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A DUALITY FOR INVOLUTIVE BISEMILATTICES

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Abstract. We establish a duality between the category of involutive bisemilattices and the category of semilattice inverse systems of Stone spaces, using Stone duality from one side and the representation of involutive bisemilattices as Płonka sum of Boolean algebras, from the other. Furthermore, we show that the dual space of an involutive bisemilattice can be axiomatized as a GR space with involution, a generalization of the spaces introduced by Gierz and Romanowska, equipped with an involution as additional operation.

1. Introduction

It is a common trend in mathematics to study dualities for general algebraic structures and, in particular, for those arising from mathematical logic. The first step towards this direction traces back to the pioneering work by Stone for Boolean algebras [24]. Later on, Stone duality has been extended to the more general case of distributive lattices by Priestley [19]. The two above mentioned are the prototypical examples of dualities obtained via *dualizing* objects and will be both recalled and constructively used in the present work.

These kind of dualities have an intrinsic value: they are indeed a way of describing the very same mathematical object from two different perspectives, the target category and its dual. More generally, dualities between algebraic structures and corresponding topological spaces may open the way to applications as algebraic problems can possibly be translated into topological ones, or new insights can be obtained via the representation of a particular algebra as an algebra of continuos functions over a certain space (for a more detailed exposition of applications see [3, 2]).

The starting point of our analysis is the duality established by Gierz and Romanowska [4] between distributive bisemilattices and compact totally disconnected partially ordered left normal bands with constants, which, for sake of compactness, we refer to as GR spaces. The duality is obtained via the usual strategy of finding a suitable candidate to play the role of dualizing, or schizofrenic, object. However, the relevance of the result lies mainly in the use, for the first time, of Płonka sums as an essential tool for stating the duality.

Our aim is to provide a duality between the categories of involutive bisemilattices and those topological spaces, here christened as GR spaces with involution. The

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former consists of a class of algebras introduced and extensively studied in [1] as algebraic semantics (although not equivalent) for paraconsistent weak Kleene logic. The logical interests around these structures is relatively recent; on the other hand, it is easily checked that involutive bisemilattices, as introduced in [1], are equivalent to the regularization of the variety of Boolean algebras, axiomatized by Płonka [15, 16].[1] For this reason, involutive bisemilattices are strictly connected to Boolean algebras as they are representable as Płonka sums of Boolean algebras.

The present work consists of two main results. On the one hand, taking advantage of the Płonka sums representation in terms of Boolean algebras and Stone duality, we are able to describe the dual space of an involutive bisemilattice as semilattice inverse systems of Stone spaces (Theorem 4.8). On the other hand, we generalize Gierz and Romanowska duality by considering GR spaces with involution as an additional operation (Theorem 4.17). As a byproduct of our analysis we get a topological description of *semilattice inverse systems* of Stone spaces (Corollary 4.18).
The paper is structured as follows. In Section 2 we summarize all the necessary notions and known results about bisemilattices, Gierz and Romanowska duality and involutive bisemilattices. In Section 3 we the categories of *semilattice* direct and inverse systems, proving that, when constructed using dually equivalent categories, they are also dually equivalent. In Section 4, we introduce GR spaces with involution and prove the main results. Finally, in Section 5 we make some considerations about categories admitting both topological duals and a representation in terms of Płonka sums. By using Priestley duality, we then extend our results to the category of distributive bisemilattices.

2. Preliminaries

A *distributive bisemilattice* is an algebra $\mathbf{A} = \langle A, +, \cdot \rangle$ of type $\langle 2, 2 \rangle$ such that both $+$ and $\cdot$ are idempotent, associative and commutative operations and, moreover, $+$ ($\cdot$ respectively) distributes over $\cdot$ ($+$ respectively). Distributive bisemilattices, originally called "quasi-lattices", have been introduced by Płonka [13]; nowadays, similar structures are studied in a more general setting under the name of Birkhoff systems (see [6], [7]). Throughout the paper we will refer to these algebras simply as *bisemilattices*. Observe that every distributive lattice is an example of bisemilattice and every semilattice is a bisemilattice, where the two operations coincide. Any bisemilattice induces two different partial orders, namely $x \leq_{\cdot} y$ iff $x \cdot y = x$ and $x \leq_{+} y$ iff $x + y = y$.

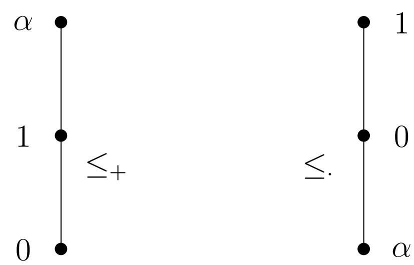
Example 2.1. The 3-element algebra $\mathbf{3} = \langle \{0, 1, \alpha\}, \cdot, + \rangle$, whose operations are defined by the so-called *weak Kleene tables* (given below), is the most prominent example of bisemilattice, as it generates the variety of (distributive) bisemilattices [9].

[1] The authors were not aware of some of the mentioned results by Płonka when writing [1].

$\cdot$	0	α	1
0	0	α	0
α	α	α	α
1	0	α	1

$+$	0	α	1
0	0	α	1
α	α	α	α
1	1	α	1

The two partial orders induced by **3** are displayed in the following Hasse diagrams:



A duality for bisemilattices has been established in [4], by using **3** as *dualizing* object. We recall here all the notions needed to state the main result.
A *left normal band* is an idempotent semigroup $\langle A, *\rangle$ satisfying the additional identity $x*(y*z) \approx x*(z*y)$, which is weak form of commutativity. A left normal band can be equipped with a partial order.

Definition 2.2. A *partially ordered left normal band* is an algebra $\mathbf{A} = \langle A, *, \leq\rangle$ such that

i) $\langle A, *\rangle$ is a left normal band
ii) $\langle A, \leq\rangle$ is a partially ordered set
iii) if $x \leq y$ then $x*z \leq y*z$ and $z*x \leq z*y$
iv) $x*y \leq x$

In any partially ordered left normal band it is possible to define a second partial order via $*$ and $\leq$: $a \sqsubseteq b$ iff $a*b \leq b$ and $b*a = b$. A partially ordered left normal band may be also extended by adding constants.

Definition 2.3. A *partially ordered left normal band with constants* is an algebra $\mathbf{A} = \langle A, *, \leq, c_0, c_1, c_\alpha\rangle$ such that $\langle A, *, \leq\rangle$ is a partially ordered left normal band and c_0, c_1 and c_α are constants satisfying

(1) $x*c_\alpha = c_\alpha * x = c_\alpha$
(2) $x*c_0 = x*c_1 = x$
(3) $c_0 \sqsubseteq x \leq c_1$ and $c_\alpha \leq x \sqsubseteq c_\alpha$
(4) if $c_0 * x = c_1 * x$ then $x = c_\alpha$

Definition 2.4. A *GR space* is a structure $\mathbf{A} = \langle A, *, \leq, c_0, c_1, c_\alpha, \tau\rangle$, such that $\langle A, *, \leq, c_0, c_1, c_\alpha\rangle$ is a partially ordered left normal band with constants and τ is a topology making $* : A \times A \to A$ a continous map and $\langle A, \leq, \tau\rangle$ is a compact *totally order disconnected* space[2].

Example 2.5. The support set of $\mathbf{3}$, namely $\{0, 1, \alpha\}$ equipped with the discrete topology, where $\leq \equiv \leq_{\cdot}$, $c_0 = 0$, $c_1 = 1$, $c_\alpha = \alpha$ and $*$ is defined as follows:

$$a * b = \begin{cases} a & \text{if } b \neq \alpha \\ b & \text{otherwise} \end{cases}$$

is a GR space (it is not difficult to check that operation $a * b = a + a \cdot b = a \cdot (a + b)$ and that the induced order $\sqsubseteq$ coincides with $\leq_{+}$).

