$ $m \models \phi$.

(1.2) If $\mathcal{M}$ is a preferential structure, and X is a set of $\mathcal{L}-$models for a classical propositional language $\mathcal{L}$, or a set of pairs $\langle m, i\rangle$, where the m are such models, we call $\mathcal{M}$ a *classical preferential structure* or *model*.

(2) *Validity* in a preferential structure, or the *semantical consequence relation* defined by such a structure:
Let $\mathcal{M}$ be as above.
We define:
$T \models_{\mathcal{M}} \phi$ iff $\mu_{\mathcal{M}}(M(T)) \models \phi$, i.e., $\mu_{\mathcal{M}}(M(T)) \subseteq M(\phi)$.

(3) $\mathcal{M}$ will be called *definability preserving* iff for all $X \in \boldsymbol{D}_{\mathcal{L}}$ $\mu_{\mathcal{M}}(X) \in \boldsymbol{D}_{\mathcal{L}}$.

As $\mu_{\mathcal{M}}$ is defined on $\boldsymbol{D}_{\mathcal{L}}$, but need by no means always result in some new definable set, this is (and reveals itself as a quite strong) additional property.

(Note that the X in above definition is any set, intuitively $x \in \mathcal{Y}$, where $\mathcal{Y}$ is some set of sets as in Definition 3.3 (page 411).)

Definition 3.5. Let $\mathcal{Y} \subseteq \mathcal{P}(U)$. (In applications to logic, $\mathcal{Y}$ will be $\boldsymbol{D}_{\mathcal{L}}$.)

A preferential structure $\mathcal{M}$ is called $\mathcal{Y}-$smooth iff for every $X \in \mathcal{Y}$ every element $x \in X$ is either minimal in X or above an element, which is minimal in X. More precisely:

(1) The version without copies:
If $x \in X \in \mathcal{Y}$, then either $x \in \mu(X)$ or there is $x' \in \mu(X).x' \prec x$.

(2) The version with copies:
If $x \in X \in \mathcal{Y}$, and $\langle x, i\rangle \in \mathcal{U}$, then either there is no $\langle x', i'\rangle \in \mathcal{U}$, $x' \in X$, $\langle x', i'\rangle \prec \langle x, i\rangle$ or there is $\langle x', i'\rangle \in \mathcal{U}$, $\langle x', i'\rangle \prec \langle x, i\rangle$, $x' \in X$, s.t. there is no $\langle x'', i''\rangle \in \mathcal{U}$, $x'' \in X$, with $\langle x'', i''\rangle \prec \langle x', i'\rangle$.
(Writing down all details here again might make it easier to read applications of the definition later on.)

When considering the models of a language $\mathcal{L}$, $\mathcal{M}$ will be called *smooth* iff it is $\boldsymbol{D}_\mathcal{L}-$smooth ; $\boldsymbol{D}_\mathcal{L}$ is the default.

Obviously, the richer the set $\mathcal{Y}$ is, the stronger the condition $\mathcal{Y}-$smoothness will be.

FACT 3.2. Let $\prec$ be an irreflexive, binary relation on X, then the following two conditions are equivalent:

(1) There is Ω and an irreflexive, total, binary relation $\prec'$ on Ω and a function $f : X \to \Omega$ s.t. $x \prec y \Leftrightarrow f(x) \prec' f(y)$ for all $x, y \in X$.

(2) Let $x, y, z \in X$ and $x \perp y$ wrt. $\prec$ (i.e., neither $x \prec y$ nor $y \prec x$), then $z \prec x \Rightarrow z \prec y$ and $x \prec z \Rightarrow y \prec z$.

PROOF. (1) $\Rightarrow$ (2): Let $x \perp y$, thus neither $f(x) \prec' f(y)$ nor $f(y) \prec' f(x)$, but then $f(x) = f(y)$. Let now $z \prec x$, so $f(z) \prec' f(x) = f(y)$, so $z \prec y$. $x \prec z \Rightarrow y \prec z$ is similar.

(2) $\Rightarrow$ (1): For $x \in X$ let $[x] := \{x' \in X : x \perp x'\}$, and $\Omega := \{[x] : x \in X\}$. For $[x], [y] \in \Omega$ let $[x] \prec' [y] :\Leftrightarrow x \prec y$. This is well-defined: Let $x \perp x'$, $y \perp y'$ and $x \prec y$, then $x \prec y'$ and $x' \prec y'$. Obviously, $\prec'$ is an irreflexive, total binary relation. Define $f : X \to \Omega$ by $f(x) := [x]$, then $x \prec y \Leftrightarrow [x] \prec' [y] \Leftrightarrow f(x) \prec' f(y)$. ■

DEFINITION 3.6. We call an irreflexive, binary relation $\prec$ on X, which satisfies (1) (equivalently (2)) of Fact 3.2 (page 414), ranked. By abuse of language, we also call a preferential structure $\langle X, \prec \rangle$ ranked, iff $\prec$ is.

FACT 3.3. If $\prec$ on X is ranked, and free of cycles, then $\prec$ is transitive.

PROOF. Let $x \prec y \prec z$. If $x \perp z$, then $y \succ z$, resulting in a cycle of length 2. If $z \prec x$, then we have a cycle of length 3. So $x \prec z$. ■

We give a generalized abstract nonsense result, taken from [5], which must be part of the folklore:

LEMMA 3.4. *Given a set X and a binary relation R on X, there exists a total preorder (i.e., a total, reflexive, transitive relation) S on X that extends R such that*

$\forall x, y \in X(xSy, ySx \Rightarrow xR^*y)$

where R^* is the reflexive and transitive closure of R.

PROOF. Define $x \equiv y$ iff xR^*y and yR^*x. The relation $\equiv$ is an equivalence relation. Let $[x]$ be the equivalence class of x under $\equiv$. Define $[x] \preceq [y]$ iff xR^*y. The definition of $\preceq$ does not depend on the representatives x and y chosen. The relation $\preceq$ on equivalence classes is a partial order. Let $\leq$ be any total order on these equivalence classes that extends $\preceq$. Define xSy iff $[x] \leq [y]$. The relation S is total (since $\leq$ is total) and transitive (since $\leq$ is transitive) and is therefore a total preorder. It extends R by the definition of $\preceq$ and the fact that $\leq$ extends $\preceq$. Suppose now xSy and ySx. We have $[x] \leq [y]$ and $[y] \leq [x]$ and therefore $[x] = [y]$ by antisymmetry. Therefore $x \equiv y$ and xR^*y. ■

To put AGM revision into context, we cite the main definitions and results, essentially from [1].

Definition 3.7. We present in parallel the logical and the semantic (or purely algebraic) side. For the latter, we work in some fixed universe U, and the intuition is $U = M_{\mathcal{L}}$, $X = M(K)$, etc., so, e.g., $A \in K$ becomes $X \subseteq B$, etc.

(For reasons of readability, we omit most caveats about definability.)

$K_\perp$ will denote the inconsistent theory.

We consider two functions, - and $*$, taking a deductively closed theory and a formula as arguments, and returning a (deductively closed) theory on the logics side. The algebraic counterparts work on definable model sets. It is obvious that $(K-1)$, $(K*1)$, $(K-6)$, $(K*6)$ have vacuously true counterparts on the semantical side. Note that K (X) will never change, everything is relative to fixed K (X). $K*\phi$ is the result of revising K with ϕ. $K-\phi$ is the result of subtracting enough from K to be able to add $\neg\phi$ in a reasonable way, called contraction.

Moreover, let $\leq_K$ be a relation on the formulas relative to a deductively closed theory K on the formulas of $\mathcal{L}$, and $\leq_X$ a relation on $\mathcal{P}(U)$ or a suitable subset of $\mathcal{P}(U)$ relative to fixed X. When the context is clear, we simply write $\leq$. $\leq_K$ $(\leq_X)$ is called a relation of epistemic entrenchment for K (X).

