# The variety generated by all the ordinal sums of perfect MV-chains 

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#### Abstract

We present the logic $\mathrm{BL}_{\text {Chang }}$, an axiomatic extension of BL (see Háj98) whose corresponding algebras form the smallest variety containing all the ordinal sums of perfect MV-chains. We will analyze this logic and the corresponding algebraic semantics in the propositional and in the first-order case. As we will see, moreover, the variety of $\mathrm{BL}_{\text {Chang }}$-algebras will be strictly connected to the one generated by Chang's MV-algebra (that is, the variety generated by all the perfect MV-algebras): we will also give some new results concerning these last structures and their logic.


## 1 Introduction

MV-algebras were introduced in [Cha58] as the algebraic counterpart of Łukasiewicz (infinite-valued) logic. During the years these structures have been intensively studied (for a hystorical overview, see [Cig07]): the book [CDM99] is a reference monograph on this topic.

Perfect MV-algebras were firstly studied in [BDL93] as a refinement of the notion of local MV-algebras: this analysis was expanded in [DL94], where it was also shown that the class of perfect MV-algebras $\operatorname{Perf}(M V)$ does not form a variety, and the variety generated by $\operatorname{Per} f(M V)$ is also generated by Chang's MV-algebra (see Section 2.2 for the definition). Further studies, about this variety and the associated logic have been done in [BDG07a, BDG07b].

On the other side, Basic Logic BL and its correspondent variety, BL-algebras, were introduced in [Háj98]: Łukasiewicz logic results to be one of the axiomatic extensions of BL and MV-algebras can also be defined as a subclass of BL-algebras. Moreover, the connection between MV-algebras and BL-algebras is even stronger: in fact, as shown in AM03], every ordinal sum of MV-chains is a BL-chain.

For these reasons one can ask if there is a variety of BL-algebras whose chains are (isomorphic to) ordinal sums of perfect MV-chains: even if the answer to this question
is negative, we will present the smallest variety (whose correspondent logic is called $\mathrm{BL}_{\text {Chang }}$ ) containing this class of BL-chains.

As we have anticipated in the abstract, there is a connection between the variety of $\mathrm{BL}_{\text {Chang }}$-algebras and the one generated by Chang's MV-algebra. In fact the first-one is axiomatized (over the variety of BL-algebras) with an equation that, over MV-algebras, is equivalent to the one that axiomatize the variety generated by Chang MV-algebras: however, the two equations are not equivalent, over BL.

The paper is structured as follows: in Section 2 we introduce the necessary logical and algebraic background: moreover some basic results about perfect MV-algebras and other structures will be listed. In Section 3 we introduce the main theme of the article: the variety of $\mathrm{BL}_{\text {Chang }}$ and the associated logic. The analysis will be done in the propositional case: completeness results, algebraic and logical properties and also some results about the variety generated by Chang's MV-algebra. We conclude with Section 4 , where we will analyze the first-order versions of $\mathrm{BL}_{\text {Chang }}$ and $Ł_{\text {Chang }}$ : for the first-one the completeness results will be much more negative.

To conclude, we list the main results.

- $\mathrm{BL}_{\text {Chang }}$ enjoys the finite strong completeness (but not the strong one) w.r.t. $\omega \mathscr{V}$, where $\omega \mathscr{V}$ represents the ordinal sum of $\omega$ copies of the disconnected rotation of the standard cancellative hoop.
- $Ł_{\text {Chang }}$ (the logic associated to the variety generated by Chang's MV-algebra) enjoys the finite strong completeness (but not the strong one) w.r.t. $\mathscr{V}, \mathscr{V}$ being the disconnected rotation of the standard cancellative hoop.
- There are two BL-chains $\mathscr{A}, \mathscr{B}$ that are strongly complete w.r.t., respectively $\biguplus_{\text {Chang }}$ and $\mathrm{BL}_{\text {Chang }}$.
- Every $Ł_{\text {Chang }}$-chain that is strongly complete w.r.t. $Ł_{\text {Chang }}$ is also strongly complete w.r.t $Ł_{\text {Chang }} \forall$.
- There is no $\mathrm{BL}_{\text {Chang }}$-chain to be complete w.r.t. $\mathrm{BL}_{\text {Chang }} \forall$.


## 2 Preliminaries

### 2.1 Basic concepts

Basic Logic BL was introduced by P. Hájek in Háj98]. It is based over the connectives $\{\&, \rightarrow, \perp\}$ and a denumerable set of variables $V A R$. The formulas are defined inductively, as usual (see [Háj98] for details).

Other derived connectives are the following.
negation: $\neg \varphi:=\varphi \rightarrow \perp$; verum or top: $\top:=\neg \perp$; meet: $\varphi \wedge \psi:=\varphi \&(\varphi \rightarrow \psi)$; join: $\varphi \vee \psi:=((\varphi \rightarrow \psi) \rightarrow \psi) \wedge((\psi \rightarrow \varphi) \rightarrow \varphi)$.

BL is axiomatized as follows.

$$
\begin{align*}
& (\varphi \rightarrow \psi) \rightarrow((\psi \rightarrow \chi) \rightarrow(\varphi \rightarrow \chi))  \tag{A1}\\
& (\varphi \& \psi) \rightarrow \varphi  \tag{A2}\\
& (\varphi \& \psi) \rightarrow(\psi \& \varphi)  \tag{A3}\\
& (\varphi \&(\varphi \rightarrow \psi)) \rightarrow(\psi \&(\psi \rightarrow \varphi))  \tag{A4}\\
& (\varphi \rightarrow(\psi \rightarrow \chi)) \rightarrow((\varphi \& \psi) \rightarrow \chi)  \tag{A5a}\\
& ((\varphi \& \psi) \rightarrow \chi) \rightarrow(\varphi \rightarrow(\psi \rightarrow \chi))  \tag{A5b}\\
& ((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow(((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)  \tag{A6}\\
& \perp \rightarrow \varphi \tag{A7}
\end{align*}
$$