We call $\mathfrak{DB}$ the category of bisemilattices (whose morphisms are homomorphisms of bisemilattices) and $\mathfrak{GR}$ the category of GR spaces (whose morphisms are continuous maps preserving $*$, constants and the order). The above mentioned duality is stated as follows:

Theorem 2.6. [4, Theorem 7.5] *The categories* $\mathfrak{DB}$ *and* $\mathfrak{GR}$ *are dual to each other under the invertible functor* $\mathrm{Hom_b}(-, \mathbf{3}) : \mathfrak{DB} \to \mathfrak{GR}$ *and its inverse* $\mathrm{Hom_{GR}}(-, \mathbf{3}) : \mathfrak{GR} \to \mathfrak{DB}$*.*

In detail, given a bisemilattice $\mathbf{S}$, its dual GR space is $\widehat{\mathbf{S}} = \mathrm{Hom_b}(\mathbf{S}, \mathbf{3})$, i.e. the space of the homomorphisms (of bisemilattices) from $\mathbf{S}$ to $\mathbf{3}$. Analogously, if $\mathbf{A}$ is a GR space, then the dual is given by $\widehat{\mathbf{A}} = \mathrm{Hom_{GR}}(\mathbf{A}, \mathbf{3})$, the bisemilattice of morphisms of $\mathfrak{GR}$.

The isomorphism between $\mathbf{S}$ and $\widehat{\widehat{\mathbf{S}}}$ is given by:

$$\varepsilon_S : \mathbf{S} \to \widehat{\widehat{\mathbf{S}}}, x \mapsto \varepsilon_S(x), \varepsilon_S(x)(\varphi) = \varphi(x), \tag{1}$$

for every $x \in S$ and $\varphi \in \widehat{S}$.

Analogously, for $\mathbf{A}$ and $\widehat{\widehat{\mathbf{A}}}$, the isomorphism is given by:

$$\delta_A : \mathbf{A} \to \widehat{\widehat{\mathbf{A}}}, x \mapsto \delta_A(x), \delta_A(x)(\varphi) = \varphi(x), \tag{2}$$

for every $x \in A$ and $\varphi \in \widehat{A}$.

The class of *involutive bisemilattices* has been introduced in [1] as the most suitable candidate to be the algebraic counterpart of PWK logic.

Definition 2.7. An *involutive bisemilattice* is an algebra $\mathbf{B} = \langle B, \cdot, +, ', 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ satisfying:

I1. $x + x \approx x$;

I2. $x + y \approx y + x$;

[2]A topological space is totally order disconnected if (1) $\{(a, b) \in A \times A : a \leq b\}$ is closed; (2) if $a \not\leq b$ then there is an open and closed lower set U such that $b \in U$ and $a \notin U$.

I3. $x + (y + z) \approx (x + y) + z$;
I4. $(x')' \approx x$;
I5. $x \cdot y \approx (x' + y')'$;
I6. $x \cdot (x' + y) \approx x \cdot y$;
I7. $0 + x \approx x$;
I8. $1 \approx 0'$.

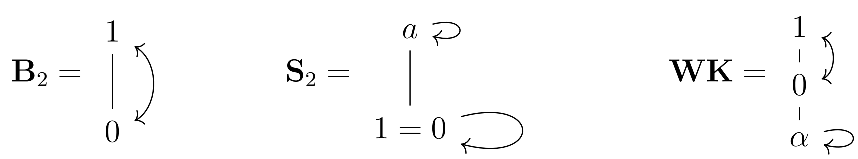
We denote the variety of involutive bisemilattices by $\mathcal{IBSL}$.

Every involutive bisemilattice has, in particular, the structure of a join semilattice with zero, in virtue of axioms (I1)–(I3) and (I7). More than that, it is possible to prove [1, Proposition 20] that $\cdot$ distributes over $+$ and viceversa, therefore the reduct $\langle B, +, \cdot \rangle$ is a bisemilattice. Notice that, in virtue of axioms (I5) and (I8), the operations $\cdot$ and 1 are completely determined by $+$, $'$, and 0. It is not difficult to check that every involutive bisemilattice has also the structure of a meet semilattice with 1, and that the equations $x+y \approx (x' \cdot y')'$, $x+y \approx x+(x' \cdot y)$ are satisfied. There are different equivalent ways to define involutive bisemilattices: it is not difficult to check that $\mathcal{IBSL}$ corresponds to the regularization of the variety of Boolean algebras described in [17].

Example 2.8. Every Boolean algebra, in particular the 2-element Boolean algebra $\mathbf{B}_2$, is an involutive bisemilattice. Also, the 2-element semilattice with zero, which we call $\mathbf{S}_2$, endowed with identity as its unary fundamental operation, is an involutive bisemilattice. The most prominent example of involutive bisemilattice is the 3-element algebra $\mathbf{WK}$, which is obtained by expanding the language of $\mathbf{3}$ with an involution behaving as follows:

$'$	
1	0
α	α
0	1

Upon considering the partial order $\leq$. induced by the product in its bisemilattice reduct, it becomes a 3-element chain with α as its bottom element. $\mathbf{B}_2$, $\mathbf{S}_2$ and $\mathbf{WK}$ can be represented by means of the following Hasse diagrams (the dashes represent the order, while the arrows represent the negation):



It is not difficult to verify that $\mathbf{B}_2$ is a subalgebra of $\mathbf{WK}$, while $\mathbf{S}_2$ is a quotient.

Although the algebra **WK** allows to define the logic PWK (upon setting $\{1,\alpha\}$ as designated values), its relevance is a consequence also of the fact that it generates the variety $\mathcal{IBSL}$, [1, Corollary 31]. This result can be also proved, observing that involutive bisemilattices coincide with the regularization of Boolean algebras, axiomatized in [16], and, due to [10], the only subdirectly irreducible members of the class are $\mathbf{B}_2$, $\mathbf{S}_2$ and **WK**.

As the main focus of this paper is introducing a duality for involutive bisemilattices, it is useful to recall here the definition of dual categories. We assume the reader is familiar with the concepts of category and morphism (in a category).

Definition 2.9. Two categories $\mathfrak{C}$ and $\mathfrak{D}$ are *equivalent* provided there exist two covariant functors, $\mathcal{F}:\mathfrak{C}\to\mathfrak{D}$ and $\mathcal{G}:\mathfrak{D}\to\mathfrak{C}$ such that $\mathcal{G}\circ\mathcal{F}=id_{\mathfrak{C}}$ and $\mathcal{F}\circ\mathcal{G}=id_{\mathfrak{D}}$.

Whenever the functors considered in the above definition are *controvariant* (instead of covariant), the two categories $\mathfrak{C}$ and $\mathfrak{D}$ are said to be *dually equivalent* or, briefly, *duals*.

3. The categories of semilattice inverse and direct systems

In this section we are going to describe a very general procedure to construct dualities for algebraic structures admitting a Płonka sum representation.

For our purposes, we need to strengthen the well known concepts of inverse and direct system of a category, hence we introduce the notions of semilattice inverse and semilattice direct systems in a very direct and way. For sake of simplicity, we opt for presenting these topics following the current trend in algebraic topology (see [11] for details).

Definition 3.1. Let $\mathfrak{C}$ be an arbitrary category, a *semilattice inverse system* in the category $\mathfrak{C}$ is a tern $\mathcal{X}=\langle X_i,p_{ii'},I\rangle$ such that

(i) I is a join semilattice with lower bound;
(ii) for each $i\in I$, X_i is an object in $\mathfrak{C}$;
(iii) $p_{ii'}:X_{i'}\to X_i$ is a morphism of $\mathfrak{C}$, for each pair $i\leqslant i'$, satisfying that p_{ii} is the identity in X_i and such that $i\leq i'\leq i''$ implies $p_{ii'}\circ p_{i'i''}=p_{ii''}$.

I is called the *index set* of the system $\mathcal{X}$, X_i are the *terms* and $p_{ii'}$ are referred to as *bonding morphisms* of $\mathcal{X}$. For convention, we indicate with $\vee$ the semilattice operation on I, $\leq$ the induced order and i_0 the lower bound in I.

The only difference making an inverse system a semilattice inverse system is the requirement on the index set to be a semilattice with lower bound instead of a directed preorder.

