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On Paraconsistent Weak Kleene Logic: Axiomatisation and Algebraic Analysis

Abstract. Paraconsistent Weak Kleene logic (PWK) is the 3-valued logic with two designated values defined through the weak Kleene tables. This paper is a first attempt to investigate PWK within the perspective and methods of abstract algebraic logic (AAL). We give a Hilbert-style system for PWK and prove a normal form theorem. We examine some algebraic structures for PWK, called *involutive bisemilattices*, showing that they are distributive as bisemilattices and that they form a variety, $\mathcal{IBSL}$, generated by the 3-element algebra **WK**; we also prove that every involutive bisemilattice is representable as the Płonka sum over a direct system of Boolean algebras. We then study PWK from the viewpoint of AAL. We show that $\mathcal{IBSL}$ is not the equivalent algebraic semantics of any algebraisable logic and that PWK is neither protoalgebraic nor selfextensional, not assertional, but it is truth-equational. We fully characterise the deductive filters of PWK on members of $\mathcal{IBSL}$ and the reduced matrix models of PWK. Finally, we investigate PWK with the methods of second-order AAL—we describe the class $\mathsf{Alg}(\mathrm{PWK})$ of PWK-algebras, algebra reducts of basic full generalised matrix models of PWK, showing that they coincide with the quasivariety generated by **WK**—which differs from $\mathcal{IBSL}$—and explicitly providing a quasiequational basis for it.

Keywords: Paraconsistent Weak Kleene Logic, Three-valued logics, Bisemilattices, Płonka sums, Abstract algebraic logic.

1. Introduction

In his *Introduction to Metamathematics* [26, § 64], S.C. Kleene distinguishes between a "strong sense" and a "weak sense" of propositional connectives when partially defined predicates are present. Each of these meanings is made explicit via certain 3-valued truth tables, which have become widely known as *strong Kleene tables* and *weak Kleene tables*, respectively. If the elements of the base set are labelled as $0, {}^1\!/_2, 1$, the strong tables for conjunction, disjunction and negation are displayed below:

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$\wedge$	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$\neg$	
1	0
1/2	1/2
0	1

The weak tables for the same connectives, on the other hand, are given by:

$\wedge$	0	1/2	1
0	0	1/2	0
1/2	1/2	1/2	1/2
1	0	1/2	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1/2
1	1	1/2	1

$\neg$	
1	0
1/2	1/2
0	1

Each set of tables naturally gives rise to two options for building a many-valued logic, depending on whether we choose to consider only 1 as a designated value, or 1 together with the "middle" value 1/2. Thus, there are four logics in the Kleene family:[1]

- Strong Kleene logic [26, § 64], given by the strong Kleene tables with 1 as a designated value;
- The Logic of Paradox, LP [33], given by the strong Kleene tables with 1, 1/2 as designated values;
- Bochvar's logic [6], given by the weak Kleene tables with 1 as a designated value;
- Paraconsistent Weak Kleene logic, PWK [21,35], given by the weak Kleene tables with 1, 1/2 as designated values.

The first three logics have all but gone unnoticed by mathematicians, philosophers, and computer scientists. Strong Kleene logic has applications in artificial intelligence as a model of partial information [1] and nonmonotonic reasoning [39], and in philosophy as a bedrock logic for Kripke's theory of truth and other related proposals [14]; the theory of *Kleene algebras*, moreover, has stirred a considerable amount of interest in general algebra [27]. LP has been fervently supported by Graham Priest in the context of a dialetheic approach to the truth-theoretical and set-theoretical paradoxes, and has enjoyed an enduring popularity that made it the object of intense

[1] Here we treat the expression "Kleene family" informally and we do not intend to be exhaustive. There are other logics that could also be considered within the family of Kleene logics, defined by using two or more of these matrices (see for instance [16]).

study both on the proof-theoretical and on the semantical level [34]. And even Bochvar's logic, while not the biggest game in the 3-valued town, is still touched on in several papers and books (see e.g. [4, Ch. 5]).

In terms of sheer impact, PWK is the "ugly duckling" in the family of Kleene logics. Essentially introduced by Halldén [21] and, in a completely independent way, by Prior [35], it is often passed over in silence in the main reviews on finite-valued logics. Most of the extant literature concerns the philosophical interpretation of the third value [3,7,13,21,40] and a discussion of the so-called *contamination principle* (any sentence containing a subsentence evaluated at $1/2$ is itself evaluated at $1/2$), as well as proof systems of various kinds [3,10,11,15]. An important study on PWK as a consequence relation is [9], to be analysed later in this paper. It has also been noticed early on that the negation and constant-free reduct of the 3-element algebra **WK**, which is defined by the weak Kleene tables, is an instance of a *distributive bisemilattice*, a notion on which there is a burgeoning literature (see the references in Section 3.2 below)—actually, the variety of distributive bisemilattices is generated by this reduct. Yet, despite this intriguing connection to algebra, virtually no paper has viewed PWK in the perspective of Algebraic Logic.[2] A partial exception is [15], but a careful assessment of the results in this paper is made difficult by issues with the similarity type of the algebras and logics it considers, and by the authors' failure to adopt the language and concepts of mainstream abstract algebraic logic (AAL).

The aim of this paper is to give a contribution towards filling this gap, so as to surmise that the ugly duckling might actually be a gorgeous swan. In Section 2 we introduce PWK formally, give a Hilbert-style system for it, and prove a normal form theorem. In Section 3 we provide our readers with the necessary background on the important algebraic construction of Płonka sums and on bisemilattices. In Section 4 we examine some algebraic structures for PWK, called *involutive bisemilattices*. Among other results, we show that involutive bisemilattices are always distributive as bisemilattices, that **WK** generates the variety $\mathcal{IBSL}$ of involutive bisemilattices, and that every involutive bisemilattice is representable as the Płonka sum over a direct system of Boolean algebras; moreover, we axiomatise relative to $\mathcal{IBSL}$ its nontrivial subvarieties, namely, Boolean algebras and lower-

[2] This makes a sharp contrast with LP, which has been thoroughly studied under this aspect [36,38].

bounded semilattices. Finally, in Section 5, we study PWK by recourse to the toolbox of AAL. We show that $\mathcal{IBSL}$ is not the equivalent algebraic semantics of any algebraisable logic and that PWK is neither protoalgebraic nor selfextensional, nor assertional, but it is truth-equational. We fully characterise the deductive filters of PWK on members of $\mathcal{IBSL}$ and the reduced matrix models of PWK. Finally, we investigate PWK with the methods of second-order AAL—we describe the intrinsic variety of PWK, $\mathbb{V}$(PWK), and the classes Alg^*(PWK) and Alg(PWK). We prove that $\mathsf{Alg}^*(\mathrm{PWK}) \subsetneq \mathsf{Alg}(\mathrm{PWK}) \subsetneq \mathbb{V}(\mathrm{PWK}) = \mathcal{IBSL}$, Alg^*(PWK) is not a generalised quasivariety, while Alg(PWK) is the quasivariety generated by **WK**, and explicitly provide a quasiequational basis for it.

2. Paraconsistent Weak Kleene Logic

We start by fixing some terms and notation. Given a similarity type ν, the absolutely free algebra $\mathbf{Fm}$ of type ν over a countably infinite set X of generators will be called the *formula algebra* of type ν; its members will be equivalently called ν-*terms* or ν-*formulas* (with ν suppressed when clear from the context) and referred to by the symbols $t, s, \ldots$ or $\alpha, \beta, \ldots$ Members of X will be called (propositional) *variables* and referred to by the symbols $x, y, \ldots$ or $p, q, \ldots$ Ordered pairs of ν-formulas are called ν-*equations*, and will be written in the form $\alpha \approx \beta$ instead of $\langle \alpha, \beta \rangle$. A *logic* of type ν is a pair $\mathrm{L} = \langle \mathbf{Fm}, \vdash_{\mathrm{L}} \rangle$, where $\mathbf{Fm}$ is the formula algebra of type ν, and $\vdash_{\mathrm{L}}$ is a substitution-invariant consequence relation over $\mathbf{Fm}$.

2.1. Preliminaries

As we hinted in our introduction, Paraconsistent Weak Kleene logic can be semantically defined as the logic $\mathrm{PWK} = \langle \mathbf{Fm}_1, \vDash_{\mathsf{PWK}} \rangle$, where:

- $\mathbf{Fm}_1$ is the formula algebra of type $(2, 2, 1, 0, 0)$, namely, of the type containing the connectives $\wedge, \vee, \neg, 0, 1$;
- PWK is the matrix[3] $\langle \mathbf{WK}, \{1, 1/2\} \rangle$, where $\mathbf{WK}$ is the algebra whose universe is $\{1, 1/2, 0\}$ and whose operations are given by the following tables:

[3] The concepts of *logical matrix* and *matrix consequence* are defined in Section 5.1.

$\wedge$	0	$1/2$	1
0	0	$1/2$	0
$1/2$	$1/2$	$1/2$	$1/2$
1	0	$1/2$	1

$\vee$	0	$1/2$	1
0	0	$1/2$	1
$1/2$	$1/2$	$1/2$	$1/2$
1	1	$1/2$	1

$\neg$	
1	0
$1/2$	$1/2$
0	1

Thus, for every $\Gamma \cup \{\alpha\} \subseteq Fm_1$,

$$\begin{aligned}\Gamma \vDash_{\mathbf{PWK}} \alpha \iff & \text{ for every valuation } v \text{ on } \mathbf{WK}, \\ & v[\Gamma] \subseteq \{1/2, 1\} \text{ implies } v(\alpha) \in \{1/2, 1\}.\end{aligned}$$

Since **PWK** is a finite matrix, the logic PWK is *finitary*, i.e., $\Gamma \vDash_{\mathbf{PWK}} \alpha$ if and only if there is a finite subset $\Delta \subseteq \Gamma$ such that $\Delta \vDash_{\mathbf{PWK}} \alpha$. Moreover, the matrix $\langle \mathbf{B}_2, \{1\}\rangle$, where $\mathbf{B}_2$ is the 2-element Boolean algebra, is a submatrix of **PWK**, and therefore PWK is included in classical propositional logic, CL. In spite of the fact that PWK is actually weaker than CL, as we will see, both logics have the same theorems. Indeed, if $\nvDash_{\mathbf{PWK}} \alpha$, then there is a valuation v on $\mathbf{WK}$ such that $v(\alpha) = 0$. Without loss of generality, we can assume that v sends the variables that are not in α to 0. By looking at the tables, it is easy to see that for every variable p in α, $v(p) \neq 1/2$, whence v is actually a valuation on $\mathbf{B}_2$ such that $v(\alpha) = 0$. Thus, $\nvdash_{\mathrm{CL}} \alpha$.

PWK has been thoroughly investigated by Ciuni and Carrara in [9], where a characterisation theorem for PWK is proved.[4]

THEOREM 1 ([9, Thm. 14]). *For all $\Gamma \cup \{\alpha\} \subseteq Fm_1$, we have that $\Gamma \vDash_{\mathbf{PWK}} \alpha$ if and only if there is a finite (possibly empty) subset $\Delta \subseteq \Gamma$ such that* $\mathrm{var}(\Delta) \subseteq \mathrm{var}(\alpha)$ *and* $\Delta \vdash_{\mathrm{CL}} \alpha$.

The preceding theorem has this immediate consequence:

COROLLARY 2. *If α and β are two formulas of Fm_1 such that* $\mathrm{var}(\alpha) = \mathrm{var}(\beta)$, *then*

$$\alpha \dashv\vDash_{\mathbf{PWK}} \beta \iff \alpha \dashv\vdash_{\mathrm{CL}} \beta.$$

2.2. An Axiomatisation of PWK

Existing proof systems for PWK or for some of its linguistic variants include sequent calculi [11] and tableaux systems [3,10,15]. In what follows, we present a more traditional Hilbert system for this logic. More precisely, we introduce a new logic HPWK by means of a Hilbert-style calculus and then

[4]For special cases of the next Theorem, see also [29] or [11, Thm. 8].

show that HPWK exactly coincides with PWK. Throughout the rest of this paper, $\alpha \to \beta$ is used as shorthand for $\neg\alpha \vee \beta$.

Definition 3. HPWK is the logic $\langle \mathbf{Fm}_1, \vdash_{\mathrm{HPWK}} \rangle$, where $\vdash_{\mathrm{HPWK}}$ is the derivability relation of the deductive system with the following axioms and inference rules:

A1. $(\alpha \vee \alpha) \to \alpha$;

A2. $\alpha \to (\alpha \vee \beta)$;

A3. $(\alpha \vee \beta) \to (\beta \vee \alpha)$;

A4. $(\alpha \to \beta) \to ((\gamma \vee \alpha) \to (\gamma \vee \beta))$;

A5. $(\alpha \wedge \beta) \to \neg(\neg\alpha \vee \neg\beta)$;

A6. $\neg(\neg\alpha \vee \neg\beta) \to (\alpha \wedge \beta)$;

A7. $\alpha \to 1$;

A8. $0 \to \alpha$;

$$[\mathrm{RMP}]\ \frac{\alpha \qquad \alpha \to \beta}{\beta} \qquad \text{provided that } \mathrm{var}(\alpha) \subseteq \mathrm{var}(\beta).$$

Henceforth, we ambiguously use HPWK both for the logic we just defined and for the Hilbert calculus that yields its consequence relation. Notice that the only difference between HPWK and CL is the proviso that constrains Modus Ponens (RMP means *Restricted Modus Ponens*).[5] It is immediate to check that PWK fails to satisfy Modus Ponens but satisfies this restricted version. This is in contrast with LP, where not even RMP is valid, since $p, p \to (p \wedge q) \nvDash_{\mathrm{LP}} p \wedge q$. Because of this inferential restriction, HPWK turns out to be weaker than CL. Nevertheless, we prove in the next proposition that both logics have the same theorems.

Proposition 4. *For any $\alpha \in Fm_1$, $\vdash_{\mathrm{HPWK}} \alpha$ if and only if $\vdash_{\mathrm{CL}} \alpha$.*

Proof. If $\vdash_{\mathrm{HPWK}} \alpha$, then there is a proof D of α that uses only the axioms (A1)–(A8) and RMP. Since (A1)–(A8) are classical theorems, and RMP is an instance of the usual Modus Ponens, D also counts as a proof of α in the deductive system for CL given by (A1)–(A8) and Modus Ponens.

[5] Actually, there is nothing special about our choice of the axioms (A1)–(A8)—we could have picked any other set of axioms that, together with Modus Ponens, yields a complete Hilbert system for CL, with the caveat that the working language is $\wedge, \vee, \neg, 0, 1$; see [22], [23], or [24] concerning the importance of the language when choosing a certain set of axioms for a particular logic.

All we have to prove now is that we can transform any proof $D = \langle \alpha_1, \dots, \alpha_n \rangle$ of α in this deductive system for CL into another proof that only uses RMP. We will proceed by induction on the length n of D. If $n = 1$, then $\alpha = \alpha_1$ is an axiom, and there is nothing to prove. Let us now assume that $n > 1$, and that $\alpha = \alpha_n$ follows from α_i and $\alpha_j = \alpha_i \to \alpha$ by Modus Ponens, with $i, j < n$. By the induction hypothesis, we have proofs of α_i and α_j in HPWK. Let $\langle \beta_1, \dots, \beta_m \rangle$ be the result of concatenating these two proofs, in such a way that $\beta_s = \alpha_i$ and $\beta_m = \alpha_j = \alpha_i \to \alpha = \beta_s \to \alpha$. This is still a proof in HPWK. Consider a substitution σ fixing all the variables of α and sending all the variables of $\beta_1, \dots, \beta_m$ that are not contained among the variables of α to some particular variable of α (or to 0, if α has no variable). Thus, $\langle \sigma(\beta_1), \dots, \sigma(\beta_m) \rangle$ is still a proof in HPWK. Now, $\sigma(\beta_m) = \sigma(\beta_s \to \alpha) = \sigma(\beta_s) \to \sigma(\alpha) = \sigma(\beta_s) \to \alpha$, since σ fixes the variables of α. Moreover, $\mathrm{var}(\sigma(\beta_s)) \subseteq \mathrm{var}(\alpha)$, by our choice of σ. Therefore, α follows by an application of RMP to $\sigma(\beta_s)$ and $\sigma(\beta_m)$, and thus $\langle \sigma(\beta_1), \dots, \sigma(\beta_m), \alpha \rangle$ is a proof of α in HPWK. ■

One can readily see that PWK fails conjunction elimination in general and, in particular, absorption, i.e., $\alpha \wedge (\alpha \vee \beta) \nvDash_{\mathrm{PWK}} \alpha$. Nonetheless, we can derive in HPWK certain linguistic restrictions of conjunction elimination and a further weak form of Modus Ponens. All these features of the logic have algebraic counterparts, as we will see in Section 4.