Modus ponens is the only inference rule:

$$
\begin{equation*}
\frac{\varphi \quad \varphi \rightarrow \psi}{\psi} . \tag{MP}
\end{equation*}
$$

Among the extensions of BL (logics obtained from it by adding other axioms) there is the well known Łukasiewicz (infinitely-valued) logic Ł, that is, BL plus

$$
\neg \neg \varphi \rightarrow \varphi .
$$

(INV)
On Łukasiewicz logic we can also define a strong disjunction connective (in the following sections, we will introduce a strong disjunction connective, for BL, that will be proved to be equivalent to the following, over $Ł$ )

$$
\varphi \curlyvee \psi:=\neg(\neg \varphi \& \neg \psi) .
$$

The notations $\varphi^{n}$ and $n \varphi$ will indicate $\underbrace{\varphi \& \ldots \& \varphi}_{n \text { times }}$ and $\underbrace{\varphi \gamma \cdots \gamma \varphi}_{n \text { times }}$.
Given an axiomatic extension L of BL, a formula $\varphi$ and a theory $T$ (a set of formulas), the notation $T \vdash_{L} \varphi$ indicates that there is a proof of $\varphi$ from the axioms of $L$ and the ones of $T$. The notion of proof is defined like in classical case (see [Háj98]).

We now move to the semantics: for all the unexplained notions of universal algebra, we refer to [BS81 Grä08].

Definition 2.1. $A$ BL-algebra is an algebraic structure of the form $\mathscr{A}=\langle A, *, \Rightarrow, \sqcap, \sqcup, 0,1\rangle$ such that

- $\langle A, \sqcap, \sqcup, 0,1\rangle$ is a bounded lattice, where 0 is the bottom and 1 the top element.
- $\langle A, *, 1\rangle$ is a commutative monoid.
- $\langle *, \Rightarrow\rangle$ forms a residuated pair, i.e.

$$
\begin{equation*}
z * x \leq y \quad \text { iff } \quad z \leq x \Rightarrow y \tag{res}
\end{equation*}
$$

it can be shown that the only operation that satisfies (res) is $x \Rightarrow y=\max \{z$ : $z * x \leq y\}$.

- $\mathscr{A}$ satisfies the following equations

$$
\begin{align*}
& (x \Rightarrow y) \sqcup(y \Rightarrow x)=1  \tag{pl}\\
& x \sqcap y=x *(x \Rightarrow y) . \tag{div}
\end{align*}
$$

Two important types of BL-algebras are the followings.

- A BL-chain is a totally ordered BL-algebra.
- A standard BL-algebra is a BL-algebra whose support is $[0,1]$.

Notation: in the following, with $\sim x$ we will indicate $x \Rightarrow 0$.
Definition 2.2. An MV-algebra is a BL-algebra satisfying

$$
\begin{equation*}
x=\sim \sim x . \tag{inv}
\end{equation*}
$$

A well known example of $M V$-algebra is the standard $M V$-algebra $[0,1]_{ \pm}=$ $\langle[0,1], *, \Rightarrow$, min, $\max , 0,1\rangle$, where $x * y=\max (0, x+y-1)$ and $x \Rightarrow y=\min (1,1-x+$ $y)$.

In every MV-algebra we define the algebraic equivalent of $\curlyvee$, that is

$$
x \oplus y:=\sim(\sim x * \sim y) .
$$

The notations (where $x$ is an element of some BL-algebra) $x^{n}$ and $n x$ will indicate $\underbrace{x * \cdots * x}_{n \text { times }}$ and $\underbrace{x \oplus \cdots \oplus x}_{n \text { times }}$.

Given a BL-algebra $\mathscr{A}$, the notion of $\mathscr{A}$-evaluation is defined in a truth-functional way (starting from a map $v: V A R \rightarrow A$, and extending it to formulas), for details see [Háj98].

Consider a BL-algebra $\mathscr{A}$, a theory $T$ and a formula $\varphi$. With $\mathscr{A} \models \varphi(\mathscr{A}$ is a model of $\varphi$ ) we indicate that $v(\varphi)=1$, for every $\mathscr{A}$-evaluation $v ; \mathscr{A} \models T$ denotes that $\mathscr{A} \models \psi$, for every $\psi \in T$. Finally, the notation $T \models_{\mathscr{A}} \varphi$ means that if $\mathscr{A} \models T$, then $\mathscr{A} \models \varphi$.

A BL-algebra $\mathscr{A}$ is called L-algebra, where L is an axiomatic extension of BL, whenever $\mathscr{A}$ is a model for all the axioms of L .

Definition 2.3. Let $L$ be an axiomatic extension of $B L$ and $K$ a class of L-algebras. We say that L is strongly complete (respectively: finitely strongly complete, complete) with respect to $K$ if for every set $T$ of formulas (respectively, for every finite set $T$ of formulas, for $T=\emptyset$ ) and for every formula $\varphi$ we have

$$
T \vdash_{L} \varphi \quad \text { iff } \quad T \models_{K} \varphi .
$$

### 2.2 Perfect MV-algebras, hoops and disconnected rotations

We recall that Chang's $M V$-algebra ([Cha58]) is a BL-algebra of the form

$$
C=\left\langle\left\{a_{n}: n \in \mathbb{N}\right\} \cup\left\{b_{n}: n \in \mathbb{N}\right\}, *, \Rightarrow, \sqcap, \sqcup, b_{0}, a_{0}\right\rangle .
$$

Where for each $n, m \in \mathbb{N}$, it holds that $b_{n}<a_{m}$, and, if $n<m$, then $a_{m}<a_{n}, b_{n}<b_{m}$; moreover $a_{0}=1, b_{0}=0$ (the top and the bottom element).

