# Book Reviews

Gottlob Frege, **Basic Laws of Arithmetic**. Translated and edited by P. A. Ebert and M. Rossberg, with C. Wright, with an appendix by R. T. Cook. Oxford: Oxford University Press, 2013, pp. 680. ISBN: 978-0-19-928174-9 (hardback) $100.

The publication of a complete English translation of both volumes of Gottlob Frege's magnum opus, *Grundgesetze der Arithmetik*, has been one of the most anticipated moments in academic logic and philosophy for many years. It has finally arrived. It is the result of a decade of hard work by a team of researchers assembled by Crispin Wright and led by Philip Ebert and Marcus Rossberg. It is a unique, impressive and important accomplishment.

It would be hard to overstate the central importance of *Grundgesetze* for Frege's career and its eventual impact. His groundbreaking contributions to logic began with 1879's *Begriffsschrift* in which he laid out his second-order function calculus. This was the first appearance of modern quantification theory, and arguably, it was the most important advance in logic since Aristotle. In his next book, *Die Grundlagen der Arithmetik* (1884), Frege informally made the case for his logicist understanding of numbers and criticized rival views. It is unquestionably one of the most important works ever written on the philosophy of mathematics. Yet, there can be little doubt that Frege himself would have seen these two earlier books as mere preparatory studies for the larger project he sought to bring to culmination in *Grundgesetze*. Therein, he was to utilize an enhanced version of his logical system to present "gapless" proofs of the core principles of arithmetic from logical foundations. Even Frege's influential articles from the early 1890s such as "Über Sinn und Bedeutung" ("On Sense and Reference") and "Über Begriff und Gegenstand" ("On Concept and Object")—now considered classics in the philosophy of language literature—are best understood in relation to *Grundgesetze*. In particular, they elaborate on changes to Frege's views on the theory of meaning and logical syntactic analysis which prompted him to abandon an earlier draft of the larger work.

Nowhere has interest in and demand for Frege's writings been greater than in the English-speaking academic community. Although Montgomery Furth published a translation of Part I of *Grundgesetze* in 1964, and portions of the remainder have appeared in collections, the lack of a full translation has been sorely felt. Yet, even the briefest perusal of the content makes it clear why it took so long for one to be produced. Faithfully reproducing Frege's unique two-dimensional logical notation, which dominates most of the pages, would have been an almost insurmountable obstacle until very recently. Even now, two new highly sophisticated LaTeX packages had to be created: J. J. Green's `fge` package which provides a font containing unique symbols found in *Grundgesetze* and the `grundgesetze` package for drawing the subcomponent/supercomponent branches which make up the logical structure of Frege's propositions. The latter builds upon Josh Parsons's earlier `begriff` package. As someone who occasionally has to typeset Frege's notation in LaTeX, I can testify that the creation of these packages, which are now publicly available

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as free open-source software, will serve as a lasting contribution of the translation project even beyond the translation itself.

Volume I of *Grundgesetze* was published in 1893, and contains Part I of the work and the beginning of Part II. Volume II was published in 1903; it contains the remainder of Part II (picking up right where it left off), and proceeding into Part III. However, Part III ends abruptly; Frege had clearly intended to continue it in his planned Volume III. However, as is now well known, while Volume II was in the process of being typeset, Frege learned of the inconsistency of his core logic from a letter sent by Bertrand Russell. He hastily added an afterword to the volume. Nonetheless, the project was derailed and Frege never completed Volume III. Indeed, it is not clear how much of it he had written; anything he had done was destroyed along with other portions of his Nachlaß during World War II. The translation binds the two volumes together into a single book. However, it matches its pagination to the original volumes, and hence the page numbers "reset" when volume II begins (and then again for the appendices). This certainly has its advantages, but could also potentially cause confusion, for, e.g., citation purposes. It also means that different pages have larger or smaller margins depending on how much space the translation took relative to the original German. Also like the original, the translation switches back and forth between a two-column layout and a single-column layout depending on the width of the formal propositions involved. Fortunately, one gets accustomed to these quirks quickly.

The first volume opens with a foreword in which Frege discusses the overall project, and apologizes for a delay in its publication. He mentions two factors: the changes to his views on meaning and logical segmentation mentioned earlier, and the chilly overall reception of his work. The foreword also contains some harsh polemics against what Frege calls (I: p. XIV) "the ruinous incursion of psychology into logic", singling out the work of Benno Erdmann for criticism.

Part I, "Exposition of the concept-script", follows. Therein, Frege explains his two-dimensional logical notation, and the syntactic rules of his language including the distinction between object names and function names of various "levels". He further introduces his basic laws (axioms), inference rules, and his policies for introducing new symbols by definition. Included among these of course is his notorious Basic Law V, which states that functions $F$ and $G$ have the same value-range (or in the case of concepts, the same extension) just in case they have the same value for every argument. He goes on to define certain important functions and derives certain important logical theorems. Throughout, Frege discusses his conception of reference, distinguishing it from sense, and presents an argument to the effect that every correctly formed name of his language has a unique reference, including under this the contention that every proposition has a unique truth-value. Perhaps the most convoluted part of this discussion is Frege's attempts to explain how it is that expressions involving his value-range notation $\dot{\epsilon}(\ldots\epsilon\ldots)$ are known to have reference given that Law V only fixes the truth-value of identity statements between two such expressions. Frege himself (see [4, p. 132]) came to regard this argument as faulty after learning of the inconsistency of his system, which he blamed on Law V. Given the renewed interest in so-called "abstraction principles" in the philosophy of mathematics, including Law V, and worries about their adequacy in light of Frege's own "Julius Caesar objection", these passages will no doubt continue to generate discussion.

