The variety generated by all the ordinal sums of perfect MV-chains

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"To my friend Erika, and to her invaluable talent in finding surprisingly deep connections among poetry, art, philosophy and logic."

Abstract

We present the logic BL_{Chang} , an axiomatic extension of BL (see [Háj98]) whose corresponding algebras form the smallest variety containing all the ordinal sums of perfect MV-chains. We will analyze this logic and the corresponding algebraic semantics in the propositional and in the first-order case. As we will see, moreover, the variety of BL_{Chang} -algebras will be strictly connected to the one generated by Chang's MV-algebra (that is, the variety generated by all the perfect MV-algebras): we will also give some new results concerning these last structures and their logic.

1 Introduction

MV-algebras were introduced in [Cha58] as the algebraic counterpart of Łukasiewicz (infinite-valued) logic. During the years these structures have been intensively studied (for a hystorical overview, see [Cig07]): the book [CDM99] is a reference monograph on this topic.

Perfect MV-algebras were firstly studied in [BDL93] as a refinement of the notion of local MV-algebras: this analysis was expanded in [DL94], where it was also shown that the class of perfect MV-algebras Perf(MV) does not form a variety, and the variety generated by Perf(MV) is also generated by Chang's MV-algebra (see Section 2.2 for the definition). Further studies, about this variety and the associated logic have been done in [BDG07a, BDG07b].

On the other side, Basic Logic BL and its correspondent variety, BL-algebras, were introduced in [Háj98]: Łukasiewicz logic results to be one of the axiomatic extensions of BL and MV-algebras can also be defined as a subclass of BL-algebras. Moreover, the connection between MV-algebras and BL-algebras is even stronger: in fact, as shown in [AM03], every ordinal sum of MV-chains is a BL-chain.

For these reasons one can ask if there is a variety of BL-algebras whose chains are (isomorphic to) ordinal sums of perfect MV-chains: even if the answer to this question

is negative, we will present the smallest variety (whose correspondent logic is called BL_{Chang}) containing this class of BL-chains.

As we have anticipated in the abstract, there is a connection between the variety of BL_{Chang} -algebras and the one generated by Chang's MV-algebra. In fact the first-one is axiomatized (over the variety of BL-algebras) with an equation that, over MV-algebras, is equivalent to the one that axiomatize the variety generated by Chang MV-algebras: however, the two equations are *not* equivalent, over BL.

The paper is structured as follows: in Section 2 we introduce the necessary logical and algebraic background: moreover some basic results about perfect MV-algebras and other structures will be listed. In Section 3 we introduce the main theme of the article: the variety of BL_{Chang} and the associated logic. The analysis will be done in the propositional case: completeness results, algebraic and logical properties and also some results about the variety generated by Chang's MV-algebra. We conclude with Section 4, where we will analyze the first-order versions of BL_{Chang} and Ł_{Chang}: for the first-one the completeness results will be much more negative.

To conclude, we list the main results.

- BL_{Chang} enjoys the finite strong completeness (but not the strong one) w.r.t. $\omega \mathcal{V}$, where $\omega \mathcal{V}$ represents the ordinal sum of ω copies of the disconnected rotation of the standard cancellative hoop.
- L_{Chang} (the logic associated to the variety generated by Chang's MV-algebra) enjoys the finite strong completeness (but not the strong one) w.r.t. \mathcal{V} , \mathcal{V} being the disconnected rotation of the standard cancellative hoop.
- There are two BL-chains \mathscr{A}, \mathscr{B} that are strongly complete w.r.t., respectively \mathbb{E}_{Chang} and \mathbb{BL}_{Chang} .
- Every Ł_{Chang}-chain that is strongly complete w.r.t. Ł_{Chang} is also strongly complete w.r.t Ł_{Chang}∀.
- There is no BL_{Chang} -chain to be complete w.r.t. $BL_{Chang} \forall$.

2 Preliminaries

2.1 Basic concepts

Basic Logic BL was introduced by P. Hájek in [Háj98]. It is based over the connectives $\{\&, \rightarrow, \bot\}$ and a denumerable set of variables *VAR*. The formulas are defined inductively, as usual (see [Háj98] for details).

Other derived connectives are the following.

negation: $\neg \varphi := \varphi \rightarrow \bot$; *verum* or *top*: $\top := \neg \bot$; *meet*: $\varphi \land \psi := \varphi \& (\varphi \rightarrow \psi)$; *join*: $\varphi \lor \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \land ((\psi \rightarrow \varphi) \rightarrow \varphi)$.

BL is axiomatized as follows.