Definition 3.2. Given two semilattice inverse systems $\mathcal{X}=\langle X_i,p_{ii'},I\rangle$ and $\mathcal{Y}=\langle Y_j,q_{jj'},J\rangle$, a *morphism* between $\mathcal{X}$ and $\mathcal{Y}$ is a pair (φ,f_j) such that

i) $\varphi:J\to I$ is a semilattice homomorphism;
ii) for each $j\in J$, $f_j:X_{\varphi(j)}\to Y_j$ is a morphism in $\mathfrak{C}$, such that whenever $j\leq j'$, then the diagram in Fig.3 commutes.

$$\begin{tikzcd}
X_{\varphi(j)} \arrow[d, "f_j"'] & X_{\varphi(j')} \arrow[l, "p_{\varphi(j)\varphi(j')}"'] \arrow[d, "f_{j'}"] \\
Y_j & Y_{j'} \arrow[l, "q_{jj'}"]
\end{tikzcd}$$

FIGURE 1. The commuting diagram defining morphisms of semilattice inverse systems.

Notice that, for morphisms of semilattice inverse systems, the assumption that $\varphi : J \to I$ is a (semilattice) homomorphism implies that whenever $j \leq j'$ then $\varphi(j) \leq \varphi(j')$. Given three semilattice inverse systems $\mathcal{X} = \langle X_i, p_{ii'}, I\rangle$, $\mathcal{Y} = \langle Y_j, q_{jj'}, J\rangle$, $\mathcal{Z} = \langle Z_k, r_{kk'}, K\rangle$, the composition of morphisms is defined in the same way as for inverse systems.

Lemma 3.3. *The composition of morphisms between semilattice inverse systems is a morphism.*

Proof. Let $(\varphi, f_j) : \mathcal{X} \to \mathcal{Y}$, $(\psi, g_k) : \mathcal{Y} \to \mathcal{Z}$, then $(\chi, h_k) = (\psi, g_k)(\varphi, f_j) : \mathcal{X} \to \mathcal{Z}$ is $\chi = \varphi\psi$, $h_k = g_k f_{\chi(k)}$. χ is the composition of two (semilattice) homomorphisms, hence it is a semilattice homomorphism. The claim follows from the commutativity of the following diagram (we omitted the indexes for the maps p, q, r, f, g to make the notation less cumbersome)

$$\begin{tikzcd}
X_{\chi(k)} \arrow[d, "f"'] & X_{\chi(k')} \arrow[l, "p"'] \arrow[d, "f"] \\
Y_{\psi(k)} \arrow[d, "g"'] & Y_{\psi(k')} \arrow[l, "q"] \arrow[d, "g"] \\
Z_k & Z_{k'} \arrow[l, "r"]
\end{tikzcd}$$

$\square$

Proposition 3.4. *Let $\mathfrak{C}$ an arbitrary category. Then* Sem-inv-$\mathfrak{C}$ *is the category whose objects are semilattice inverse systems in $\mathfrak{C}$ with morphisms as defined above.*

Proof. The composition of morphisms between systems is associative and the identity morphism is $(1_I, 1_i)$, where $1_I : I \to I$ is the identity homomorphism on I and $1_i : X_i \to X_i$ is the identity morphism in the category $\mathfrak{C}$. $\square$

The category of *semilattice direct systems* of a given category $\mathfrak{C}$ is obtained by reversing morphisms of Sem-inv-$\mathfrak{C}$ as follows:

Definition 3.5. Let $\mathfrak{C}$ be an arbitrary category. A *semilattice direct system* in $\mathfrak{C}$ is a triple $\mathbb{X} = \langle X_i, p_{ii'}, I\rangle$ such that

(i) I is a join semilattice with least element.
(ii) X_i is an object in $\mathfrak{C}$, for each $i \in I$;
(iii) $p_{ii'} : X_i \to X_{i'}$ is a morphism of $\mathfrak{C}$, for each pair $i \leqslant i'$, satisfying that p_{ii} is the identity in X_i and such that $i \leq i' \leq i''$ implies $p_{i'i''} \circ p_{ii'} = p_{ii''}$.

We call I, X_i, the index set and the terms of the direct system, respectively, while we refer to $p_{ii'}$ as *transition morphisms* to stress the crucial difference with respect to inverse systems.

A morphism between two semilattice direct systems $\mathbb{X}$ and $\mathbb{Y}$ is a pair $(\varphi, f_i) : \mathbb{X} \to \mathbb{Y}$ s. t.

i) $\varphi : I \to J$ is a semilattice homomorphism
ii) $f_i : X_i \to Y_{\varphi(i)}$ is a morphism of $\mathfrak{C}$, making the following diagram commutative for each $i, i' \in I$, $i \leq i'$:

$$\begin{tikzcd} X_i \arrow[r, "p_{ii'}"] \arrow[d, "f_i"'] & X_{i'} \arrow[d, "f_{i'}"] \\ Y_{\varphi(i)} \arrow[r, "q_{\varphi(i)\varphi(i')}"'] & Y_{\varphi(i')} \end{tikzcd}$$

FIGURE 2. The commuting diagram defining morphisms of semilattice direct systems.

The composition of two morphisms is defined as $(f_i, \varphi)(g_j, \psi) = (h_i, \chi)$,

$$\chi = \psi\varphi, \qquad h_i = g_{\varphi(i)} f_i : X_i \to Z_{\chi(i)}.$$

It is easily verified that the composition (h_i, χ) is a morphism and it is associative and that the element $(1_I, 1_i)$, where $1_I : I \to I$ is the identity map on I and $1_i : X_i \to X_i$ is the identity morphism in $\mathfrak{C}$, is the identity morphism between semilattice direct systems. Therefore semilattice direct systems form a category which we will call Sem-dir-$\mathcal{C}$.

In the remaining part of this section we aim to show that the categories of semilattice direct and semilattice inverse systems of dually equivalent categories are dually equivalent. In order to do that, given a controvariant functor $\mathcal{F}\colon \mathfrak{C} \to \mathfrak{D}$ between two categories $\mathfrak{C}$ and $\mathfrak{D}$, we define a new functor $\widetilde{\mathcal{F}}\colon$ Sem-dir-$\mathfrak{C} \to$ Sem-inv-$\mathfrak{D}$ as follows

$$\widetilde{\mathcal{F}}(\mathbb{X}) := \langle \mathcal{F}(X_i), \mathcal{F}(p_{ii'}), I\rangle \qquad \widetilde{\mathcal{F}}(\varphi, f_i) := (\varphi, \mathcal{F}(f_i)), \tag{3}$$

where $\mathbb{X} = \langle X_i, p_{ii'}, I\rangle$ is an object and (φ, f_i) a morphism in the category Sem-dir-$\mathfrak{C}$.

Similarly, whenever $\mathcal{G}\colon \mathfrak{D} \to \mathfrak{C}$ is a controvariant functor, then we define $\widetilde{\mathcal{G}}\colon$ Sem-inv-$\mathfrak{D} \to$ Sem-dir-$\mathfrak{C}$ as in (3). The crucial point is proving that the new maps are indeed functors.

Lemma 3.6. *Let $\mathcal{F} : \mathfrak{C} \to \mathfrak{D}$ be a controvariant functor between two categories $\mathfrak{C}$ and $\mathfrak{D}$. Then:*

(1) $\widetilde{\mathcal{F}}$ is a controvariant functor between Sem-dir-$\mathfrak{C}$ and Sem-inv-$\mathfrak{D}$;

(2) $\widetilde{\mathcal{G}}$ is a controvariant functor between Sem-inv-$\mathfrak{C}$ and Sem-dir-$\mathfrak{D}$.

Proof. Proof of (1) and (2) are essentially analogous, so we just give the details of (1): the reader can check that they can easily adapted to prove (2).
Assume that $\mathbb{X} = \langle X_i, p_{ii'}, I\rangle$ is an object in Sem-dir-$\mathfrak{C}$. We first show that $\widetilde{\mathcal{F}}(\mathbb{X})$ is an object in Sem-inv-$\mathfrak{D}$, namely it satisfies conditions (i), (ii), (iii) of Definition 3.1. Recall that, by (3), $\widetilde{\mathcal{F}}(\mathbb{X}) := \langle \mathcal{F}(X_i), \mathcal{F}(p_{ii'}), I\rangle$.