The operation $*$ is defined as follows, for each $n, m \in \mathbb{N}$ :

$$
b_{n} * b_{m}=b_{0}, b_{n} * a_{m}=b_{\max (0, n-m)}, a_{n} * a_{m}=a_{n+m}
$$

Definition 2.4 ([|BDL93]). Let $\mathscr{A}$ be an MV-algebra and let $x \in \mathscr{A}$ : with ord $(x)$ we mean the least (positive) natural $n$ such that $x^{n}=0$. If there is no such $n$, then we set $\operatorname{ord}(x)=\infty$.

- An MV-algebra is called loca 1 if for every element $x$ it holds that $\operatorname{ord}(x)<\infty$ or $\operatorname{ord}(\sim x)<\infty$.
- An MV-algebra is called perfect iffor every element $x$ it holds that $\operatorname{ord}(x)<\infty$ iff $\operatorname{ord}(\sim x)=\infty$.

An easy consequence of this definition is that every perfect MV-algebra cannot have a negation fixpoint.

With Perfect $(M V)$ and $\operatorname{Local}(M V)$ we will indicate the class of perfect and local MV-algebras. Moreover, given a BL-algebra $\mathscr{A}$, with $\mathbf{V}(\mathscr{A})$ we will denote the variety generated by $\mathscr{A}$.

Theorem 2.1 ([BDL93]). Every MV-chain is local.
Clearly there are local MV-algebras that are not perfect: $[0,1]_{£}$ is an example.
Now, in [DL94] it is shown that
Theorem 2.2.

- $\mathbf{V}(C)=\mathbf{V}(\operatorname{Perfect}(M V))$,
- Perfect $(M V)=\operatorname{Local}(M V) \cap \mathbf{V}(C)$.

It follows that the class of chains in $\mathbf{V}(C)$ coincides with the one of perfect MVchains. Moreover

Theorem 2.3 ([DL94]). An MV-algebra is in the variety $\mathbf{V}(C)$ iff it satisfies the equation $(2 x)^{2}=2\left(x^{2}\right)$.

As shown in [BDG07a], the logic correspondent to this variety is axiomatized as Ł plus $(2 \varphi)^{2} \leftrightarrow 2\left(\varphi^{2}\right)$ : we will call it $Ł_{\text {Chang }}$.

[^0]We now recall some results about hoops
Definition 2.5 ([Fer92, BF00]). A hoop is a structure $\mathscr{A}=\langle A, *, \Rightarrow, 1\rangle$ such that $\langle A, *, 1\rangle$ is a commutative monoid, and $\Rightarrow$ is a binary operation such that

$$
x \Rightarrow x=1, \quad x \Rightarrow(y \Rightarrow z)=(x * y) \Rightarrow z \quad \text { and } \quad x *(x \Rightarrow y)=y *(y \Rightarrow x)
$$

In any hoop, the operation $\Rightarrow$ induces a partial order $\leq$ defined by $x \leq y$ iff $x \Rightarrow y=$ 1. Moreover, hoops are precisely the partially ordered commutative integral residuated monoids (pocrims) in which the meet operation $\sqcap$ is definable by $x \sqcap y=x *(x \Rightarrow y)$. Finally, hoops satisfy the following divisibility condition:

$$
\begin{equation*}
\text { If } x \leq y \text {, then there is an element } z \text { such that } z * y=x \text {. } \tag{div}
\end{equation*}
$$

We recall a useful result.
Definition 2.6. Let $\mathscr{A}$ and $\mathscr{B}$ be two algebras of the same language. Then we say that

- $\mathscr{A}$ is a partial subalgebra of $\mathscr{B}$ if $A \subseteq B$ and the operations of $\mathscr{A}$ are the ones of $\mathscr{A}$ restricted to $A$. Note that $A$ could not be closed under these operations (in this case these last ones will be undefined over some elements of A): in this sense $\mathscr{A}$ is a partial subalgebra.
- $\mathscr{A}$ is partially embeddable into $\mathscr{B}$ when every finite partial subalgebra of $\mathscr{A}$ is embeddable into $\mathscr{B}$. Generalizing this notion to classes of algebras, we say that a class $K$ of algebras is partially embeddable into a class $M$ if every finite partial subalgebra of a member of $K$ is embeddable into a member of $M$.

Definition 2.7. A bounded hoop is a hoop with a minimum element; conversely, an unbounded hoop is a hoop without minimum.

Let $\mathscr{A}$ be a bounded hoop with minimum a: with $\mathscr{A}^{+}$we mean the (partial) subalgebra of $\mathscr{A}$ defined over the universe $A^{+}=\{x \in A: x>x \Rightarrow a\}$.

A hoop is Wajsberg iff it satisfies the equation $(x \Rightarrow y) \Rightarrow y=(y \Rightarrow x) \Rightarrow x$.
A hoop is cancellative iff it satisfies the equation $x=y \Rightarrow(x * y)$.
Proposition 2.1 ([Fer92 BF00, AFM07]). Every cancellative hoop is Wajsberg. Totally ordered cancellative hoops coincide with unbounded totally ordered Wajsberg hoops, whereas bounded Wajsberg hoops coincide with (the 0-free reducts of) MValgebras.

We now recall a construction introduced in [Jen03] (and also used in [EGHM03, NEG05), called disconnected rotation.