Part II is entitled "Proofs of the basic laws of cardinal number". Therein, Frege provides definitions of certain cardinal numbers, including zero and one, as well as the predecessor relation and its ancestral. He uses these to prove a series of results equivalent with the Peano-Dedekind axioms for number theory. He furthermore discusses the infinite cardinal number he writes as $\infty$, which is essentially Cantor's $\aleph_0$, and derives a number of a results about it, including that the extensions of concepts which have this cardinality are all and only those which can be ordered in a progression. He provides a general theory of series and lays the foundation for the possibility of definitions by recursion. Thanks largely to Crispin Wright's work [12], it is now known that very similar proofs of the Peano-Dedekind axioms can be given in a second-order logic without Frege's problematic Law V if one adopts Hume's Principle (the principle that the number belonging to concepts $F$ and $G$ are the same if and only if $F$ and $G$ can be put in 1–1 correspondence) as an axiom. This result has come to be known as "Frege's Theorem". Frege himself in effect derives Hume's Principle from Law V and proceeds from there, though he does at times make inessential further appeals to Law V. Closer examination of Part II of *Grundgesetze* may help answer the question as to whether or not Frege himself was explicitly aware of what he in effect had shown—-the possibility of basing number theory on Hume's Principle on its own—and if so, whether he would have considered it of any importance. (For work along these lines, see [6, chap. 2].)

Part III is entitled "The Real Numbers", and has so far been perhaps the least known and least discussed part of *Grundgesetze*. This is no doubt in part because Part III was left unfinished; indeed, Frege never even makes it to an informal, much less a technically precise, definition of real numbers. It is clear that he understands them as ratios (themselves a kind of extension of relations) between magnitudes (also themselves a kind of extension of relations). He sketches briefly the route whereby we are to understand what a magnitude is (by describing what it is for a concept to have an extension whose members can be arranged by magnitude), but never fully completes his definitions. The technical material near the end of the volume deals only with preliminary notions such as that of positival classes, their limits and positive classes. We are left to speculate how Frege might have finished his account (for such attempts, see, e.g., [11] and [2, chaps. 19–22]).

What is perhaps most striking about the published portion of Part III is that Frege devotes more than half of it, over 90 pages, to informal philosophical discussion and polemics, which stands in stark contrast to most of the rest of *Grundgesetze*. Of course, Frege had already published an informal account of his views of (finite and infinite) cardinal numbers in the *Grundlagen*, and criticized rival views. He had not done so for the theory of real numbers, and some philosophical discussion about different treatments of real numbers might not have been unexpected at this point. However, Frege seems instead to be seizing on the opportunity to criticize broadly competing developments in the philosophy of mathematics generally, including work by Weierstrass and Cantor. He saves his harshest criticisms for the "game formalism" found in the works of Eduard Heine and Frege's Jena colleague Carl Thomae. His attacks are long and often repetitive, and are seemingly out of place given that relatively little of the discussion involves real numbers in particular. They are also rather uncharitable; Frege often seizes upon imprecision in expression to argue that his opponents of guilty of serious confusion between words and their meanings. No doubt there is some validity to his criticisms, but nonetheless it is hard not to recognize in the tone of these discussions one of the possible reasons for the generally negative reception of Frege's work during his lifetime.

One portion of interest in this discussion comes in §§146–147 in which Frege contrasts his own understanding of how it is we come to recognize the existence of an object, or more specifically, a value-range, in common between functions having the same value for every argument, with other theorists understanding of "creative" definitions through which we "create" apparently "new" mathematical entities. There Frege insists that we must take our ability to recognize this common "object" as an axiom, and that this axiom—viz., Basic Law V—can in no way be regarded as a definition or as something brought about by a stipulative and/or creative act. Again, since Law V takes the form of an "abstraction principle", these passages deserve special attention for the light they may shed on his general attitude about such principles, which has been the subject of some controversy.