(i) is clearly satisfied as I is a semilattice with lower bound;

(ii) Since X_i is an object in $\mathfrak{C}$ and $\mathcal{F}$ a functor, $\mathcal{F}(X_i)$ is an object of $\mathfrak{D}$;

(iii) Let $i \leq i'$. Then there exists a morphism $p_{ii'} : X_i \to X_{i'}$ of $\mathfrak{C}$ such that p_{ii} is the identity in X_i and moreover, if $i \leq i' \leq i''$ then $p_{i'i''} \circ p_{ii'} = p_{ii''}$. Since $\mathcal{F}$ is a controvariant functor, $\mathcal{F}(p_{ii'}) : \mathcal{F}(X_{i'}) \to \mathcal{F}(X_i)$ is a morphism of $\mathfrak{D}$. Moreover, $\mathcal{F}(p_{ii}) = \mathcal{F}(1_{\mathfrak{C}}) = 1_{\mathfrak{D}}$ and compositions are obviously preserved.

This shows that $\widetilde{\mathcal{F}}(\mathbb{X})$ is an object. Now, suppose that (φ, f_i) is a morphism in Sem-dir-$\mathfrak{C}$ between two objects $\mathbb{X} = \langle X_i, p_{ii'}, I\rangle$, $\mathbb{Y} = \langle Y_j, p_{jj'}, J\rangle$. We show that $\widetilde{\mathcal{F}}(\varphi, f_i) := (\varphi, \mathcal{F}(f_i))$ is a morphism from $\widetilde{\mathcal{F}}(\mathbb{Y}) = \langle \mathcal{F}(Y_j), \mathcal{F}(q_{jj'}), J\rangle$ to $\widetilde{\mathcal{F}}(\mathbb{X}) = \langle \mathcal{F}(X_i), \mathcal{F}(p_{ii'}), I\rangle$ (which also assures that $\widetilde{\mathcal{F}}$ is controvariant). We check that $\widetilde{\mathcal{F}}(\varphi, f_i)$ satisfies the properties i) and ii) in Definition 3.2.

i) clearly holds as $\varphi : I \to J$ is a semilattice homorphism from the index set of $\widetilde{\mathcal{F}}(\mathbb{X})$ to the index set of $\widetilde{\mathcal{F}}(\mathbb{Y})$;

ii) For every $i \in I$, $f_i : X_i \to Y_{\varphi(i)}$ is a morphism in $\mathfrak{C}$ making the Diagram in Fig.1 commutative. Therefore, $\mathcal{F}(f_i) : \mathcal{F}(Y_{\varphi(i)}) \to \mathcal{F}(X_i)$ is a morphism in $\mathfrak{D}$. Suppose that $i \leq i'$, for some $i, i' \in I$, then $p_{ii'} : X_i \to X_{i'}$ and $\mathcal{F}(p_{ii'}) : \mathcal{F}(X_{i'}) \to \mathcal{F}(X_i)$ is the correspondent morphism in $\mathfrak{D}$. Since φ is a semilattice homomorphism, we have $\varphi(i) \leq \varphi(i')$, and $\widetilde{\mathcal{F}}(\mathbb{X})$ is a semilattice inverse system, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(Y_{\varphi(i)}) \arrow[d, "\mathcal{F}(f_i)"'] & & \mathcal{F}(Y_{\varphi(i')}) \arrow[ll, "\mathcal{F}(q_{\varphi(i)\varphi(i')})"'] \arrow[d, "\mathcal{F}(f_{i'})"] \\
\mathcal{F}(X_i) & & \mathcal{F}(X_{i'}) \arrow[ll, "\mathcal{F}(p_{ii'})"]
\end{tikzcd}$$

This conclude our claim. $\square$

Notice that the statement of Lemma 3.6 is false when considering *covariant* functors instead of controvariant as shown by the following example.

Example 3.7. Let $\mathfrak{C}$ be an algebraic category, $\mathfrak{Set}$ the category of sets and $\mathcal{F} \colon \mathfrak{C} \to \mathfrak{Set}$ the forgetful functor. For any object $\mathbb{X} = \langle X_i, p_{ii'}, I \rangle$ in Sem-dir-$\mathfrak{C}$, $\widetilde{\mathcal{F}}(\mathbb{X})$ is not an object in Sem-inv-$\mathfrak{Set}$. Indeed, for any two indexes such that $i \leq i'$, we have a morphism in $\mathfrak{C}$, $p_{ii'} \colon X_i \to X_{i'}$; since $\mathcal{F}$ is covariant, $\widetilde{\mathcal{F}}(p_{ii'}) = \mathcal{F}(p_{ii'})$ is a function (a morphism in $\mathfrak{Set}$) from $\mathcal{F}(X_i)$ to $\mathcal{F}(X_{i'})$, hence it does not fulfill condition (iii) in Definition 3.1.

Theorem 3.8. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be dually equivalent categories. Then* Sem-dir-$\mathfrak{C}$ *and* Sem-inv-$\mathfrak{D}$ *are dually equivalent.*

Proof. By hypothesis we have two controvariant functors $\mathcal{F}$ and $\mathcal{G}$

$$\begin{tikzcd}
\mathfrak{C} \arrow[r, bend left, "\mathcal{F}"] & \mathfrak{D} \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}$$

such that such that $\mathcal{G} \circ \mathcal{F} = id_{\mathfrak{C}}$ and $\mathcal{F} \circ \mathcal{G} = id_{\mathfrak{D}}$. By Lemma 3.6 we have controvariant functors $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$

$$\begin{tikzcd}
\text{Sem-dir-}\mathfrak{C} \arrow[r, bend left, "\widetilde{\mathcal{F}}"] & \text{Sem-inv-}\mathfrak{D} \arrow[l, bend left, "\widetilde{\mathcal{G}}"]
\end{tikzcd}$$

We only need to check that the compositions $\widetilde{\mathcal{G}} \circ \widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}} \circ \widetilde{\mathcal{G}}$ are the identities in the categories Sem-dir-$\mathfrak{C}$ and Sem-inv-$\mathfrak{D}$, respectively. Let $\mathbb{X} = \langle X_i, p_{ii'}, I\rangle$ be an object in Sem-dir-$\mathfrak{C}$. Then

$$\widetilde{\mathcal{G}}(\widetilde{\mathcal{F}}(\mathbb{X})) = \widetilde{\mathcal{G}}(\langle \mathcal{F}(X_i), \mathcal{F}(p_{ii'}), I\rangle) = \langle \mathcal{G} \circ \mathcal{F}(X_i), \mathcal{G} \circ \mathcal{F}(p_{ii'}), I\rangle = \langle X_i, p_{ii'}, I\rangle,$$

where the last equality is obtained by $\mathcal{G} \circ \mathcal{F} = id_{\mathfrak{C}}$. It is analogous to verify that $\widetilde{\mathcal{F}} \circ \widetilde{\mathcal{G}}$ is the identity. □

The above result somehow resembles *semilattice-based dualities* establish by Romanowska and Smith in [21, 22], where the authors essentially show how to lift a duality between two categories, in particular an algebraic category and its dual representation spaces, to a duality involving the correspondent semilattice representations. The duality in Theorem 3.8 is also "based" on certain semilattice systems. However, the two approaches are characterized by a substantial difference: Romanowska and Smith indeed consider, from one side, the semilattice sum of an algebraic category but, on the other, the semilattice representation of the dual spaces, and thus the duality, is obtained by *dualizing* the semilattice of the index sets. In order to achieve this, they rely on the duality due to Hofmann, Mislove and Stralka [8] for semilattices (see also [3] for details). This means, that the semilatttice representation of the dual spaces (of the considered categories) is constructed via compact topological semilattices with 0 which carries the Boolean topology (namely makes the space compact, Hausdorff and totally disconnected).

4. The category of Involutive Bisemilattices and its dual

Płonka introduced [12, 14, 16] a construction to build algebras out of semilattice systems of algebras[3], see also [18, 23]

Definition 4.1. Let $\mathbb{A} = \langle \mathbf{A}_i, \varphi_{ii'}, I\rangle$ be a semilattice direct system of algebras $\mathbf{A}_i = \langle A_i, f_t\rangle$ of a fixed type ν, then the *Płonka sum* over $\mathbb{A}$ is the algebra $\mathcal{P}_l(\mathbb{A}) = \langle \bigsqcup_I A_i, g_t^{\mathcal{P}}\rangle$, whose universe is the disjoint union and the operations $g_t^{\mathcal{P}}$ are defined as follows: for every n-ary $g_t \in \nu$, and $a_1, \dots, a_n \in \bigsqcup_I A_i$, where $n \geqslant 1$ and $a_r \in A_{i_r}$, we set $j = i_1 \vee \dots \vee i_n$ and define

$$g_t^{\mathcal{P}}(a_1, \dots, a_n) = g_t^{\mathbf{A}_j}(\varphi_{i_1 j}(a_1), \dots, \varphi_{i_n j}(a_n)).$$

In case ν contains constants, then, for every constant $g \in \nu$, we define $g^{\mathcal{P}} = g^{\mathbf{A}_{i_0}}$.