PROPOSITION 5. *The following rules are derivable in* HPWK*:*

$$[\wedge E_1]\ \frac{\alpha \wedge \beta}{\alpha} \quad \textit{if } \mathrm{var}(\beta) \subseteq \mathrm{var}(\alpha); \qquad [\wedge I]\ \frac{\alpha \quad \beta}{\alpha \wedge \beta}\ ;$$

$$[\wedge E_2]\ \frac{\alpha \wedge \beta}{\beta} \quad \textit{if } \mathrm{var}(\alpha) \subseteq \mathrm{var}(\beta); \qquad [\mathrm{WMP}]\ \frac{\alpha \wedge (\alpha \to \beta)}{\alpha \wedge \beta}\ .$$

PROOF. If $\mathrm{var}(\beta) \subseteq \mathrm{var}(\alpha)$, then

$$[\mathrm{RMP}]\ \dfrac{\alpha \wedge \beta \qquad [\text{Prop. } 4]\ \dfrac{\vdots}{\alpha \wedge \beta \to \alpha}}{\alpha}$$

This shows that $\wedge E_1$ is derivable, and the proof for $\wedge E_2$ is similar. For $\wedge I$, we have:

$$
\dfrac{\beta \qquad \dfrac{\alpha \qquad \dfrac{\vdots}{\alpha \to (\beta \to \alpha \wedge \beta)}\,[\text{Prop. } 4]}{\beta \to \alpha \wedge \beta}\,[\text{RMP}]}{\alpha \wedge \beta}\,[\text{RMP}]
$$

The derivability of WMP is a straightforward consequence of the fact that $(\alpha \wedge (\alpha \to \beta)) \to (\alpha \wedge \beta)$ is a theorem of CL and $\text{var}(\alpha \wedge (\alpha \to \beta)) = \text{var}(\alpha \wedge \beta)$. ■

Notice that, in the presence of $\wedge I$, WMP is indeed a weak form of Modus Ponens. The consequent of WMP cannot be replaced by β, since this would imply an unrestricted conjunction elimination. Actually, a kind of converse to the previous proposition is available, whence the axioms (A1)–(A8) and rules $\wedge I$, WMP, $\wedge E_2$ provide an alternate axiomatisation of HPWK.

PROPOSITION 6. *The rule RMP is derivable in any deductive system including the rules $\wedge I$, WMP and $\wedge E_2$.*

PROOF. If $\text{var}(\alpha) \subseteq \text{var}(\beta)$, RMP is derivable as follows:

$$
[\wedge E_2]\ \dfrac{[\text{WMP}]\ \dfrac{[\wedge I]\ \dfrac{\alpha \qquad \alpha \to \beta}{\alpha \wedge (\alpha \to \beta)}}{\alpha \wedge \beta}}{\beta}
$$
■

We are now ready to show that HPWK yields a Hilbert-style axiomatisation of PWK.

THEOREM 7. HPWK = PWK.

PROOF. It is easy to check, by direct inspection, that for every axiom α of HPWK and for every valuation v on **WK**, $v(\alpha) \in \{1, 1/2\}$. Moreover, if $\text{var}(\alpha) \subseteq \text{var}(\beta)$ and v is a valuation such that $v(\beta) = 0$, then $v(p) \neq 1/2$ for every $p \in \text{var}(\beta)$, and hence for every $p \in \text{var}(\alpha)$. Therefore, $v(\alpha) \in \{0, 1\}$. Thus, if $v(\alpha) = 1$, then $v(\alpha \to \beta) = 0$. This proves that the rule RMP is sound with respect to $\vDash_{\mathbf{PWK}}$. Therefore, $\Sigma \vdash_{\text{HPWK}} \alpha$ implies $\Sigma \vDash_{\mathbf{PWK}} \alpha$.

For the other direction, suppose that $\Sigma \vDash_{\mathbf{PWK}} \alpha$. By Theorem 1, there is a finite subset $\Delta \subseteq \Sigma$ such that $\Delta \vdash_{\text{CL}} \alpha$ and $\text{var}(\Delta) \subseteq \text{var}(\alpha)$. If $\Delta = \emptyset$, then $\vdash_{\text{HPWK}} \alpha$, by virtue of Proposition 4, whereby we get $\Sigma \vdash_{\text{HPWK}} \alpha$. Otherwise, let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. By the Deduction Theorem, $\vdash_{\text{CL}} \alpha_1 \to (\alpha_2 \to (\cdots \to (\alpha_n \to \alpha) \cdots))$. Thus, $\vdash_{\text{HPWK}} \alpha_1 \to (\alpha_2 \to (\cdots \to (\alpha_n \to \alpha) \cdots))$, by Proposition 4, and since $\text{var}(\alpha_1, \ldots, \alpha_n) \subseteq \text{var}(\alpha)$, by several applications of RMP we obtain $\{\alpha_1, \ldots, \alpha_n\} \vdash_{\text{HPWK}} \alpha$, so $\Sigma \vdash_{\text{HPWK}} \alpha$. ■

Hereafter, the label PWK will refer to the logic that is the object of Theorem 7, irrespective of its syntactic or semantic characterisation, and $\vdash_{\mathrm{PWK}}$ will be used as a symbol for its consequence relation.

To round off this section, we establish a normal form theorem for PWK. As usual, a *literal* is either a variable or the negation of such, and a *disjunctive clause* is a (finite) disjunction of literals. The Conjunctive Normal Form (CNF) Theorem for CL ensures that every formula is interderivable with a conjunction of disjunctive clauses. By using the deduction theorem for CL and the rule RMP, we can prove that such a theorem holds for PWK as well. But this result is of little avail, since in PWK conjunction does not simplify and then we cannot replicate, the classical proof that leads from the CNF theorem to the fact that every formula is interderivable with the *set* of the disjunctive clauses of its conjunctive normal form. To make some headway, we need to relax the notion of clause in such a way as to take good care of the variables involved. An *elementary contradiction* is a formula of the form $p \wedge \neg p$, where p is a variable. A *clause* is a finite disjunction of literals and elementary contradictions.

Proposition 8. *Every formula α with variables is interderivable in* PWK *with a conjunction of clauses $\beta_1, \ldots, \beta_n$ such that, for every i,* $\mathrm{var}(\beta_i) = \mathrm{var}(\alpha)$. *Moreover, α is interderivable in* PWK *with the set of the clauses of its conjunctive normal form, that is*

$$\alpha \dashv\vdash_{\mathrm{PWK}} \{\beta_1, \ldots, \beta_n\}.$$

Proof. Let $\gamma_1 \wedge \cdots \wedge \gamma_n$ be a conjunctive normal form of α in CL. Thus, every γ_i is a finite disjunction of literals. The classical theorem also ensures that the variables of the normal form are the same as the variables of α. For every i, let $\beta_i = \gamma_i \vee (p_{i1} \wedge \neg p_{i1}) \vee \cdots \vee (p_{im_i} \wedge \neg p_{im_i})$, where $\{p_{i1}, \ldots, p_{im_i}\}$ is the set of variables of α that are missing in γ_i. It is obvious that $\beta_i \dashv\vdash_{\mathrm{CL}} \gamma_i$, and therefore $\alpha \dashv\vdash_{\mathrm{CL}} \beta_1 \wedge \cdots \wedge \beta_n$. Thus, since $\mathrm{var}(\alpha) = \mathrm{var}(\beta_1 \wedge \cdots \wedge \beta_n)$, by Corollary 2 we have that $\alpha \dashv\vdash_{\mathrm{PWK}} \beta_1 \wedge \cdots \wedge \beta_n$. Finally, since the variables of every clause are always the same, several applications of $\wedge E_1$, $\wedge E_2$, and $\wedge I$ would give us that $\beta_1 \wedge \cdots \wedge \beta_n \dashv\vdash_{\mathrm{PWK}} \{\beta_1, \ldots, \beta_n\}$, and we are done with our proof. ■

3. Płonka Sums and Bisemilattices

3.1. Płonka Sums

In the late 1960's, J. Płonka [30,32] devised a powerful construction whereby one can represent algebras meeting particular requirements as sums of a certain kind of algebras with stronger properties, in such a way as to enable transfer of important information from the summands onto the sum. A Płonka sum of a system of algebras is, in a sense, a fibration indexed by a join semilattice and whose fibres are the algebras of the system. Thus, one can think of it as a new algebra within which the algebras of the system live separately. Here, we briefly abridge the main ingredients of this method.

A *direct system* of algebras consists of a compatible family of homomorphism of algebras of the same type indexed by a join semilattice,[6] that is, a pair $\mathtt{T} = \langle(\varphi_{ij} : i \leqslant j), \mathbf{I}\rangle$ such that:

1. $\mathbf{I} = \langle I, \leqslant\rangle$ is a join semilattice;
2. $\varphi_{ij} : \mathbf{A}_i \to \mathbf{A}_j$ is a homomorphism, for each $i \leqslant j$, satisfying that φ_{ii} is the identity in $\mathbf{A}_i$ and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$;
3. If $i, j \in I$ are different, then $\mathbf{A}_i$ and $\mathbf{A}_j$ are disjoint.[7]

If $\mathtt{T} = \langle(\varphi_{ij} : i \leqslant j), \mathbf{I}\rangle$ is a direct system of algebras of type ν, then the *Płonka sum* over $\mathtt{T}$ is the algebra $\mathbf{T} = \langle\bigcup_I A_i, \{g^{\mathbf{T}} : g \in \nu\}\rangle$, where each $g^{\mathbf{T}}$ is defined as follows: for every n-ary $g \in \nu$, and $a_1, \ldots, a_n \in T$, where $n \geqslant 1$ and $a_r \in A_{i_r}$, we set $j = i_1 \vee \cdots \vee i_n$ and define

$$g^{\mathbf{T}}(a_1, \ldots, a_n) = g^{\mathbf{A}_j}(\varphi_{i_1 j}(a_1), \ldots, \varphi_{i_n j}(a_n)).$$

In case ν contains constants, we need to assume that $\mathbf{I}$ is actually a semilattice with a bottom element $\perp$. In that case, for every constant $g \in \nu$, we define $g^{\mathbf{T}} = g^{\mathbf{A}_\perp}$.

Example 9. Let $\mathbf{A}_i = \mathbf{B}_2$ (the 2-element Boolean algebra), and let $\mathbf{A}_j$ and $\mathbf{A}_k$ be isomorphic copies of the 4-element Boolean algebra, with their elements labelled as follows:

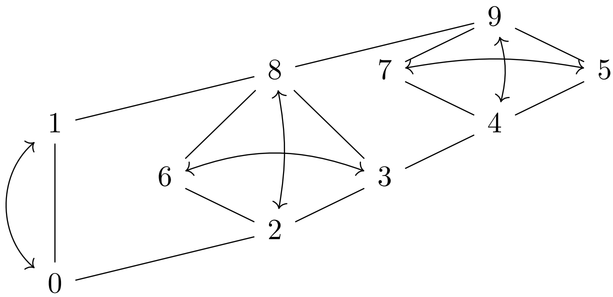
$$\mathbf{A}_j = \begin{tikzcd}[column sep=small] & 8 & \\ 6 \arrow[ur, dash] \arrow[dr, dash] & & 3 \arrow[ul, dash] \arrow[dl, dash] \\ & 2 & \end{tikzcd} \qquad \mathbf{A}_k = \begin{tikzcd}[column sep=small] & 9 & \\ 7 \arrow[ur, dash] \arrow[dr, dash] & & 5 \arrow[ul, dash] \arrow[dl, dash] \\ & 4 & \end{tikzcd}$$

Let $\mathbf{I}$ be the partial order with three elements $i < j < k$, and $\mathtt{T}$ the directed system over $\mathbf{I}$ in which the only nontrivial homomorphism is $\varphi_{jk} : \mathbf{A}_j \to \mathbf{A}_k$, given by $\varphi_{jk}(3) = 4$ (and therefore $\varphi_{jk}(6) = 9$). Hence the Płonka sum $\mathbf{T}$ over this direct system is given by the following diagram, where arrows

[6] Actually, direct systems are in general defined over posets with the property that any two elements have an upper bound, but we are interested only in those that are indexed by a join semilattice.

[7] This is just a technical requirement that simplifies the proofs, but not a true restriction.

describe the behaviour of negation and the other lines describe the order given by the operation $\vee^{\mathbf{T}}$:



It turns out that Płonka sums are importantly related to the presence of certain functions on the algebras that are to be represented. Let in fact $\mathbf{A}$ be a ν-algebra. A function $f : A^2 \to A$ is a *partition function* in $\mathbf{A}$ if the following conditions are satisfied for all $a, b, c, a_i \in A$ and for all $g \in \nu$ of arity $n \geqslant 1$, and every constant[8] $h \in \nu$:

P1. $f(a, a) = a$;

P2. $f(a, f(b, c)) = f(a, f(c, b)) = f(f(a, b), c)$;

P3. $f(g(\overline{a}), b) = g(f(a_1, b), \ldots, f(a_n, b))$;

P4. $f(b, g(\overline{a})) = f(b, g(f(b, a_1), \ldots, f(b, a_n)))$;

P5. $g(\overline{a}) = f(g(\overline{a}), a_i)$ for $1 \leqslant i \leqslant n$;

P6. $f(a, g(a, \ldots, a)) = a$;

P7. $f(a, h^{\mathbf{A}}) = a$.

If $\mathbf{A}$ is such that f is a partition function in $\mathbf{A}$, the relation $\theta \subseteq A^2$ defined as follows:

$$a \,\theta\, b \iff f(a, b) = a \text{ and } f(b, a) = b,$$

is an equivalence on A such that if $(A_i : i \in I)$ is the family of the equivalence classes in A/θ, then for every $i \in I$, A_i is the universe of an algebra of the same type—called a *Płonka fibre* of $\mathbf{A}$—in which all the nonnullary operations are defined as the restrictions of the corresponding ones of $\mathbf{A}$, and for every constant $h \in \nu$, $h^{\mathbf{A}_i} = f(h^{\mathbf{A}}, b)$, where $b \in A_i$ is an arbitrary

[8]The solution suggested by Płonka in [30] of considering constants as unary operations actually does not work, and a slight modification of the notion of a *partition function* is needed in order to accommodate algebras with constants in their types. Indeed, adding condition (P7) to Płonka's definition solves the problem.

element. Upon defining, for $i, j \in I$, $i \leqslant j$ if and only if there exist $a \in A_i, b \in A_j$ such that $f(b,a) = b$, and $\varphi_{ij}(a) = f(a,b)$, where b is arbitrary in the fibre A_j, $\mathbf{I} = \langle I, \leqslant \rangle$ becomes a join semilattice, the pair

$$\mathtt{T} = \langle (\varphi_{ij} : i \leqslant j), \mathbf{I} \rangle$$

a direct system, and the Płonka sum over $\mathtt{T}$ is a representation of $\mathbf{A}$. Thus, if $\mathcal{V}$ is a variety of type ν, t is a binary term of the same type, $\mathbf{A} \in \mathcal{V}$ and

$$\mathcal{V}' = \mathsf{Mod}(\mathrm{Eq}(\mathcal{V}) \cup \{t(x,y) \approx x\}),$$

whenever the term operation $t^{\mathbf{A}}$ is a partition function in $\mathbf{A}$, the above construction yields a Płonka sum representation of $\mathbf{A}$, whose fibres belong to $\mathcal{V}'$. Conversely, it can be checked that if $\mathbf{A}$ is isomorphic to the Płonka sum over a direct system of algebras from $\mathcal{V}'$, $t^{\mathbf{A}}$ is a partition function in $\mathbf{A}$. In sum, we obtain:

Theorem 10 ([30, Thm. III]). *Let $\mathcal{V}$ be a variety of type ν, let t be a binary term of the same type, and let $\mathbf{A} \in \mathcal{V}$. Moreover, let*

$$\mathcal{V}' = \mathsf{Mod}(\mathrm{Eq}(\mathcal{V}) \cup \{t(x,y) \approx x\}).$$

Then $t^{\mathbf{A}}$ is a partition function in $\mathbf{A}$ if and only if $\mathbf{A}$ is isomorphic to the Płonka sum over a direct system of algebras from $\mathcal{V}'$.

Recall that an equation $t \approx s$ in a given type is *regular* if the terms t and s have the same variables. We make a note of the following remarkable property of Płonka sums.

Lemma 11 ([30, Thm. I]). *Let $\mathbf{T}$ be the Płonka sum over a direct system $\mathtt{T} = \langle (\varphi_{ij} : i \leqslant j), \mathbf{I} \rangle$.*

1. *If a regular equation $t \approx s$ is satisfied in every algebra of the system, it is also satisfied in $\mathbf{T}$.*
2. *If an equation $t \approx s$ is satisfied in $\mathbf{T}$, then it is satisfied in every algebra of the system. Moreover, if $\mathbf{I}$ has at least two elements, then $t \approx s$ is a regular equation.*

3.2. Bisemilattices

Bisemilattices (also called *quasi-lattices* in the literature) were introduced by J. Płonka in [31] as a common generalisation of lattices and semilattices, and also as the motivating example for his direct system construction of the previous subsection. Over the years, they attracted the attention of such algebraists as Balbes, Kalman, and Romanowska. Outstanding papers about bisemilattices include [2,20,25,28].

A *bisemilattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee \rangle$ of type $(2,2)$ such that the reducts $\langle A, \wedge \rangle$ and $\langle A, \vee \rangle$ are semilattices. It is called *distributive* in case $\wedge$ distributes over $\vee$ and $\vee$ distributes over $\wedge$. A bisemilattice, in other words, falls short of being a lattice in that the absorption identities may fail. It is readily seen that the negation and constant-free reduct $\mathbf{WK}_0$ of the algebra $\mathbf{WK}$ is a distributive bisemilattice, and clearly every (distributive) lattice is a (distributive) bisemilattice. Semilattices can be identified with bisemilattices satisfying the identity $x \wedge y \approx x \vee y$, and of course they are distributive.

The variety of bisemilattices will be denoted by $\mathcal{BSL}$, while $\mathcal{DBSL}$ will refer to the distributive subvariety of $\mathcal{BSL}$. Hereafter, we will mainly focus on $\mathcal{DBSL}$, although some of the results that follow also hold for the larger variety $\mathcal{BSL}$.