Definition 2.8. Let $\mathscr{A}$ be a cancellative hoop. We define an algebra, $\mathscr{A}^{*}$, called the disconnected rotation of $\mathscr{A}$, as follows. Let $\mathscr{A} \times\{0\}$ be a disjoint copy of $A$. For every $a \in A$ we write $a^{\prime}$ instead of $\langle a, 0\rangle$. Consider $\left\langle A^{\prime}=\left\{a^{\prime}: a \in A\right\}, \leq\right\rangle$ with the inverse order and let $A^{*}:=A \cup A^{\prime}$. We extend these orderings to an order in $A^{*}$ by putting
$a^{\prime}<b$ for every $a, b \in A$. Finally, we take the following operations in $A^{*}: 1:=1_{\mathscr{A}}$, $0:=1^{\prime}, \sqcap_{\mathscr{A}^{*}}, \sqcup_{\mathscr{A}^{*}}$ as the meet and the join with respect to the order over $A^{*}$. Moreover,

$$
\begin{aligned}
& \sim_{\mathscr{A}^{*}} a:= \begin{cases}a^{\prime} \quad \text { if } a \in A \\
b & \text { if } a=b^{\prime} \in A^{\prime}\end{cases} \\
& a *_{\mathscr{A}^{*}} b:= \begin{cases}a *_{\mathscr{A}} b & \text { if } a, b \in A \\
\sim_{\mathscr{A}^{*}}\left(a \Rightarrow_{\mathscr{A}^{\prime}} \sim_{\mathscr{A}^{*}} b\right) & \text { if } a \in A, b \in A^{\prime} \\
\sim_{\mathscr{A}^{*}}\left(b \Rightarrow_{\mathscr{A}^{\prime}} \sim_{\mathscr{A}^{*}} a\right) & \text { if } a \in A^{\prime}, b \in A \\
0 & \text { if } a, b \in A^{\prime}\end{cases} \\
& a \Rightarrow_{\mathscr{A}^{*}} b:= \begin{cases}a \Rightarrow_{\mathscr{A}^{\prime}} b & \text { if } a, b \in A \\
\sim_{\mathscr{A}^{*}}\left(a *_{\mathscr{A}^{*}} \sim_{\mathscr{A}^{*}} b\right) & \text { if } a \in A, b \in A^{\prime} \\
1 & \text { if } a \in A^{\prime}, b \in A \\
\left(\sim_{\mathscr{A}^{*}} b\right) \Rightarrow_{\mathscr{A}^{\prime}}\left(\sim_{\mathscr{A}^{*}} a\right) & \text { if } a, b \in A^{\prime} .\end{cases}
\end{aligned}
$$

Theorem 2.4 ([NEG05], theorem 9]). Let $\mathscr{A}$ be an MV-algebra. The followings are equivalent:

- A is a perfect MV-algebra.
- A is isomorphic to the disconnected rotation of a cancellative hoop.

To conclude the section, we present the definition of ordinal sum.
Definition 2.9 ([|AM03]). Let $\langle I, \leq\rangle$ be a totally ordered set with minimum 0 . For all $i \in I$, let $\mathscr{A}_{i}$ be a hoop such that for $i \neq j, A_{i} \cap A_{j}=\{1\}$, and assume that $\mathscr{A}_{0}$ is bounded. Then $\bigoplus_{i \in I} \mathscr{A}_{i}\left(\right.$ the ordinal sum of the family $\left.\left(\mathscr{A}_{i}\right)_{i \in I}\right)$ is the structure whose base set is $\bigcup_{i \in I} A_{i}$, whose bottom is the minimum of $\mathscr{A}_{0}$, whose top is 1 , and whose operations are

$$
\begin{aligned}
x \Rightarrow y & = \begin{cases}x \Rightarrow^{\mathscr{A}_{i}} y & \text { if } x, y \in A_{i} \\
y & \text { if } \exists i>j\left(x \in A_{i} \text { and } y \in A_{j}\right) \\
1 & \text { if } \exists i<j\left(x \in A_{i} \backslash\{1\} \text { and } y \in A_{j}\right)\end{cases} \\
x * y & = \begin{cases}x *^{\mathscr{A}_{i}} y & \text { if } x, y \in A_{i} \\
x & \text { if } \exists i<j\left(x \in A_{i} \backslash\{1\}, y \in A_{j}\right) \\
y & \text { if } \exists i<j\left(y \in A_{i} \backslash\{1\}, x \in A_{j}\right)\end{cases}
\end{aligned}
$$

When defining the ordinal sum $\bigoplus_{i \in I} \mathscr{A}_{i}$ we will tacitly assume that whenever the condition $A_{i} \cap A_{j}=\{1\}$ is not satisfied for all $i, j \in I$ with $i \neq j$, we will replace the $\mathscr{A}_{i}$ by isomorphic copies satisfying such condition. Moreover if all $\mathscr{A}_{i}$ 's are isomorphic to some $\mathscr{A}$, then we will write $I \mathscr{A}$, instead of $\bigoplus_{i \in I} \mathscr{A}_{i}$. Finally, the ordinal sum of two hoops $\mathscr{A}$ and $\mathscr{B}$ will be denoted by $\mathscr{A} \oplus \mathscr{B}$.

Note that, since every bounded Wajsberg hoop is the 0-free reduct of an MValgebra, then the previous definition also works with these structures.

Theorem 2.5 (AM03, theorem 3.7]). Every BL-chain is isomorphic to an ordinal sum whose first component is an MV-chain and the others are totally ordered Wajsberg hoops.

Note that in [Bus04] it is presented an alternative and simpler proof of this result.

## 3 The variety of $\mathrm{BL}_{\text {Chang }}$-algebras

Consider the following connective

$$
\varphi \underline{\vee} \psi:=((\varphi \rightarrow(\varphi \& \psi)) \rightarrow \psi) \wedge((\psi \rightarrow(\varphi \& \psi)) \rightarrow \varphi)
$$

Call $\uplus$ the algebraic operation, over a BL-algebra, corresponding to $\underline{\vee}$; we have that
Lemma 3.1. In every MV-algebra the following equation holds

$$
x \uplus y=x \oplus y .
$$

Proof. It is easy to check that $x \uplus y=x \oplus y$, over $[0,1]_{M V}$, for every $x, y \in[0,1]$.
We now analyze this connective in the context of Wajsberg hoops.
Proposition 3.1. Let $\mathscr{A}$ be a linearly ordered Wajsberg hoop. Then