Involutive bisemilattices, as well as bisemilattices admits a representation as Płonka sums over a semilattice sistem of Boolean algebras (this was already proved in [15, 16]).

Theorem 4.2 ([1, Thm. 46])**.**

1) *If $\mathbb{A}$ is a semilattice direct system of Boolean algebras, then the $\mathcal{P}_l(\mathbb{A})$ is an involutive bisemilattice.*

[3]It is essential to have a semilattice with lower bound (instead of a pointless semilattice) when working with algebras having constants, see [16].

2) If $\mathbf{B}$ *is an involutive bisemilattice, then* $\mathbf{B}$ *is isomorphic to the Płonka sum over a semilattice direct system of Boolean algebras*[4].

The above result states that every involutive bisemilattice admits a unique representation as Płonka sum of Boolean algebras. We summarize here the categories we are dealing with

Category	**Objects**	**Morphisms**
$\mathfrak{BA}$	Boolean Algebras	Homomorphisms of $\mathcal{BA}$
$\mathfrak{IBSL}$	Involutive bisemilattices	Homomorphisms of $\mathcal{IBSL}$
Sem-dir-$\mathfrak{BA}$	semilattice direct systems of B.A.	Morphisms of Sem-dir-$\mathfrak{BA}$
$\mathfrak{SA}$	Stone spaces	continuous maps
Sem-inv-$\mathfrak{SA}$	semilattice inverse systems of Stone sp.	Morphisms of Sem-inv-$\mathfrak{SA}$

Theorem 4.2 states that the objects of the category $\mathfrak{IBSL}$ are isomorphic to the objects of the category Sem-dir-$\mathfrak{BA}$. We aim at proving more, namely that they are also equivalent as categories. In order to establish this, we prove the following auxiliary lemmata.

Lemma 4.3. *Let* $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I\rangle$ *and* $\mathbb{B} = \langle \mathbf{B}_j, q_{jj'}, J\rangle$ *be semilattice direct systems of Boolean algebras. Let* $\mathbf{A} = \mathcal{P}_l(\mathbb{A})$, $\mathbf{B} = \mathcal{P}_l(\mathbb{B})$ *and* $h : \mathbf{A} \to \mathbf{B}$ *a homomorphism, then for any* $i \in I$ *there exists a* $j \in J$ *such that*

(1) $h(A_i) \subseteq B_j$

(2) $h_{|A_i}$ *is a Boolean homomorphism from* $\mathbf{A}_i$ *into* $\mathbf{B}_j$

Proof. (1) As first notice that, from the construction of Płonka sums, we have that for any $x \in A_i$, also $x' \in A_i$. Consequently, for any $h(x) \in B_j$, for a certain $j \in J$, then also $h(x)' \in B_j$. Let $a \in A_i$ for some $i \in I$, then there exists a $j \in J$ such that $h(a) \in B_j$. Therefore $h(0_{A_i}) = h(a \wedge a') = h(a) \wedge h(a') = h(a) \wedge h(a)' = 0_{B_j}$, where the last equality holds since $h(a)$ and $h(a)'$ belong to the same Boolean algebra B_j. Similarly, $h(1_{A_i}) = h(a \vee a') = h(a) \vee h(a') = h(a) \vee h(a)' = 1_{B_j}$.

We now have to prove that for any $a \in A_i$, with $a \neq 0_{A_i}$ we have that $h(a) \in B_j$. Suppose, by contradiction, that $a \in A_i$, and $h(a) \in B_k$, with $j \neq k$. Then $0_{B_j} = h(0_{A_i}) = h(a \wedge a') = h(a) \wedge h(a') = h(a) \wedge h(a)' = 0_{B_k}$, which is impossible, as, by construction $B_j \cap B_k = \emptyset$, hence, necessarily $h(A_i) \subseteq B_j$.

(2) follows from the fact that h preserves joins, meets and complements by definition and we already proved that $h(0_{A_i}) = 0_{B_j}$ and $h(1_{A_i}) = 1_{B_j}$. □

Theorem 4.2 together with Lemma 4.3 state that $\mathcal{IBSL}-$homomorphisms are nothing but homomorphisms between the correspondent (unique) Płonka sum representations. The statement of Lemma 4.3 can be exposed more precisely saying that there exists a map $\varphi : I \to J$ such that for every homomorphism $h : \mathcal{P}_l(\mathbb{A}) \to \mathcal{P}_l(\mathbb{B})$,

[4]The form of the semilattice direct system used in the Płonka sum representation is not needed for the purposes of this work. For more details, the reader could refer to [1] or [17].

$h(A_i) \subseteq B_{\varphi(i)}$. It is not difficult to prove that such map is actually a semilattice homomorphism.

Lemma 4.4. *Let* $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I\rangle$ *and* $\mathbb{B} = \langle \mathbf{B}_j, q_{jj'}, J\rangle$ *be semilattice direct systems of Boolean algebras. Let* $\mathbf{A} = \mathcal{P}_l(\mathbb{A})$, $\mathbf{B} = \mathcal{P}_l(\mathbb{B})$, $h : \mathbf{A} \to \mathbf{B}$ *a homomorphism and* $\varphi_h : I \to J$ *such that* $h(A_i) \subseteq B_{\varphi(i)}$. *Then* φ_h *is a semilattice homomorphism.*

Proof. Let $a_1 \in A_i$ and $a_2 \in A_{i'}$, with $i, i' \in I$; by definition of $\mathcal{P}_l(\mathbb{A})$, $a_1 \wedge a_2 \in A_{i\vee i'}$ and $h(a_1) \in B_{\varphi_h(i)}$, $h(a_2) \in B_{\varphi_h(i')}$, then $h(a_1 \wedge a_2) = h(a_1) \wedge h(a_2) \in B_{\varphi_h(i)\vee\varphi_h(i')}$. But since $h(a_1 \wedge a_2) \in B_{\varphi_h(i\vee i')}$, then necessarily $\varphi_h(i \vee i') = \varphi_h(i) \vee \varphi_h(i')$, i.e. φ_h is a semilattice homorphism. □

Lemma 4.5. *Let* $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I\rangle$ *and* $\mathbb{B} = \langle \mathbf{B}_j, q_{jj'}, J\rangle$ *be semilattice direct systems of Boolean algebras and* (φ, f_i) *a morphism from* $\mathbb{A}$ *to* $\mathbb{B}$. *Then* $h\colon \mathcal{P}_l(\mathbb{A}) \to \mathcal{P}_l(\mathbb{B})$, *defined as*

$$h(a) := f_i(a),$$

where $i \in I$ *is the index such that* $a \in A_i$, *is a homorphism of involutive bisemilattices.*

Proof. The map h is well defined for every $i \in I$, as by assumption f_i is homomorphism of Boolean algebras. We only have to check that h is compatible with all the operations of an involutive bisemilattice. To simplify the notation we set $\mathbf{A} = \mathcal{P}_l(\mathbb{A})$, $\mathbf{B} = \mathcal{P}_l(\mathbb{B})$.

As regards the constants (we give details of one of them only), let i_0 be the lower bound in I (if follows that $\varphi(i_0)$ is the lower bound in J), then $h(0) = f_{i_0}(0) = 0_{\varphi(i_0)} = 0$. Similarly, for negation, assume $a \in A_i$, for some $i \in I$: $h(\neg a) = f_i(\neg a) = \neg f_i(a) = \neg h(a)$.