Although full absorption fails to hold in $\mathcal{DBSL}$, appropriate restrictions of this principle are indeed satisfied:

Lemma 12. *Every distributive bisemilattice satisfies the following identities:*

$$x \vee y \vee (x \wedge y) \approx x \vee y;$$
$$x \wedge y \wedge (x \vee y) \approx x \wedge y;$$
$$x \vee (x \wedge y) \vee (y \wedge z) \approx x \vee (y \wedge z);$$
$$x \wedge (x \vee y) \wedge (y \vee z) \approx x \wedge (y \vee z).$$

Recall that a *left normal band* is an idempotent semigroup satisfying the identity $xyz \approx xzy$. Upon defining $x \odot y = x \vee (x \wedge y)$, every $\mathbf{A} \in \mathcal{DBSL}$ has a left normal band term reduct $\langle A, \odot \rangle$. Moreover, in every distributive bisemilattice $\mathbf{A} = \langle A, \wedge, \vee \rangle$, the relations

$$a \leqslant b \iff a \vee b = b \qquad \text{and} \qquad a \leqslant' b \iff a \wedge b = a$$

are both semilattice orderings of $\mathbf{A}$. These partial orders coincide if and only if $\mathbf{A}$ is a distributive lattice, and are dual to each other if and only if $\mathbf{A}$ is a semilattice.

The variety $\mathcal{DBSL}$ is generated by the 3-element algebra $\mathbf{WK}_0$. In fact:

Theorem 13. [25] *The only nontrivial subdirectly irreducible distributive bisemilattices are:*

- $\mathbf{WK}_0$*;*
- *the 2-element distributive lattice* $\mathbf{D}_2$*;*
- *the 2-element semilattice* $\mathbf{S}_2$*.*

Since $\mathbf{D}_2, \mathbf{S}_2 \leqslant \mathbf{WK}_0$*, then* $\mathcal{DBSL} = \mathbb{V}(\mathbf{WK}_0)$*.*

Thus, the only nontrivial subvarieties of $\mathcal{DBSL}$ are the variety $\mathcal{DL} = \mathbb{V}(\mathbf{D}_2)$ of distributive lattices, axiomatised relative to $\mathcal{DBSL}$ by the equation $x \odot y \approx x$, and the variety $\mathcal{SL} = \mathbb{V}(\mathbf{S}_2)$ of semilattices, axiomatised relative to $\mathcal{DBSL}$ by the equation $x \wedge y \approx x \vee y$, or equivalently by $x \odot y \approx y \odot x$.

If $\mathbf{A}$ is a distributive bisemilattice, $g(a, b) = a \odot b$ is a partition function in $\mathbf{A}$, whence Theorem 10 applies and we have that:

THEOREM 14 ([31, Thm. 3]). *Any distributive bisemilattice is isomorphic to the Płonka sum over a direct system of distributive lattices.*

Recalling the concept of regular identity from the previous subsection, Theorem 14 and Lemma 11 readily imply that:

THEOREM 15. *$\mathcal{DBSL}$ is the variety satisfying exactly the regular* $(2, 2)$-*identities satisfied by $\mathcal{DL}$.*

4. Involutive Bisemilattices

In this section, we attempt to identify a suitable candidate to play the role of an algebraic counterpart of the logic PWK. The first obvious *desideratum* that any such class of algebras has to meet is the presence, in its type, of an operation symbol for *negation*. In [15] Finn and Grigolia, and independently Brzozowski in [8], introduce, under the same name of *De Morgan bisemilattices*, an expansion of $\mathcal{DBSL}$ by an involutive negation operation that obeys the De Morgan laws. It turns out that this concept is too weak in several respects for our needs, for it does not retain enough of the Boolean structure of **WK**. As a consequence, here we go for a stronger notion—called *involutive bisemilattice*—that satisfies an additional constraint, namely I6, an algebraic counterpart of weak Modus Ponens, WMP (see Proposition 5).

4.1. Definition and Basic Results

DEFINITION 16. An *involutive bisemilattice* is an algebra $\mathbf{B}=\langle B, \wedge, \vee, \neg, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ satisfying:

I1. $x \vee x \approx x$;

I2. $x \vee y \approx y \vee x$;

I3. $x \vee (y \vee z) \approx (x \vee y) \vee z$;

I4. $\neg\neg x \approx x$;

I5. $x \wedge y \approx \neg(\neg x \vee \neg y)$;

I6. $x \wedge (\neg x \vee y) \approx x \wedge y$;

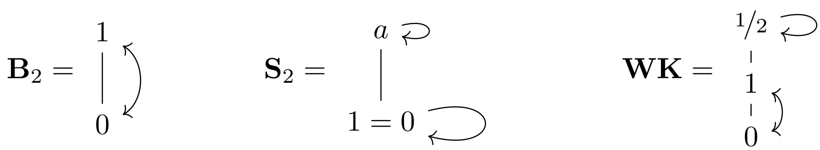
I7. $0 \vee x \approx x$;

I8. $1 \approx \neg 0$.

Thus, the class of involutive bisemilattices is an equational class, which we denote by $\mathcal{IBSL}$.

One can readily see that every involutive bisemilattice has, in particular, the structure of a join semilattice with zero, by virtue of axioms (I1)–(I3) and (I7). More than that, the negation and constant-free reduct of an arbitrary involutive bisemilattice is a bisemilattice, whence the label we have chosen is not a misnomer. Notice that, by virtue of axioms (I5) and (I8), the operations $\wedge$ and 1 are completely determined by $\vee$, $\neg$, and 0.

Example 17. Every Boolean algebra, in particular the 2-element Boolean algebra $\mathbf{B}_2$, is an involutive bisemilattice. Also, the 2-element semilattice with zero, which we continue to call $\mathbf{S}_2$ with a slight abuse, endowed with identity as its unary fundamental operation, is an involutive bisemilattice. But our example of primary interest is the algebra $\mathbf{WK}$. Upon considering the partial order $\leqslant$ induced by join in its bisemilattice reduct, it becomes a 3-element chain with $1/2 = \neg 1/2$ as its top element. We can represent these algebras by means of the following diagrams (the dashes represent the order induced by join, while the arrows represent the negation):



It is not difficult to check that every involutive bisemilattice has also the structure of a meet semilattice with 1, and that the equations

$$x \vee y \approx \neg(\neg x \wedge \neg y) \tag{I9}$$

$$x \vee y \approx x \vee (\neg x \wedge y) \tag{I10}$$

are satisfied. In fact, given $\mathbf{B} \in \mathcal{IBSL}$ and $a, b \in B$,

$$a \vee b = \neg\neg(\neg\neg a \vee \neg\neg b) = \neg(\neg a \wedge \neg b),$$

by virtue of (I4) and (I5), and

$$\begin{aligned} a \vee (\neg a \wedge b) &= \neg(\neg a \wedge \neg(\neg a \wedge b)) = \neg(\neg a \wedge \neg(\neg\neg\neg a \wedge \neg\neg b)) \\ &= \neg(\neg a \wedge (\neg\neg a \vee \neg b)) = \neg(\neg a \wedge \neg b) = a \vee b, \end{aligned}$$

by virtue of (I4), (I6), and (I9). Notice that (I9) and (I10) are the result of swapping $\vee$ and $\wedge$ in axioms (I5) and (I6), respectively. Given any property

(P) of type $(2, 2, 1, 0, 0)$, we call the property (P') that results from swapping $\vee$ and $\wedge$, as well as 0 and 1, the *dual* of (P). Hence, we can establish the following duality principle:

PROPOSITION 18. *Given an involutive bisemilattice* $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$, *the algebra* $\mathbf{B}^{\partial} = \langle B, \vee, \wedge, \neg, 1, 0\rangle$ *is also an involutive bisemilattice, and moreover the map* $\neg : \mathbf{B} \to \mathbf{B}^{\partial}$ *is an isomorphism. Therefore, given any property* (P) *in the language* $\{\wedge, \vee, \neg, 0, 1\}$, *we have that* (P) *is true in all involutive bisemilattices if and only if its dual property* (P') *is also such.*

Proposition 18 is a very useful tool, since it entitles us to spare ourselves the trouble of proving half of the equations that are valid in all involutive bisemilattices. We will use it without any further mention.

PROPOSITION 19. *In every involutive bisemilattice* $\mathbf{B}$ *the following equations are satisfied:*

1. $x \vee (x \wedge y) \approx x \vee (y \wedge \neg y)$;
2. $(x \wedge \neg x) \vee \neg(x \wedge \neg x) \approx x \vee \neg x$;
3. $x \vee (x \wedge y) \approx x \wedge (x \vee y)$;
4. $x \vee (y \wedge z) \approx x \vee ((x \vee y) \wedge z)$.

PROOF. Let $a, b, c \in B$. Then:

(1)
$$\begin{aligned} a \vee (a \wedge b) &= a \vee (\neg a \wedge a \wedge b) && \text{(I10)} \\ &= a \vee (\neg a \wedge (a \vee \neg b) \wedge b) && \text{(I6)} \\ &= a \vee (\neg a \wedge \neg b \wedge b) && \text{(I6)} \\ &= a \vee (\neg b \wedge b) && \text{(I10)} \\ &= a \vee (b \wedge \neg b) \end{aligned}$$

(2)
$$\begin{aligned} (a \wedge \neg a) \vee \neg(a \wedge \neg a) &= (a \wedge \neg a) \vee \neg a \vee a \\ &= a \vee (\neg a \wedge a) \vee \neg a \\ &= a \vee a \vee \neg a && \text{(I10)} \\ &= a \vee \neg a \end{aligned}$$

(3)
$$\begin{aligned} a \vee (a \wedge b) &= a \vee (b \wedge \neg b) && \text{Prop. 19.(1)} \\ &= (a \vee (b \wedge \neg b)) \wedge (a \vee (b \wedge \neg b)) \\ &= (a \vee (b \wedge \neg b)) \wedge (a \vee (b \wedge \neg b) \vee (b \wedge \neg b)) \\ &= (a \vee (b \wedge \neg b)) \wedge ((b \wedge \neg b) \vee \neg(b \wedge \neg b)) && \text{Prop.19.(1)} \\ &= (a \vee (b \wedge \neg b)) \wedge (b \vee \neg b) && \text{Prop.19.(2)} \end{aligned}$$

$$
\begin{aligned}
&= (b \vee \neg b) \wedge (a \vee \neg(b \vee \neg b)) && \\
&= a \wedge (b \vee \neg b) && \text{(I6)} \\
&= a \wedge (a \vee b) && \text{Prop.19.(1)}
\end{aligned}
$$

(4)
$$
\begin{aligned}
a \vee (b \wedge c) &= a \vee (\neg a \wedge b \wedge c) && \text{(I10)} \\
&= a \vee (\neg a \wedge (a \vee b) \wedge c) && \text{(I6)} \\
&= a \vee ((a \vee b) \wedge c) && \text{(I10)}
\end{aligned}
$$

■

Proposition 20. *If* $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ *is an involutive bisemilattice, then* $\langle B, \wedge, \vee\rangle$ *is a distributive bisemilattice, that is, the equation*

$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$$

and its dual are satisfied.

Proof. Let $a, b, c \in B$. Then:

$$
\begin{aligned}
a \vee (b \wedge c) &= a \vee (b \wedge c) \vee (b \wedge c) && \\
&= a \vee (b \wedge (a \vee c)) \vee (b \wedge c) && \text{Prop.19.(4)} \\
&= a \vee (b \wedge c) \vee (b \wedge (a \vee c)) && \\
&= a \vee (b \wedge c) \vee (b \wedge (a \vee (b \wedge c))) && \text{Prop.19.(4)} \\
&= (a \vee (b \wedge c)) \vee ((a \vee (b \wedge c)) \wedge b) && \\
&= (a \vee (b \wedge c)) \wedge ((a \vee (b \wedge c)) \vee b) && \text{Prop.19.(3)} \\
&= (a \vee (b \wedge c)) \wedge (b \vee a \vee (b \wedge c)) && \\
&= (a \vee (b \wedge c)) \wedge (b \vee a \vee ((a \vee b) \wedge c)) && \text{Prop.19.(4)} \\
&= (a \vee (b \wedge c)) \wedge ((a \vee b) \vee ((a \vee b) \wedge c)) && \\
&= (a \vee (b \wedge c)) \wedge (a \vee b) \wedge (a \vee b \vee c) && \text{Prop.19.(3)} \\
&= (a \vee ((a \vee b) \wedge c)) \wedge (a \vee b) \wedge (a \vee b \vee c) && \text{Prop.19.(4)} \\
&= (a \vee c) \wedge (a \vee b) \wedge (a \vee b \vee c) && \text{Prop.19.(4)} \\
&= (a \vee c) \wedge (a \vee b \vee ((a \vee b) \wedge c)) && \text{Prop.19.(3)} \\
&= (a \vee c) \wedge (a \vee b \vee (a \wedge c)) && \text{Prop.19.(4)} \\
&= (a \vee c) \wedge (a \vee (a \wedge c) \vee b), &&
\end{aligned}
$$

and by Proposition 19.(3)–(4), $(a \vee c) \wedge (a \vee (a \wedge c) \vee b) = (a \vee c) \wedge ((a \wedge (a \vee c)) \vee b) = (a \vee b) \wedge (a \vee c)$. ■

Every involutive bisemilattice is equipped with the two partial orderings $\leqslant$ and $\leqslant'$ from Subsection 3.2, for which 0 and 1 are a bottom and a top element, respectively. In general, these two orderings are different, and therefore involutive bisemilattices, in spite of their having distributive lattice reducts, may well contain $\mathbf{M}_3$ or $\mathbf{N}_5$ as sublattices. In fact, any semilattice with zero $\langle B, \vee, 0\rangle$ makes an instance of an involutive bisemilattice $\langle B, \vee, \vee, \neg, 0, 0\rangle$, in which negation is the identity,[9] and hence both the pentagon and the diamond, as semilattices, can be considered involutive bisemilattices.

Not surprisingly, the two orderings of an involutive bisemilattice are intertwined, since $\wedge$ is completely determined by $\vee$ and $\neg$.

Lemma 21. *Let* $\mathbf{B}$ *be an involutive bisemilattice. Then, for every* $a, b \in B$,

$$a \leqslant b \iff \neg b \leqslant' \neg a.$$

Proof. If $a, b \in B$, then using (I9) we have:

$$\begin{aligned} a \leqslant b \iff a \vee b = b \iff \neg(\neg a \wedge \neg b) = b \iff \neg a \wedge \neg b \\ = \neg b \iff \neg b \leqslant' \neg a. \end{aligned}$$ ■

Corollary 22. *In every semilattice, considered as an involutive bisemilattice, the order* $\leqslant'$ *is the dual of the order* $\leqslant$.

4.2. $\mathcal{IBSL}$ is Generated by WK

Our next goal is to prove that the class $\mathcal{IBSL}$ is the variety generated by the involutive bisemilattice $\mathbf{WK}$. We start by showing that, given an involutive bisemilattice $\mathbf{B}$, the involution does not interchange elements of the interval $[0, 1] = \{a \in B : 0 \leqslant a \leqslant 1\}$ with elements outwith this interval.

Proposition 23. *Let* $\mathbf{B}$ *be an involutive bisemilattice. Then, for all* $a \in B$, *it holds:*

$$0 \leqslant a \leqslant 1 \iff 0 \leqslant \neg a \leqslant 1.$$

Proof. Notice that we only have to prove one implication, because the other follows from the properties of negation. If $a \leqslant 1$, then $\neg a \wedge 0 = 0$, and thus

$$\neg a \vee 1 = (\neg a \vee 1) \wedge 1 = (\neg a \vee 1) \wedge (0 \vee 1) = (\neg a \wedge 0) \vee 1 = 0 \vee 1 = 1.$$

That is, $\neg a \leqslant 1$.

[9] Hereafter, duplicate occurrences of $\vee$ and 0 in the type will be omitted whenever we consider semilattices with zero as involutive bisemilattices.

Given an involutive bisemilattice $\mathbf{B}$, we tweak a construction by Kalman [25] and define on $\mathbf{B}$ a congruence Φ and a family of congruences Δ_a, for every $a \in B$, that will prove useful in our demonstration that $\mathcal{IBSL}$ is the variety generated by $\mathbf{WK}$:

$$\Phi = \{\langle a, b\rangle \in B^2 : a = b \text{ or } a, b \nleqslant 1\}, \tag{*}$$

$$\Delta_a = \{\langle b, c\rangle \in B^2 : a \vee b = a \vee c \text{ and } a \vee \neg b = a \vee \neg c\}. \tag{**}$$

Lemma 24. *Given an involutive bisemilattice* $\mathbf{B}$*, the relations* Φ *and* Δ_a*, for every* $a \in B$*, are congruences.*

Proof. It is easy to check that they are equivalence relations. We show now that they are compatible with the operations $\vee$ and $\neg$, and therefore also with $\wedge$.
Φ is a congruence:

($\vee$) Let $\langle a, b\rangle, \langle c, d\rangle \in \Phi$. If $a = b$ and $c = d$, then trivially $a \vee c = b \vee d$, and therefore $\langle a \vee c, b \vee d\rangle \in \Phi$. Otherwise, suppose that $a \neq b$, whence $a \nleqslant 1$ and $b \nleqslant 1$. Notice that $a \leqslant a \vee c$, and therefore if $a \vee c \leqslant 1$, we would have that $a \leqslant 1$, a contradiction, whence $a \vee c \nleqslant 1$. Analogously, we have that $b \vee d \nleqslant 1$. Thus, $\langle a \vee c, b \vee d\rangle \in \Phi$.

($\neg$) Let $\langle a, b\rangle \in \Phi$. If $a = b$, our conclusion readily follows. Otherwise, we have $a \nleqslant 1$ and $b \nleqslant 1$, and hence by Proposition 23, $\neg a \nleqslant 1$ and $\neg b \nleqslant 1$, whence $\langle \neg a, \neg b\rangle \in \Phi$.