Table 4 (page 420), "AGM theory revision", presents "rationality postulates" for contraction (-), rationality postulates revision $(*)$ and epistemic entrenchment. In AGM tradition, K will be a deductively closed theory, ϕ, ψ formulas. Accordingly, X will be the set of models of a theory, A, B the model sets of formulas.

In the further development, formulas ϕ etc. may sometimes also be full theories. As the transcription to this case is evident, we will not go into details.

Definition 3.8. We define the collective and the individual variant of choosing the closest elements in the second operand by two operators, $|, \uparrow : \mathcal{P}(U) \times \mathcal{P}(U) \rightarrow \mathcal{P}(U)$:

Let d be a distance or pseudo-distance.

$X \mid Y := \{y \in Y : \exists x_y \in X.\forall x' \in X, \forall y' \in Y(d(x_y, y) \leq d(x', y')\}$

(the collective variant, used in theory revision)

and

$X \uparrow Y := \{y \in Y : \exists x_y \in X.\forall y' \in Y(d(x_y, y) \leq d(x_y, y')\}$

(the individual variant, used for counterfactual conditionals and theory update).

Thus, $A \mid_d B$ is the subset of B consisting of all $b \in B$ that are closest to A. Note that, if A or B is infinite, $A \mid_d B$ may be empty, even if A and B are not empty. A condition assuring nonemptiness will be imposed when necessary.

Remark 3.5.

(1) Note that $(X \mid 7)$ and $(X \mid 8)$ express a central condition for ranked structures: If we note $X \mid .$ by $f_X(.)$, we then have: $f_X(A) \cap B \neq \emptyset \Rightarrow f_X(A \cap B) = f_X(A) \cap B$.

(2) It is trivial to see that AGM revision cannot be defined by an individual distance (see Definition 3.8 (page 415)): Suppose $X \mid Y := \{y \in Y : \exists x_y \in X(\forall y' \in Y.d(x_y, y) \leq d(x_y, y'))\}$. Consider a, b, c. $\{a, b\} \mid \{b, c\} = \{b\}$ by $(X \mid 3)$ and $(X \mid 4)$, so $d(a, b) < d(a, c)$. But on the other hand $\{a, c\} \mid \{b, c\} = \{c\}$, so $d(a, b) > d(a, c)$, *contradiction.*

Proposition 3.6. We refer here to Table 5 (page 421), "AGM interdefinability". Contraction, revision, and epistemic entrenchment are interdefinable by the following equations, i.e., if the defining side has the respective properties, so will the defined side. (See [1].)

Speaking in terms of distance defined revision, $X \mid A$ is the set of those $a \in A$, which are closest to X, and $X \ominus A$ is the set of y which are either in X, or in $\boldsymbol{C}(A)$ and closest to X among those in $\boldsymbol{C}(A)$.

The reader is referred to [1] for more explanation and motivation.

Table 1. Logical rules, definitions and connections Part I

Logical rules, definitions and connections Part I			
Logical rule		Corr.	Model set
Basics			
(SC) Supraclassicality $\alpha \vdash \beta \Rightarrow \alpha \mid\!\sim \beta$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	$\Rightarrow$ (4.1) $\Leftarrow$ (4.2)	$(\mu \subseteq)$ $f(X) \subseteq X$
(REF) Reflexivity $T \cup \{\alpha\} \mid\!\sim \alpha$			
(LLE) Left Logical Equivalence $\vdash \alpha \leftrightarrow \alpha', \alpha \mid\!\sim \beta \Rightarrow$ $\alpha' \mid\!\sim \beta$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$		
(RW) Right Weakening $\alpha \mid\!\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow$ $\alpha \mid\!\sim \beta'$	(RW) $T \mid\!\sim \beta, \vdash \beta \rightarrow \beta' \Rightarrow$ $T \mid\!\sim \beta'$		
(wOR) $\alpha \mid\!\sim \beta,\ \alpha' \vdash \beta \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta$	(wOR) $\overline{\overline{T}} \cap \overline{T'} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ (3.1) $\Leftarrow$ (3.2)	(μwOR) $f(X \cup Y) \subseteq f(X) \cup Y$
$(disjOR)$ $\alpha \vdash \neg\alpha',\ \alpha \mid\!\sim \beta,$ $\alpha' \mid\!\sim \beta \Rightarrow \alpha \vee \alpha' \mid\!\sim \beta$	$(disjOR)$ $\neg Con(T \cup T') \Rightarrow$ $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ (2.1) $\Leftarrow$ (2.2)	$(\mu disjOR)$ $X \cap Y = \emptyset \Rightarrow$ $f(X \cup Y) \subseteq f(X) \cup f(Y)$
(CP) Consistency Preservation $\alpha \mid\!\sim \bot \Rightarrow \alpha \vdash \bot$	(CP) $T \mid\!\sim \bot \Rightarrow T \vdash \bot$	$\Rightarrow$ (5.1) $\Leftarrow$ (5.2)	$(\mu\emptyset)$ $f(X) = \emptyset \Rightarrow X = \emptyset$
			$(\mu\emptyset fin)$ $X \neq \emptyset \Rightarrow f(X) \neq \emptyset$ for finite X
	(AND_1) $\alpha \mid\!\sim \beta \Rightarrow \alpha \not\mid\!\sim \neg\beta$		
	(AND_n) $\alpha \mid\!\sim \beta_1, \ldots, \alpha \mid\!\sim \beta_{n-1} \Rightarrow$ $\alpha \not\mid\!\sim (\neg\beta_1 \vee \ldots \vee \neg\beta_{n-1})$		
(AND) $\alpha \mid\!\sim \beta, \alpha \mid\!\sim \beta' \Rightarrow$ $\alpha \mid\!\sim \beta \wedge \beta'$	(AND) $T \mid\!\sim \beta, T \mid\!\sim \beta' \Rightarrow$ $T \mid\!\sim \beta \wedge \beta'$		
(CCL) Classical Closure	(CCL) $\overline{\overline{T}}$ classically closed		
(OR) $\alpha \mid\!\sim \beta, \alpha' \mid\!\sim \beta \Rightarrow$ $\alpha \vee \alpha' \mid\!\sim \beta$	(OR) $\overline{\overline{T}} \cap \overline{\overline{T'}} \subseteq \overline{\overline{T \vee T'}}$	$\Rightarrow$ (1.1) $\Leftarrow$ (1.2)	(μOR) $f(X \cup Y) \subseteq f(X) \cup f(Y)$
$\overline{\overline{\alpha \wedge \alpha'}} \subseteq \overline{\overline{\overline{\alpha}} \cup \{\alpha'\}}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$	$\Rightarrow$ (6.1) $\Leftarrow (\mu dp) + (\mu \subseteq)$ (6.2) $\not\Leftarrow -(\mu dp)$ (6.3) $\Leftarrow (\mu \subseteq)$ (6.4) $T' = \phi$	(μPR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$
		$\Leftarrow$ (6.5) $T' = \phi$	$(\mu PR')$ $f(X) \cap Y \subseteq f(X \cap Y)$
(CUT) $T \mid\!\sim \alpha; T \cup \{\alpha\} \mid\!\sim \beta \Rightarrow$ $T \mid\!\sim \beta$	(CUT) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} \subseteq \overline{\overline{T}}$	$\Rightarrow$ (7.1) $\Leftarrow$ (7.2)	(μCUT) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(X) \subseteq f(Y)$