As for binary operations (we consider $\wedge$ only as the case of $\vee$ is analogous), assume $a \in A_i$, $b \in A_j$ and set $k = i \vee j$:

$$\begin{gathered} h(a \wedge b)) = h(p_{ik}(a) \wedge^{\mathbf{A}_k} p_{jk}(b)) = f_k(p_{ik}(a) \wedge^{\mathbf{A}_k} p_{jk}(b)) = \\ = f_k(p_{ik}(a)) \wedge^{\mathbf{B}_{\varphi(k)}} f_k(p_{jk}(b)) = q_{\varphi(i)\varphi(k)}(f_i(a)) \wedge^{\mathbf{B}_{\varphi(k)}} q_{\varphi(j)\varphi(k)}(f_j(b)) = \\ = (f_i(a) \wedge f_j(b)) = (h(a) \wedge h(b)), \end{gathered}$$

where the equality in the second line is justified by the commutativity of the following diagram, which holds for $i, j \leq k$, as, by assumption, (φ, f_i) is morphism in Semdir-$\mathfrak{C}$).

$$\begin{tikzcd} \mathbf{A}_i \arrow[r, "p_{ik}"] \arrow[d, "f_i"'] & \mathbf{A}_k \arrow[d, "f_k"] \\ \mathbf{B}_{\varphi(i)} \arrow[r, "q_{\varphi(i)\varphi(k)}"'] & \mathbf{B}_{\varphi(k)} \end{tikzcd}$$

□

Theorem 4.6. *The categories $\mathfrak{IBSL}$ and Sem-dir-$\mathfrak{BA}$ are equivalent.*

Proof. The equivalence is proved by defining the following functors:

$$\begin{tikzcd} \mathfrak{IBSL} \arrow[rr, bend left, "\mathcal{F}"] & & \text{Sem-dir-}\mathfrak{BA} \arrow[ll, bend left, "\mathcal{G}"] \end{tikzcd}$$

$\mathcal{F}$ associates to an involutive bisemilattices $\mathbf{A} \cong \mathcal{P}_l(\mathbb{A})$ (due to Theorem 4.2), the semilattice direct system of Boolean algebras $\mathbb{A}$. On the other hand, for a semilattice direct system of Boolean algebras $\mathbb{A}$, we define $\mathcal{G}(\mathbb{A}) := \mathcal{P}_l(\mathbb{A})$.

We firstly check that $\mathcal{F}$ and $\mathcal{G}$ are controvariant functors.

Let $\mathbf{A}, \mathbf{B} \in \mathcal{IBSL}$ such that $\mathbf{A} \cong \mathcal{P}_l(\mathbb{A})$ and $\mathbf{B} \cong \mathcal{P}_l(\mathbb{B})$, with $\mathbb{A} = \langle \mathbf{A}_i, p_{ii'}, I \rangle$ and $\mathbb{B} = \langle \mathbf{B}_j, q_{jj'}, J \rangle$ semilattice direct systems of Boolean algebras. Then, for every IBSL-homomorphism $h\colon \mathbf{A} \to \mathbf{B}$, we define $\mathcal{F}(h) := (\varphi_h, h_{|A_i})$, where φ_h is the semilattice homomorphism defined in Lemma and $h_{|A_i}$ is the restriction of h on the Boolean components $\mathbf{A}_i$ of the Płonka sum corresponding to $\mathbf{A}$. Lemmas 4.3 4.4 guarantee that φ_h is a semilattice homomorphism and that $h_{|A_i}$ is a Boolean homomorphism in each component $\mathbf{A}_i$ of the Płonka sum $\mathcal{P}_l(\mathbb{A})$. Moreover, the following diagram is commutative for each $i \leq i'$ (notice that $i \leq i'$ implies $\varphi_h(i) \leq \varphi_h(i')$)

$$\begin{tikzcd} \mathbf{A}_i \arrow[r, "p_{ii'}"] \arrow[d, "h_{|A_i}"'] & \mathbf{A}_{i'} \arrow[d, "h_{|A_{i'}}"] \\ \mathbf{B}_{\varphi_h(i)} \arrow[r, "q_{\varphi_h(i)\varphi_h(i')}"'] & \mathbf{B}_{\varphi_h(i')} \end{tikzcd}$$

Therefore $\mathcal{F}(h)$ is a morphism from $\mathbb{A}$ to $\mathbb{B}$, showing that $\mathcal{F}$ is a covariant functor.

As concern $\mathcal{G}$, by Theorem 4.2 we know that $\mathcal{P}_l(\mathbb{A})$ is an involutive bisemilattice. Moreover, if $(\varphi, f_i)\colon \mathbb{A} \to \mathbb{B}$ is a morphism between the semilattice direct systems of Boolean algebras $\mathbb{A}$ and $\mathbb{B}$, then set $\mathcal{G}(\varphi, f_i) := h$, as defined in Lemma 4.5, which assures that $\mathcal{G}(\varphi, f_i)$ is a homomorphism from $\mathcal{P}_l(\mathbb{A})$ to $\mathcal{P}_l(\mathbb{B})$.

We are left with verifying that the composition of the two functors gives the identity in both categories. Let $\mathbf{A} \in \mathcal{IBSL}$, such that $\mathbf{A} \cong \mathcal{P}_l(\mathbb{A})$then

$$\mathcal{G}(\mathcal{F}(\mathbf{A})) = \mathcal{G}(\mathcal{F}(\mathcal{P}_l(\mathbb{A}))) = \mathcal{G}(\mathbb{A}) = \mathcal{P}_l(\mathbb{A}) = \mathbf{A};$$

$$\mathcal{F}(\mathcal{G}(\mathbb{A})) = \mathcal{F}(\mathcal{P}_l(\mathbb{A})) = \mathbb{A}.$$

□

Recall that a *Stone space* is topological space which is compact, Hausdorff and totally disconnected. Stone spaces can be viewed as a category, which we refer to as $\mathfrak{SA}$, with continuous maps as morphisms.

It is well known that the category of Stone spaces is the dual of the category of Boolean algebras [24]. The above statement, combined with Theorem 3.8, gives immediately the following

Corollary 4.7. *The categories Sem-dir-*$\mathfrak{BA}$ *and Sem-inv-*$\mathfrak{SA}$ *are dually equivalent.*

By Theorem 4.6, $\mathfrak{IBSL}$ is equivalent to the category of semilattice direct systems of Boolean algebras. Due to Corollary 4.7, we have then a first abstract characterization of the dual category of $\mathfrak{IBSL}$.

Theorem 4.8. *The category Sem-inv-*$\mathfrak{SA}$ *and* $\mathfrak{IBSL}$ *are dually equivalent.*

Theorem 4.8 gives a description of the dual category of involutive bisemilattices in terms of Stone spaces, i.e. the dual category of Boolean algebras, objects coming into play due to the representation Theorem 4.2.

The above theorem together with Theorem 4.2 should be compared with the following theorem due to Haimo [5], where *direct limits* are considered instead of Płonka sums. In the following statement, $\lim_{\longrightarrow}$, $\lim_{\longleftarrow}$ denote the direct and inverse limit, respectively.

Theorem 4.9 ([5], Th. 9)**.** *Let* $\{\mathbf{A}_i\}$ *be a direct system of Boolean algebras and* $\{\mathbf{A}_i^*\}$ *the corresponding family of Stone spaces. Then*

$$(\lim_{\longrightarrow} \mathbf{A}_i)^* \cong \lim_{\longleftarrow} \mathbf{A}_i^*.$$

In Theorem 4.17 (see below) we will give a *concrete* topological axiomatization of the dual space of an involutive bisemilattice via Gierz and Romanowska duality (see Theorem 2.6). Logically motivated by the fact that involutive bisemilattices are intrinsically characterized by an involutive negation, we aim at recovering a unary operation in the dual space: we expand GR spaces to GR spaces with involution.

Definition 4.10. A *GR space with involution* is a GR space $\mathbf{G}$ with a continous map $\neg : G \to G$ such that for any $a \in G$:

- **G1**. $\neg(\neg a) = a$
- **G2**. $\neg(a * b) = \neg a * \neg b$
- **G3**. if $a \leq b$ then $\neg b \sqsubseteq \neg a$
- **G4**. $\neg c_0 = c_1$, $\neg c_1 = c_0$ and $\neg c_\alpha = c_\alpha$
- **G5**. The space $\mathrm{Hom}_{\mathrm{GR}}(\mathbf{A}, \mathbf{3})$ (see Section 2) equipped with natural involution $\neg$, i.e. $\neg\varphi(a) = (\varphi(\neg a))'$ satisfies $\varphi \cdot (\neg\varphi + \psi) = \psi \cdot \varphi$, where operations are defined pointwise;
- **G6**. there exist $\varphi_0, \varphi_1 \in \mathrm{Hom}_{\mathrm{GR}}(\mathbf{A}, \mathbf{3})$ such that $\neg\varphi_0 = \varphi_1$ and $\varphi + \varphi_0 = \varphi$, for each $\varphi \in \mathrm{Hom}_{\mathrm{GR}}(\mathbf{A}, \mathbf{3})$.