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
 (A1)

$$\varphi \& \psi) \to \varphi \tag{A2}$$

$$\varphi \& \psi) \to (\psi \& \varphi) \tag{A3}$$

$$\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi)) \tag{A4}$$

$$\begin{array}{ll} (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) & (A1) \\ (\varphi \& \psi) \to \varphi & (A2) \\ (\varphi \& \psi) \to (\psi \& \varphi) & (A3) \\ (\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi)) & (A4) \\ (\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi) & (A5a) \\ ((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi)) & (A5b) \\ ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) & (A6) \end{array}$$

$$(\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi)) \tag{A5b}$$

$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$$
 (A6)

$$\bot \to \varphi \tag{A7}$$

Modus ponens is the only inference rule:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}.$$
 (MP)

Among the extensions of BL (logics obtained from it by adding other axioms) there is the well known Łukasiewicz (infinitely-valued) logic Ł, that is, BL plus

$$\neg \neg \varphi \rightarrow \varphi.$$
 (INV)

On Łukasiewicz logic we can also define a strong disjunction connective (in the following sections, we will introduce a strong disjunction connective, for BL, that will be proved to be equivalent to the following, over Ł)

$$\varphi \lor \psi := \neg (\neg \varphi \& \neg \psi).$$

The notations φ^n and $n\varphi$ will indicate $\underbrace{\varphi\&\ldots\&\varphi}_{n \text{ times}}$ and $\underbrace{\varphi\uparrow\cdots\uparrow\varphi}_{n \text{ times}}$. Given an axiomatic extension L of BL, a formula φ and a theory T (a set of formulas), the notation $T \vdash_L \varphi$ indicates that there is a proof of φ from the axioms of L and the ones of T. The notion of proof is defined like in classical case (see [Háj98]).

We now move to the semantics: for all the unexplained notions of universal algebra, we refer to [BS81, Grä08].

Definition 2.1. A BL-algebra is an algebraic structure of the form $\mathscr{A} = \langle A, *, \Rightarrow, \sqcap, \sqcup, 0, 1 \rangle$ such that

- $\langle A, \Box, \sqcup, 0, 1 \rangle$ is a bounded lattice, where 0 is the bottom and 1 the top element.
- $\langle A, *, 1 \rangle$ is a commutative monoid.
- $\langle *, \Rightarrow \rangle$ forms a residuated pair, i.e.

$$z * x \le y \quad iff \quad z \le x \Rightarrow y,$$
 (res)

it can be shown that the only operation that satisfies (res) is $x \Rightarrow y = \max\{z : z \in \mathbb{N}\}$ $z * x \leq y$.

• A satisfies the following equations

$$(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1 \tag{pl}$$

$$x \sqcap y = x * (x \Rightarrow y). \tag{div}$$

Two important types of BL-algebras are the followings.

- A BL-chain is a totally ordered BL-algebra.
- A standard BL-algebra is a BL-algebra whose support is [0,1].

Notation: in the following, with $\sim x$ we will indicate $x \Rightarrow 0$.

Definition 2.2. An MV-algebra is a BL-algebra satisfying

$$x = \sim \sim x.$$
 (inv)

A well known example of MV-algebra is the standard MV-algebra $[0,1]_L = \langle [0,1], *, \Rightarrow, \min, \max, 0,1 \rangle$, where $x * y = \max(0, x + y - 1)$ and $x \Rightarrow y = \min(1, 1 - x + y)$.

In every MV-algebra we define the algebraic equivalent of Υ , that is

$$x \oplus y := \sim (\sim x \ast \sim y).$$

The notations (where x is an element of some BL-algebra) x^n and nx will indicate $x * \cdots * x$ and $x \oplus \cdots \oplus x$.

Consider a BL-algebra \mathscr{A} , a theory T and a formula φ . With $\mathscr{A} \models \varphi$ (\mathscr{A} is a model of φ) we indicate that $v(\varphi) = 1$, for every \mathscr{A} -evaluation v; $\mathscr{A} \models T$ denotes that $\mathscr{A} \models \psi$, for every $\psi \in T$. Finally, the notation $T \models_{\mathscr{A}} \varphi$ means that if $\mathscr{A} \models T$, then $\mathscr{A} \models \varphi$.

A BL-algebra \mathscr{A} is called L-algebra, where L is an axiomatic extension of BL, whenever \mathscr{A} is a model for all the axioms of L.

Definition 2.3. Let *L* be an axiomatic extension of *BL* and *K* a class of *L*-algebras. We say that *L* is strongly complete (respectively: finitely strongly complete, complete) with respect to *K* if for every set *T* of formulas (respectively, for every finite set *T* of formulas, for $T = \emptyset$) and for every formula φ we have

$$T \vdash_L \varphi$$
 iff $T \models_K \varphi$.

n times n times

Given a BL-algebra \mathscr{A} , the notion of \mathscr{A} -evaluation is defined in a truth-functional way (starting from a map $v : VAR \to A$, and extending it to formulas), for details see [Háj98].

2.2 Perfect MV-algebras, hoops and disconnected rotations

We recall that Chang's MV-algebra ([Cha58]) is a BL-algebra of the form

$$C = \langle \{a_n : n \in \mathbb{N}\} \cup \{b_n : n \in \mathbb{N}\}, *, \Rightarrow, \sqcap, \sqcup, b_0, a_0 \rangle.$$

Where for each $n, m \in \mathbb{N}$, it holds that $b_n < a_m$, and, if n < m, then $a_m < a_n$, $b_n < b_m$; moreover $a_0 = 1$, $b_0 = 0$ (the top and the bottom element).