Δ_a is a congruence:

($\vee$) Suppose that $\langle b, c\rangle, \langle d, e\rangle \in \Delta_a$. Thus, we have

$$a \vee (b \vee d) = (a \vee b) \vee (a \vee d) = (a \vee c) \vee (a \vee e) = a \vee (c \vee e).$$

Moreover, $a \vee \neg(b \vee d) = a \vee (\neg b \wedge \neg d) = (a \vee \neg b) \wedge (a \vee \neg d) = (a \vee \neg c) \wedge (a \vee \neg e) = a \vee (\neg c \wedge \neg e) = a \vee \neg(c \vee e)$. Thus, $\langle b \vee d, c \vee e\rangle \in \Delta_a$.

($\neg$) Considering $\langle b, c\rangle \in \Delta_a$, it readily follows from the definition of Δ_a and axiom (I4) that $\langle \neg b, \neg c\rangle \in \Delta_a$. ■

Lemma 25. *Given an involutive bisemilattice* $\mathbf{B}$*, we have that* $\Delta_a = \Delta$ *if and only if* $a = 0$*.*

Proof. ($\Leftarrow$) Notice that, if $\langle b, c\rangle \in \Delta_0$ then $b = 0 \vee b = 0 \vee c = c$. Thus, $\Delta_0 = \Delta$. ($\Rightarrow$) Suppose that $\Delta_a = \Delta$. First observe that, for every $b \in B$, trivially $a \vee (a \vee b) = a \vee b$. Furthermore, by (I10) we have $a \vee \neg(a \vee b) = a \vee (\neg a \wedge \neg b) = a \vee \neg b$. Thus, for every $b \in B$, we have that $\langle a \vee b, b\rangle \in \Delta_a = \Delta$, and therefore $a \vee b = b$. In particular, for $b = 0$ we have $a = a \vee 0 = 0$. ■

Lemma 26. *Let* **B** *be an involutive bisemilattice. If* $a \in B$*, then*

$$\Delta_{\neg a} = \{\langle b, c\rangle \in B^2 : a \wedge b = a \wedge c \text{ and } a \wedge \neg b = a \wedge \neg c\}.$$

Proof. For every $b, c \in B$, we have that:

$$\begin{aligned} \langle b, c\rangle \in \Delta_{\neg a} &\iff \neg a \vee b = \neg a \vee c \text{ and } \neg a \vee \neg b = \neg a \vee \neg c \\ &\iff \neg(\neg a \vee b) = \neg(\neg a \vee c) \text{ and } \neg(\neg a \vee \neg b) = \neg(\neg a \vee \neg c) \\ &\iff a \wedge \neg b = a \wedge \neg c \text{ and } a \wedge b = a \wedge c. \end{aligned}$$

■

Next, given an involutive bisemilattice **B**, and two arbitrary elements $a, b \in B$, the following binary operation $\odot$ on B is well-defined by Proposition 19.(3):

$$a \odot b = a \wedge (a \vee b) = a \vee (a \wedge b).$$

Lemma 27. *Given an involutive bisemilattice* **B***, it holds that:*

$$\Delta_a \cap \Delta_{\neg a} = \{\langle b, c\rangle \in B^2 : b \odot a = c \odot a \text{ and } \neg b \odot a = \neg c \odot a\}.$$

Proof. ($\subseteq$) Let $\langle b, c\rangle \in \Delta_a \cap \Delta_{\neg a}$. Then, in particular, $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$. So,

$$\begin{aligned} b \odot a &= b \wedge (b \vee a) = b \wedge (c \vee a) = (b \wedge c) \vee (b \wedge a) \\ &= (b \wedge c) \vee (a \wedge c) = (b \vee a) \wedge c = (a \vee c) \wedge c = c \odot a. \end{aligned}$$

Since Δ_a and $\Delta_{\neg a}$ are congruences, our assumption implies that $\langle \neg b, \neg c\rangle \in \Delta_a \cap \Delta_{\neg a}$, and therefore we also have that $\neg b \odot a = \neg c \odot a$. ($\supseteq$) First of all, observe that for every $a, b \in B$, we have:

$$a \wedge (b \odot a) = a \wedge (b \vee (b \wedge a)) = (a \wedge b) \vee (a \wedge b \wedge a) = a \wedge b.$$

Therefore, if $b \odot a = c \odot a$, then $a \wedge b = a \wedge (b \odot a) = a \wedge (c \odot a) = a \wedge c$. If moreover $\neg b \odot a = \neg c \odot a$, then we obtain that $a \wedge \neg b = a \wedge \neg c$, and therefore $\langle b, c\rangle \in \Delta_{\neg a}$. By a similar argument, using the fact that for every $a, b \in B$, $a \vee (b \odot a) = a \vee b$, we obtain that if $b \odot a = c \odot a$ and $\neg b \odot a = \neg c \odot a$, then $\langle b, c\rangle \in \Delta_a$. ■

Lemma 28. *Let* **B** *be a subdirectly irreducible involutive bisemilattice. Then* $[0, 1] = \{0, 1\}$.

Proof. For the nontrivial inclusion, observe that if $a \leqslant 1$, then by Proposition 23 we have that $\neg a \leqslant 1$, and hence $a \wedge 0 = 0$. Therefore, for every $b \in B$, we have that

$$b \odot a = b \wedge (b \vee a) = (b \vee 0) \wedge (b \vee a) = b \vee (0 \wedge a) = b \vee 0 = b.$$

Hence, if $\langle b, c\rangle \in \Delta_a \cap \Delta_{\neg a}$, by Lemma 27 we have that $b = b \odot a = c \odot a = c$. Thus, $\Delta_a \cap \Delta_{\neg a} = \Delta$. Being $\mathbf{B}$ subdirectly irreducible, we have that $\Delta_a = \Delta$ or $\Delta_{\neg a} = \Delta$. Thus, by Lemma 25, we obtain that $a = 0$ or $\neg a = 0$. That is, $a = 0$ or $a = 1$, as was to be proven. ■

We can now state the main result of this subsection and its corollary, namely, that the only subdirectly irreducible involutive bisemilattices are $\mathbf{WK}$, the 2-element Boolean algebra $\mathbf{B}_2$, and the 2-element semilattice with zero $\mathbf{S}_2$, whereby it follows that $\mathcal{IBSL}$ is the variety generated by $\mathbf{WK}$.

Theorem 29. *The only nontrivial subdirectly irreducible bisemilattices are* $\mathbf{WK}$*,* $\mathbf{S}_2$*, and* $\mathbf{B}_2$*, up to isomorphism.*

Proof. Let $\mathbf{B}$ be a nontrivial subdirectly irreducible involutive bisemilattice and let $C = B \setminus \{0, 1\}$. If $C = \emptyset$ then $B = \{0, 1\}$, and since $\mathbf{B}$ is nontrivial, then $0 \neq 1$. Hence, $\mathbf{B} = \mathbf{B}_2$. If $C \neq \emptyset$, consider the congruence $\theta = \Phi \cap \bigcap_{a \in C} \Delta_a$, with an eye to showing that $\theta = \Delta$. Suppose that $\langle b, c\rangle \in \theta$. Then, in particular $\langle b, c\rangle \in \Phi$. If $b \in \{0, 1\}$, then $b = c$, by virtue of the definition of Φ. Analogously if $c \in \{0, 1\}$. If on the other hand $b, c \in C$, then $\langle b, c\rangle \in \Delta_b \cap \Delta_{\neg b}$, and $\langle b, c\rangle \in \Delta_c \cap \Delta_{\neg c}$. Thus, by Lemma 27, we obtain that $c \leqslant c \vee (c \wedge b) = c \odot b = b \odot b = b$, and symmetrically $b \leqslant c$. That is, in this case also $b = c$, whence $\theta = \Delta$. By Lemma 25, none of the Δ_a's, for $a \in C$, is Δ, and since $\mathbf{B}$ is subdirectly irreducible, then it should be $\Phi = \Delta$. By Lemma 28, the set $\{a \in B : a \leqslant 1\}$ is $\{0, 1\}$, and since we are assuming that $C \neq \emptyset$, then necessarily we have that $T = \{a \in B : a \nleqslant 1\}$ is nonempty. But the congruence Φ identifies all the elements of T and therefore T is a singleton, since $\Phi = \Delta$. Thus, we only have two possibilities: $0 = 1$, and therefore $\mathbf{B} = \mathbf{S}_2$, or $0 \neq 1$, in which case $\mathbf{B} = \mathbf{WK}$. ■

Corollary 30. $\mathcal{IBSL}$ *is the variety generated by* $\mathbf{WK}$*.*

Proof. The result follows immediately from Theorem 29 and the fact that $\mathbf{B}_2$ is a subalgebra of $\mathbf{WK}$, and $\mathbf{S}_2$ is the quotient of $\mathbf{WK}$ by the congruence Δ_1, which is the smallest congruence identifying 0 and 1. ■

Corollary 31. *The only nontrivial proper subvarieties of* $\mathcal{IBSL}$ *are the disjoint varieties* $\mathcal{BA}$ *of Boolean algebras and* $\mathcal{SL}$ *of semilattices with zero.*

4.3. The Structure of Involutive Bisemilattices

Our next goal is to describe in some detail the structure theory of involutive bisemilattices. In particular, we will axiomatise the subvarieties $\mathcal{BA}$ and $\mathcal{SL}$ relative to $\mathcal{IBSL}$. Moreover, we will provide a representation of involutive bisemilattices as Płonka sums of Boolean algebras.

We gather from the proof of Lemma 28 that, given an involutive bisemilattice $\mathbf{B}$ and $a \in B$, if $a \leqslant 1$ then for every b, we have that $b \wedge (b \vee a) = b$ and also that $a \wedge 0 = 0$, and therefore $a \wedge \neg a = a \wedge (\neg a \vee 0) = a \wedge 0 = 0$. That is to say, for the elements of the segment $[0, 1]$, the absorption law is satisfied and $\neg a$ is a complement of a. Consequently, the segment $[0, 1]$ is a Boolean subuniverse of $\mathbf{B}$. We will however establish a more general result. In order to get it, we have to fix some notation.

Definition 32. Given an involutive bisemilattice $\mathbf{B}$, we say that an element $c \in B$ is *Boolean* if $c \in [0, 1]$. We say that c is *positive* if $1 \leqslant c$, and denote the set of positive elements of $\mathbf{B}$ by $P(\mathbf{B})$. Moreover, we say that c is *fix* if $\neg c = c$.

As we will see, every fix element is positive, while there are elements that are neither Boolean nor positive. For a start, we develop a little further the arithmetic of involutive bisemilattices.

Proposition 33. *In every involutive bisemilattice* $\mathbf{B}$ *the following equations are satisfied:*

1. $x \vee \neg x \vee y \approx y \vee \neg y \vee x$;
2. $x \vee \neg x \approx 1 \vee x$;
3. $1 \vee x \approx 1 \vee \neg x$.

Proof. Let $a, b \in B$. By two applications of (I9), we obtain:

$$a \vee \neg a \vee b = a \vee (\neg a \wedge \neg b) \vee b = a \vee \neg b \vee b = b \vee \neg b \vee a.$$

By taking $y = 0$ in (1), we obtain $x \vee \neg x = x \vee \neg x \vee 0 = 0 \vee \neg 0 \vee x = 1 \vee x$. (3) is a consequence of the symmetry of (2). ∎

Remark 34. Notice that by Proposition 33.(2),

$$\Delta_1 = \{\langle b, c\rangle : b \vee \neg b = c \vee \neg c\} = \{\langle b, c\rangle : b \wedge \neg b = c \wedge \neg c\}.$$

We make a note of this fact for later use.

Proposition 33 grants characterisations for the positive elements and the fix elements of any involutive bisemilattice $\mathbf{B}$: an element is positive if and only if it is greater than its negate, and therefore an element c is fix if and only if $c, \neg c \in P(\mathbf{B})$. In particular, all fix elements are positive.

Corollary 35. *For every involutive bisemilattice* $\mathbf{B}$ *and every* $a \in B$, *we have that:*

1. *the element* $a \vee \neg a$ *is positive;*

2. *a is positive if and only if $\neg a \leqslant a$;*
3. *a is fix if and only if a and $\neg a$ are positive.*

PROOF. In virtue of Proposition 33.(2), we have that $a \vee \neg a = 1 \vee a$, which is obviously positive. For the same reason,

$$1 \leqslant a \iff 1 \vee a = a \iff a \vee \neg a = a \iff \neg a \leqslant a.$$

(3) is an immediate consequence of (2). ∎

REMARK 36. Notice that it follows from the second part of the previous corollary that the set of positive elements of an involutive bisemilattice $\mathbf{B}$ is equationally definable:

$$P(\mathbf{B}) = \{c \in B : c \vee \neg c = c\}.$$

Next, let us widen our focus from the segment $[0, 1]$ to an arbitrary segment of the form $[\neg c, c] = \{x \in B : \neg c \leqslant x \leqslant c\}$ for some positive element c in an involutive bisemilattice $\mathbf{B}$. It turns out that all such segments are universes of Boolean algebras, and that they partition B.

LEMMA 37. *Let $\mathbf{B}$ be an involutive bisemilattice. For every $a \in B$, there exists a unique positive element $c_a \in B$ such that $a \in [\neg c_a, c_a]$. Moreover, $c_a = a \vee \neg a$.*

PROOF. Let $a \in B$ and consider $c_a = a \vee \neg a$. First, we prove that $\neg c_a \leqslant a \leqslant c_a$. Obviously, $a \leqslant a \vee \neg a$. For the other inequality, notice that

$$a \vee \neg c_a = a \vee \neg(a \vee \neg a) = a \vee (\neg a \wedge a) = a \vee a = a,$$

by virtue of (I9). Consider now a positive $c \in B$ such that $\neg c \leqslant a \leqslant c$. Then,

$$\neg c_a = \neg(a \vee \neg a) = \neg a \wedge a = \neg a \wedge (a \vee \neg c) = \neg a \wedge \neg c = \neg(a \vee c) = \neg c,$$

whence it follows that $c = c_a$. ∎

LEMMA 38. *Let $\mathbf{B}$ be an involutive bisemilattice and $c \in B$ a positive element. Then, we have that:*

1. *For every $a \in B$, $c \vee a = c \vee \neg a$.*
2. *If $a \leqslant c$ and $\neg c \leqslant b$, then $b \odot a = b$.*

PROOF. In virtue of Proposition 33.(3), we have that for every $a \in B$, $c \vee a = (c \vee 1) \vee a = c \vee (1 \vee a) = c \vee (1 \vee \neg a) = (c \vee 1) \vee \neg a = c \vee \neg a$, since $1 \leqslant c$. For (2), suppose that c is positive, $a \leqslant c$ and $\neg c \leqslant b$. It follows that $b \odot a = b \wedge (b \vee a) = (b \vee \neg c) \wedge (b \vee a) = b \vee (\neg c \wedge a) = b \vee \neg(c \vee \neg a) = b \vee \neg(c \vee a) = b \vee \neg c = b$. ∎

Proposition 39. *Let* $\mathbf{B}$ *be an involutive bisemilattice. For every positive* $c \in B$*, the segment* $[\neg c, c]$ *is the universe of a Boolean algebra* $\mathbf{C}$ *under the restrictions of the nonnullary operations of* $\mathbf{B}$*, and with* $0^{\mathbf{C}} = \neg c$*,* $1^{\mathbf{C}} = c$*.*

Proof. Let c be as in the statement. Notice that, for every $a \in B$, $c_{\neg a} = \neg a \vee \neg\neg a = a \vee \neg a = c_a$. Hence, by Lemma 37, if $a \in [\neg c, c]$, then $c = c_a = c_{\neg a}$, and therefore $\neg a \in [\neg c_{\neg a}, c_{\neg a}] = [\neg c, c]$. That is to say, $[\neg c, c]$ is closed under negation, and since it is obviously closed under $\vee$, it is closed under $\wedge$ as well. Since $\neg c$ and c are the bottom and the top of $[\neg c, c]$, respectively, $\mathbf{C}$ is a bounded involutive bisemilattice, whence all we have to prove is that the absorption laws are satisfied and that for every $a \in [\neg c, c]$, $\neg a$ is its complement in $\mathbf{C}$. If we are given $a, b \in [\neg c, c]$, then $a \leqslant c$ and $\neg c \leqslant b$, and thus by virtue of Lemma 38, $b \vee (b \wedge a) = b \odot a = b$. Therefore the absorption laws are satisfied and the two orderings $\leqslant$ and $\leqslant'$ coincide. Finally, by Lemma 37, we have that if $a \in [\neg c, c]$, then $c = c_a = a \vee \neg a$, and hence $\neg c = \neg c_a = \neg(a \vee \neg a) = a \wedge \neg a$, which proves that $\neg a$ is the complement of a. ∎

It follows from Lemma 37 and Proposition 39 that every involutive bisemilattice can be viewed as a union of Boolean blocks. The next Proposition provides criteria to determine when a given $\mathbf{B} \in \mathcal{IBSL}$ is made up by a unique block, that is, when it is a Boolean algebra itself.

Proposition 40. *Let* $\mathbf{B}$ *an involutive bisemilattice. The following statements are equivalent:*

1. $\mathbf{B}$ *is a Boolean algebra.*
2. 1 *is the maximum of* $\mathbf{B}$ *with respect to the ordering* $\leqslant$*.*
3. $\mathbf{B}$ *satisfies* $x \vee \neg x \approx 1$*.*

Proof. If $\mathbf{B}$ is a Boolean algebra, then obviously it satisfies $x \leqslant 1$. For the other implication, just notice that (2) is equivalent to $B = [0, 1]$, but $[0, 1]$ is the universe of a Boolean algebra by virtue of Proposition 39. The second and the third statement are equivalent by virtue of Proposition 33.(2). ∎

Let us list some additional properties of fix elements and positive elements, in order to get a better grasp of the structure of an involutive bisemilattice. In the next lemma we show that the set of fix elements of an involutive bisemilattice is closed upwards, with respect to the ordering $\leqslant$.