Example 4.11. **WK** equipped with discrete topology is the canonical example of GR space with involution.

Definition 4.12. $\mathfrak{IGR}$ is the category whose objects are GR spaces with involution and whose morphisms are GR-morphisms preserving involution.

Given a GR space with involution $\mathbf{G}$, we can consider its GR space reduct (simply its involution free reduct), call it $\mathbf{A}$, which can be associated to the dual distributive bisemilattice $\widehat{\mathbf{A}} = \mathrm{Hom}_{\mathrm{GR}}(\mathbf{A}, \mathbf{3})$. Aiming at turning it into an involutive bisemilattice, we define an involution on $\widehat{\mathbf{A}}$ as follows:

$$\neg\Phi(a) = (\Phi(\neg a))',$$

for each $\Phi \in \widehat{\mathbf{A}}$ and $a \in G$, where $\neg$ and $'$ are the involutions of $\mathbf{G}$ and $\mathbf{WK}$, respectively.

Adopting the same idea, given an arbitrary involutive bisemilattice $\mathbf{I}$, we consider its bisemilattice reduct $\mathbf{S} = \langle I, +, \cdot \rangle$, which is distributive [1, Proposition 20], and therefore can be associated to its dual GR space, $\widehat{\mathbf{S}} = \mathrm{Hom}_{\mathrm{b}}(\mathbf{S}, \mathbf{3})$ (see Section 2). The bisemilattice $\mathbf{3}$ turns into $\mathbf{WK}$ just by adding the usual involution and the constants 0, 1, so it makes sense to define an involution on $\widehat{\mathbf{S}}$ as:

$$\neg\varphi(x) = (\varphi(x'))',$$

for any $\varphi \in \widehat{\mathbf{S}}$ and $x \in S$.

Lemma 4.13. *Let* $\mathbf{G}$ *be a GR-space with involution and* $\mathbf{A}$ *its GR-space reduct. Then, if* $\Phi \in \widehat{\mathbf{A}}$ *then* $\neg\Phi \in \widehat{\mathbf{A}}$*. Moreover,* $\widehat{\mathbf{G}} = \langle \widehat{\mathbf{A}}, \neg \rangle$ *is an involutive bisemilattice.*

Proof. Assuming that Φ is a morphism of GR spaces, we have to verify that also $\neg\Phi$ is, i.e. that it is a continuos map, preserving operation $*$, constants and the order $\leq$. Observe that $\neg\Phi$ is continuous as it is the composition of continuous maps.

Concerning operations and constants, we have:
$\neg\Phi(a * b) = (\Phi\neg(a * b))' = (\Phi(\neg a * \neg b))' = (\Phi(\neg a) * \Phi(\neg b))' = (\Phi(\neg a))' * (\Phi(\neg b))' = \neg\Phi(a) * \neg\Phi(b)$.
$\neg\Phi(c_0) = (\Phi(\neg c_0))' = (\Phi(c_1))' = 1' = 0$.

Similarly, $\neg\Phi(c_1) = (\Phi(\neg c_1))' = (\Phi(c_0))' = 0' = 1$ and $\neg\Phi(c_\alpha) = (\Phi(\neg c_\alpha))' = (\Phi(c_\alpha))' = \alpha' = \alpha$.

As for the order, let $a \leq b$, but then $\neg b \sqsubseteq \neg a$. Since Φ preserve both the orders, $\Phi(\neg b) \leq_+ \Phi(\neg a)$, thus $(\Phi(\neg a))' \leq_\cdot (\Phi(\neg a))'$, i.e. $\neg\Phi(a) \leq \neg\Phi(b)$.

To prove that $\widehat{G}$ is an involutive bisemilattice, we have to check that conditions **I1** to **I8** of Definition 2.7 hold for $\widehat{\mathbf{G}}$. Clearly, **I1**, **I2** and **I3** hold as $\widehat{\mathbf{A}}$ is a distributive bisemilattice, while **I6**, **I7** and **I8** hold by definition. For the remaining ones, let $\Phi, \Psi \in \widehat{\mathbf{A}}$ and $a \in A$.
I4. $\neg(\neg\Phi(a)) = \neg\Phi(\neg(a))' = \Phi(\neg\neg a)'' = \Phi(a)$.
I5. $\neg(\Phi + \Psi)(a) = (\Phi + \Psi(\neg a))' = (\Phi(\neg a) + \Psi(\neg a))' = ('\Phi(\neg a))' \cdot (\Psi(\neg a))' = \neg\Phi(a) \cdot \neg\Psi(a)$. ◻

Proposition 4.14. $\mathbf{G} \cong \hat{\hat{\mathbf{G}}}$.

Proof. We make good use of the duality established in [4], from which it follows $\mathbf{A} \cong \widehat{\widehat{\mathbf{A}}}$, where $\mathbf{A}$ is the GR space reduct of $\mathbf{G}$. To prove our claim we only have to prove that the isomorphism, given by (2), $\delta_{_A}(x)(\varphi) = \varphi(x)$, for $x \in A$ and $\varphi \in \widehat{\mathbf{A}}$, preserve the involution. This is easily checked, indeed

$$(\neg\delta_{_A}(x))(\varphi) = (\delta_{_A}(x)(\neg\varphi))' = (\neg\varphi(x))' = (\varphi(\neg x))'' = \varphi(\neg x).$$

□

Lemma 4.15. *Let* $\mathbf{I} \in \mathcal{IBSL}$ *with* $\mathbf{S}$ *its bisemilattice reduct. If* $\varphi \in \widehat{\mathbf{S}}$ *then* $\neg\varphi \in \widehat{\mathbf{S}}$. *Moreover,* $\widehat{\mathbf{I}} = \langle \widehat{\mathbf{S}}, \neg \rangle$ *is a GR space with involution.*

Proof. Suppose that $\varphi \in \widehat{\mathbf{S}}$, i.e. it is a map preserving sum and multiplication. It suffices to verify that also $\neg\varphi$ preserves the two operations. $\neg\varphi(x + y) = (\varphi(x + y)')' = (\varphi(x' \cdot y'))' = (\varphi(x') \cdot \varphi(y'))' = (\varphi(x'))' + (\varphi(y'))' = \neg\varphi(x) + \neg\varphi(y)$. For multiplication the proof runs analogously.

For the second part, by [4], we have that $\widehat{\mathbf{S}}$ is a GR space, thus we only have to check that $\neg$ has the required properties. Let $\varphi, \psi \in \widehat{\mathbf{S}}$ and $x \in S$; properties **G1** – **G4** can be easily verified as follows:

$$\neg(\neg\varphi(x)) = \neg(\varphi(x'))' = (\varphi(x''))'' = \varphi(x).$$

$$\neg(\varphi * \psi)(x) = (\varphi * \psi(x'))' = (\varphi(x') * \varphi(x'))' = (\varphi(x'))' * (\psi(x'))' = \neg\varphi(x) * \neg\psi(x).$$

Let $\varphi \leq \psi$, i.e. $\varphi(x) \leq_{\cdot} \psi(x)$ for each $x \in S$. In particular $\varphi(x') \leq_{\cdot} \psi(x')$, thus $(\psi(x'))' \leq_{+} (\varphi(x'))'$, i.e. $\neg\psi \sqsubseteq \neg\varphi$.

Let φ_0, φ_1 and φ_α the constant homorphisms (of bisemilattices) on 0, 1 and α, respectively. $\neg\varphi_0(x) = (\varphi_0(x'))' = 0' = 1 = \varphi_1(x)$; $\neg\varphi_1(x) = (\varphi_1(x'))' = 1' = 0 = \varphi_0(x)$; $\neg\varphi_\alpha(x) = (\varphi_\alpha(x'))' = \alpha' = \alpha = \varphi_\alpha(x)$.