- If $\mathscr{A}$ is unbounded (i.e. a cancellative hoop), then $x \uplus y=1$, for every $x, y \in \mathscr{A}$.
- If $\mathscr{A}$ is bounded, let $a$ be its minimum. Then, by defining $\sim x:=x \Rightarrow a$ and $x \oplus y=\sim(\sim x * \sim y)$ we have that $x \oplus y=x \uplus y$, for every $x, y \in \mathscr{A}$

Proof. An easy check.
Now, since the variety of cancellative hoops is generated by its linearly ordered members (see [EGHM03]), then we have that

Corollary 3.1. The equation $x \uplus y=1$ holds in every cancellative hoop.
We now characterize the behavior of $\uplus$ for the case of BL-chains.
Proposition 3.2. Let $\mathscr{A}=\bigoplus_{i \in I} \mathscr{A}_{i}$ be a BL-chain. Then

$$
x \uplus y= \begin{cases}x \oplus y, & \text { if } x, y \in \mathscr{A}_{i} \text { and } \mathscr{A}_{i} \text { is bounded } \\ 1, & \text { if } x, y \in \mathscr{A}_{i} \text { and } \mathscr{A}_{i} \text { is unbounded } \\ \max (x, y), & \text { otherwise } .\end{cases}
$$

for every $x, y \in \mathscr{A}$.
Proof. If $x, y$ belong to the same component of $\mathscr{A}$, then the result follows from Lemma 3.1 and Proposition 3.1. For the case in which $x$ and $y$ belong to different components of $\mathscr{A}$, this is a direct computation.

Remark 3.1. From the previous proposition we can argue that $\uplus$ is a good approximation, for BL, of what that $\oplus$ represents for MV-algebras. Note that a similar operation was introduced in [ABM09]: the main difference with respect to $\uplus$ is that, when $x$ and y belong to different components of a BL-chain, then the operation introduced in [ABM09] holds 1.

In the following, for every element $x$ of a BL-algebra, with the notation $\bar{n} x$ we will denote $\underbrace{x \uplus \cdots \uplus x}_{n \text { times }}$; analogously $\bar{n} \varphi$ means $\underbrace{\varphi \vee \cdots \underline{V} \varphi}_{n \text { times }}$.

Definition 3.1. We define $B L_{\text {Chang }}$ as the axiomatic extension of BL, obtained by adding

$$
\begin{equation*}
(\overline{2} \varphi)^{2} \leftrightarrow \overline{2}\left(\varphi^{2}\right) . \tag{cha}
\end{equation*}
$$

That is, writing it in extended form

$$
\left(\varphi^{2} \rightarrow\left(\varphi^{2} \& \varphi^{2}\right) \rightarrow \varphi^{2}\right) \leftrightarrow\left(\left(\varphi \rightarrow \varphi^{2}\right) \rightarrow \varphi\right)^{2} .
$$

Clearly the variety corresponding to $\mathrm{BL}_{\text {Chang }}$ is given by the class of BL-algebras satisfying the equation $(\overline{2} x)^{2}=\overline{2}\left(x^{2}\right)$.

Moreover,
Definition 3.2. We will call pseudo-perfect Wajsberg hoops those Wajsberg hoops satisfying the equation $(\overline{2} x)^{2}=\overline{2}\left(x^{2}\right)$.

Remark 3.2. Thanks to Lemma 3.1 we have that

$$
\vdash_{E}\left((\overline{2} \varphi)^{2} \leftrightarrow \overline{2}\left(\varphi^{2}\right)\right) \leftrightarrow\left((2 \varphi)^{2} \leftrightarrow 2\left(\varphi^{2}\right)\right),
$$

that is, if we add $(\overline{2} \varphi)^{2} \leftrightarrow \overline{2}\left(\varphi^{2}\right)$ or $(2 \varphi)^{2} \leftrightarrow 2\left(\varphi^{2}\right)$ to $\ell$, then we obtain the same logic $Ł_{\text {Chang }}$.

These formulas, however are not equivalent over BL: see Remark 3.3 for details.
Theorem 3.1. Every totally ordered pseudo-perfect Wajsberg hoop is a totally ordered cancellative hoop or (the 0-free reduct of) a perfect MV-chain.

More in general, the variety of pseudo-perfect Wajsberg hoops coincides with the class of the 0-free subreducts of members of $\mathbf{V}(C)$.

Proof. In [EGHM03] it is shown that the variety of Wajsberg hoops coincides with the class of the 0 -free subreducts of MV-algebras. The results easily follow from this fact and from Proposition 2.1, Theorem 2.3 and Definition 3.2.

As a consequence, we have
Theorem 3.2. Let $\mathbb{W H}, \mathbb{C H}, p s \mathbb{W} \mathbb{H}$ be, respectively, the varieties of Wajsberg hoops, cancellative hoops, pseudo-perfect Wajsberg hoops. Then we have that

$$
\mathbb{C H} \subset p s \mathbb{W} \mathbb{H} \subset \mathbb{W} \mathbb{H} .
$$

Proof. An easy consequence of Theorem 3.1.
The first inclusion follows from the fact that $p s W H H$ contains all the totally ordered cancellative hoops and hence the variety generated by them. For the second inclusion note that, for example, the 0 -free reduct of $[0,1]_{£}$ belongs to $\mathbb{W} H \mathbb{H} \backslash p s \mathbb{W} H$.

We now describe the structure of $\mathrm{BL}_{\text {Chang }}$-chains, with an analogous of the Theorem 2.5 for BL-chains.

Theorem 3.3. Every $B L_{\text {Chang-chain }}$ is isomorphic to an ordinal sum whose first component is a perfect $M V$-chain and the others are totally ordered pseudo-perfect Wajsberg hoops.