Lemma 41. *Let* $\mathbf{B}$ *be an involutive bisemilattice. Then, we have:*

1. *If* c *is a fix element, then for every* $a \in B$*,* $c \wedge a = c \vee a$*.*

2. *If c is a fix element and $c \leqslant a$, then a is also a fix element.*

Proof. For (1), suppose that c is a fix element of $\mathbf{B}$, and pick an arbitrary $a \in B$. Then,

$$c \vee a = c \vee (\neg c \wedge a) = c \vee (c \wedge a) = c \wedge (c \vee a) = c \wedge (\neg c \vee a) = c \wedge a.$$

Regarding (2), if c is fix, then for every $a \in B$

$$c \leqslant a \iff c \vee a = a \iff c \wedge a = a \iff \neg c \vee \neg a = \neg a \iff \neg c \leqslant \neg a \iff c \leqslant \neg a.$$

Hence, if $c \leqslant a$, then $c \leqslant \neg a$, and since c is fix then, in particular, it is positive by virtue of Corollary 35. Therefore, both a and $\neg a$ are positive, and again by Corollary 35, a is a fix element. ■

We enumerate below a number of characterisations of $\mathcal{SL}$ as a subvariety of $\mathcal{IBSL}$.

Proposition 42. *Let $\mathbf{B}$ an involutive bisemilattice. The following statements are equivalent:*

1. $\mathbf{B}$ *satisfies* $x \wedge y \approx x \vee y$.
2. $\mathbf{B}$ *satisfies* $\neg x \approx x$.
3. $\mathbf{B}$ *satisfies* $1 \approx 0$.
4. $\mathbf{B}$ *satisfies* $x \odot y \approx x \wedge y$.

Proof. (1) $\Rightarrow$ (2) If $\mathbf{B}$ satisfies $x \wedge y \approx x \vee y$, then for every $a, b \in B$ we have $a \vee (\neg a \vee b) = a \vee (\neg a \wedge b) = a \vee b$. In particular $a \vee \neg a = a \vee \neg a \vee a = a \vee a = a$, that is $\neg a \leqslant a$. Since this is true for all $a \in B$, then we also have that $a = \neg\neg a \leqslant \neg a$, whence $a = \neg a$.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (4) If $1 = 0$, then for every $a, b \in B$, we have that $a \odot b = a \vee (a \wedge b) = (a \wedge 1) \vee (a \wedge b) = a \wedge (1 \vee b) = a \wedge (0 \vee b) = a \wedge b$.

(4) $\Rightarrow$ (1) Suppose that $\mathbf{B}$ satisfies $x \odot y \approx x \wedge y$, and pick $a \in B$. Then, $a = a \vee 0 = a \vee (\neg a \wedge 0) = a \vee (\neg a \odot 0) = a \vee (\neg a \wedge (\neg a \vee 0)) = a \vee \neg a$. That is, a is positive. Since a is arbitrary, we also have that $\neg a$ is positive, and hence every element of $\mathbf{B}$ is fix. Thus, for every $a, b \in B$, $a \wedge b = a \vee b$, by virtue of Lemma 41. ■

Corollary 43. *Let $\mathbf{B}$ be an involutive bisemilattice, and let Δ_1 be the congruence defined in $(**)$. The quotient algebra $\mathbf{B}/\Delta_1$ is isomorphic to the semilattice $\langle P(\mathbf{B}), \vee, 1\rangle$ of the positive elements of $\mathbf{B}$.*

Proof. First we notice that, by virtue of Proposition 42, $\mathbf{B}/\Delta_1$ is a semilattice with bottom element $0/\Delta_1$, because $\langle 1,0\rangle \in \Delta_1$, and therefore $\neg(0/\Delta_1) = \neg 0/\Delta_1 = 1/\Delta_1 = 0/\Delta_1$. Remark further that, if a and b are positive elements such that $a/\Delta_1 = b/\Delta_1$, then $a = 1 \vee a = 1 \vee b = b$. That is, the projection $\mathbf{B} \to \mathbf{B}/\Delta_1$ gives a bijection between the positive elements of $\mathbf{B}$ and the elements of $\mathbf{B}/\Delta_1$, respecting $\vee$ and sending 1 to the bottom element $0/\Delta_1$ of $\mathbf{B}/\Delta_1$. That is to say, the map $\pi : \langle P(\mathbf{B}), \vee, 1\rangle \to \mathbf{B}/\Delta_1$ defined by $\pi(a) = a/\Delta_1$ is an isomorphism. ∎

Although we could produce a direct proof of the next result, we choose to put to good use the preceding corollary as an example of its usefulness.

Lemma 44. *Let $\mathbf{B}$ be an involutive bisemilattice. Then we have that:*

1. *If a and b are positive, then also $a \wedge b = a \vee b$ is positive.*
2. *If $1 \leqslant a \leqslant b$, then $\neg a \leqslant \neg b$.*

Proof. For (1), select $a, b \in P(\mathbf{B})$ and, given any $c \in B$, denote by $[c]$ the equivalence class c/Δ_1. Then, $[a \wedge b] = [a] \wedge [b] = [a] \vee [b] = [a \vee b]$, since $\mathbf{B}/\Delta_1$ is a semilattice. Thus, $a \wedge b = (1 \vee a) \wedge (1 \vee b) = 1 \vee (a \wedge b) = 1 \vee (a \vee b) = a \vee b$. As for (2), if a and b are positive then

$$a \leqslant b \iff a \vee b = b \iff a \wedge b = b \iff \neg a \vee \neg b = \neg b \iff \neg a \leqslant \neg b.$$

∎

We now present the main result of this section, according to which every Płonka sum over a direct system of Boolean algebras, indexed by a semilattice with zero, is an involutive bisemilattice, and every involutive bisemilattice admits a representation as a Płonka sum of Boolean algebras.[10]

Theorem 45.

1. *If $\mathtt{T} = \langle(\varphi_{ij} : i \leqslant j), \mathbf{I}\rangle$ is a direct system of Boolean algebras, then the Płonka sum $\mathbf{T}$ over $\mathtt{T}$ is an involutive bisemilattice.*
2. *If $\mathbf{B}$ is an involutive bisemilattice, then $\mathbf{B}$ is isomorphic to the Płonka sum over the direct system $\mathtt{T} = \langle(\varphi_{cd} : c \leqslant d), \langle P(\mathbf{B}), \leqslant\rangle\rangle$, where the homomorphism $\varphi_{cd} : [\neg c, c] \to [\neg d, d]$ is given by $\varphi_{cd}(a) = \neg d \vee a$.*

Proof. (1) This is a direct consequence of Lemma 11 and the axiomatisation of involutive bisemillatices. Indeed, all the equations in Definition 16

[10] For a more refined representation of a class of algebras that includes $\mathcal{IBSL}$ in terms of involutorial Płonka sums of algebras satisfying a generalised version of the absorption law, see [12].

are regular equations satisfied in every Boolean algebra, and therefore they are satisfied in $\mathbf{T}$ as well. That is, $\mathbf{T}$ is an involutive bisemilattice.

(2) Let $\mathbf{B}$ be an involutive bisemilattice. We know that $f(a,b) = a \odot b$ is a partition function on its bisemilattice reduct; hence, in order to show that it is a partition function on $\mathbf{B}$, it suffices to verify that for all $a, b \in B$ the following conditions are satisfied:

$$\neg a \odot b = \neg(a \odot b); \qquad \neg a \odot a = \neg a; \qquad a \odot 0 = a;$$
$$b \odot \neg a = b \odot \neg(b \odot a); \qquad a \odot \neg a = a; \qquad a \odot 1 = a.$$

For the first one, notice that, by Proposition 19.(1), $\neg a \odot b = \neg a \vee (\neg a \wedge b) = \neg a \vee (b \wedge \neg b) = \neg a \vee (\neg a \wedge \neg b) = \neg(a \wedge (a \vee b)) = \neg(a \odot b)$. The rest are immediate to do resorting to Definition 16 and Proposition 20. Therefore, by Theorem 10, $\mathbf{B}$ is representable as the Płonka sum over a direct system of involutive bisemilattices satisfying $x \odot y \approx x$, and thus such that for all a,

$$1 = 1 \odot a = 1 \vee (1 \wedge a) = 1 \vee a.$$

These Płonka fibres are Boolean by Proposition 40. By Remark 34, we have that

$$\Delta_1 = \{\langle b, c\rangle : b \vee \neg b = c \vee \neg c\} = \{\langle b, c\rangle : b \wedge \neg b = c \wedge \neg c\}.$$

Furthermore, $\Delta_1 = \{\langle b, c\rangle : b \odot c = b \text{ and } c \odot b = c\}$. In fact, by Lemmas 37 and 38, if $b \wedge \neg b \leqslant b, c \leqslant b \vee \neg b$, then $b \odot c = b$ and $c \odot b = c$; conversely, if $b \odot c = b$ and $c \odot b = c$, then by Proposition 19.(1) $b \vee (c \wedge \neg c) = b$ and $c \vee (b \wedge \neg b) = c$. So $b \in [c \wedge \neg c, c \vee \neg c]$ and $c \in [b \wedge \neg b, b \vee \neg b]$, whence our conclusion follows by Lemma 37. So, the proof of Theorem 10 implies that the semilattice of indices in our direct system can be taken to be $\mathbf{B}/\Delta_1$, which is isomorphic to the semilattice $\langle P(\mathbf{B}), \vee, 1\rangle$ of the positive elements of $\mathbf{B}$ by Corollary 43, and by Lemma 37 each fibre has the form $[\neg c, c]$, for c a positive element. Finally, the proof of Theorem 10 implies further that $\varphi_{cd}(a) = a \odot d$, and by Proposition 19.(1) and Corollary 35.(2),

$$a \odot d = a \vee (a \wedge d) = a \vee (d \wedge \neg d) = \neg d \vee a. \qquad \blacksquare$$

Corollary 46. *$\mathcal{IBSL}$ is the variety satisfying exactly the regular* $(2, 2, 1, 0, 0)$*-identities satisfied by $\mathcal{BA}$.*

Proof. This is an immediate consequence of Theorem 45 and Lemma 11. $\blacksquare$

For example, there are two different ways of equationally expressing the fact that 1 is the top element of a Boolean algebra: $x \wedge 1 \approx x$ and $x \vee 1 \approx 1$.

The former identity is regular, and therefore is satisfied by every involutive bisemilattice, while the latter is not, and therefore it fails in every involutive bisemilattice with more than one Boolean fibre, that is, in every involutive bisemilattice that is not a Boolean algebra.

Recall from Corollary 31 that the only nontrivial subvarieties of $\mathcal{IBSL}$ are $\mathcal{BA} = \mathbb{V}(\mathbf{B}_2)$, $\mathcal{SL} = \mathbb{V}(\mathbf{S}_2)$, and $\mathcal{IBSL}$ itself. This means that any involutive bisemilattice $\mathbf{B}$ falls under one of the following three cases, in terms of its Płonka sum representation:

- $\mathbf{B}$ has only one Boolean fibre. In that case, $\mathbf{B}$ is a Boolean algebra.
- All the Boolean fibres of $\mathbf{B}$ are trivial. In this case, $\mathbf{B}$ satisfies $x \vee y \approx x \wedge y$, and hence it is a semilattice.
- $\mathbf{B}$ has at least two nontrivial Boolean fibres. If so, consider $\mathbf{B}/\Phi$, where Φ is the congruence defined in $(*)$. Notice that $[0, 1]$ cannot be trivial, for suppose otherwise: then $0 = 1$ and hence all the Boolean fibres would be also trivial. Recall that Φ identifies all the elements that are not in $[0, 1]$, and therefore $\mathbf{B}/\Phi$ turns out to be the Płonka sum of a nontrivial Boolean algebra and a trivial Boolean algebra. Therefore, $\mathbf{WK}$ is isomorphic to a subalgebra of $\mathbf{B}/\Phi$, and hence the only subvariety of $\mathcal{IBSL}$ containing $\mathbf{B}$ is $\mathcal{IBSL}$ itself.

5. Algebraic Study of PWK

5.1. Abstract Algebraic Logic

We recap now some notions from AAL that will be used in the rest of this section. Standard references for the material that follows include [5, 17, 19].

Given $\mathbf{Fm}$ of type ν, a *formula-equation transformer* is a map $\tau : Fm \to \mathcal{P}(Fm^2)$ such that, given a variable $p \in X$, $\tau(p)$ is a set of ν-equations in the single variable p, and for every $\alpha \in Fm$, $\tau(\alpha)$ is the result of uniformly replacing, in each member of $\tau(p)$, the variable p by the formula α; an *equation-formula transformer* is a map $\rho : Fm^2 \to \mathcal{P}(Fm)$ such that for any basic equation $p \approx q \in X^2$, $\rho(p, q)$ is a set of ν-formulas in the variables p, q, and for every equation $\alpha \approx \beta$, the set $\rho(\alpha, \beta)$ is the result of uniformly replacing, in each member of $\rho(p, q)$, the variables p, q by the formulas α, β, respectively. Given any function $f : A \to B$, and any subset $D \subseteq A$, we denote by $f[D]$ the set $\{f(d) : d \in D\}$, and thus, for every set of formulas Γ, $\tau[\Gamma] = \bigcup\{\tau(\gamma) : \gamma \in \Gamma\}$.

If $\mathcal{K}$ is a class of algebras of type ν, and L is a logic of the same type, $\mathcal{K}$ is called an *algebraic semantics* for L if there exists a formula-equation transformer τ s.t., for all $\Gamma \cup \{\alpha\} \subseteq Fm$,

$$\Gamma \vdash_{\mathrm{L}} \alpha \iff \tau[\Gamma] \vDash_{\mathcal{K}} \tau(\alpha).$$

$\mathcal{K}$ is said to be *equivalent* to L if there exists an equation-formula transformer ρ that inverts τ, meaning that for all $\alpha \approx \beta \in Fm^2$,

$$\alpha \approx \beta \dashv\vDash_{\mathcal{K}} \tau[\rho(\alpha, \beta)].$$

A logic L is said to be *algebraisable* if and only if it has an equivalent algebraic semantics $\mathcal{K}$. By virtue of [5, Cor. 2.11], given any such $\mathcal{K}$, the largest equivalent algebraic semantics for L may be identified with the quasivariety $\mathcal{Q}$ generated by $\mathcal{K}$; we use the expression *equivalent quasivariety semantics* to refer to this class. If $\mathcal{Q}$ is a variety, we call it an *equivalent variety semantics* for L.

One of the classical results of AAL is the Isomorphism Theorem for algebraisable logics, which we will use later on. In order to state this theorem we need to introduce some extra terminology. Let ν be a similarity type. A *ν-matrix* (or simply a *matrix*, when ν is understood) is a pair $\mathsf{M} = \langle \mathbf{A}, F \rangle$, where $\mathbf{A}$ is an algebra of type ν and F is a subset of A. The algebra $\mathbf{A}$ and the set F are called the *algebraic reduct* and the *filter* of M, respectively. For any matrix $\mathsf{M} = \langle \mathbf{A}, F \rangle$ and $\Gamma \cup \{\alpha\} \subseteq Fm$, let $\vDash_{\mathsf{M}}$ be the relation defined by

$$\Gamma \vDash_{\mathsf{M}} \alpha \iff \text{for every valuation } v \text{ on } \mathbf{A},\ v[\Gamma] \subseteq F \text{ implies } v(\alpha) \in F.$$

If M is a class of matrices, then $\Gamma \vDash_M \alpha$ if $\Gamma \vDash_{\mathsf{M}} \alpha$, for every $\mathsf{M} \in M$. If L is a logic of type ν and $\mathbf{A}$ an algebra of the same type, a subset F of A is called an L-*filter*, or just a *deductive filter* when L is understood, if $\Gamma \vdash_{\mathrm{L}} \alpha$ implies $\Gamma \vDash_{\langle \mathbf{A}, F \rangle} \alpha$, for all $\Gamma \cup \{\alpha\} \subseteq Fm$. The elements in F are said to be *designated*. The set of all L-filters of an algebra $\mathbf{A}$ is denoted by $\mathcal{F}\mathrm{i}_{\mathrm{L}}\, \mathbf{A}$. If F is an L-filter, then the matrix $\langle \mathbf{A}, F \rangle$ is called a *matrix model* of L. Given an algebra $\mathbf{A}$ and a set $F \subseteq A$, it is not difficult to see that there is a largest congruence $\theta = \boldsymbol{\Omega}^{\mathbf{A}} F$ on $\mathbf{A}$ such that F is a union of θ-cosets. We call it the *Leibniz congruence* of F, and the natural map $\boldsymbol{\Omega}^{\mathbf{A}}_{\mathrm{L}}$ that assigns to every L-filter F its corresponding Leibniz congruence $\boldsymbol{\Omega}^{\mathbf{A}} F$ is called the *Leibniz operator* on $\mathbf{A}$. The *$\mathcal{K}$-relative congruences* of an algebra $\mathbf{A}$ are those congruences θ such that $\mathbf{A}/\theta \in \mathcal{K}$. The set of $\mathcal{K}$-relative congruences of $\mathbf{A}$ is denoted by $\mathrm{Co}_{\mathcal{K}}\, \mathbf{A}$. Both $\mathcal{F}\mathrm{i}_{\mathrm{L}}\, \mathbf{A}$ and $\mathrm{Co}_{\mathcal{K}}\, \mathbf{A}$ are lattice-ordered by inclusion.