Table 2. Logical rules, definitions and connections Part II

Logical rules, definitions and connections Part II			
Logical rule		Corr.	Model set
Cumulativity			
(wCM) $\alpha \mid\sim \beta, \alpha' \vdash \alpha, \alpha \wedge \beta \vdash \alpha' \Rightarrow$ $\alpha' \mid\sim \beta$			
(CM_2) $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow \alpha \wedge \beta \not\vdash \neg \beta'$			
(CM_n) $\alpha \mid\sim \beta_1, \ldots, \alpha \mid\sim \beta_n \Rightarrow$ $\alpha \wedge \beta_1 \wedge \ldots \wedge \beta_{n-1} \not\vdash \neg \beta_n$			
(CM) Cautious Monotony $\alpha \mid\sim \beta, \alpha \mid\sim \beta' \Rightarrow$ $\alpha \wedge \beta \mid\sim \beta'$	(CM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	$\Rightarrow$ (8.1) $\Leftarrow$ (8.2)	(μCM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$
or $(ResM)$ Restricted Monotony $T \mid\sim \alpha, \beta \Rightarrow T \cup \{\alpha\} \mid\sim \beta$		$\Rightarrow$ (9.1) $\Leftarrow$ (9.2)	$(\mu ResM)$ $f(X) \subseteq A \cap B \Rightarrow$ $f(X \cap A) \subseteq B$
(CUM) Cumulativity $\alpha \mid\sim \beta \Rightarrow$ $(\alpha \mid\sim \beta' \Leftrightarrow \alpha \wedge \beta \mid\sim \beta')$	(CUM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	$\Rightarrow$ (11.1) $\Leftarrow$ (11.2)	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$
	$(\subseteq \supseteq)$ $T \subseteq \overline{\overline{T'}}, T' \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T'}} = \overline{\overline{T}}$	$\Rightarrow$ (10.1) $\Leftarrow$ (10.2)	$(\mu \subseteq \supseteq)$ $f(X) \subseteq Y, f(Y) \subseteq X \Rightarrow$ $f(X) = f(Y)$
Rationality			
$(RatM)$ Rational Monotony $\alpha \mid\sim \beta, \alpha \not\mid\sim \neg \beta' \Rightarrow$ $\alpha \wedge \beta' \mid\sim \beta$	$(RatM)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} \supseteq \overline{\overline{\overline{T'}} \cup T}$	$\Rightarrow$ (12.1) $\Leftarrow$ (μdp) (12.2) $\not\Leftarrow$ $-(\mu dp)$ (12.3) $\Leftarrow$ $T = \phi$ (12.4)	$(\mu RatM)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) \subseteq f(Y) \cap X$
	$(RatM =)$ $Con(T \cup \overline{\overline{T'}}), T \vdash T' \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{\overline{T'}} \cup T}$	$\Rightarrow$ (13.1) $\Leftarrow$ (μdp) (13.2) $\not\Leftarrow$ $-(\mu dp)$ (13.3) $\Leftarrow$ $T = \phi$ (13.4)	$(\mu =)$ $X \subseteq Y, X \cap f(Y) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$
	$(Log =')$ $Con(\overline{\overline{T'}} \cup T) \Rightarrow$ $\overline{\overline{T \cup T'}} = \overline{\overline{\overline{T'}} \cup T}$	$\Rightarrow$ (14.1) $\Leftarrow$ (μdp) (14.2) $\not\Leftarrow$ $-(\mu dp)$ (14.3) $\Leftarrow$ $T = \phi$ (14.4)	$(\mu =')$ $f(Y) \cap X \neq \emptyset \Rightarrow$ $f(Y \cap X) = f(Y) \cap X$
(DR) $\alpha \vee \beta \mid\sim \gamma \Rightarrow$ $\alpha \mid\sim \gamma$ or $\beta \mid\sim \gamma$	$(Log \parallel)$ $\overline{\overline{T \vee T'}}$ is one of $\overline{\overline{T}}$, or $\overline{\overline{T'}}$, or $\overline{\overline{T}} \cap \overline{\overline{T'}}$ (by (CCL))	$\Rightarrow$ (15.1) $\Leftarrow$ (15.2)	$(\mu \parallel)$ $f(X \cup Y)$ is one of $f(X)$, $f(Y)$ or $f(X) \cup f(Y)$
	$(Log\cup)$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\neg Con(\overline{\overline{T \vee T'}} \cup T')$	$\Rightarrow$ $(\mu \subseteq) + (\mu =)$ (16.1) $\Leftarrow$ (μdp) (16.2) $\not\Leftarrow$ $-(\mu dp)$ (16.3)	$(\mu\cup)$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) \cap Y = \emptyset$
	$(Log\cup')$ $Con(\overline{\overline{T'}} \cup T), \neg Con(\overline{\overline{T'}} \cup \overline{\overline{T}}) \Rightarrow$ $\overline{\overline{T \vee T'}} = \overline{\overline{T}}$	$\Rightarrow$ $(\mu \subseteq) + (\mu =)$ (17.1) $\Leftarrow$ (μdp) (17.2) $\not\Leftarrow$ $-(\mu dp)$ (17.3)	$(\mu\cup')$ $f(Y) \cap (X - f(X)) \neq \emptyset \Rightarrow$ $f(X \cup Y) = f(X)$
			$(\mu \in)$ $a \in X - f(X) \Rightarrow$ $\exists b \in X. a \notin f(\{a, b\})$

Table 3. Interdependencies of algebraic rules

Interdependencies of algebraic rules			
Basics			
(1.1)	(μPR)	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu PR')$
(1.2)		$\Leftarrow$	
(2.1)	(μPR)	$\Rightarrow (\mu \subseteq)$	(μOR)
(2.2)		$\Leftarrow (\mu \subseteq) + (-)$	
(2.3)		$\Rightarrow (\mu \subseteq)$	(μwOR)
(2.4)		$\Leftarrow (\mu \subseteq) + (-)$	
(3)	(μPR)	$\Rightarrow$	(μCUT)
(4)	$(\mu \subseteq) + (\mu \subseteq\supseteq) + (\mu CUM)$ $+(\mu RatM) + (\cap)$	$\not\Rightarrow$	(μPR)
Cumulativity			
(5.1)	(μCM)	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu ResM)$
(5.2)		$\Leftarrow$ (infin.)	
(6)	$(\mu CM) + (\mu CUT)$	$\Leftrightarrow$	(μCUM)
(7)	$(\mu \subseteq) + (\mu \subseteq\supseteq)$	$\Rightarrow$	(μCUM)
(8)	$(\mu \subseteq) + (\mu CUM) + (\cap)$	$\Rightarrow$	$(\mu \subseteq\supseteq)$
(9)	$(\mu \subseteq) + (\mu CUM)$	$\not\Rightarrow$	$(\mu \subseteq\supseteq)$
Rationality			
(10)	$(\mu RatM) + (\mu PR)$	$\Rightarrow$	$(\mu =)$
(11)	$(\mu =)$	$\Rightarrow$	$(\mu PR) + (\mu RatM)$
(12.1)	$(\mu =)$	$\Rightarrow (\cap) + (\mu \subseteq)$	$(\mu =')$
(12.2)		$\Leftarrow$	
(13)	$(\mu \subseteq) + (\mu =)$	$\Rightarrow (\cup)$	$(\mu \cup)$
(14)	$(\mu \subseteq) + (\mu \emptyset) + (\mu =)$	$\Rightarrow (\cup)$	$(\mu \parallel)$, $(\mu \cup')$, (μCUM)
(15)	$(\mu \subseteq) + (\mu \parallel)$	$\Rightarrow (-)$ of $\mathcal{Y}$	$(\mu =)$
(16)	$(\mu \parallel) + (\mu \in) + (\mu PR) +$ $(\mu \subseteq)$	$\Rightarrow (\cup)$ + sing.	$(\mu =)$
(17)	$(\mu CUM) + (\mu =)$	$\Rightarrow (\cup)$ + sing.	$(\mu \in)$
(18)	$(\mu CUM) + (\mu =) + (\mu \subseteq)$	$\Rightarrow (\cup)$	$(\mu \parallel)$
(19)	$(\mu PR) + (\mu CUM) + (\mu \parallel)$	$\Rightarrow$ sufficient, e.g., true in $\boldsymbol{D}_{\mathcal{L}}$	$(\mu =)$.
(20)	$(\mu \subseteq) + (\mu PR) + (\mu =)$	$\not\Rightarrow$	$(\mu \parallel)$
(21)	$(\mu \subseteq) + (\mu PR) + (\mu \parallel)$	$\not\Rightarrow$ (without $(-)$)	$(\mu =)$
(22)	$(\mu \subseteq) + (\mu PR) + (\mu \parallel) +$ $(\mu =) + (\mu \cup)$	$\not\Rightarrow$	$(\mu \in)$ (thus not representable by ranked structures)