The operation * is defined as follows, for each $n, m \in \mathbb{N}$:

$$b_n * b_m = b_0, \ b_n * a_m = b_{\max(0,n-m)}, \ a_n * a_m = a_{n+m}.$$

Definition 2.4 ([BDL93]). Let \mathscr{A} be an MV-algebra and let $x \in \mathscr{A}$: with ord(x) we mean the least (positive) natural n such that $x^n = 0$. If there is no such n, then we set $ord(x) = \infty$.

- An MV-algebra is called local¹ if for every element x it holds that $ord(x) < \infty$ or $ord(\sim x) < \infty$.
- An MV-algebra is called perfect if for every element x it holds that ord(x) < ∞ iff ord(~x) = ∞.

An easy consequence of this definition is that every perfect MV-algebra cannot have a negation fixpoint.

With Perfect(MV) and Local(MV) we will indicate the class of perfect and local MV-algebras. Moreover, given a BL-algebra \mathscr{A} , with $\mathbf{V}(\mathscr{A})$ we will denote the variety generated by \mathscr{A} .

Theorem 2.1 ([BDL93]). Every MV-chain is local.

Clearly there are local MV-algebras that are not perfect: $[0,1]_{L}$ is an example. Now, in [DL94] it is shown that

Theorem 2.2.

- $\mathbf{V}(C) = \mathbf{V}(Perfect(MV)),$
- $Perfect(MV) = Local(MV) \cap \mathbf{V}(C).$

It follows that the class of chains in $\mathbf{V}(C)$ coincides with the one of perfect MVchains. Moreover

Theorem 2.3 ([DL94]). An MV-algebra is in the variety V(C) iff it satisfies the equation $(2x)^2 = 2(x^2)$.

As shown in [BDG07a], the logic correspondent to this variety is axiomatized as Ł plus $(2\varphi)^2 \leftrightarrow 2(\varphi^2)$: we will call it $\mathcal{L}_{\text{Chang}}$.

¹Usually, the local MV-algebras are defined as MV-algebras having a unique (proper) maximal ideal. In [BDL93], however, it is shown that the two definitions are equivalent. We have preferred the other definition since it shows in a more transparent way that perfect MV-algebras are particular cases of local MV-algebras.

We now recall some results about hoops

Definition 2.5 ([Fer92, BF00]). A hoop is a structure $\mathscr{A} = \langle A, *, \Rightarrow, 1 \rangle$ such that $\langle A, *, 1 \rangle$ is a commutative monoid, and \Rightarrow is a binary operation such that

$$x \Rightarrow x = 1$$
, $x \Rightarrow (y \Rightarrow z) = (x * y) \Rightarrow z$ and $x * (x \Rightarrow y) = y * (y \Rightarrow x)$.

In any hoop, the operation \Rightarrow induces a partial order \leq defined by $x \leq y$ iff $x \Rightarrow y =$ 1. Moreover, hoops are precisely the partially ordered commutative integral residuated monoids (pocrims) in which the meet operation \sqcap is definable by $x \sqcap y = x * (x \Rightarrow y)$. Finally, hoops satisfy the following divisibility condition:

If
$$x \le y$$
, then there is an element z such that $z * y = x$. (div)

We recall a useful result.

Definition 2.6. Let \mathscr{A} and \mathscr{B} be two algebras of the same language. Then we say that

- \mathscr{A} is a partial subalgebra of \mathscr{B} if $A \subseteq B$ and the operations of \mathscr{A} are the ones of \mathscr{A} restricted to A. Note that A could not be closed under these operations (in this case these last ones will be undefined over some elements of A): in this sense \mathscr{A} is a partial subalgebra.
- A is partially embeddable into B when every finite partial subalgebra of A is embeddable into B. Generalizing this notion to classes of algebras, we say that a class K of algebras is partially embeddable into a class M if every finite partial subalgebra of a member of K is embeddable into a member of M.

Definition 2.7. A bounded *hoop is a hoop with a minimum element; conversely, an* unbounded *hoop is a hoop without minimum.*

Let \mathscr{A} *be a bounded hoop with minimum a: with* \mathscr{A}^+ *we mean the (partial) subalgebra of* \mathscr{A} *defined over the universe* $A^+ = \{x \in A : x > x \Rightarrow a\}.$

A hoop is Wajsberg iff it satisfies the equation $(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$. A hoop is cancellative iff it satisfies the equation $x = y \Rightarrow (x * y)$.

Proposition 2.1 ([Fer92, BF00, AFM07]). Every cancellative hoop is Wajsberg. Totally ordered cancellative hoops coincide with unbounded totally ordered Wajsberg hoops, whereas bounded Wajsberg hoops coincide with (the 0-free reducts of) MValgebras.

We now recall a construction introduced in [Jen03] (and also used in [EGHM03, NEG05]), called *disconnected rotation*.