It follows that every ordinal sum of perfect $M V$-chains is a $B L_{\text {Chang-chain. }}$
Proof. Thanks to Theorems 2.2 and 2.3, Remark 3.2 and Definition 3.2, we have that every MV-chain (Wajsberg hoop) satisfying the equation $(\overline{2} x)^{2}=\overline{2}\left(x^{2}\right)$ is perfect (pseudo-perfect): using these facts and Proposition 3.2 we have that a BL-chain satisfies the equation $(\overline{2} x)^{2}=\overline{2}\left(x^{2}\right)$ iff it holds true in all the components of its ordinal sum. From these facts and Theorem 2.5 we get the result.

As a consequence, we obtain the following corollaries.
Corollary 3.2. The variety of $B L_{\text {Chang-algebras contains the ones of }}$ product-algebras and Gödel-algebras: however it does not contains the variety of MValgebras.

Proof. From the previous theorem it is easy to see that the variety of $\mathrm{BL}_{\text {Chang }}$-algebras contains $[0,1]_{\Pi}$ and $[0,1]_{G}$, but not $[0,1]_{\mathrm{E}}$.

Corollary 3.3. Every finite BL Chang-chain is an ordinal sum of a finite number of copies of the two elements boolean algebra. Hence the class of finite $B L_{\text {Chang }}$-chains coincides with the one of finite Gödel chains.

For this reason it is immediate to see that the finite model property does not hold for $\mathrm{BL}_{\text {Chang }}$.

We conclude with the following remark.
Remark 3.3. - One can ask if it is possible to axiomatize the class $B L_{p e r f}$ of $B L-$ algebras, whose chains are the BL-algebras that are ordinal sum of perfect $M V$ chains: the answer, however, is negative. In fact, the class of bounded Wajsberg hoops does not form a variety: for example, it is easy to check that for every bounded pseudo-perfect Wajsberg hoop $\mathscr{A}$, its subalgebra $\mathscr{A}^{+}$(see Definition 2.7) forms a cancellative hoop. Hence $B L_{\text {perf }}$ cannot be a variety.
However, as we will see in Section 3.2, the variety of $B L_{\text {Chang-algebras }}$ is the "best approximation" of $B L_{p e r f,}$ in the sense that it is the smallest variety to contain $B L_{\text {perf. }}$

- In DSE 02 (see also CT06]) it is studied the variety, called $P_{0}$, generated by all the perfect BL-algebras (a BL-algebra $\mathscr{A}$ is perfect if, by calling $M V(\mathscr{A})$
the biggest subalgebra of $\mathscr{A}$ to be an MV-algebra, then $M V(\mathscr{A})$ is a perfect $M V$-algebra). $P_{0}$ is axiomatized with the equation

$$
\begin{equation*}
\sim\left(\left(\sim\left(x^{2}\right)\right)^{2}\right)=\left(\sim\left((\sim x)^{2}\right)\right)^{2} \tag{0}
\end{equation*}
$$

One can ask which is the relation between $P_{0}$ and the variety of $B L_{\text {Chang-algebras. }}$
 fact, an easy check shows that a BL-chain is perfect if and only if the first component of its ordinal sum is a perfect MV-chain. Hence we have:

- Every $B L_{\text {Chang-chain is }}$ a perfect BL-chain.
- There are perfect $B L$-chains that are not $B L_{\text {Chang-chains: an example is }}$ given by $C \oplus[0,1]_{ \pm}$.

Now, since the variety of $B L_{\text {Chang }}$-algebras is generated by its chains (like any variety of BL-algebras, see Haj98]), then we get the result.
Finally note that $\left(p_{0}\right)$ is equivalent to $2\left(x^{2}\right)=(2 x)^{2}$ : hence, differently to what happens over $Ł$ (see Remark 3.2), the equations $2\left(x^{2}\right)=(2 x)^{2}$ and $\overline{2}\left(x^{2}\right)=(\overline{2} x)^{2}$ are not equivalent, over BL.

### 3.1 Subdirectly irreducible and simple algebras

We begin with a general result about Wajsberg hoops.
Theorem 3.4 ([Fer92, Corollary 3.11]). Every subdirectly irreducible Wajsberg hoop is totally ordered.

As a consequence, we have:
Corollary 3.4. Every subdirectly irreducible pseudo-perfect Wajsberg hoop is totally ordered.

We now move to simple algebras.
It is shown in [Tur99, Theorem 1] that the simple BL-algebras coincide with the simple MV-algebras, that is, with the subalgebras of $[0,1]_{\mathrm{E}}$ (see [CDM99, Theorem 3.5.1]). Therefore we have:

Theorem 3.5. The only simple $B L_{\text {Chang-algebra }}$ is the two elements boolean algebra 2.

An easy consequence of this fact is that the only simple $Ł_{C h a n g}$-algebra is $\mathbf{2}$.

### 3.2 Completeness

We begin with a result about pseudo-perfect Wajsberg hoops.
Theorem 3.6. The class pMV of 0 -free reducts of perfect $M V$-chains generates $p s \mathbb{W H}$.

Proof. From Theorems 2.4 and 3.1 it is easy to check that the variety generated by $p M V$ contains all the totally ordered pseudo-perfect Wajsberg hoops.

From these facts and Corollary 3.4, we have that $p M V$ must be generic for $p s \mathbb{W H}$.

Theorem 3.7 ([|EEG $\left.\left.{ }^{+} 09\right]\right)$. Let $L$ be an axiomatic extension of BL, then $L$ enjoys the finite strong completeness w.r.t a class $K$ of L-algebras iff every countable L-chain is partially embeddable into $K$.

As shown in [Háj98] product logic enjoys the finite strong completeness w.r.t $[0,1]_{\Pi}$ and hence every countable product chain is partially embeddable into $[0,1]_{\Pi} \simeq \mathbf{2} \oplus$ $(0,1]_{C}$, with $(0,1]_{C}$ being the standard cancellative hoop (i.e. the 0 -free reduct of $[0,1]_{\Pi} \backslash\{0\}$ ). Since every totally ordered product chain is of the form $\mathbf{2} \oplus \mathscr{A}$, where $\mathscr{A}$ is a cancellative hoop (see [EGHM03]), it follows that:

Proposition 3.3. Every countable totally ordered cancellative hoop partially embeds into $(0,1]_{C}$.