Table 4. AGM theory revision

AGM theory revision			
Contraction, $K - \phi$			
$(K-1)$	$K - \phi$ is deductively closed		
$(K-2)$	$K - \phi \subseteq K$	$(X \ominus 2)$	$X \subseteq X \ominus A$
$(K-3)$	$\phi \notin K \Rightarrow K - \phi = K$	$(X \ominus 3)$	$X \not\subseteq A \Rightarrow X \ominus A = X$
$(K-4)$	$\not\vdash \phi \Rightarrow \phi \notin K - \phi$	$(X \ominus 4)$	$A \neq U \Rightarrow X \ominus A \not\subseteq A$
$(K-5)$	$K \subseteq \overline{(K - \phi) \cup \{\phi\}}$	$(X \ominus 5)$	$(X \ominus A) \cap A \subseteq X$
$(K-6)$	$\vdash \phi \leftrightarrow \psi \Rightarrow K - \phi = K - \psi$		
$(K-7)$	$(K - \phi) \cap (K - \psi) \subseteq$ $K - (\phi \wedge \psi)$	$(X \ominus 7)$	$X \ominus (A \cap B) \subseteq$ $(X \ominus A) \cup (X \ominus B)$
$(K-8)$	$\phi \notin K - (\phi \wedge \psi) \Rightarrow$ $K - (\phi \wedge \psi) \subseteq K - \phi$	$(X \ominus 8)$	$X \ominus (A \cap B) \not\subseteq A \Rightarrow$ $X \ominus A \subseteq X \ominus (A \cap B)$
Revision, $K * \phi$			
$(K*1)$	$K * \phi$ is deductively closed	-	
$(K*2)$	$\phi \in K * \phi$	$(X \mid 2)$	$X \mid A \subseteq A$
$(K*3)$	$K * \phi \subseteq \overline{K \cup \{\phi\}}$	$(X \mid 3)$	$X \cap A \subseteq X \mid A$
$(K*4)$	$\neg\phi \notin K \Rightarrow$ $\overline{K \cup \{\phi\}} \subseteq K * \phi$	$(X \mid 4)$	$X \cap A \neq \emptyset \Rightarrow$ $X \mid A \subseteq X \cap A$
$(K*5)$	$K * \phi = K_\perp \Rightarrow \vdash \neg\phi$	$(X \mid 5)$	$X \mid A = \emptyset \Rightarrow A = \emptyset$
$(K*6)$	$\vdash \phi \leftrightarrow \psi \Rightarrow K * \phi = K * \psi$	-	
$(K*7)$	$K * (\phi \wedge \psi) \subseteq$ $\overline{(K * \phi) \cup \{\psi\}}$	$(X \mid 7)$	$(X \mid A) \cap B \subseteq$ $X \mid (A \cap B)$
$(K*8)$	$\neg\psi \notin K * \phi \Rightarrow$ $\overline{(K * \phi) \cup \{\psi\}} \subseteq K * (\phi \wedge \psi)$	$(X \mid 8)$	$(X \mid A) \cap B \neq \emptyset \Rightarrow$ $X \mid (A \cap B) \subseteq (X \mid A) \cap B$
Epistemic entrenchment			
$(EE1)$	$\leq_K$ is transitive	$(EE1)$	$\leq_X$ is transitive
$(EE2)$	$\phi \vdash \psi \Rightarrow \phi \leq_K \psi$	$(EE2)$	$A \subseteq B \Rightarrow A \leq_X B$
$(EE3)$	$\forall \phi, \psi$ $(\phi \leq_K \phi \wedge \psi$ or $\psi \leq_K \phi \wedge \psi)$	$(EE3)$	$\forall A, B$ $(A \leq_X A \cap B$ or $B \leq_X A \cap B)$
$(EE4)$	$K \neq K_\perp \Rightarrow$ $(\phi \notin K$ iff $\forall \psi . \phi \leq_K \psi)$	$(EE4)$	$X \neq \emptyset \Rightarrow$ $(X \not\subseteq A$ iff $\forall B . A \leq_X B)$
$(EE5)$	$\forall \psi . \psi \leq_K \phi \Rightarrow \vdash \phi$	$(EE5)$	$\forall B . B \leq_X A \Rightarrow A = U$

Table 5. AGM interdefinability

AGM interdefinability	
$K * \phi := \overline{(K - \neg\phi)} \cup \phi$	$X \mid A := (X \ominus \boldsymbol{C}(A)) \cap A$
$K - \phi := K \cap (K * \neg\phi)$	$X \ominus A := X \cup (X \mid \boldsymbol{C}(A))$
$K - \phi := \{\psi \in K : (\phi <_K \phi \vee \psi \text{ or } \vdash \phi)\}$	$X \ominus A := \begin{cases} X & iff \quad A = U, \\ \bigcap\{B : X \subseteq B \subseteq U, A <_X A \cup B\} & otherwise \end{cases}$
$\phi \leq_K \psi :\leftrightarrow \begin{cases} \vdash \phi \wedge \psi \\ or \\ \phi \notin K - (\phi \wedge \psi) \end{cases}$	$A \leq_X B :\leftrightarrow \begin{cases} A, B = U \\ or \\ X \ominus (A \cap B) \not\subseteq A \end{cases}$

4. Construction and Proof

We change perspective a little, and work directly with a ranked relation, so we forget about the (fixed) K of revision, and have an equivalent, ranked structure. (This result is part of the folklore: for fixed K, AGM axioms describe a ranked relation on the set of all models, where model of K are at the bottom, see above.) We are then interested in an operator ν, which returns a model set $\nu(M(\phi))$, written sloppily $\nu(\phi)$, where $\nu(\phi) \cap M(\phi)$ is given by a ranked relation $<$, and $\nu(\phi) - M(\phi) := \{x \notin M(\phi) : \exists y \in \nu(\phi) \cap M(\phi)(x \lhd y)\}$, and $\lhd$ is an arbitrary subrelation of $<$.

The essential problem is to find such y, as we have only formulas to find it. In the finite case, every model can be described by a formula, i.e., there is a formula which holds in this model, and nowhere else. In the infinite case, we need a full theory to do this. Our aim is, however, to achieve characterization with rules about formulas. So we have to take a more careful approach. But this is also the challenge which makes the problem interesting. (If we had full theories, we could just look at all $Th(\{y\})$ whether $x \in \nu(Th(\{y\}))$, where x, y are models.)

There is still some more work to do, as we have to connect the two relations, and simply taking a ready representation result will not do, as we shall see.

We first introduce some notation, then a set of conditions, and formulate the representation result. Soundness will be trivial. For completeness, we construct first the ranked relation $<$, show that it does what it should do, and then the subrelation $\lhd$.