Definition 2.8. Let \mathscr{A} be a cancellative hoop. We define an algebra, \mathscr{A}^* , called the disconnected rotation of \mathscr{A} , as follows. Let $\mathscr{A} \times \{0\}$ be a disjoint copy of A. For every $a \in A$ we write a' instead of $\langle a, 0 \rangle$. Consider $\langle A' = \{a' : a \in A\}, \leq \rangle$ with the inverse order and let $A^* := A \cup A'$. We extend these orderings to an order in A^* by putting

a' < b for every $a, b \in A$. Finally, we take the following operations in A^* : $1 := 1_{\mathscr{A}}$, $0 := 1', \sqcap_{\mathscr{A}^*}, \sqcup_{\mathscr{A}^*}$ as the meet and the join with respect to the order over A^* . Moreover,

$$\sim_{\mathscr{A}^*} a := \begin{cases} a' & \text{if } a \in A \\ b & \text{if } a = b' \in A' \end{cases}$$
$$a *_{\mathscr{A}^*} b := \begin{cases} a *_{\mathscr{A}} b & \text{if } a, b \in A \\ \sim_{\mathscr{A}^*} (a \Rightarrow_{\mathscr{A}} \sim_{\mathscr{A}^*} b) & \text{if } a \in A, b \in A' \\ \sim_{\mathscr{A}^*} (b \Rightarrow_{\mathscr{A}} \sim_{\mathscr{A}^*} a) & \text{if } a \in A', b \in A \\ 0 & \text{if } a, b \in A' \end{cases}$$
$$a \Rightarrow_{\mathscr{A}^*} b := \begin{cases} a \Rightarrow_{\mathscr{A}} b & \text{if } a, b \in A \\ \sim_{\mathscr{A}^*} (a *_{\mathscr{A}^*} \sim_{\mathscr{A}^*} b) & \text{if } a \in A, b \in A' \\ 1 & \text{if } a \in A', b \in A \\ (\sim_{\mathscr{A}^*} b) \Rightarrow_{\mathscr{A}} (\sim_{\mathscr{A}^*} a) & \text{if } a, b \in A'. \end{cases}$$

Theorem 2.4 ([NEG05, theorem 9]). Let \mathscr{A} be an MV-algebra. The followings are equivalent:

- A is a perfect MV-algebra.
- *A is isomorphic to the disconnected rotation of a cancellative hoop.*

To conclude the section, we present the definition of ordinal sum.

Definition 2.9 ([AM03]). Let $\langle I, \leq \rangle$ be a totally ordered set with minimum 0. For all $i \in I$, let \mathscr{A}_i be a hoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$, and assume that \mathscr{A}_0 is bounded. Then $\bigoplus_{i \in I} \mathscr{A}_i$ (the ordinal sum of the family $(\mathscr{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$, whose bottom is the minimum of \mathscr{A}_0 , whose top is 1, and whose operations are

$$x \Rightarrow y = \begin{cases} x \Rightarrow^{\mathscr{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j(x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j(x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases}$$
$$x * y = \begin{cases} x *^{\mathscr{A}_i} y & \text{if } x, y \in A_i \\ x & \text{if } \exists i < j(x \in A_i \setminus \{1\}, y \in A_j) \\ y & \text{if } \exists i < j(y \in A_i \setminus \{1\}, x \in A_j) \end{cases}$$

When defining the ordinal sum $\bigoplus_{i \in I} \mathscr{A}_i$ we will tacitly assume that whenever the condition $A_i \cap A_j = \{1\}$ is not satisfied for all $i, j \in I$ with $i \neq j$, we will replace the \mathscr{A}_i by isomorphic copies satisfying such condition. Moreover if all \mathscr{A}_i 's are isomorphic to some \mathscr{A} , then we will write $I\mathscr{A}$, instead of $\bigoplus_{i \in I} \mathscr{A}_i$. Finally, the ordinal sum of two hoops \mathscr{A} and \mathscr{B} will be denoted by $\mathscr{A} \oplus \mathscr{B}$.

Note that, since every bounded Wajsberg hoop is the 0-free reduct of an MValgebra, then the previous definition also works with these structures. **Theorem 2.5** ([AM03, theorem 3.7]). Every BL-chain is isomorphic to an ordinal sum whose first component is an MV-chain and the others are totally ordered Wajsberg hoops.

Note that in [Bus04] it is presented an alternative and simpler proof of this result.

3 The variety of BL_{Chang}-algebras

Consider the following connective

$$\varphi \lor \psi := ((\varphi \to (\varphi \& \psi)) \to \psi) \land ((\psi \to (\varphi \& \psi)) \to \varphi)$$

Call \uplus the algebraic operation, over a BL-algebra, corresponding to \lor ; we have that

Lemma 3.1. In every MV-algebra the following equation holds

$$x \uplus y = x \oplus y.$$

Proof. It is easy to check that $x \uplus y = x \oplus y$, over $[0,1]_{MV}$, for every $x, y \in [0,1]$.

We now analyze this connective in the context of Wajsberg hoops.

Proposition 3.1. Let *A* be a linearly ordered Wajsberg hoop. Then

- If \mathscr{A} is unbounded (i.e. a cancellative hoop), then $x \uplus y = 1$, for every $x, y \in \mathscr{A}$.
- If \mathscr{A} is bounded, let a be its minimum. Then, by defining $\sim x := x \Rightarrow a$ and $x \oplus y = \sim (\sim x* \sim y)$ we have that $x \oplus y = x \boxplus y$, for every $x, y \in \mathscr{A}$

Proof. An easy check.