Theorem 47. (Isomorphism Theorem). *If* L *is an algebraisable logic with equivalent algebraic semantics* $\mathcal{K}$ *and* $\mathbf{A}$ *is an arbitrary algebra of the same type, then* $\mathcal{F}\mathrm{i}_{\mathrm{L}}\, \mathbf{A}$ *and* $\mathrm{Co}_{\mathcal{K}}\, \mathbf{A}$ *are isomorphic lattices. Moreover, the isomorphism is given by the Leibniz operator* $\boldsymbol{\Omega}^{\mathbf{A}}_{\mathrm{L}}$.

Not every logic is algebraisable, but for certain logics there is still a class of algebras that can be associated to them in a sensible way. These logics are ordered in a hierarchy according to the strength of the relationship between them and their attendant classes of algebras. This hierarchy is called the *Leibniz hierarchy*, for the rank assigned to a logic therein can be characterised by properties of the Leibniz operator. A logic L is called *protoalgebraic* if the Leibniz operator is monotone on the set of L-filters, i.e. if for all $\mathbf{A}$ and for all L-filters $F, G \subseteq A$ such that $F \subseteq G$, we have that $\boldsymbol{\Omega}^{\mathbf{A}} F \subseteq \boldsymbol{\Omega}^{\mathbf{A}} G$. Protoalgebraic logics are one of the two lowest known levels of the Leibniz hierarchy; the other one is truth-equational logics, introduced in [37]. A matrix $\mathsf{M} = \langle \mathbf{A}, F \rangle$ is said to be *reduced* if $\boldsymbol{\Omega}^{\mathbf{A}} F$ is the identity relation on A. The class of reduced models of a logic L is denoted by $\mathsf{Mod}^*(\mathrm{L})$. A logic L is *truth-equational* if the filters of the reduced models of L are defined by a set of equations. The characterisation that places truth-equational logics as part of the Leibniz hierarchy is the following: a logic is truth-equational if and only if its Leibniz operator is completely order-reflecting. Truth-equational logics have algebraic semantics, but they are not necessarily protoalgebraic. A particular and very common case of truth-equational logics are *assertional* logics, which are logics with an algebraic semantics given by a transformer of the form $\tau(x) = \{x \approx \top\}$, being $\top$ a constant term.

For an algebraisable logic L, its equivalent quasivariety semantics is given by the class of the algebra reducts of its reduced matrix models:

$$\mathsf{Alg}^*(\mathrm{L}) = \{\mathbf{A} : \text{there is a reduced model } \langle \mathbf{A}, F \rangle \text{ of L}\}.$$

For this reason it is fair to say that the "algebraic counterpart" of a protoalgebraic logic L is the class $\mathsf{Alg}^*(\mathrm{L})$. Notice that the notion of a reduced model of a logic takes into account only one filter at a time, and for nonprotoalgebraic logics this is not usually enough, since their Leibniz operators generally fail to be monotone. For these logics, the theory of *full models* [18] turns out to be more adequate.

A *g-matrix*[11] is a pair $\langle \mathbf{A}, \mathcal{C} \rangle$ such that $\mathcal{C}$ is a *closure system* on A, that is, a set of subsets of A containing A as an element and closed under arbitrary intersections. The idiosyncratic example of a g-matrix is $\langle \mathbf{Fm}, \mathcal{T}h(\mathrm{L}) \rangle$, where

[11] An equivalent way of defining a g-matrix is as a pair $\langle \mathbf{A}, C \rangle$ such that C is a *closure operator* on A, i.e., a map $\mathcal{P}(A) \to \mathcal{P}(A)$ such that for every $X \subseteq A$, $X \subseteq CX$, $CCX = CX$, and if $X \subseteq Y \subseteq A$, then $CX \subseteq CY$. Thus, a g-model of a logic L would be a g-matrix $\langle \mathbf{A}, C \rangle$ such that for every entailment $\Sigma \vdash_{\mathrm{L}} \alpha$ and every valuation v on $\mathbf{A}$, $v(\alpha) \in Cv[\Sigma]$; and the Tarski congruence of a g-matrix $\langle \mathbf{A}, C \rangle$ would be the largest congruence θ of $\mathbf{A}$ that is compatible with C, in the sense that $\langle a, b \rangle \in \theta$ implies $C\{a\} = C\{b\}$.

$\mathcal{T}h(\mathrm{L})$ is the set of all *theories* of L, that is, all the sets $\Sigma \subseteq Fm$ such that for every formula α, we have that $\Sigma \vdash_{\mathrm{L}} \alpha$ implies $\alpha \in \Sigma$. A *congruence* of a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is a congruence $\theta \in \mathrm{Co}\,\mathbf{A}$ that is *compatible* with $\mathcal{C}$, meaning that if $\langle a, b \rangle \in \theta$, then for all $T \in \mathcal{C}$, $a \in T$ if and only if $b \in T$. The *Tarski congruence* of a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is the largest congruence of the g-matrix, denoted by $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}\mathcal{C}$. In the particular case that $\mathbf{A} = \mathbf{Fm}$ and $\mathcal{C} = \mathcal{F}\mathrm{i}_{\mathrm{L}}\,\mathbf{Fm}$, the Tarski congruence is simply denoted by $\widetilde{\boldsymbol{\Omega}}\mathrm{L}$. A g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is *reduced* if and only if $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}\mathcal{C} = \Delta$. It is easy to see that, given an algebra $\mathbf{A}$ and a subset $F \subseteq A$, the set $\{F, A\}$ is a closure system on A, that $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}\{F, A\} = \boldsymbol{\Omega}^{\mathbf{A}}F$, and therefore $\langle \mathbf{A}, F \rangle$ is reduced if and only if $\langle \mathbf{A}, \{A, F\} \rangle$ is reduced. Thus, g-matrices are a generalisation of ordinary matrices. We can also generalise the concept of a model as follows. A *g-model* of a logic L is a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ such that for every $T \in \mathcal{C}$, $\langle \mathbf{A}, T \rangle$ is a model of L. Thus, a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ is a g-model of L if and only if $\mathcal{C} \subseteq \mathcal{F}\mathrm{i}_{\mathrm{L}}\,\mathbf{A}$. A g-model is a *basic full* g-model of L if $\mathcal{C} = \mathcal{F}\mathrm{i}_{\mathrm{L}}\,\mathbf{A}$. A characterisation of the Tarski congruence of a g-matrix $\langle \mathbf{A}, \mathcal{C} \rangle$ in terms of Leibniz congruences is the following:

$$\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}\mathcal{C} = \bigcap_{T \in \mathcal{C}} \boldsymbol{\Omega}^{\mathbf{A}}T.$$

Thus, for a protoalgebraic logic L, $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}\,\mathcal{F}\mathrm{i}_{\mathrm{L}}\,\mathbf{A} = \boldsymbol{\Omega}^{\mathbf{A}}F$, where F is the smallest L-filter of $\mathbf{A}$, since the Leibniz operator is monotone. Hence, in this case the class of all algebra reducts of reduced basic full g-models coincides with $\mathsf{Alg}^*(\mathrm{L})$, but in general these two classes are different. For an arbitrary logic L, its "algebraic counterpart" can be identified with the class

$$\mathsf{Alg}(\mathrm{L}) = \{\mathbf{A} : \text{the g-matrix } \langle \mathbf{A}, \mathcal{F}\mathrm{i}_{\mathrm{L}}\,\mathbf{A} \rangle \text{ is reduced}\},$$

which is called the class of L-*algebras*. If there is a variety that can be "naturally" associated to a logic L, this is its *intrinsic variety* $\mathbb{V}(\mathrm{L}) = \mathbb{V}(\mathbf{Fm}/\widetilde{\boldsymbol{\Omega}}\mathrm{L})$, although this does not have to coincide with either $\mathsf{Alg}(\mathrm{L})$ or $\mathsf{Alg}^*(\mathrm{L})$. In general, $\mathsf{Alg}^*(\mathrm{L}) \subseteq \mathsf{Alg}(\mathrm{L})$, and both classes generate the same variety, namely $\mathbb{V}(\mathsf{Alg}^*(\mathrm{L})) = \mathbb{V}(\mathsf{Alg}(\mathrm{L})) = \mathbb{V}(\mathrm{L})$, and the same quasivariety (see [18, Thm. 2.23, Cor. 2.24] or [17, § 5.4]).

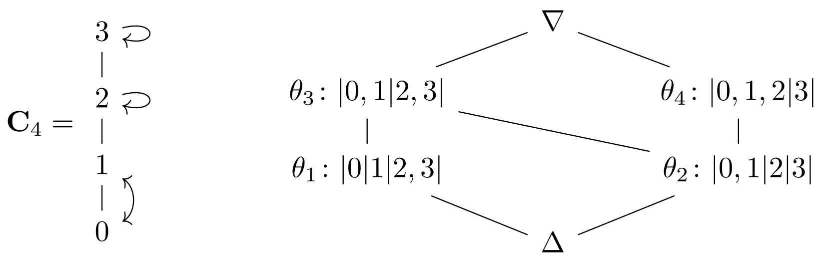
On a different tack, another set of properties of interest concerns the algebraic behaviour of interderivability, seen as a relation on the formula algebra $\mathbf{Fm}$. A logic L is *selfextensional* if and only if the interderivability relation $\dashv\vdash_{\mathrm{L}}$ is a congruence on $\mathbf{Fm}$. L is *Fregean* if for every $\Gamma \subseteq Fm$, the relation $\equiv_\Gamma$ defined by $\alpha \equiv_\Gamma \beta$ if and only if $\Gamma, \alpha \dashv\vdash_{\mathrm{L}} \Gamma, \beta$ is a congruence on $\mathbf{Fm}$. Obviously, if a logic is Fregean, in particular it is selfextensional.

5.2. PWK in the Leibniz and in the Frege Hierarchies

It is not inappropriate to wonder whether the variety $\mathcal{IBSL}$ is the actual algebraic counterpart of the logic PWK. Such a guess stands to reason, for PWK is the logic defined by the matrix **PWK** with **WK** as an underlying algebra, and $\mathcal{IBSL}$ is the variety generated by **WK**. More to the point, we could ask whether PWK is algebraisable and, if so, whether $\mathcal{IBSL}$ is its equivalent variety semantics. This question will be presently answered in the negative—indeed, $\mathcal{IBSL}$ is not the equivalent algebraic semantics of any algebraisable logic. And furthermore, PWK is not algebraisable, since it is not even protoalgebraic. Further down the line, we will observe that PWK is not selfextensional either, and therefore it is non-Fregean. However, let us start with the first of the results we have just announced.

Theorem 48. *$\mathcal{IBSL}$ is not the equivalent algebraic semantics of any algebraisable logic* L.

Proof. Consider the involutive bisemilattice $\mathbf{C}_4$ given by the diagram below, where on the right-hand side we depict its congruence lattice:



Notice that the Leibniz congruence $\boldsymbol{\Omega}^{\mathbf{C}_4}\{2\}$ is θ_2. Furthermore, it can readily be checked by inspection that $\{2\}$ is the only subset $F \subseteq C_4$ such that $\boldsymbol{\Omega}^{\mathbf{C}_4}F = \theta_2$. If $\mathcal{IBSL}$ were the equivalent algebraic semantics of a logic L, then by Theorem 47, $\boldsymbol{\Omega}^{\mathbf{C}_4}$ would yield an isomorphism between $\mathcal{F}\mathrm{i}_{\mathrm{L}}\,\mathbf{C}_4$ and $\mathrm{Co}_{\mathcal{IBSL}}\,\mathbf{C}_4 = \mathrm{Co}\,\mathbf{C}_4$, since $\mathbf{C}_4 \in \mathcal{IBSL}$ and $\mathcal{IBSL}$ is a variety. Thus, $\{2\}$ would necessarily be an L-filter. Furthermore, since $\Delta \subsetneq \theta_2 = \boldsymbol{\Omega}^{\mathbf{C}_4}\{2\}$, we have that $\emptyset$, which is the only proper subset of $\{2\}$, would be an L-filter, and hence L would be purely implicational and thus theoremless. The only protoalgebraic logic with no theorems is the almost inconsistent one [18, p. 60], which is not algebraisable. Thus, there is no logic L with $\mathcal{IBSL}$ as its equivalent algebraic semantics. ■

Theorem 49. PWK *is not protoalgebraic.*

Proof. Suppose ex absurdo that PWK is protoalgebraic. Then, by the intrinsic characterisation of protoalgebraic logics (see e.g. [17, Thm. 6.7]), there would be a set $\Delta(p, q)$ of formulas in the variables p, q such that

1. $\vdash_{\mathrm{PWK}} \Delta(p, p)$,
2. $p, \Delta(p, q) \vdash_{\mathrm{PWK}} q$.

If any of the formulas of $\Delta(p, q)$ only depends on the variable p, or the variable q, then by virtue of (1) this formula is a theorem and therefore superfluous in $\Delta(p, q)$. Thus, we can assume w.l.g. that all the formulas of $\Delta(p, q)$ essentially depend on both variables. Now, consider a valuation v to the algebra **WK** such that $v(p) = 1/2$ and $v(q) = 0$. Then, $v[\{p\} \cup \Delta(p, q)] = \{1/2\}$, while $v(q) = 0$, contradicting (2). Summing up, such a $\Delta(p, q)$ cannot exist, and therefore PWK is not protoalgebraic. ■

Theorem 50. PWK *is not selfextensional, and therefore it is non-Fregean.*

Proof. All we have to show is that the interderivability relation $\dashv\vdash_{\mathrm{PWK}}$ is not a congruence of the formula algebra $\mathbf{Fm}_1$. In order to do that, notice that $\neg p \vee p \dashv\vdash_{\mathrm{PWK}} \neg q \vee q$, since all instances of the excluded middle are classical theorems, and therefore theorems of PWK. However, the valuation v to **WK** that sends q to 0 and all the remaining variables to $1/2$ is such that $v(\neg(\neg p \vee p)) = 1/2$ and $v(\neg(\neg q \vee q)) = 0$, and therefore $\neg(\neg p \vee p) \nvdash_{\mathrm{PWK}} \neg(\neg q \vee q)$. ■

Despite the foregoing result, a certain refinement of the interderivability relation in PWK is a congruence of the formula algebra $\mathbf{Fm}_1$. In fact, let $\Theta = \{\langle \alpha, \beta \rangle \in Fm_1^2 : \mathrm{var}(\alpha) = \mathrm{var}(\beta)\}$; then $\Theta \cap \dashv\vdash_{\mathrm{PWK}}$ is a congruence of $\mathbf{Fm}_1$.

5.3. Deductive Filters and Matrix Models

Although PWK is not a protoalgebraic logic, it still makes sense to characterise the class $\mathsf{Alg}^*(\mathrm{PWK})$ of the algebra reducts of reduced models of PWK. In order to do so, we need to provide a workable description of the Leibniz congruence of a PWK-filter of an arbitrary algebra of the appropriate similarity type. In the following proposition, we establish such a characterisation, very much in the same spirit as the one in [16] for Belnap's Four-Valued Logic.

Proposition 51. *Let* $\mathbf{A}$ *be an algebra of type* $(2, 2, 1, 0, 0)$ *and* $F \subseteq A$ *a* PWK*-filter. Then, for every* $a, b \in A$, $\langle a, b \rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F$ *if and only if for every* $c \in A$,

$$a \vee c \in F \iff b \vee c \in F \quad \textit{and} \quad \neg a \vee c \in F \iff \neg b \vee c \in F. \qquad \text{(Leib)}$$

Proof. If F is a PWK-filter of an algebra $\mathbf{A}$ and $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F$, then for every formula $\alpha(p, \overline{q})$ containing the variable p and any possibly other variables $\overline{q} = q_1, \ldots, q_k$, for every $a, b \in A$, and every set of parameters $\overline{c} = c_1, \ldots, c_k$ in A, we have that $\alpha^{\mathbf{A}}(a, \overline{c}) \in F$ if and only if $\alpha^{\mathbf{A}}(b, \overline{c}) \in F$. In particular, for the formulas $p \vee q$ and $\neg p \vee q$, we obtain the implications of (Leib).

Now, suppose that $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\mathbf{A}} F$. W.l.g. we can assume that there is a formula $\alpha(p, \overline{q})$ and certain parameters $\overline{c}$ in A such that $\alpha^{\mathbf{A}}(a, \overline{c}) \in F$ and $\alpha^{\mathbf{A}}(b, \overline{c}) \notin F$. Let $\beta_1 \wedge \cdots \wedge \beta_n$ be the conjunctive normal form of α given by Proposition 8. It follows that $\alpha \dashv\vdash_{\mathrm{PWK}} \{\beta_1, \ldots, \beta_n\}$, and therefore $\alpha^{\mathbf{A}}(a, \overline{c}) \in F$ if and only if for every i, $\beta_i^{\mathbf{A}}(a, \overline{c}) \in F$, and analogously for b. Therefore, there is a clause, say $\beta_1(p, \overline{q})$, such that $\beta_1^{\mathbf{A}}(a, \overline{c}) \in F$ and $\beta_1^{\mathbf{A}}(b, \overline{c}) \notin F$. Obviously, the variable p necessarily appears in β_1. If the remaining variables in β_1 are among $\overline{q} = q_1, \ldots, q_k$, $\beta_1(p, \overline{q})$ coincides with some of the following, up to equivalence:

(i) p;	(ii) $\neg p$;	(iii) $p \vee \neg p$;
(iv) $p \vee \gamma(\overline{q})$;	(v) $\neg p \vee \gamma(\overline{q})$;	(vi) $p \vee \neg p \vee \gamma(\overline{q})$.

However, the cases (iii) and (vi) can be ruled out, since these formulas are always evaluated to positive elements, and therefore in both cases we would have that $\beta_1(b, \overline{c}) \in F$. In cases (i) or (ii), one or the other of the implications in (Leib) would fail by considering $c = 0$. The same would happen in cases (iv) or (v), by considering $c = \gamma(\overline{c})$. Thus we have seen that if $\langle a, b\rangle \notin \boldsymbol{\Omega}^{\mathbf{A}} F$, then one or the other of the implications of (Leib) fails for some c, as was to be shown. ∎

Thanks to this characterisation of the Leibniz congruence, we can restrict our search of members of $\mathsf{Alg}^*(\mathrm{PWK})$ to involutive bisemilattices.