Note that, contrary to Booth et al., we work here with the strict part of the relations. This is a slight restriction, but our main interest was to deal with an infinite case by finite means. So, for what we want to do, it is perhaps not so important.

Notation 4.1.

We set
$\mu^+(X) := \nu(X) \cap X$
$\mu^-(X) := \nu(X) - X$
where $X := M(\phi)$ for some ϕ.

Recall that, intuitively, $\mu^+(X)$ is the set of $<$ $-$minimal elements of X, and $\mu^-(X) = \{x \notin X : \exists y \in \mu^+(X)(x \lhd y)\}$, where $\lhd$ is an arbitrary subrelation of $<$.

So, $\mu^-(X)$ is, roughly, everything which is underneath some minimal element of X. This motivates the properties of μ^- to be defined now.

CONDITION 4.1.

(μ^-1) $Y \cap \mu^-(X) \neq \emptyset \Rightarrow \mu^+(Y) \cap X = \emptyset$
(μ^-2) $Y \cap \mu^-(X) \neq \emptyset \Rightarrow \mu^+(X \cup Y) = \mu^+(Y)$
(μ^-3) $Y \cap \mu^-(X) \neq \emptyset \Rightarrow \mu^-(Y) \cap X = \emptyset$
(μ^-4) $\mu^+(A) \subseteq \mu^+(B) \Rightarrow \mu^-(A) \subseteq \mu^-(B)$
(μ^-5) $\mu^+(X \cup Y) = \mu^+(X) \cup \mu^+(Y) \Rightarrow \mu^-(X \cup Y) = \mu^-(X) \cup \mu^-(Y)$

(μ^-1) says that if there is $y \in Y$ which is below X, then no minimal element of Y can be in X, and (μ^-2) says that in this case the minimal elements of $X \cup Y$ have to be exactly the minimal elements of Y. But then also what is below Y, cannot be in X, (μ^-3). These conditions exclude cycles.

(μ^-4) and (μ^-5) are consequences of the existential quantifier in the intuitive meaning of μ^-.

Many of the axioms characterizing basic removal in Theorem 2.1 (page 407) are about rankedness, and thus closely follow AGM tradition, we will put them into conditions about μ^+, where we speak about rankedness. On the other hand, our axioms ($\mu^- i$) either exclude cycles, or treat the existential quantifier. So Booth's and our own sets of axioms don't have very much in common — directly, of course.

Recall for the following conditions ($\mu\emptyset$), ($\mu \subseteq$), ($\mu =$) defined in Table 1 (page 417), and in Table 2 (page 418) which we repeat here for the reader's convenience:

($\mu\emptyset$) $X \neq \emptyset \rightarrow \mu(X) \neq \emptyset$
($\mu \subseteq$) $\mu(X) \subseteq X$
($\mu =$) $X \subseteq Y$, $\mu(Y) \cap X \neq \emptyset \Rightarrow \mu(X) = \mu(Y) \cap X$.

FACT 4.1. (μ^-1) and ($\mu\emptyset$), ($\mu \subseteq$) for μ^+ imply

(1) $\mu^+(X) \cap Y \neq \emptyset \Rightarrow \mu^+(X) \cap \mu^-(Y) = \emptyset$
(2) $X \cap \mu^-(X) = \emptyset$.

PROOF. (1) Let $\mu^+(X) \cap \mu^-(Y) \neq \emptyset$, then $X \cap \mu^-(Y) \neq \emptyset$, so by ($\mu^-1$) $\mu^+(X) \cap Y = \emptyset$.

(2) Set $X := Y$, and use ($\mu\emptyset$), ($\mu \subseteq$), (μ^-1). ■

FACT 4.2. Let the following conditions ($\mu \subseteq$), ($\mu\emptyset$), ($\mu =$) for μ^+, and (μ^-1)–(μ^-5) hold for μ^+ and μ^-. Let σ, σ' be partial models, i.e. defined only for some propositional variables. Let $\sigma \subseteq \sigma'$, $M(\sigma') \cap \mu^+(X) \neq \emptyset$, then $\mu^-(M(\sigma') \cap X) \subseteq \mu^-(M(\sigma) \cap X)$. (We have antitony for μ^- in σ.)

PROOF. By $M(\sigma') \subseteq M(\sigma)$, $M(\sigma) \cap \mu^+(X) \neq \emptyset$. Thus, by ($\mu =$) for μ^+, $M(\sigma) \cap \mu^+(X) = \mu^+(M(\sigma) \cap X)$, and $M(\sigma') \cap \mu^+(X) = \mu^+(M(\sigma') \cap X)$, so by $M(\sigma') \subseteq M(\sigma)$ $\mu^+(M(\sigma') \cap X) \subseteq \mu^+(M(\sigma) \cap X)$, so by (μ^-4) $\mu^-(M(\sigma') \cap X) \subseteq \mu^-(M(\sigma) \cap X)$. ■

Consider in the following Proposition $\nu : \{M(\phi) : \phi \in F(\mathcal{L})\} \to \boldsymbol{D}_{\mathcal{L}}$.

Let $\mu^+(X) := \nu(X) \cap X$, and $\mu^-(X) := \nu(X) - X$. (Thus, as ν is definability preserving, so is μ^+.)

PROPOSITION 4.3.

$\nu : \{M(\phi) : \phi \in F(\mathcal{L})\} \to \boldsymbol{D}_{\mathcal{L}}$ is representable by $<$ and $\lhd$,

where $<$ is a smooth ranked relation, and $\lhd$ a subrelation of $<$, and $\mu^+(X)$ is the set of $<-$minimal elements of X, and $\mu^-(X) = \{x \notin X : \exists y \in \mu^+(X).(x \lhd y)\}$,

iff the following conditions hold:

$(\mu \subseteq)$, $(\mu\emptyset)$, $(\mu =)$ for μ^+, and (μ^-1) – (μ^-5) for μ^+ and μ^-.

(Note that we look only at definablity preserving ν, this is the general prerequisite. We do *not* claim that any ν satisfying the $\mu-$conditions is automatically definability preserving.)

The rest of this Section is the proof of above Proposition.

4.1. Proof

Set $\mathcal{Y} := \{M(\phi) : \phi \in F(\mathcal{L})\}$. Note that $\mathcal{Y}$ is closed under finite unions, finite intersections, and complementation.

Soundness is easy:

The first three hold for smooth ranked structures, and the others are easily verified.

We turn to Completeness. For this purpose, we first generate the ranked relation $<$, and then the subrelation $\lhd$.

4.1.1. The ranked relation $<$

There is a small problem.

The authors first thought that one may take any result for ranked structures off the shelf, plug in the other relation somehow (see the second half), and that's it. No, that is not sufficient, as the following example shows:

EXAMPLE 4.1. Suppose there is x, and a sequence x_i converging to x in the usual topology. Thus, if $x \in M(\phi)$, then there will always be some x_i in $M(\phi)$, too. Take now a ranked structure $\mathcal{Z}$, where all the x_i are strictly smaller than x, and make some y bigger than x. Consider now any formula ψ such that $x \models \psi$, $y \models \psi$. As infinitely many of the x_i will also be models of ψ, x will never be a minimal model of ψ. So we will never be able to determine that $x < y$. (If we are allowed to consider full theories, we could "see" this, of course.) So just considering minimal models will never show $x < y$. If, however, we now can also look at models smaller than y, we can see, under some prerequisites about the order, e.g. definability preservation, that x is in this set, so we know that $x < y$. Thus, we cannot just take the minimality information, construct the first relation, and then look at what is below, to determine the second relation. We have to take the second information also to construct the first relation. $\square$

Consequently, considering μ^- may give strictly more information, and we have to put in a little more work. We just patch a proof for simple ranked structures, adding information obtained through μ^-.