Both of the volumes have appendices containing tables of important theorems and definitions. The second volume has an additional afterword in which Frege responds to the discovery of the contradiction from Russell's paradox. Briefly, some concepts have value-ranges that fall under them; some do not. The concept *being a value-range of a concept which it does not fall under* has a value-range, and this value-range falls under it just in case it does not. Frege considers a more radical response of not regarding value-ranges as "objects" in the same sense as the possible arguments to the functions of which they are the value-ranges. He rejects this suggestion, calling into question the intelligibility of such "improper objects", and noting the great complications such distinctions would no doubt bring about, especially when going on to consider the need to differentiate concepts applicable to improper objects from regular concepts, and the status of *their* value-ranges, and so on. Frege blames the contradiction on his Law V, which in effect guarantees the existence of as many objects as functions on objects (mapping from object to object). However, for $n$ objects, there are $n^n$ possible mappings, and so long as as $n \geq 2$, $n^n > n$; hence there cannot be as many objects as mappings. Frege considers his value-range notation "$\grave{\epsilon}\phi(\epsilon)$" as standing for a second-level function mapping its argument function $\phi(\xi)$ to its value-range. If there are to be fewer objects than functions, and this notation is to be retained, it must not always be understood to yield different values for different (non-equivalent) functions as argument. Indeed, Frege proves a general theorem to the effect that for every second-level function $M_\beta$, there are first-level functions $F$ and $G$, which, when taken as argument to it, yield the same value, but are such that this shared value falls under the one but not the other. He proposes then that distinct functions $F$ and $G$ are to be understood as having the same value-range when they differ *only* in that this shared value-range falls under the one but not the other. An axiom to this effect, Law V′, is suggested as a replacement for Basic Law V. Unfortunately, it is now known (see [8, 10]) that Law V′ is also inconsistent in any system in which at least two objects can be proven to exist (including Frege's own system, which treats the two truth-values the True and the False as objects). It is not known whether or not Frege knew of the formal defect with his proposal, but in any case no further published work involving the revised system was to appear. Nonetheless, approaches in the vicinity of Frege's, allowing distinct (non-coextensive) concepts or functions to "share" a value-range or extension, have enjoyed some measure of success and interest (see, e.g., [1]).

The translation adds to Frege's own work a number of additional resources. There is a brief foreword by Crispin Wright discussing the history of the translation project. The editors/translators Ebert and Rossberg add a introduction discussing the history of *Grundgesetze* and elaborating upon some of their choices with regard to important technical terms. For those words for which various rival translations are possible, they seem overall to have opted for the term which has become most entrenched in the English secondary literature. For example, while a case could certainly be made for translating Frege's "Bedeutung" more simply as "meaning"—as has in fact been done in other translations (see [3, 5])—Ebert and Rossberg opt instead for "reference", given its popularity in the broader discussions of Frege's sense/reference distinction. Similarly, while I personally find Furth's choice of "course-of-values" for Frege's "Werthverlauf" more evocative, there can be little question that their choice of "value-range" is much more common in the literature. Their decisions overall are judicious. The translation as a whole is fairly close to the German, as I think it should be. They include afterwards a list of notes on particular passages where complications arose or something important is lost in translation. They also include a list of small corrections made to the German editions. A bibliography containing a list of Frege's own works and those cited by Frege follows. One might have hoped also for a list of important secondary works on Frege's foundational work, which is not found. (The editors, however, are working on editing a companion anthology which may include such a bibliography.) Finally, the book includes an excellent appendix written by Roy T. Cook entitled "How to Read *Grundgesetze*", which introduces the reader to Frege's logical system, making note of both its curious idiosyncrasies and undeniable charm. I can easily imagine this becoming *the* go-to resource for giving students what they need to know before delving into Frege's technical writings.

Working through the *Grundgesetze* as a whole reiterates a point made by Cook in the appendix: however unfamiliar Frege's notation may seem at first, its reputation for being obscure or unworkable is completely undeserved. No doubt it is typographically inconvenient and space-inefficient, but for more complicated propositions with a large number of subcomponents it avoids both the morass of brackets contemporary infix notation requires as well as as the unnaturalness and unreadability of Polish notation. Frege's exposition of his system is at times less precise than current conventions for introducing artificial languages. Nonetheless, compared to others of his time, and considering this was the first fully rigorously axiomatized logical calculus ever created, Frege's efforts must on the whole be judged as remarkably conscientious and thorough.

In light of its inconsistency, Frege's *Grundgesetze* must in *some way* be considered a kind of failure. Frege himself was one of the first to admit that he had not accomplished what he had set out to do. From a broader perspective, however, few works can be considered more successful. While Frege himself did not engage in logical meta-theory, he was the first to insist on the kind of rigorous formulation that would make such investigation possible. Frege directly influenced such philosopher-logicians as Russell, Carnap and Wittgenstein to develop their own foundational theories. Even Frege's failure has led to fruitful discussion about, as Dummett put it, "what led the serpent of inconsistency into paradise" [2, p. 209], and debates about various revisions of Law V, or of Frege's impredicative replacement rule for second-order variables, and so on. Indeed, it is impossible even to imagine what contemporary investigations into the logical foundations of mathematics would or could be like without the important influence Frege's *Grundgesetze* project has had on it. It may be a cautionary tale, but if so, it is one whose lessons will likely never lose their relevance. It is a delight finally to have a complete English translation, putting the various pieces together. It belongs on the bookshelf of any serious student of logic or the foundations or philosophy of mathematics. The timing of the release is fortuitous as it coincides closely with the release of helpful secondary guides (including, e.g., [7] and [9]). For those who have not yet made a serious study of Frege's magnum opus, there is indeed no better time.

## References


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