Theorem 52. *$\mathsf{Alg}^*(\mathrm{PWK}) \subseteq \mathcal{IBSL}$.*

Proof. Let $\langle \mathbf{A}, F\rangle$ be a reduced matrix model of PWK and let $\alpha \approx \beta$ be one of the equations in Definition 16. We need to prove that $\mathbf{A} \vDash \alpha \approx \beta$. Observe that, in every case, we have $\alpha \dashv\vdash_{\mathrm{CL}} \beta$, and thus $\alpha \vee q \dashv\vdash_{\mathrm{CL}} \beta \vee q$ and $\neg\alpha \vee q \dashv\vdash_{\mathrm{CL}} \neg\beta \vee q$, where q is a fresh variable not in $\mathrm{var}(\alpha) = \mathrm{var}(\beta)$. Hence, $\alpha \vee q \dashv\vdash_{\mathrm{PWK}} \beta \vee q$ and $\neg\alpha \vee q \dashv\vdash_{\mathrm{PWK}} \neg\beta \vee q$ by virtue of Corollary 2, because $\mathrm{var}(\alpha \vee q) = \mathrm{var}(\beta \vee q)$ and $\mathrm{var}(\neg\alpha \vee q) = \mathrm{var}(\neg\beta \vee q)$. Let v be a valuation to $\mathbf{A}$, and let c be an arbitrary element of A. Consider the valuation u such that $u(p) = v(p)$ for every variable $p \neq q$, and $u(q) = c$.

Hence,

$$v(\alpha) \vee c = u(\alpha) \vee u(q) = u(\alpha \vee q) \in F \iff v(\beta) \vee c = u(\beta) \vee u(q) \\ = u(\beta \vee q) \in F,$$

and similarly $\neg v(\alpha) \vee c \in F \iff \neg v(\beta) \vee c \in F$. Therefore, by virtue of Proposition 51, we obtain that $\langle v(\alpha), v(\beta)\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F = \Delta$, since the matrix is reduced, and hence $v(\alpha) = v(\beta)$. Since the valuation v was arbitrary, we have $\mathbf{A} \vDash \alpha \approx \beta$. Thus, $\mathbf{A} \in \mathcal{IBSL}$. ■

COROLLARY 53. *The intrinsic variety of* PWK *is* $\mathbb{V}(\text{PWK}) = \mathcal{IBSL}$.

PROOF. This is an immediate consequence of Theorem 52, Corollary 30, and the fact that $\mathbf{WK} \in \mathsf{Alg}^*(\text{PWK})$, since these imply that

$$\mathbb{V}(\text{PWK}) = \mathbb{V}(\mathsf{Alg}^*(\text{PWK})) \subseteq \mathcal{IBSL} = \mathbb{V}(\mathbf{WK}) \subseteq \mathbb{V}(\mathsf{Alg}^*(\text{PWK})).$$ ■

We will prove that the inclusion of Theorem 52 is actually proper. But before that, we need a firmer grasp of the PWK-filters of involutive bisemilattices, which the next Proposition will help us to build.

PROPOSITION 54. *Let* $\mathbf{B}$ *be an involutive bisemilattice.* $F \subseteq B$ *is a* PWK-*filter of* B *if and only if:*

F1. $P(\mathbf{B}) \subseteq F$;

F2. $a \in F, a \leqslant b \Rightarrow b \in F$;

F3. $a, b \in F \Rightarrow a \wedge b \in F$.

PROOF. Suppose that F is a PWK-filter of an involutive bisemilattice $\mathbf{B}$. Let moreover $c \in P(\mathbf{B})$ and let v be a valuation s.t. $v(p) = c$. Since $p \vee \neg p$ is a theorem of PWK, $c = c \vee \neg c = v(x \vee \neg x) \in F$. Suppose now that $a \leqslant b$ and $a \in F$. Since $p \vdash_{\text{CL}} p \vee q$, by virtue of Theorem 1, it follows that $p \vdash_{\text{PWK}} p \vee q$. Thus, considering a valuation v such that $v(p) = a$ and $v(q) = b$, since $v(p) = a \in F$, then also $b = a \vee b = v(p \vee q) \in F$. Assume finally that $a, b \in F$. By the rule $\wedge I$ we have that $p, q \vdash_{\text{PWK}} p \wedge q$. Thus, for a valuation v sending p to a and q to b, we would have $a \wedge b = v(p \wedge q) \in F$.

Conversely, suppose that F contains $P(\mathbf{B})$, is closed upwards, and is closed under $\wedge$. It can be readily checked that for every valuation v, $v(\alpha) \in P(\mathbf{B}) \subseteq F$ for any axiom α in Definition 3. Thus, it will suffice to prove that F is also closed w.r.t. the rule RMP. Suppose that we have two formulas α and β such that $v(\alpha) \in F$ and $v(\alpha \to \beta) \in F$, and moreover $\text{var}(\alpha) \subseteq \text{var}(\beta)$, with an eye to showing that $v(\beta) \in F$. Our assumption entails that

$$v(\alpha) \wedge v(\beta) = v(\alpha) \wedge (\neg v(\alpha) \vee v(\beta)) = v(\alpha) \wedge v(\alpha \to \beta) \in F.$$

Now, since $\mathrm{var}(\alpha) \subseteq \mathrm{var}(\beta)$, we have that the equation $(\alpha \wedge \beta) \vee \beta \approx \beta$, which is obviously valid in every Boolean algebra *qua* instance of the absorption law, is regular, and therefore valid in every involutive bisemilattice, by virtue of Corollary 46. So

$$(v(\alpha) \wedge v(\beta)) \vee v(\beta) = v(\beta).$$

Thus, $v(\alpha) \wedge v(\beta) \leqslant v(\beta)$, and since $v(\alpha) \wedge v(\beta) \in F$ and F is closed upwards, we also obtain $v(\beta) \in F$, as was to be proved. ■

REMARK 55. In any involutive bisemilattice $\mathbf{B}$, the set $P(\mathbf{B})$ is the smallest PWK-filter.

We also can provide a characterisation of the PWK-filters of an involutive bisemilattice in terms of its Płonka sum representation.

PROPOSITION 56. *Let* $\mathbf{B}$ *be the involutive bisemilattice that is the Płonka sum of the direct system of Boolean algebras* $\langle(\varphi_{cd} : c \leqslant d), P(\mathbf{B})\rangle$. *For every* $F \subseteq B$ *and for every* $c \in P(\mathbf{B})$, *let* $\mathbf{A}_c = [\neg c, c]$ *and* $F_c = F \cap A_c$. *Then, the following statements are equivalent:*

1. F *is a* PWK-*filter of* $\mathbf{B}$.
2. *For every* $c \in P(\mathbf{B})$, F_c *is a Boolean filter of* $\mathbf{A}_c$, *and* F *is closed upwards.*
3. *For every* $c \in P(\mathbf{B})$, F_c *is a Boolean filter of* $\mathbf{A}_c$, *and for every* $c \leqslant d$, $\varphi_{cd}[F_c] \subseteq F_d$.

Then, every PWK-*filter corresponds to a family of Boolean filters* $(F_c : c \in P(\mathbf{B}))$, *one of each* $\mathbf{A}_c$, *such that* $\varphi_{cd}[F_c] \subseteq F_d$, *provided that* $c \leqslant d$.

PROOF.

(1) $\Rightarrow$ (2) Since the nonnullary operations of every $\mathbf{A}_c$ are the restrictions of the corresponding operations of $\mathbf{B}$, and F is closed upwards, closed under $\wedge$, and contains all positive elements, then each F_c is a Boolean filter of $\mathbf{A}_c$.

(2) $\Rightarrow$ (3) Suppose that $a \in A_c$ and $c \leqslant d$. Then, $\varphi_{cd}(a) = \varphi_{cd}(a) \vee^{\mathbf{A}_d} \varphi_{cd}(a) = a \vee^{\mathbf{B}} \varphi_{cd}(a)$, and therefore $a \leqslant \varphi_{cd}(a)$. If we take $a \in F_c \subseteq F$, then $\varphi_{cd}(a) \in F \cap A_d$, since F is closed upwards, which proves that $\varphi_{cd}[F_c] \subseteq F_d$.

(3) $\Rightarrow$ (1) Suppose that F_c is a Boolean filter for every $c \in P(\mathbf{B})$ and that $\varphi_{cd}[F_c] \subseteq F_d$, whenever $c \leqslant d$. The positive elements of $\mathbf{B}$ are exactly the top elements of the algebras $\mathbf{A}_c$ of the direct system, and

therefore $F = \bigcup_{c\in P(\mathbf{B})} F_c$ contains them all. Furthermore, suppose that $a \leqslant b$, and $a \in A_c$ and $b \in A_d$, and set $x = c \vee d$. By definition, $b = a \vee^{\mathbf{B}} b = \varphi_{cx}(a) \vee^{\mathbf{A}_x} \varphi_{dx}(b) \in A_x$, and hence $b \in A_d \cap A_x$. But since the algebras of the system are pairwise disjoint, we obtain that $x = d$. Now, if $a \in F_c$, then $\varphi_{cd}(a) \in F_d$, and hence $b = \varphi_{cd}(a) \vee^{\mathbf{A}_d} b \in F_d \subseteq F$, since F_d is closed upwards. This proves that F is closed upwards. Finally, if $a \in F_c$, $b \in F_d$, and $x = c \vee d$, then $a \wedge^{\mathbf{B}} b = \varphi_{cx}(a) \wedge^{\mathbf{A}_x} \varphi_{dx}(b) \in F_x \subseteq F$. This is because both $\varphi_{cx}(a)$ and $\varphi_{dx}(b)$ belong to F_x, by our hypotheses, and F_x is a Boolean filter, and hence closed under $\wedge$. ■

LEMMA 57. *Let* $\mathbf{B}$ *be an involutive bisemilattice,* $\mathbf{A}_c$ *any of its Boolean fibres,* F *a* PWK*-filter, and* $F_c = F \cap A_c$*. Then,* $A_c^2 \cap \boldsymbol{\Omega}^{\mathbf{B}} F = \boldsymbol{\Omega}^{\mathbf{A}_c} F_c$*.*

PROOF. In order to prove that $A_c^2 \cap \boldsymbol{\Omega}^{\mathbf{B}} F \subseteq \boldsymbol{\Omega}^{\mathbf{A}_c} F_c$, suppose that $a, b \in \mathbf{A}_c$ are such that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{B}} F$. Hence, for every $x \in B$, $a \vee x \in F$ if and only if $b \vee x \in F$, according to Proposition 51. By taking $x = \neg a$, we obtain that $b \vee \neg a \in F$, since $a \vee \neg a \in F$. Analogously, we can see that $a \vee \neg b \in F$. Now, $b \vee \neg a, a \vee \neg b \in A_c$, whence $b \vee \neg a, a \vee \neg b \in F_c$. This implies that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}_c} F_c$, since F_c is a filter of the Boolean algebra $\mathbf{A}_c$.

For the other inclusion, consider $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}_c} F_c$. We will prove that $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{B}} F$ by recourse to Proposition 51. Pick then an arbitrary element $x \in B$. Since F_c is a filter of the Boolean algebra $\mathbf{A}_c$, we have that $b \vee \neg a, a \vee \neg b \in F_c \subseteq F$, and therefore also $b \vee \neg a \vee x, a \vee \neg b \vee x \in F$. If we suppose that $a \vee x \in F$, then

$$(a \wedge b) \vee x = (a \wedge (\neg a \vee b)) \vee x = (a \vee x) \wedge (\neg a \vee b \vee x) \in F,$$

and since $a, b \in \mathbf{A}_c$, we have $a \wedge b \leqslant b$, and hence $(a \wedge b) \vee x \leqslant b \vee x$, whence we obtain that $b \vee x \in F$. By symmetry, if $b \vee x \in F$, then $a \vee x \in F$. Similarly, if $\neg a \vee x \in F$, then $(\neg a \wedge \neg b) \vee x \in F$, and again, since $\neg a, \neg b \in \mathbf{A}_c$, then $\neg a \wedge \neg b \leqslant \neg b$, whence $(\neg a \wedge \neg b) \vee x \leqslant \neg b \vee x$, and therefore $\neg b \vee x \in F$. The other implication follows by symmetry. ■

LEMMA 58. *If* $\langle \mathbf{B}, F\rangle$ *is a reduced model of* PWK*, then* $\mathbf{B}$ *is an involutive bisemilattice and* F *is the set* $P(\mathbf{B})$ *of positive elements of* $\mathbf{B}$*.*

PROOF. If $\langle \mathbf{B}, F\rangle$ is a reduced model of PWK, then by Theorem 52, $\mathbf{B}$ is an involutive bisemilattice and F is a PWK-filter, which implies that $P(\mathbf{B}) \subseteq F$. Putting once more to good use Proposition 51, let us now prove that, whenever $a \in F$, $\langle a, c_a\rangle \in \boldsymbol{\Omega}^{\mathbf{B}} F$, where $c_a = a \vee \neg a$. Since $a \in F$ by hypothesis and $c_a \in P(\mathbf{B}) \subseteq F$, we have that for every $c \in B$, $a \vee c, c_a \vee c \in F$, whence the first implication of (Leib) is trivial. For the second implication, notice that $\neg c_a \leqslant \neg a$, whereby for every $c \in B$, $\neg c_a \vee c \in F$ implies $\neg a \vee c \in F$. Let us suppose now that for a certain $c \in B$, $\neg a \vee c \in F$. Hence,

$$\neg c_a \vee c = (\neg a \wedge a) \vee c = (\neg a \vee c) \wedge (a \vee c) \in F,$$

since the former conjunct is in F by hypothesis, and the latter is in F because $a \in F$. Thus, for all $a \in F$, $\langle a, c_a \rangle \in \boldsymbol{\Omega}^{\mathbf{B}} F = \Delta$, since $\langle \mathbf{B}, F \rangle$ is reduced, and therefore $a = c_a \in P(\mathbf{B})$, as we wanted to prove. ■

We can now finally state the result we have been after.

Theorem 59. $\mathbf{B} \in \mathsf{Alg}^*(\mathrm{PWK})$ *if and only if* $\mathbf{B}$ *is an involutive bisemilattice and for every* $a < b$ *positive elements, there is* $c \in B$ *such that*

$$1 \leqslant \neg b \vee c \quad \textit{but} \quad 1 \nleqslant \neg a \vee c.$$

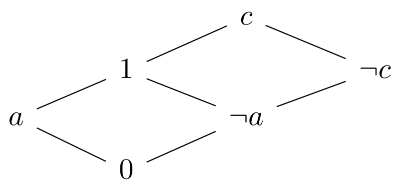
Moreover, $\langle \mathbf{B}, F \rangle \in \mathsf{Mod}^*(\mathrm{PWK})$ *if and only if* $\mathbf{B}$ *is an involutive bisemilattice satisfying the above condition and* $F = P(\mathbf{B})$.

Proof. Suppose that $\mathbf{B} \in \mathsf{Alg}^*(\mathrm{PWK})$. Therefore, there is a PWK-filter F such that $\langle \mathbf{B}, F \rangle$ is a reduced model of PWK, whence Lemma 58 implies that $\mathbf{B}$ is an involutive bisemilattice and $F = P(\mathbf{B})$. Suppose now for the sake of argument that $\langle a, b \rangle \in \boldsymbol{\Omega}^{\mathbf{B}} P(\mathbf{B})$, with $a \neq b$. Hence, a and b belong to different fibres, for if $a, b \in [\neg c, c]$, then by Lemma 57 $\langle a, b \rangle \in [\neg c, c]^2 \cap \boldsymbol{\Omega}^{\mathbf{B}} P(\mathbf{B}) = \boldsymbol{\Omega}^{[\neg c, c]}\{c\} = \Delta_{[\neg c, c]}$, and we would get $a = b$, a contradiction. Now, since a and b belong to different fibres, then $a \vee 1$ and $b \vee 1$ will be also different, although they will be congruent modulo $\boldsymbol{\Omega}^{\mathbf{B}} P(\mathbf{B})$. So, we can assume that a and b are positive. Moreover, either $a \vee b \neq a$ or $a \vee b \neq b$—assume w.l.g. the former. Then, $a = a \vee a$ and $a \vee b$ will be positive and congruent modulo $\boldsymbol{\Omega}^{\mathbf{B}} P(\mathbf{B})$. So, we can further assume that $1 \leqslant a < b$.

By Proposition 51, for every $c \in B$, $a \vee c \in P(\mathbf{B})$ if and only if $b \vee c \in P(\mathbf{B})$, which is trivial because a and b are positive; and also for every $c \in B$, $\neg a \vee c \in P(\mathbf{B})$ if and only if $\neg b \vee c \in P(\mathbf{B})$. Since $a \leqslant b$, and $a, b \in P(\mathbf{B})$, then $\neg a \leqslant \neg b$, and therefore $\neg a \vee c \in P(\mathbf{B})$ implies $\neg b \vee c \in P(\mathbf{B})$. Thus, the only nontrivial implication is that for all $c \in B$, if $\neg b \vee c \in P(\mathbf{B})$, then $\neg a \vee c \in P(\mathbf{B})$.