Now, since the variety of cancellative hoops is generated by its linearly ordered members (see [EGHM03]), then we have that

Corollary 3.1. *The equation* $x \uplus y = 1$ *holds in every cancellative hoop.*

We now characterize the behavior of \uplus for the case of BL-chains.

Proposition 3.2. Let $\mathscr{A} = \bigoplus_{i \in I} \mathscr{A}_i$ be a BL-chain. Then

$$x \uplus y = \begin{cases} x \oplus y, & \text{if } x, y \in \mathcal{A}_i \text{ and } \mathcal{A}_i \text{ is bounded} \\ 1, & \text{if } x, y \in \mathcal{A}_i \text{ and } \mathcal{A}_i \text{ is unbounded} \\ \max(x, y), & \text{otherwise.} \end{cases}$$

for every $x, y \in \mathscr{A}$ *.*

Proof. If *x*, *y* belong to the same component of \mathscr{A} , then the result follows from Lemma 3.1 and Proposition 3.1. For the case in which *x* and *y* belong to different components of \mathscr{A} , this is a direct computation.

Remark 3.1. From the previous proposition we can argue that \uplus is a good approximation, for BL, of what that \oplus represents for MV-algebras. Note that a similar operation was introduced in [ABM09]: the main difference with respect to \uplus is that, when x and y belong to different components of a BL-chain, then the operation introduced in [ABM09] holds 1.

In the following, for every element *x* of a BL-algebra, with the notation $\overline{n}x$ we will denote $x \uplus \cdots \uplus x$; analogously $\overline{n}\varphi$ means $\varphi \lor \cdots \lor \varphi$.

Definition 3.1. We define BL_{Chang} as the axiomatic extension of BL, obtained by adding

$$(\overline{2}\boldsymbol{\varphi})^2 \leftrightarrow \overline{2}(\boldsymbol{\varphi}^2).$$
 (cha)

That is, writing it in extended form

$$(\boldsymbol{\varphi}^2 \to (\boldsymbol{\varphi}^2 \& \boldsymbol{\varphi}^2) \to \boldsymbol{\varphi}^2) \leftrightarrow ((\boldsymbol{\varphi} \to \boldsymbol{\varphi}^2) \to \boldsymbol{\varphi})^2.$$

Clearly the variety corresponding to BL_{Chang} is given by the class of BL-algebras satisfying the equation $(\overline{2}x)^2 = \overline{2}(x^2)$.

Moreover,

Definition 3.2. We will call pseudo-perfect Wajsberg hoops those Wajsberg hoops satisfying the equation $(\overline{2}x)^2 = \overline{2}(x^2)$.

Remark 3.2. Thanks to Lemma 3.1 we have that

$$\vdash_{L} ((\overline{2}\boldsymbol{\varphi})^{2} \leftrightarrow \overline{2}(\boldsymbol{\varphi}^{2})) \leftrightarrow ((2\boldsymbol{\varphi})^{2} \leftrightarrow 2(\boldsymbol{\varphi}^{2})),$$

that is, if we add $(\overline{2}\varphi)^2 \leftrightarrow \overline{2}(\varphi^2)$ or $(2\varphi)^2 \leftrightarrow 2(\varphi^2)$ to *L*, then we obtain the same logic *L*_{Chang}.

These formulas, however are not equivalent over BL: see Remark 3.3 for details.

Theorem 3.1. Every totally ordered pseudo-perfect Wajsberg hoop is a totally ordered cancellative hoop or (the 0-free reduct of) a perfect MV-chain.

More in general, the variety of pseudo-perfect Wajsberg hoops coincides with the class of the 0-free subreducts of members of V(C).

Proof. In [EGHM03] it is shown that the variety of Wajsberg hoops coincides with the class of the 0-free subreducts of MV-algebras. The results easily follow from this fact and from Proposition 2.1, Theorem 2.3 and Definition 3.2. \Box

As a consequence, we have

Theorem 3.2. Let WH, CH, psWH be, respectively, the varieties of Wajsberg hoops, cancellative hoops, pseudo-perfect Wajsberg hoops. Then we have that

$$\mathbb{CH} \subset ps\mathbb{WH} \subset \mathbb{WH}.$$

Proof. An easy consequence of Theorem 3.1.

The first inclusion follows from the fact that psWH contains all the totally ordered cancellative hoops and hence the variety generated by them. For the second inclusion note that, for example, the 0-free reduct of $[0,1]_{\rm L}$ belongs to $WH \setminus psWH$.

We now describe the structure of BL_{Chang} -chains, with an analogous of the Theorem 2.5 for BL-chains.

Theorem 3.3. Every BL_{Chang} -chain is isomorphic to an ordinal sum whose first component is a perfect MV-chain and the others are totally ordered pseudo-perfect Wajsberg hoops.

It follows that every ordinal sum of perfect MV-chains is a BL_{Chang}-chain.