Theorem 3.8. Every countable perfect $M V$-chain partially embeds into $\mathscr{V}=(0,1]_{C}^{*}$ (i.e. the disconnected rotation of $\left.(0,1]_{C}\right)$.

Proof. Immediate from Proposition 3.3 and Theorem 2.4.
Corollary 3.5. The logic $Ł_{\text {Chang }}$ is finitely strongly complete w.r.t. $\mathscr{V}$.
Theorem 3.9. $B L_{\text {Chang }}$ enjoys the finite strong completeness w.r.t. $\omega \mathscr{V}$. As a consequence, the variety of $B L_{\text {Chang-algebras is generated by the class of all ordinal sums of }}$ perfect MV-chains and hence is the smallest variety to contain this class of algebras.

Proof. Thanks to Theorem 3.7 it is enough to show that every countable $\mathrm{BL}_{\text {Chang }}$-chain partially embeds into $\omega \mathscr{V}$ (i.e. the ordinal sum of " $\omega$ copies" of $\mathscr{V}$ ). This fact, however, follows immediately from Proposition 3.3 and Theorems 3.3 and 3.8.

But we cannot obtain a stronger result: in fact
Theorem 3.10. $B L_{\text {Chang }}$ is not strongly complete w.r.t. $\omega \mathscr{V}$.
Proof. Suppose not: from the results of [ $\mathrm{CEG}^{+} 09$, Theorem 3.5] this is equivalent to claim that every countable $\mathrm{BL}_{\text {Chang }}$-chain embeds into $\omega \mathscr{V}$. But, this would imply that every countable totally ordered cancellative hoop embeds into $(0,1]_{C}$ : this means that every countable product-chain embeds into $[0,1]_{\Pi}$, that is product logic is strongly complete w.r.t $[0,1]_{\Pi}$. As it is well known (see [Háj98, Corollary 4.1.18]), this is false.

With an analogous proof we obtain
Theorem 3.11. $Ł_{\text {Chang }}$ is not strongly complete w.r.t. $\mathscr{V}$
However, thanks to [Mon11 Theorem 3] we can claim
Theorem 3.12. There exist a $Ł_{\text {Chang }}$-chain $\mathscr{A}$ and a $B L_{\text {Chang }}$-chain $\mathscr{B}$ such that $Ł_{\text {Chang }}$ is strongly complete w.r.t. $\mathscr{A}$ and $B L_{\text {Chang }}$ is strongly complete w.r.t. $\mathscr{B}$.

Problem 3.1. Which can be some concrete examples of such $\mathscr{A}$ and $\mathscr{B}$ ?

## 4 First-order logics

We assume that the reader is acquainted with the formalization of first-order logics, as developed in [Háj98, CH10].

Briefly, we work with (first-order) languages without equality, containing only predicate and constant symbols: as quantifiers we have $\forall$ and $\exists$. The notions of terms and formulas are defined inductively like in classical case.

As regards to semantics, given an axiomatic extension L of BL we restrict to L chains: the first-order version of $L$ is called $\mathrm{L} \forall$ (see HAáj98, CH10] for an axiomatization). A first-order $\mathscr{A}$-interpretation ( $\mathscr{A}$ being an L-chain) is a structure $\mathbf{M}=$ $\left\langle M,\left\{r_{P}\right\}_{p \in \mathbf{P}},\left\{m_{c}\right\}_{c \in \mathbf{C}}\right\rangle$, where $M$ is a non-empty set, every $r_{P}$ is a fuzzy $\operatorname{ariety}(P)$-ary relation, over $M$, in which we interpretate the predicate $P$, and every $m_{c}$ is an element of $M$, in which we map the constant $c$.

Given a map $v: V A R \rightarrow M$, the interpretation of $\|\varphi\|_{\mathbf{M}, v}^{\mathscr{A}}$ in this semantics is defined in a Tarskian way: in particular the universally quantified formulas are defined as the infimum (over $\mathscr{A}$ ) of truth values, whereas those existentially quantified are evaluated as the supremum. Note that these inf and sup could not exist in $\mathscr{A}$ : an $\mathscr{A}$-model $\mathbf{M}$ is called safe if $\|\varphi\|_{\mathbf{M}, v}$ is defined for every $\varphi$ and $v$.

A model is called witnessed if the universally (existentially) quantified formulas are evaluated by taking the minimum (maximum) of truth values in place of the infimum (supremum): see [Háj07] $\mathrm{CH06}, \mathrm{CH10]}$ for details.

The notions of soundness and completeness are defined by restricting to safe models (even if in some cases it is possible to enlarge the class of models: see [BM09]): see [Háj98, CH10, CH06] for details.

We begin with a positive result about $\biguplus_{\text {Chang }} \forall$.
Definition 4.1. Let $L$ be an axiomatic extension of $B L$. With $L \forall^{w}$ we define the extension of $L \forall$ with the following axioms

$$
\begin{align*}
& (\exists y)(\varphi(y) \rightarrow(\forall x) \varphi(x)) \\
& (\exists y)((\exists x) \varphi(x) \rightarrow \varphi(y)) . \tag{Cヨ}
\end{align*}
$$

Theorem 4.1 ([CH06, Proposition 6]). $£ \forall$ coincides with $Ł \forall^{w}$, that is $\ell \forall \vdash(\mathbf{C} \forall)(\mathrm{C} \exists)$.