Therefore, we have seen that $\langle \mathbf{B}, F \rangle$ is a reduced model of PWK if and only if $\mathbf{B}$ is an involutive bisemilattice, $F = P(\mathbf{B})$, and for every positive elements $a < b$ there is c such that $\neg b \vee c \in P(\mathbf{B})$ but $\neg a \vee c \notin P(\mathbf{B})$, as we wanted to show. ■

Example 60. The class $\mathsf{Alg}^*(\mathrm{PWK})$ is not closed under quotients or subalgebras, and therefore it is not even a generalised quasivariety. Indeed, consider the involutive bisemilattice $\mathbf{B}$ given by the following diagram:



This corresponds to a Płonka sum of a 4-element Boolean algebra and a 2-element Boolean algebra, with a homomorphism from the former to the latter. Therefore, by the characterisation we have just given, $\langle \mathbf{B}, P(\mathbf{B})\rangle$ is a reduced model of PWK. However, its subalgebra $\mathbf{A}$ with universe $A = \{0, 1, \neg c, c\}$, representable as a Płonka sum of two 2-element Boolean algebras with an isomorphism between them, is not in Alg^*(PWK), since $\langle \mathbf{A}, \{1, c\}\rangle$ is not reduced, and $\{1, c\} = P(\mathbf{A})$. Observe further that $\mathbf{B}/\Delta_1 \cong \mathbf{S}_2$, where Δ_1 is the congruence defined by (∗∗), and $\mathbf{S}_2$ is obviously not in Alg^*(PWK), since it has two fix elements.

Observe that the precedent example shows an algebra $\mathbf{B} \in \mathsf{Alg}^*$(PWK) such that the set $P(\mathbf{B})$ has two elements. Furthermore, $\langle \mathbf{B}, P(\mathbf{B})\rangle$ is a reduced model of PWK by virtue of Theorem 59. Thus, we conclude that PWK cannot be assertional. Nevertheless, we can prove that PWK is truth-equational, a result that finishes the classification of PWK within the Leibniz hierarchy.

Theorem 61. PWK *is truth-equational.*

Proof. This is an immediate consequence of Theorem 59 and the fact that the set of positive elements of an involutive bisemilattice is equationally definable, as we mentioned in Remark 36. ■

5.4. The Generalised Matrix Semantics of PWK

Given that PWK is not protoalgebraic, it comes as no surprise that the class Alg^*(PWK) is a little unwieldy (recall that it is not even a generalised quasivariety, as we saw in Example 60). In light of our introductory remarks, we can expect to be better off with Alg(PWK), which, as argued above, has to be considered as the proper algebraic counterpart of the nonprotoalgebraic logic PWK. We now prove that these two classes are different from each other and from the variety $\mathcal{IBSL}$.

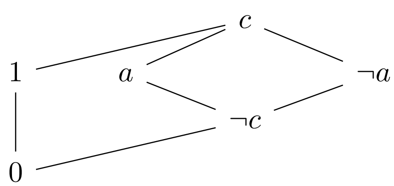
Theorem 62. $\mathsf{Alg}^*(\mathrm{PWK}) \subsetneq \mathsf{Alg}(\mathrm{PWK}) \subsetneq \mathcal{IBSL}$.

Proof. The general theory of AAL dictates that $\mathsf{Alg}^*(\mathrm{PWK}) \subseteq \mathsf{Alg}(\mathrm{PWK})$, and the varieties they generate are the same, namely the intrinsic variety of PWK, which we saw in Corollary 53 coincides with $\mathcal{IBSL}$. Therefore

$\mathsf{Alg}^*(\mathrm{PWK}) \subseteq \mathsf{Alg}(\mathrm{PWK}) \subseteq \mathcal{IBSL}$. Now, the involutive bisemilattice $\mathbf{S}_2$ has only one filter, which is S_2 itself, and therefore

$$\widetilde{\boldsymbol{\Omega}}^{\mathbf{S}_2}\, \mathcal{F}\mathrm{i}_{\mathrm{PWK}}\, \mathbf{S}_2 = \boldsymbol{\Omega}^{\mathbf{S}_2} S_2 = \nabla,$$

which is not the identity. Hence, $\mathbf{S}_2 \notin \mathsf{Alg}(\mathrm{PWK})$. Finally, consider the involutive bisemilattice $\mathbf{B}$ given by the following diagram, corresponding to the Płonka sum of a 2-element Boolean algebra and a 4-element Boolean algebra, with an embedding from the former to the latter:



$\langle \mathbf{B}, P(\mathbf{B})\rangle$ is not reduced, since $\boldsymbol{\Omega}^{\mathbf{B}} P(\mathbf{B})$ is the congruence that identifies 1 and c, 0 and $\neg c$, and nothing else. Thus, $\mathbf{B} \notin \mathsf{Alg}^*(\mathrm{PWK})$. On the other hand, the set $F = \{1, \neg c, a, \neg a, c\}$ is a PWK-filter of $\mathbf{S}$ and $\boldsymbol{\Omega}^{\mathbf{B}} F$ is the congruence identifying all the elements $\neg c, a, \neg a, c$, and nothing else. The Tarski congruence $\widetilde{\boldsymbol{\Omega}}^{\mathbf{B}}\, \mathcal{F}\mathrm{i}_{\mathrm{PWK}}\, \mathbf{B}$ is included in the intersection of these two congruences, which is the identity. Therefore, $\langle \mathbf{B}, \mathcal{F}\mathrm{i}_{\mathrm{PWK}}\, \mathbf{B}\rangle$ is reduced, whence $\mathbf{B} \in \mathsf{Alg}(\mathrm{PWK})$. ■

As we mentioned before, if a logic L is protoalgebraic, then $\mathsf{Alg}^*(\mathrm{PWK}) = \mathsf{Alg}(\mathrm{PWK})$. Thus, Theorem 62 yields a proof that PWK is not protoalgebraic different than that in Theorem 49. Another consequence of this theorem is that $\mathsf{Alg}^*(\mathrm{PWK})$ is not a quasivariety, since, as already recalled, for any logic L, $\mathsf{Alg}^*(\mathrm{L})$ and $\mathsf{Alg}(\mathrm{L})$ generate the same quasivariety, and were it the case that $\mathsf{Alg}^*(\mathrm{PWK})$ is a quasivariety, we would get $\mathsf{Alg}^*(\mathrm{PWK}) = \mathsf{Alg}(\mathrm{PWK})$. Finally, we also observe that $\mathsf{Alg}(\mathrm{PWK})$ cannot be a variety, since it contains $\mathbf{WK}$ but it fails to contain $\mathbf{S}_2$. Also, by Corollary 31, the only proper subvarieties of $\mathcal{IBSL}$ are Boolean algebras and semilattices, and $\mathsf{Alg}(\mathrm{PWK})$ is clearly neither.

To end this section, we show that $\mathsf{Alg}(\mathrm{PWK})$ is not a far cry from $\mathcal{IBSL}$. Indeed, notice that in any involutive bisemilattice with fix elements, all of them are situated near the top, so to speak, since the set they form is closed upwards (see Lemma 41). Moreover, as truth values, all of them are indistinguishable. Thus, it is only natural that the Tarski congruence identifies all the fix elements, and therefore an involutive bisemilattice with more than one fix element cannot be the algebra reduct of a basic full model of PWK.

More than that, the above—which can be expressed by a quasiequation—gives us a characterisation of the class $\mathsf{Alg}(\mathrm{PWK})$, and hence $\mathsf{Alg}(\mathrm{PWK})$ turns out to be a quasivariety which is not a variety. This is a very unusual situation, as there are few natural examples in the literature of a logic L such that $\mathsf{Alg}(\mathrm{L})$ is a quasivariety and not a variety.

THEOREM 63. $\mathsf{Alg}(\mathrm{PWK})$ *is the quasivariety of involutive bisemilattices satisfying the quasiequation*

$$\neg x \approx x \;\&\; \neg y \approx y \Rightarrow x \approx y.$$

PROOF. If $\mathbf{B}$ is an involutive bisemilattice with some fix element, then

$$\theta = \{\langle a, b\rangle : a = b \text{ or } a \text{ and } b \text{ are fix}\}$$

is a congruence that identifies exactly the fix elements of $\mathbf{B}$ and nothing else. Since all the PWK-filters of $\mathbf{B}$ contain all the fix elements, θ is compatible with all of them, and therefore $\theta \subseteq \widetilde{\boldsymbol{\Omega}}^{\mathbf{B}}\, \mathcal{F}\mathrm{i}_{\mathrm{PWK}}\, \mathbf{B}$. Thus, if there is more than one fix element in $\mathbf{B}$, the g-matrix $\langle \mathbf{B}, \mathcal{F}\mathrm{i}_{\mathrm{PWK}}\, \mathbf{B}\rangle$ is not reduced, whence $\mathbf{B} \notin \mathsf{Alg}(\mathrm{PWK})$.

For the other implication, let us assume that $\mathbf{B}$ has at most one fix element. This means that $\mathbf{B}$ is a Płonka sum of nontrivial Boolean algebras (with the possible exception of one fibre). We want to prove that $\widetilde{\boldsymbol{\Omega}} = \widetilde{\boldsymbol{\Omega}}^{\mathbf{B}}\, \mathcal{F}\mathrm{i}_{\mathrm{PWK}}\, \mathbf{B}$ is the identity. With an eye to finding a contradiction, let us assume that there are $a, b \in B$, such that $a \neq b$ and $\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}$. Mimicking the argument in Theorem 59, it can be established that a, b belong to different fibres and that we can assume, without loss of generality, that $1 \leqslant a < b$. Consider, for every positive element $c \in P(\mathbf{B})$, the set F_c defined as $F_c = [\neg c, c]$ if $b \leqslant c$, and $F_c = \{c\}$ otherwise. Given two positive elements $c \leqslant d$, we only have two possibilities: either $b \leqslant d$, and then $\varphi_{cd}[F_c] \subseteq [\neg d, d] = F_d$, or $b \nleqslant d$, and hence $b \nleqslant c$ and $\varphi_{cd}[F_c] = \varphi_{cd}[\{c\}] = \{d\} \subseteq F_d$. Thus, by virtue of Proposition 56, $F = \bigcup F_c$ is a PWK-filter of $\mathbf{B}$.

Consider the relation $\theta = \{\langle e, f\rangle \in B^2 : e = f \text{ or } \neg b \leqslant e, f\}$. It is not difficult to prove that θ is a congruence of $\mathbf{B}$ compatible with F, and therefore, $\theta \subseteq \boldsymbol{\Omega}^{\mathbf{B}} F$. Notice that $\langle b, \neg b\rangle \in \theta \subseteq \boldsymbol{\Omega}^{\mathbf{B}} F$, so $\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}} \subseteq \boldsymbol{\Omega}^{\mathbf{B}} F$ implies that $\langle a, \neg a\rangle \in \boldsymbol{\Omega}^{\mathbf{B}} F$. Since we assumed that a was positive and $b \nleqslant a$, then we have that $F_a = \{a\}$. Thus, $\langle a, \neg a\rangle \in \boldsymbol{\Omega}^{\mathbf{B}} F$ and $a \in F$ imply that $\neg a \in F \cap [\neg a, a] = F_a = \{a\}$, and therefore $\neg a = a$. That is, a is fix, which implies that b is also fix, because $a \leqslant b$. Yet, $a \neq b$, contradicting the assumption that $\mathbf{B}$ has at most one fix element. ■

THEOREM 64. $\mathsf{Alg}(\mathrm{PWK})$ *is the quasivariety generated by* $\mathbf{WK}$.

Proof. We have seen that $\mathsf{Alg}(\mathrm{PWK})$ is a quasivariety, and since $\mathbf{WK} \in \mathsf{Alg}^*(\mathrm{PWK}) \subseteq \mathsf{Alg}(\mathrm{PWK})$, all we have to prove is that every nontrivial involutive bisemilattice with at most one fix element is embeddable into a power of $\mathbf{WK}$. First, for any given Boolean algebra $\mathbf{A}$, we define the involutive bisemilattice $\widehat{\mathbf{A}}$ to be the Płonka sum of the 2-element family containing $\mathbf{A}$ and the trivial algebra $\mathbf{0}$, with the trivial homomorphism $\mathbf{A} \to \mathbf{0}$. Intuitively, we obtain $\widehat{\mathbf{A}}$ by just adding to $\mathbf{A}$ one fix element on top of all the other elements of $\mathbf{A}$. For uniformity and to avoid confusion, we can call this new element ω. Now, given a direct system of Boolean algebras $\mathtt{T} = \langle(\varphi_{ij} : i \leqslant j), \mathbf{I}\rangle$, we will prove that the Płonka sum $\mathbf{T}$ over $\mathtt{T}$ is embeddable into the product $\prod_I \widehat{\mathbf{A}}_i$. Indeed, consider the function $\eta : T \to \prod_I \widehat{A}_i$ defined as follows: for every $i \in I$, and every $a \in A_i$, $\eta(a) = (\eta(a)_j : j \in I)$, where for every $j \in I$,

$$\eta(a)_j = \begin{cases} \varphi_{ij}(a) & \text{if } i \leqslant j, \\ \omega & \text{otherwise.} \end{cases} \tag{†}$$

The function η is injective, because if $a, b \in A_i$, then $\eta(a) = \eta(b)$ implies $a = \varphi_{ii}(a) = \eta(a)_i = \eta(b)_i = \varphi_{ii}(b) = b$; and if $a \in A_i$ and $b \in A_j$, with $i \neq j$, then either $i \nleqslant j$ or $j \nleqslant i$, and therefore $\eta(a)_j = \omega \neq b = \eta(b)_j$ or $\eta(b)_i = \omega \neq a = \eta(a)_i$. Moreover, $\eta(0)_j = 0^{\mathbf{A}_j}$, for every $j \in I$, and hence $\eta(0)$ is the bottom of $\prod_I \widehat{\mathbf{A}}_i$. Furthermore, it is easy to see that for every $i \in I$, $a \in A_i$, and $j \in I$, we have that $\eta(\neg a)_j = \neg\eta(a)_j$, since if $i \leqslant j$, then $\eta(\neg a)_j = \varphi_{ij}(\neg a) = \neg\varphi_{ij}(a) = \neg\eta(a)_j$, and otherwise $\eta(\neg a)_j = \omega = \neg\omega = \neg\eta(a)_j$. Finally, given $i, j \in I$, $a \in A_i$, $b \in A_j$, and $k = i \vee j$, we have that for every $l \in I$, if $k \leqslant l$,

$$\begin{aligned} \eta(a \vee^{\mathbf{T}} b)_l &= \eta(\varphi_{ik}(a) \vee^{\mathbf{A}_k} \varphi_{jk}(b))_l = \varphi_{kl}(\varphi_{ik}(a) \vee^{\mathbf{A}_k} \varphi_{jk}(b)) \\ &= \varphi_{kl}(\varphi_{ik}(a)) \vee^{\mathbf{A}_l} \varphi_{kl}(\varphi_{jk}(b)) = \varphi_{il}(a) \vee^{\mathbf{A}_l} \varphi_{jl}(b) \\ &= \varphi_{il}(a) \vee^{\widehat{\mathbf{A}}_l} \varphi_{jl}(b) = \eta(a)_l \vee^{\widehat{\mathbf{A}}_l} \eta(b)_l. \end{aligned}$$

On the other hand, if $k \nleqslant l$, then $i \nleqslant l$ or $j \nleqslant l$, or both. Let us assume $i \nleqslant l$ and $j \leqslant l$. We would have:

$$\eta(a \vee^{\mathbf{T}} b)_l = \eta(\varphi_{ik}(a) \vee^{\mathbf{A}_k} \varphi_{jk}(b))_l = \omega = \omega \vee^{\widehat{\mathbf{A}}_l} \varphi_{jl}(b) = \eta(a)_l \vee^{\widehat{\mathbf{A}}_l} \eta(b)_l.$$

The other two cases are analogous. We have proved that $\eta : \mathbf{T} \to \prod_I \widehat{\mathbf{A}}_i$ is an embedding. In case $\mathbf{T}$ does not have any fix element, the Boolean algebra $\mathbf{A}_i$ is nontrivial, for every $i \in I$. If $\mathbf{T}$ has exactly one fix element, and it is not the trivial algebra, then one can readily see that there is $m \in I$ such that $\mathbf{A}_m = \mathbf{0}$, I has more than one element, and for every $i \in I$, $i \neq m$, $\mathbf{A}_i \neq \mathbf{0}$, and hence $i < m$. Thus, we can prove that the function $\eta : \mathbf{T} \to$

$\prod_{I\setminus\{m\}} \widehat{A}_i$ defined by (†) is again an embedding. Hence, we have proved that every nontrivial involutive bisemilattice with at most one fix element can be embedded in a product of "extended" Boolean algebras of the form $\widehat{\mathbf{A}}$, where $\mathbf{A}$ is nontrivial. Now, if $\mathbf{A}$ and $\mathbf{B}$ are Boolean algebras such that $\mathbf{A}$ is embeddable into $\mathbf{B}$, then it is not difficult to see that $\widehat{\mathbf{A}}$ is embeddable into $\widehat{\mathbf{B}}$. So, since every nontrivial Boolean algebra $\mathbf{A}$ is embeddable into a power $\mathbf{B}_2^\kappa$, with $\kappa > 0$, every nontrivial involutive bisemilattice with at most one fix element can be embedded into a product of "extended" nontrivial powers of $\mathbf{B}_2$. All that remains to prove is that for every $\kappa > 0$, the algebra $\widehat{\mathbf{B}_2^\kappa}$ is embeddable into a power of $\mathbf{WK}$. The required embedding is given by $\rho : \widehat{\mathbf{B}_2^\kappa} \to \mathbf{WK}^\kappa$, where for every $a \in \widehat{B_2^\kappa}$,

$$\rho(a) = \begin{cases} a & \text{if } a \neq \omega, \\ \overline{1/2} & \text{otherwise,} \end{cases}$$

where $\overline{1/2}$ is the sequence constantly equal to $1/2$. ■

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