We follow closely the strategy of the proof of 3.10.11 in [7]. We will, however, change notation at one point: the relation R in [7] is called $\preceq$ here. The proof goes over several steps, which we will enumerate.

Note that by Fact 3.1 (page 411), taken from [7], see also [3], $(\mu \parallel)$, $(\mu\cup)$, $(\mu\cup')$, $(\mu =')$ hold for μ^+, as the prerequisites about the domain are valid.

(1) To generate the ranked relation $<$, we first define two relations, $\preceq_1$ and $\preceq_2$, where $\preceq_1$ is the usual one for ranked structures, as defined in the proof of 3.10.11 of [7]. The relation $<$ will be defined in step (4) below.

$a \preceq_1 b$ iff $a \in \mu^+(X)$, $b \in X$ for some X, or $a = b$, and

$a \preceq_2 b$ iff $a \in \mu^-(X)$, $b \in X$ for some X.

Finally, we set $a \preceq b$ iff $a \preceq_1 b$ or $a \preceq_2 b$.

(2) Obviously, $\preceq$ is reflexive, we show that $\preceq$ is transitive by looking at the four different cases.

(2.1) In [7], it was shown that $a \preceq_1 b \preceq_1 c \Rightarrow a \preceq_1 c$. For completeness' sake, we repeat the argument: Suppose $a \preceq_1 b$, $b \preceq_1 c$, let $a \in \mu^+(A)$, $b \in A$, $b \in \mu^+(B)$, $c \in B$. We show $a \in \mu^+(A \cup B)$. By $(\mu \parallel)$ $a \in \mu^+(A \cup B)$ or $b \in \mu^+(A \cup B)$. Suppose $b \in \mu^+(A \cup B)$, then $\mu^+(A \cup B) \cap A \neq \emptyset$, so by $(\mu =)$ $\mu^+(A \cup B) \cap A = \mu^+(A)$, so $a \in \mu^+(A \cup B)$.

(2.2) Suppose $a \preceq_1 b \preceq_2 c$, we show $a \preceq_1 c$: Let $c \in Y$, $b \in \mu^-(Y) \cap X$, $a \in \mu^+(X)$. Consider $X \cup Y$. As $X \cap \mu^-(Y) \neq \emptyset$, by $(\mu^- 2)$ $\mu^+(X \cup Y) = \mu^+(X)$, so $a \in \mu^+(X \cup Y)$ and $c \in X \cup Y$, so $a \preceq_1 c$.

(2.3) Suppose $a \preceq_2 b \preceq_2 c$, we show $a \preceq_2 c$: Let $c \in Y$, $b \in \mu^-(Y) \cap X$, $a \in \mu^-(X)$. Consider $X \cup Y$. As $X \cap \mu^-(Y) \neq \emptyset$, by $(\mu^- 2)$ $\mu^+(X \cup Y) = \mu^+(X)$, so by $(\mu^- 4)$ $\mu^-(X \cup Y) = \mu^-(X)$, so $a \in \mu^-(X \cup Y)$ and $c \in X \cup Y$, so $a \preceq_2 c$.

(2.4) Suppose $a \preceq_2 b \preceq_1 c$, we show $a \preceq_2 c$: Let $c \in Y$, $b \in \mu^+(Y) \cap X$, $a \in \mu^-(X)$. Consider $X \cup Y$. As $\mu^+(Y) \cap X \neq \emptyset$, $\mu^+(X) \subseteq \mu^+(X \cup Y)$.

(Proof: By $(\mu \parallel)$, $\mu^+(X \cup Y) = \mu^+(X) \parallel \mu^+(Y)$, so, if $\mu^+(X) \not\subseteq \mu^+(X \cup Y)$, then $\mu^+(X) \cap \mu^+(X \cup Y) = \emptyset$, so $\mu^+(X) \cap (X \cup Y - \mu^+(X \cup Y)) \neq \emptyset$ by $(\mu\emptyset)$, so by $(\mu\cup')$ $\mu^+(X \cup Y) = \mu^+(Y)$. But if $\mu^+(Y) \cap X = \mu^+(X \cup Y) \cap X \neq \emptyset$, $\mu^+(X) = \mu^+(X \cup Y) \cap X$ by $(\mu =)$, so $\mu^+(X) \cap \mu^+(X \cup Y) \neq \emptyset$, *contradiction.*)

So $\mu^-(X) \subseteq \mu^-(X \cup Y)$ by (μ^-4), so $c \in X \cup Y$, $a \in \mu^-(X \cup Y)$, and $a \preceq_2 c$.

(3) We next prove the following two properties:

(3.1) $a \in \mu^+(A)$, $b \in A - \mu^+(A) \Rightarrow b \not\preceq a$.

(3.2) $a \in \mu^-(A)$, $b \in A \Rightarrow b \not\preceq a$.

Proof of (3.1):

(a) Case $\preceq_1$.

$\neg(b \preceq_1 a)$ was shown in [7], we repeat again the argument: Suppose there is B s.t. $b \in \mu^+(B)$, $a \in B$. Then by $(\mu\cup)$ $\mu^+(A \cup B) \cap B = \emptyset$, and by $(\mu\cup')$ $\mu^+(A \cup B) = \mu^+(A)$, but $a \in \mu^+(A) \cap B$, *contradiction.*

(b) Case $\preceq_2$.

Suppose there is B s.t. $a \in B$, $b \in \mu^-(B)$. But $A \cap \mu^-(B) \neq \emptyset$ implies $\mu^+(A) \cap B = \emptyset$ by (μ^-1).

Proof of (3.2):

(a) Case $\preceq_1$.

Suppose $b \preceq_1 a$, so there is B s.t. $a \in B$, $b \in \mu^+(B)$, so $B \cap \mu^-(A) \neq \emptyset$, so $\mu^+(B) \cap A = \emptyset$ by (μ^-1).

(b) Case $\preceq_2$.

Suppose $b \preceq_2 a$, so there is B s.t. $a \in B$, $b \in \mu^-(B)$, so $B \cap \mu^-(A) \neq \emptyset$, so $\mu^-(B) \cap A = \emptyset$ by (μ^-3).

(4) Let, by Lemma 3.4 (page 414), S be a total, transitive, reflexive relation on $U := M_{\mathcal{L}}$ which extends $\preceq$ s.t. $xSy, ySx \Rightarrow x \preceq y$ (recall that $\preceq$ is transitive and reflexive). But note that we "lose ignorance" here, as we make arbitrary decisions. (The authors think that one should keep in mind that one took such arbitrary decisions. This is, e.g., important to distinguish the situation of classical completeness, where we consider *all* models from, e.g., the situation of general preferential structures, where we have *one* universal model. It is good to keep such fine distinctions in mind - they might become useful one day.) Define $a < b$ iff aSb, but not bSa. If $a \perp b$ (i.e. neither $a < b$ nor $b < a$), then, by totality of S, aSb and bSa.

First, $<$ is ranked: If $c < a \perp b$, then by transitivity of S cSb, but if bSc, then again by transitivity of S aSc. Similarly for $c > a \perp b$.

(5) It remains to show that $<$ represents μ^+ and is $\mathcal{Y}-$smooth:

Let $a \in A - \mu^+(A)$. By $(\mu\emptyset)$, $\exists b \in \mu^+(A)$, so $b \preceq_1 a$, but by case (3.1) above $a \not\preceq b$, so bSa, but not aSb, so $b < a$, so $a \in A - \mu_<(A)$. Let $a \in \mu^+(A)$, then for all $a' \in A$ $a \preceq a'$, so aSa', so there is no $a' \in A$ $a' < a$, so $a \in \mu_<(A)$.