An immediate consequence is:
Corollary 4.1. Let $L$ be an axiomatic extension of $\ell$. Then $L \forall$ coincides with $L \forall^{w}$.
Theorem 4.2 ([CH06, Theorem 8]). Let L be an axiomatic extension of BL. Then $L \forall^{w}$ enjoys the strong witnessed completeness with respect to the class $K$ of $L$-chains, i.e.

$$
T \vdash_{L \forall^{w}} \varphi \quad \text { iff } \quad\|\varphi\|_{\mathbf{M}}^{\mathscr{A}}=1,
$$

for every theory $T$, formula $\varphi$, algebra $\mathscr{A} \in K$ and witnessed $\mathscr{A}$-model $\mathbf{M}$ such that $\|\psi\|_{\mathbf{M}}^{\mathscr{A}}=1$ for every $\psi \in T$.

Lemma 4.1 ([Mon11] Lemma 1]). Let L be an axiomatic extension of BL, let $\mathscr{A}$ be an L-chain, let $\mathscr{B}$ be an L-chain such that $A \subseteq B$ and let $\mathbf{M}$ be a witnessed $\mathscr{A}$-structure. Then for every formula $\varphi$ and evaluation $v$, we have $\|\varphi\|_{\mathbf{M}, v}^{\mathscr{A}}=\|\varphi\|_{\mathbf{M}, v}^{\mathscr{B}}$.

Theorem 4.3. $T h e r e$ is a $Ł_{\text {Chang }}$-chain such that $Ł_{\text {Chang }} \forall$ is strongly complete w.r.t. it. More in general, every $Ł_{\text {Chang }}$-chain that is strongly complete w.r.t $Ł_{\text {Chang }}$ is also strongly complete w.r.t. $Ł_{\text {Chang }} \forall$.

Proof. An adaptation of the proof for the analogous result, given in [Mon11. Theorem 16], for $Ł \forall$.

From Theorem 3.12 we know that there is a $Ł_{\text {Chang }}$-chain $\mathscr{A}$ strongly complete w.r.t. $\biguplus_{\text {Chang }}$ : from [ $\mathrm{CEG}^{+} 09$, Theorem 3.5] this is equivalent to claim that every countable $Ł_{\text {Chang }}$-chain embeds into $\mathscr{A}$. We show that $\mathscr{A}$ is also strongly complete w.r.t. $Ł_{\text {Chang }} \forall$.

Suppose that $T \not \mathrm{E}_{\text {Chang }} \forall \varphi$. Thanks to Corollary 4.1 and Theorem 4.2 there is a countable $Ł_{\text {Chang }}$-chain $\mathscr{C}$ and a witnessed $\mathscr{C}$-model $\mathbf{M}$ such that $\|\psi\|_{\mathbf{M}}^{\mathscr{C}}=1$, for every $\psi \in T$, but $\|\varphi\|_{M}^{\mathscr{C}}<1$. Finally, from Lemma 4.1 we have that $\|\psi\|_{M}^{\mathscr{M}}=1$, for every $\psi \in T$ and $\|\varphi\|_{\mathbf{M}}^{\mathscr{A}}=\|\varphi\|_{\mathbf{M}}^{\mathscr{G}}<1$ : this completes the proof.

For $\mathrm{BL}_{\text {Chang }} \forall$, however, the situation is not so good.
Theorem 4.4. $B L_{\text {Chang }} \forall$ cannot enjoy the completeness w.r.t. a single $B L_{\text {Chang }}$-chain.
Proof. The proof is an adaptation of the analogous result given in [Mon11] Theorem 17] for BL $\forall$.

Let $\mathscr{A}$ be a BL ${\text { Chang-chain: call } \mathscr{A}_{0} \text { its first component. We have three cases }}_{\text {con }}$

- $\mathscr{A}_{0}$ is finite: from Theorem 3.3 we have that $\mathscr{A}_{0}=\mathbf{2}$ and hence $\mathscr{A} \models(\neg \neg x) \rightarrow$ $(\neg \neg x)^{2}$. However $\mathscr{V} \not \vDash(\neg \neg x) \rightarrow(\neg \neg x)^{2}$, where $\mathscr{V}$ is the chain introduced in Section 3.2, and hence $\mathscr{A}$ cannot be complete w.r.t. BL $_{\text {Chang }} \forall$.
- $\mathscr{A}_{0}$ is infinite and dense. As shown in [Mon11. Theorem 17] the formula $(\forall x) \neg \neg P(x) \rightarrow \neg \neg(\forall x) P(x)$ is a tautology in every BL-chain whose first component is infinite and densely ordered: hence we have that $\mathscr{A} \models(\forall x) \neg \neg P(x) \rightarrow$ $\neg \neg(\forall x) P(x)$. However it is easy to check that this formula fails in $[0,1]_{G}$ : take a $[0,1]_{G_{G}}$-model $\mathbf{M}$ with $M=(0,1]$ and such that $r_{P}(m)=m$. Hence, from Corollary 3.2, it follows that $\mathrm{BL}_{\text {Chang }} \forall \forall(\forall x) \neg \neg P(x) \rightarrow \neg \neg(\forall x) P(x)$.
- $\mathscr{A}_{0}$ is infinite and not dense. As shown in [Mon11. Theorem 17] the formula $(\forall x) \neg \neg P(x) \rightarrow \neg \neg(\forall x) P(x) \vee \neg(\forall x) P(x) \rightarrow((\forall x) P(x))^{2}$ is a tautology in every BL-chain whose first component is infinite and not densely ordered: hence we have that $\mathscr{A} \models(\forall x) \neg \neg P(x) \rightarrow \neg \neg(\forall x) P(x) \vee \neg(\forall x) P(x) \rightarrow((\forall x) P(x))^{2}$. Also in this case, however, this formula fails in $[0,1]_{G}$, using the same model $\mathbf{M}$ of the previous case.


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[^0]:    ${ }^{1}$ Usually, the local MV-algebras are defined as MV-algebras having a unique (proper) maximal ideal. In [BDL93], however, it is shown that the two definitions are equivalent. We have preferred the other definition since it shows in a more transparent way that perfect MV-algebras are particular cases of local MV-algebras.