Proof. Thanks to Theorems 2.2 and 2.3, Remark 3.2 and Definition 3.2, we have that every MV-chain (Wajsberg hoop) satisfying the equation $(\overline{2}x)^2 = \overline{2}(x^2)$ is perfect (pseudo-perfect): using these facts and Proposition 3.2 we have that a BL-chain satisfies the equation $(\overline{2}x)^2 = \overline{2}(x^2)$ iff it holds true in all the components of its ordinal sum. From these facts and Theorem 2.5 we get the result.

As a consequence, we obtain the following corollaries.

Corollary 3.2. The variety of BL_{Chang}-algebras contains the ones of product-algebras and Gödel-algebras: however it does not contains the variety of MV-algebras.

Proof. From the previous theorem it is easy to see that the variety of BL_{Chang} -algebras contains $[0,1]_{\Pi}$ and $[0,1]_G$, but not $[0,1]_E$.

Corollary 3.3. Every finite BL_{Chang} -chain is an ordinal sum of a finite number of copies of the two elements boolean algebra. Hence the class of finite BL_{Chang} -chains coincides with the one of finite Gödel chains.

For this reason it is immediate to see that the finite model property does not hold for BL_{Chang}.

We conclude with the following remark.

Remark 3.3. • One can ask if it is possible to axiomatize the class BL_{perf} of BLalgebras, whose chains are the BL-algebras that are ordinal sum of perfect MVchains: the answer, however, is negative. In fact, the class of bounded Wajsberg hoops does not form a variety: for example, it is easy to check that for every bounded pseudo-perfect Wajsberg hoop \mathcal{A} , its subalgebra \mathcal{A}^+ (see Definition 2.7) forms a cancellative hoop. Hence BL_{perf} cannot be a variety.

However, as we will see in Section 3.2, the variety of BL_{Chang} -algebras is the "best approximation" of BL_{perf} , in the sense that it is the smallest variety to contain BL_{perf} .

• In [DSE⁺02] (see also [CT06]) it is studied the variety, called P₀, generated by all the perfect BL-algebras (a BL-algebra A is perfect if, by calling MV(A)

the biggest subalgebra of \mathscr{A} to be an MV-algebra, then $MV(\mathscr{A})$ is a perfect MV-algebra). P_0 is axiomatized with the equation

$$\sim ((\sim (x^2))^2) = (\sim ((\sim x)^2))^2.$$
 (p₀)

One can ask which is the relation between P_0 and the variety of BL_{Chang} -algebras. The answer is that the variety of BL_{Chang} -algebras is strictly contained in P_0 . In fact, an easy check shows that a BL-chain is perfect if and only if the first component of its ordinal sum is a perfect MV-chain. Hence we have:

- Every BL_{Chang}-chain is a perfect BL-chain.
- There are perfect BL-chains that are not BL_{Chang} -chains: an example is given by $C \oplus [0,1]_L$.

Now, since the variety of BL_{Chang} -algebras is generated by its chains (like any variety of *BL*-algebras, see [Háj98]), then we get the result.

Finally note that (p_0) is equivalent to $2(x^2) = (2x)^2$: hence, differently to what happens over *L* (see Remark 3.2), the equations $2(x^2) = (2x)^2$ and $\overline{2}(x^2) = (\overline{2}x)^2$ are not equivalent, over *BL*.

3.1 Subdirectly irreducible and simple algebras

We begin with a general result about Wajsberg hoops.

Theorem 3.4 ([Fer92, Corollary 3.11]). *Every subdirectly irreducible Wajsberg hoop is totally ordered.*

As a consequence, we have:

Corollary 3.4. Every subdirectly irreducible pseudo-perfect Wajsberg hoop is totally ordered.

We now move to simple algebras.

It is shown in [Tur99, Theorem 1] that the simple BL-algebras coincide with the simple MV-algebras, that is, with the subalgebras of $[0,1]_{L}$ (see [CDM99, Theorem 3.5.1]). Therefore we have:

Theorem 3.5. The only simple BL_{Chang} -algebra is the two elements boolean algebra 2.

An easy consequence of this fact is that the only simple L_{Chang} -algebra is 2.

3.2 Completeness

We begin with a result about pseudo-perfect Wajsberg hoops.

Theorem 3.6. The class pMV of 0-free reducts of perfect MV-chains generates psWH.

Proof. From Theorems 2.4 and 3.1 it is easy to check that the variety generated by pMV contains all the totally ordered pseudo-perfect Wajsberg hoops.

From these facts and Corollary 3.4, we have that pMV must be generic for psWH.

Theorem 3.7 ([CEG⁺09]). Let *L* be an axiomatic extension of *BL*, then *L* enjoys the finite strong completeness w.r.t a class *K* of *L*-algebras iff every countable *L*-chain is partially embeddable into *K*.

As shown in [Háj98] product logic enjoys the finite strong completeness w.r.t $[0,1]_{\Pi}$ and hence every countable product chain is partially embeddable into $[0,1]_{\Pi} \simeq \mathbf{2} \oplus (0,1]_C$, with $(0,1]_C$ being the standard cancellative hoop (i.e. the 0-free reduct of $[0,1]_{\Pi} \setminus \{0\}$). Since every totally ordered product chain is of the form $\mathbf{2} \oplus \mathscr{A}$, where \mathscr{A} is a cancellative hoop (see [EGHM03]), it follows that:

Proposition 3.3. *Every countable totally ordered cancellative hoop partially embeds into* $(0,1]_C$.