Finally, $\mu^+(A) \neq \emptyset$, all $x \in \mu^+(A)$ are minimal in A as we just saw, and for $a \in A - \mu^+(A)$ there is $b \in \mu^+(A)$, $b \preceq_1 a$, so the structure is smooth.

4.1.2. The subrelation $\lhd$

We now construct the subrelation $\lhd$ and show that $\lhd$ represents μ^-.

As this part of the proof is a bit long, and the technique perhaps somewhat unusual, we first give a detailed overview. We start with some $x \in \mu^-(X)$. Our task is to find suitable $y \in \mu^+(X)$, for which we can define $x \lhd y$. y is a model in a possibly infinite language, so y is described by a possibly infinite sequence of true/false for each propositional variable. Alternatively, y is described by a possibly infinite sequence of p_i or $\neg p_i$, $i < \kappa$ for some arbitrary, fixed, enumeration of the variables of the language. We will construct inductively such a sequence of decisions for each variable. As we can work only with (finite) formulas, we have access only to finite fragments of this sequence. We will see that this suffices. During construction, we will preserve inductively conditions (1) and (2) below, which will assure that the construction process is possible, and also, in the end, that y has the desired properties. At some points in the construction, we might choose p_i, but also $\neg p_i$, in this case we use an arbitrary model m to determine our strategy.

We turn to the construction of y. We take arbitrary $x \in \mu^-(X)$, and have to choose suitable $y \in \mu^+(X)$ to make $x \lhd y$. We first have to show that, if $\mu^-(X) \neq \emptyset$, then $\mu^+(X) \neq \emptyset$. (We are indebted to one of the referees for insisting that this should be shown properly.) If $X \neq \emptyset$, then $\mu^+(X) \neq \emptyset$ by $(\mu\emptyset)$ for μ^+. So the only possible case where it might fail is $X = \emptyset$. We have to show that $\mu^-(\emptyset) = \emptyset$. This goes as follows: $\mu^+(\emptyset) = \emptyset$ by $(\mu \subseteq)$ for μ^+, so $\mu^+(\emptyset) \subseteq \mu^+(B)$ for all B, so by (μ^-4) $\mu^-(\emptyset) \subseteq \mu^-(B)$ for all B. Take arbitrary Z, and consider (μ^-3), setting $X := Y := \mu^-(Z)$. Then $\mu^-(Z) \cap \mu^-(\mu^-(Z)) \neq \emptyset \Rightarrow \mu^-(Z) \cap \mu^-(\mu^-(Z)) = \emptyset$, contradiction, so $\mu^-(Z) \cap \mu^-(\mu^-(Z)) = \emptyset$. By the above $\mu^-(\emptyset) \subseteq \mu^-(Z) \cap \mu^-(\mu^-(Z)) = \emptyset$, and we are done.

The problem — if it is one — is that we can use only *finite* amounts of information to find such y. y, as a model, is a, possibly infinite, function $\tau : v(\mathcal{L}) \to \{t, f\}$. We construct τ inductively, using *finite* subsequences $\sigma \subseteq \tau$. (The authors consider this technique sufficiently interesting — it is, as a matter of fact, the reason why we think this extension of the result of [2] to the infinite case is publishable.) We do *not* (just) construct the finite subsequences σ mentioned below, but we construct the perhaps very big sequence τ, of which the σ may be just tiny fragments. The point is that those fragments suffice to construct τ, in a way they assure coherence of τ.

Suppose we found for $x \in \mu^-(X)$ such $y \in \mu^+(X)$ and made $x \lhd y$. By $\mu^+(X) \subseteq X$ and the definition of the relation $\preceq_2$ above, we know that $x \preceq y$. By (3.2) above, we know that $y \not\preceq x$, and by the definition of the relation $<$, we know that $x < y$, so $\lhd$ is a subrelation of $<$. (We need not know more details about the relation $\lhd$ to be constructed now, so we can do this part of the proof now.)

Take an arbitrary enumeration of the propositional variables of $\mathcal{L}$, p_i : $i < \kappa$, κ some (possibly infinite and very big) cardinal. We will inductively decide for p_i or $\neg p_i$. σ etc. will denote a finite subsequence of the choices made so far, i.e. $\sigma = \pm p_{i_0}, \ldots, \pm p_{i_n}$ for some $n < \omega$. ($\pm p$ will denote here some choice, either p or $\neg p$, which we will make inductively.)

Given such σ, $M(\sigma) := M(\pm p_{i_0}) \cap \ldots \cap M(\pm p_{i_n})$. $\sigma + \sigma'$ will be the union of two such sequences, this is again one such sequence.

Take an arbitrary model m for $\mathcal{L}$, i.e. a function $m : v(\mathcal{L}) \to \{t, f\}$. We will use this model as a "strategy", which will tell us how to decide, if we have some choice.

We determine y by an inductive process, essentially cutting away $\mu^+(X)$ around y. We choose p_i or $\neg p_i$ preserving the following conditions inductively:

For all finite sequences σ as above we have:

(1) $M(\sigma) \cap \mu^+(X) \neq \emptyset$,
(2) $x \in \mu^-(X \cap M(\sigma))$.

For didactic reasons, we do the case p_0 separately.

Recall that $(\mu\emptyset)$ holds for μ^+ by prerequisite. Consider p_0. Either $M(p_0) \cap \mu^+(X) \neq \emptyset$, or $M(\neg p_0) \cap \mu^+(X) \neq \emptyset$, or both. (Recall: $(\mu\emptyset)$ holds for μ^+, $X \neq \emptyset$, so $\mu^+(X) \neq \emptyset$, but any element is either in $M(p_0)$ or in $M(\neg p_0)$.) If e.g. $M(p_0) \cap \mu^+(X) \neq \emptyset$, but $M(\neg p_0) \cap \mu^+(X) = \emptyset$, then we have no choice, and we take p_0. In the opposite case, we take $\neg p_0$. E.g. in the first case, $\mu^+(X \cap M(p_0)) = \mu^+(X)$, so $x \in \mu^-(X \cap M(p_0))$ by $(\mu^- 4)$. If both intersections are non-empty, then by $(\mu^- 5)$ $x \in \mu^-(X \cap M(p_0))$ or $x \in \mu^-(X \cap M(\neg p_0))$, or both. Only in the last case, we use our strategy to decide whether to choose p_0 or $\neg p_0$: if $m(p_0) = t$, we choose p_0, if not, we choose $\neg p_0$. $\sigma(0)$ will be p_0 or $\neg p_0$. Obviously, (1) and (2) above are satisfied for the finite sequence $\sigma = \langle \pm p_0 \rangle$.

Suppose we have chosen p_i or $\neg p_i$ for all $i < \alpha$, i.e. defined a partial function τ from $v(\mathcal{L})$ to $\{t, f\}$, and the induction hypotheses (1) and (2) hold for all finite $\sigma \subseteq \tau$. We show now that we can extend τ to τ', which chooses additionally p_α or $\neg p_\alpha$, such that all finite $\sigma \subseteq \tau'$ still satisfy conditions (1) and (2).

Consider p_α, $\alpha < \kappa$.

We have to show that we can choose $\pm p_\alpha$ in a way such that the induction hypotheses (1) and (2) still hold for all finite subsequences σ of $\tau + \langle \pm p_\alpha \rangle$. The new cases to consider involve the choice of $\pm p_\alpha$, of course.

If there is no finite subsequence σ of the choices done so far s.t. $M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) = \emptyset$, then p_α is a candidate. Likewise for $\neg p_\alpha$.