Theorem 3.8. Every countable perfect MV-chain partially embeds into $\mathscr{V} = (0,1]_C^*$ (i.e. the disconnected rotation of $(0,1]_C$).

Proof. Immediate from Proposition 3.3 and Theorem 2.4.

Corollary 3.5. The logic L_{Chang} is finitely strongly complete w.r.t. \mathcal{V} .

Theorem 3.9. BL_{Chang} enjoys the finite strong completeness w.r.t. $\omega \mathcal{V}$. As a consequence, the variety of BL_{Chang} -algebras is generated by the class of all ordinal sums of perfect MV-chains and hence is the smallest variety to contain this class of algebras.

Proof. Thanks to Theorem 3.7 it is enough to show that every countable BL_{Chang} -chain partially embeds into $\omega \mathcal{V}$ (i.e. the ordinal sum of " ω copies" of \mathcal{V}). This fact, however, follows immediately from Proposition 3.3 and Theorems 3.3 and 3.8.

But we cannot obtain a stronger result: in fact

Theorem 3.10. *BL*_{*Chang*} *is not strongly complete w.r.t.* $\omega \mathcal{V}$.

Proof. Suppose not: from the results of [CEG⁺09, Theorem 3.5] this is equivalent to claim that every countable BL_{Chang}-chain embeds into $\omega \mathcal{V}$. But, this would imply that every countable totally ordered cancellative hoop embeds into $(0,1]_C$: this means that every countable product-chain embeds into $[0,1]_{\Pi}$, that is product logic is strongly complete w.r.t $[0,1]_{\Pi}$. As it is well known (see [Háj98, Corollary 4.1.18]), this is false.

With an analogous proof we obtain

Theorem 3.11. L_{Chang} is not strongly complete w.r.t. \mathscr{V}

However, thanks to [Mon11, Theorem 3] we can claim

Theorem 3.12. There exist a L_{Chang} -chain \mathscr{A} and a BL_{Chang} -chain \mathscr{B} such that L_{Chang} is strongly complete w.r.t. \mathscr{A} and BL_{Chang} is strongly complete w.r.t. \mathscr{B} .

Problem 3.1. Which can be some concrete examples of such \mathcal{A} and \mathcal{B} ?

4 First-order logics

We assume that the reader is acquainted with the formalization of first-order logics, as developed in [Háj98, CH10].

Briefly, we work with (first-order) languages without equality, containing only predicate and constant symbols: as quantifiers we have \forall and \exists . The notions of terms and formulas are defined inductively like in classical case.

As regards to semantics, given an axiomatic extension L of BL we restrict to Lchains: the first-order version of L is called L \forall (see [Háj98, CH10] for an axiomatization). A first-order \mathscr{A} -interpretation (\mathscr{A} being an L-chain) is a structure $\mathbf{M} = \langle M, \{r_P\}_{P \in \mathbf{P}}, \{m_c\}_{c \in \mathbf{C}} \rangle$, where *M* is a non-empty set, every r_P is a fuzzy *ariety*(*P*)-ary relation, over *M*, in which we interpretate the predicate *P*, and every m_c is an element of *M*, in which we map the constant *c*.

Given a map $v: VAR \to M$, the interpretation of $\|\varphi\|_{\mathbf{M},v}^{\mathscr{A}}$ in this semantics is defined in a Tarskian way: in particular the universally quantified formulas are defined as the infimum (over \mathscr{A}) of truth values, whereas those existentially quantified are evaluated as the supremum. Note that these inf and sup could not exist in \mathscr{A} : an \mathscr{A} -model **M** is called *safe* if $\|\varphi\|_{\mathbf{M},v}^{\mathscr{A}}$ is defined for every φ and v.

A model is called *witnessed* if the universally (existentially) quantified formulas are evaluated by taking the minimum (maximum) of truth values in place of the infimum (supremum): see [Háj07, CH06, CH10] for details.

The notions of soundness and completeness are defined by restricting to safe models (even if in some cases it is possible to enlarge the class of models: see [BM09]): see [Háj98, CH10, CH06] for details.

We begin with a positive result about $\mathcal{L}_{\text{Chang}} \forall$.

Definition 4.1. Let *L* be an axiomatic extension of *BL*. With $L \forall^w$ we define the extension of $L \forall$ with the following axioms

$$(\exists y)(\boldsymbol{\varphi}(y) \to (\forall x)\boldsymbol{\varphi}(x))$$
 (C \forall)

$$(\exists y)((\exists x)\varphi(x) \to \varphi(y)).$$
 (C \exists)

Theorem 4.1 ([CH06, Proposition 6]). $L \forall$ *coincides with* $L \forall^w$, *that is* $L \forall \vdash (C \forall), (C \exists)$.

An immediate consequence is:

Corollary 4.1. Let *L* be an axiomatic extension of *Ł*. Then $L\forall$ coincides with $L\forall^w$.

Theorem 4.2 ([CH06, Theorem 8]). Let *L* be an axiomatic extension of BL. Then $L \forall^w$ enjoys the strong witnessed completeness with respect to the class *K* of *L*-chains, i.e.

$$T \vdash_{L \forall w} \boldsymbol{\varphi} \quad iff \quad \|\boldsymbol{\varphi}\|_{\mathbf{M}}^{\mathscr{A}} = 1,$$

for every theory *T*, formula φ , algebra $\mathscr{A} \in K$ and witnessed \mathscr{A} -model **M** such that $\|\psi\|_{\mathbf{M}}^{\mathscr{A}} = 1$ for every $\psi \in T$.

Lemma 4.1 ([Mon11, Lemma 1]). Let *L* be an axiomatic extension of *BL*, let \mathscr{A} be an *L*-chain, let \mathscr{B} be an *L*-chain such that $A \subseteq B$ and let **M** be a witnessed \mathscr{A} -structure. Then for every formula φ and evaluation *v*, we have $\|\varphi\|_{\mathbf{M},v}^{\mathscr{A}} = \|\varphi\|_{\mathbf{M},v}^{\mathscr{B}}$.

Theorem 4.3. There is a L_{Chang} -chain such that $L_{Chang} \forall$ is strongly complete w.r.t. it. More in general, every L_{Chang} -chain that is strongly complete w.r.t L_{Chang} is also strongly complete w.r.t. $L_{Chang} \forall$.

Proof. An adaptation of the proof for the analogous result, given in [Mon11, Theorem 16], for $L\forall$.

From Theorem 3.12 we know that there is a \mathcal{L}_{Chang} -chain \mathscr{A} strongly complete w.r.t. \mathcal{L}_{Chang} : from [CEG⁺09, Theorem 3.5] this is equivalent to claim that every countable \mathcal{L}_{Chang} -chain embeds into \mathscr{A} . We show that \mathscr{A} is also strongly complete w.r.t. \mathcal{L}_{Chang} \forall .

Suppose that $T \not\vdash_{\mathbf{L}_{\mathrm{Chang}}} \varphi$. Thanks to Corollary 4.1 and Theorem 4.2 there is a countable $\mathbf{L}_{\mathrm{Chang}}$ -chain \mathscr{C} and a witnessed \mathscr{C} -model \mathbf{M} such that $\|\psi\|_{\mathbf{M}}^{\mathscr{C}} = 1$, for every $\psi \in T$, but $\|\varphi\|_{\mathbf{M}}^{\mathscr{C}} < 1$. Finally, from Lemma 4.1 we have that $\|\psi\|_{\mathbf{M}}^{\mathscr{C}} = 1$, for every $\psi \in T$ and $\|\varphi\|_{\mathbf{M}}^{\mathscr{C}} = \|\varphi\|_{\mathbf{M}}^{\mathscr{C}} < 1$: this completes the proof.

For $BL_{Chang} \forall$, however, the situation is not so good.

Theorem 4.4. BL_{Chang} \forall cannot enjoy the completeness w.r.t. a single BL_{Chang} -chain.

Proof. The proof is an adaptation of the analogous result given in [Mon11, Theorem 17] for BL \forall .

Let \mathscr{A} be a BL_{Chang}-chain: call \mathscr{A}_0 its first component. We have three cases

- \mathscr{A}_0 is finite: from Theorem 3.3 we have that $\mathscr{A}_0 = \mathbf{2}$ and hence $\mathscr{A} \models (\neg \neg x) \rightarrow (\neg \neg x)^2$. However $\mathscr{V} \not\models (\neg \neg x) \rightarrow (\neg \neg x)^2$, where \mathscr{V} is the chain introduced in Section 3.2, and hence \mathscr{A} cannot be complete w.r.t. $\operatorname{BL}_{\operatorname{Chang}} \forall$.
- A₀ is infinite and dense. As shown in [Mon11, Theorem 17] the formula (∀x)¬¬P(x) → ¬¬(∀x)P(x) is a tautology in every BL-chain whose first component is infinite and densely ordered: hence we have that A ⊨ (∀x)¬¬P(x) → ¬¬(∀x)P(x). However it is easy to check that this formula fails in [0,1]_G: take a [0,1]_G-model M with M = (0,1] and such that r_P(m) = m. Hence, from Corollary 3.2, it follows that BL_{Chang}∀∀ (∀x)¬¬P(x) → ¬¬(∀x)P(x).
- \mathscr{A}_0 is infinite and not dense. As shown in [Mon11, Theorem 17] the formula $(\forall x) \neg \neg P(x) \rightarrow \neg \neg (\forall x)P(x) \lor \neg (\forall x)P(x) \rightarrow ((\forall x)P(x))^2$ is a tautology in every BL-chain whose first component is infinite and not densely ordered: hence we have that $\mathscr{A} \models (\forall x) \neg \neg P(x) \rightarrow \neg \neg (\forall x)P(x) \lor \neg (\forall x)P(x) \rightarrow ((\forall x)P(x))^2$. Also in this case, however, this formula fails in $[0, 1]_G$, using the same model **M** of the previous case.

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