One of p_α or $\neg p_\alpha$ is a candidate: Suppose not, then there are σ and σ' subsequences of the choices done so far, and $M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) = \emptyset$ and $M(\sigma') \cap M(\neg p_\alpha) \cap \mu^+(X) = \emptyset$. Consider the finite subsequence $\sigma + \sigma'$. But now $M(\sigma + \sigma') \cap \mu^+(X) = M(\sigma) \cap M(\sigma') \cap \mu^+(X) \subseteq M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) \cup M(\sigma') \cap M(\neg p_\alpha) \cap \mu^+(X) = \emptyset$, contradicting (1) of the induction hypothesis for $\sigma + \sigma'$. So the induction hypothesis (1) will hold again for all finite subsequences σ of $\tau + \langle \pm p_\alpha \rangle$.

Let $\sigma \subseteq \sigma'$ be two finite subsequences of $\tau + \langle p_\alpha \rangle$ if p_α is a candidate, or $\tau + \langle \neg p_\alpha \rangle$ if $\neg p_\alpha$ is a candidate. Thus by (1) $M(\sigma') \cap \mu^+(X) \neq \emptyset$, so by Fact 4.2 (page 423) $\mu^-(M(\sigma') \cap X) \subseteq \mu^-(M(\sigma) \cap X)$.

If we have only one candidate left, say e.g. p_α, then for each sufficiently big sequence σ $M(\sigma) \cap M(\neg p_\alpha) \cap \mu^+(X) = \emptyset$, thus for such σ $\mu^+(M(\sigma) \cap M(p_\alpha) \cap X) = M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) = M(\sigma) \cap \mu^+(X) = \mu^+(M(\sigma) \cap X)$, and thus by $(\mu^- 4)$ $\mu^-(M(\sigma) \cap M(p_\alpha) \cap X) = \mu^-(M(\sigma) \cap X)$. So for all sufficiently big finite σ $x \in \mu^-(M(\sigma) \cap X) = \mu^-(M(\sigma) \cap M(p_\alpha) \cap X)$ by (2) for σ. But p_α is a candidate, so as we just saw, for any $\sigma'' \subseteq \sigma$ $\mu^-(M(\sigma) \cap M(p_\alpha) \cap X) \subseteq \mu^-(M(\sigma'') \cap M(p_\alpha) \cap X)$, and $x \in \mu^-(M(\sigma'') \cap M(p_\alpha) \cap X)$ for any finite σ'', so (2) holds again.

Suppose now that we have two candidates, thus for p_α and $\neg p_\alpha$ and each σ $M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) \neq \emptyset$ and $M(\sigma) \cap M(\neg p_\alpha) \cap \mu^+(X) \neq \emptyset$. Thus, as we just saw, $\mu^-(M(\sigma') \cap X) \subseteq \mu^-(M(\sigma) \cap X)$ for $\sigma \subseteq \sigma'$ where σ, σ' might involve p_α or $\neg p_\alpha$, as both are candidates. By the same kind of argument as above we see that either for p_α or for $\neg p_\alpha$, or for both, and for all finite subsequences σ $x \in \mu^-(M(\sigma) \cap M(p_\alpha) \cap X)$ or $x \in \mu^-(M(\sigma) \cap M(\neg p_\alpha) \cap X)$. If not, there are σ and σ' and $x \notin \mu^-(M(\sigma) \cap M(p_\alpha) \cap X) \supseteq \mu^-(M(\sigma + \sigma') \cap M(p_\alpha) \cap X)$ and $x \notin \mu^-(M(\sigma') \cap M(\neg p_\alpha) \cap X) \supseteq \mu^-(M(\sigma + \sigma') \cap M(\neg p_\alpha) \cap X)$, but $\mu^-(M(\sigma + \sigma') \cap X) = \mu^-(M(\sigma + \sigma') \cap M(p_\alpha) \cap X) \cup \mu^-(M(\sigma + \sigma') \cap M(\neg p_\alpha) \cap X)$, so $x \notin \mu^-(M(\sigma + \sigma') \cap X)$, contradicting the induction hypothesis (2).

If we can choose both, we let the strategy decide, as for p_0.

So induction hypotheses (1) and (2) will hold again for $\tau + < \pm p_\alpha >$.

This inductive procedure finally gives a complete description of some model y (relative to the strategy!), and we set $x \lhd y$.

We now show that $\lhd$ represents μ^-.

We have to show: for all $Y \in \mathcal{Y}$ $x \in \mu^-(Y) \Leftrightarrow x \in \mu_\lhd(Y) :\Leftrightarrow \exists y \in \mu^+(Y).x \lhd y$. Recall that x was chosen in $\mu^-(Y)$ at the beginning, so $x \notin Y$, and we constructed y with $x \lhd y$, not x for a given y.

"$\Rightarrow$":

As we will do above construction for all Y, it suffices to show that $y \in \mu^+(X)$, we do this now:

If $y \notin \mu^+(X)$, then $Th(y)$ is inconsistent with $Th(\mu^+(X))$, as μ^+ is definability preserving, so by classical compactness there is a suitable finite sequence σ with $M(\sigma) \cap \mu^+(X) = \emptyset$, but this was excluded by the induction hypothesis (1). So $y \in \mu^+(X)$.

"$\Leftarrow$":

Conversely, if the y constructed above is in $\mu^+(Y)$, then x has to be in $\mu^-(Y)$, as we show now:

Remember that by construction *for this* $x \in \mu^-(X)$, $y \in \mu^+(Y)$.

Suppose $y \in \mu^+(Y)$, but $x \notin \mu^-(Y)$. So $y \in \mu^+(Y)$ and $y \in \mu^+(X)$, and $Y = M(\phi)$ for some ϕ, so there will be a suitable finite sequence σ s.t. for all σ' with $\sigma \subseteq \sigma'$ $M(\sigma') \cap X \subseteq M(\phi) = Y$, and by our construction $x \in \mu^-(M(\sigma') \cap X)$ by hypothesis (2). As $y \in \mu^+(X) \cap \mu^+(Y) \cap (M(\sigma') \cap X)$ by $(\mu =)$ and $M(\sigma') \cap X \subseteq Y$, $\mu^+(M(\sigma') \cap X) \subseteq \mu^+(Y)$, so by (μ^-4) $\mu^-(M(\sigma') \cap X) \subseteq \mu^-(Y)$, so $x \in \mu^-(Y)$, *contradiction.*

We do now this construction for all strategies. Obviously, this does not modify our results.

This finishes the completeness proof. $\square$

As we postulated definability preservation, there are no problems to translate the result into logic. (Note that ν was applied to formula defined model sets, but the resulting sets were perhaps theory defined model sets.)

4.1.3. Comment

This comment is for readers familiar with [6].

One might try a construction similar to the one for Counterfactual Conditionals, see [6], and try to patch together several ranked structures, one for each K on the left, to obtain a general distance, by repeating elements.

So we would have different "copies" of A, say A_i, more precisely of its elements, and the natural definition seems to be: $A * \phi \vdash \psi$ iff for all i $A_i * \phi \vdash \psi$, so $A \mid B = \bigcup\{A_i \mid B : i \in I\}$.

But this does not work: Take $A := \{a, a', a''\}$, $B := \{b, b'\}$, with $A \mid B := \{b, b'\}$, and $a \mid B = a' \mid B = a'' \mid B = \{b\}$. Then for all copies of the singletons, the result cannot be empty, but must be $\{b\}$. But $A \mid B$ can only be a "partial" union of the $x \mid B$, $x \in A$, so it must be $\{b\}$ for all copies of A, contradiction.

(Alternative definitions with copies fail too, but no systematic investigation was done.)

5. Conclusion

Booth et al. have shown that one can do many things in revision with two relations. They also gave a representation result for the finite case.

Our contribution is to have given a representation for the interesting infinite case, where only information about formula defined model sets is given. To prove our result, we used a - in this context probably novel — technique, constructing the required models from finite fragments.

Acknowledgements. The authors are very much endebted to two very careful referees, who not only made the paper readable, but also corrected many negligences.

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