In order to prove **G5** and **G6**, it is enough to show that $\mathbf{I} \cong \widehat{\widehat{\mathbf{I}}}$. Recall that the bisemilattice reduct $\mathbf{S}$ of $\mathbf{I}$ is isomorphic to $\widehat{\widehat{\mathbf{S}}}$ under the isomorphism given by (1), namely $\varepsilon_{_S}(x)(\varphi) = \varphi(x)$, for every $\varphi \in \widehat{S}$ and $x \in S$. The map $\varepsilon_{_S}$ is obviously a homomorphism of bisemilattices and a bijection from $\mathbf{I} \setminus \{0, 1\}$ to $\widehat{\widehat{\mathbf{I}}} \setminus \{\Phi_0, \Phi_1\}$, where by Φ_0, Φ_1 we indicate the constants in $\widehat{\widehat{\mathbf{I}}}$. This map can be extended to a bijection from $\mathbf{I}$ to $\widehat{\widehat{\mathbf{I}}}$, by setting $\varepsilon_{_S}(0) = \Phi_0$ and $\varepsilon_{_S}(1) = \Phi_1$. We have to prove that Φ_0 and Φ_1 indeed play the role of the constants in $\widehat{\widehat{\mathbf{I}}}$ and that $\varepsilon_{_S}$ also preserves involution.

We start with the latter task:

$$(\neg\varepsilon_S(x))(\varphi) = (\varepsilon_S(x)(\neg\varphi))' = (\neg\varphi(x))' = (\varphi(x'))'' = \varphi(x').$$

Regarding the constants, we only need to prove that $\neg\Phi_0 = \Phi_1$ and $\Psi + \Phi_0 = \Psi$, for each $\Psi \in \widehat{\widehat{\mathbf{I}}}$. Indeed, for any $\varphi \in \widehat{\mathbf{I}}$, one has:

$$\neg\Phi_0(\varphi) = \neg\varepsilon_S(0)(\varphi) = \varphi(0') = \varphi(1) = \varepsilon_S(1)(\varphi) = \Phi_1(\varphi).$$

Finally, due to the surjectivity of ε_S, for any $\Psi \in \widehat{\widehat{\mathbf{I}}}$, there exists $x \in I$ such that $\Psi = \varepsilon_S(x)$. Therefore $\Psi(\varphi) = \varepsilon_S(x)(\varphi) = \varepsilon_S(x+0)(\varphi) = \varphi(x+0) = \varphi(x) + \varphi(0) = \varepsilon_S(x)(\varphi) + \varepsilon_S(0)(\varphi) = (\Psi + \Phi_0)(\varphi)$ and we are done. $\square$

In order to prove Theorem 4.17 we are only left with proving that the functors $\mathrm{Hom_b}(-, \mathbf{WK}) : \mathfrak{IBSL} \to \mathfrak{IGR}$ and $\mathrm{Hom}_{\mathrm{GR}}(-, \mathbf{WK}) : \mathfrak{IGR} \to \mathfrak{IBSL}$ are controvariant (we consider just the first functor as for the other the proof runs analogously).

Proposition 4.16. *Let $f : \mathbf{I} \to \mathbf{L}$ be a morphism of $\mathfrak{IBSL}$, then it induces a morphism of $\mathfrak{IGR}$ $f^* : \widehat{\mathbf{L}} \to \widehat{\mathbf{I}}$, where $\widehat{\mathbf{L}}$, $\hat{\mathbf{I}}$ are the dual spaces of $\mathbf{L}$ and $\mathbf{I}$, respectively.*

Proof. f^* is defined in the usual way, i.e. $f^*(\widehat{j})(i) = \widehat{j}(f(i))$, for each $i \in \mathbf{I}$ and $\widehat{j} \in \widehat{\mathbf{J}}$. It suffices to prove that f^* preserves involution, namely $f^*(\neg\widehat{j}) = \neg f^*(\widehat{j})$, for all $j \in J$:

$$(\neg f^*(\widehat{j}))(i) = \neg\widehat{j}(f(i)) = f^*(\neg\widehat{j})(i),$$

$\square$

Surprisingly enough, we have established that semilattice inverse systems of Stone spaces are nothing but GR spaces with involution.

Theorem 4.17. *The categories of GR spaces with involution and $\mathfrak{IBSL}$ are dually equivalent.*

Corollary 4.18. *The category Sem-inv-$\mathfrak{SA}$ is equivalent to the category of GR spaces with involution.*

Corollary 4.18 highlights an interesting as well as unexpected topological properties of Stone spaces. Indeed the category of (semilattice) inverse systems of Stone spaces which deals with a possibly infinite family of them can be described by a specific class of topological spaces, namely GR spaces with involution.

5. Final comments and remarks

It is natural to wonder whether the content of Theorem 4.17 may be extended to other algebraic categories admitting topological duals such as bisemilattices and GR spaces. Indeed, recall that bisemilattices are Płonka sums of distributive lattices, according to the following

Theorem 5.1. [13, Th. 3] *An algebra* $\mathbf{B}$ *is a bisemilattice iff it is the Płonka sum over a semilattice direct system of distributive lattices.*

A *Priestley space* is an ordered topological space, i.e. a set X equipped with a partial order $\leq$ and a topology τ, such that $\langle X, \tau\rangle$ is compact and, for $x \nleq y$ there exists a clopen up-set U such that $x \in U$ and $y \notin U$. The category of Priestley spaces, $\mathfrak{PS}$, is the category whose objects are Priestley spaces and morphisms are continuos maps preserving the ordering.

The category of Priestley spaces is the dual of the category of distributive lattices [19], [20].

Let us call $\mathfrak{BSL}$ the category of bisemilattices (objects are bisemilattices, morphisms homomorphisms of bisemilattices). It follows from our analysis and Theorem 5.1 that the objects in $\mathfrak{BSL}$ are the same as in Sem-dir-$\mathfrak{DL}$, where $\mathfrak{DL}$ stands for the category of distributive lattices. We claim that the two categories of $\mathfrak{BSL}$ and Sem-dir-$\mathfrak{DL}$ are indeed equivalent. This can be shown using the same strategy applied in Section 4.

Lemma 5.2. *Let* $\mathbf{L}$ *and* $\mathbf{M}$ *be two bisemilattices, the Płonka sums over the semilattice direct systems of distributive lattices* $\mathbb{L} = \langle L_i, \varphi_{i,i'}, I\rangle$ *and* $\mathbb{M} = \langle M_j, \varphi_{j,j'}, J\rangle$, *and let* $h : \mathbf{L} \to \mathbf{M}$ *be a homomorphism. Then, for any* $i \in I$, *there exists a* $j \in J$ *such that* $h(L_i) \subseteq M_j$.
Moreover, there exists a semilattice homomorphism $\varphi : I \to J$, *for every homorphism* $h : \mathcal{P}_l(\mathbb{L}) \to \mathcal{P}_l(\mathbb{M})$, $h(A_i) \subseteq B_{\varphi(i)}$.

Proof. Let $a, b \in L_i$: we claim that $h(a), h(b) \in \mathbf{M}_j$, for some $j \in J$. Two cases may arise: either a, b are comparable with respect to the order $\leq$ of L_i or they are not. Suppose a and b are comparable: let $a \leq b$ and suppose that $h(a) \in M_j$, $h(b) \in M_{j'}$ with $j \neq j'$. Then, $h(a) = h(a \wedge b) = h(a) \wedge h(b) \in M_{j\vee j'}$ (by definition of operations in the Płonka sum), therefore $j = j \vee j'$. On the other hand, $h(b) = h(a \vee b) = h(a) \vee h(b) \in M_{j\vee j'}$. Thus $j = j'$.

The case of $b < a$ can be proved anologously.

Suppose now that a is not comparable with b, namely $a \nleq b$ and $b \nleq a$. Clearly $a \wedge b \leq a \vee b$, hence, reasoning as above, $h(a \vee b)$ and $h(a \wedge b)$ will belong to the same M_j for some $j \in J$. Now, both a and b are comparable with $a \wedge b$ and $a \vee b$, hence necessarily $h(a) \in M_j$ and $h(b) \in M_j$. Therefore $h(L_i) \in M_j$.

The proof of the second statement runs analogously as for Lemma 4.4. □

Remark 5.3. It is not difficult to check that the statement of Lemma 4.5 can be proven analogously when considering semilattice direct systems of distributive lattices, instead of Boolean algebras, and morphisms between them.

As consequence of Theorem 5.1, Lemma 5.2 and Remark 5.3, we get

Theorem 5.4. *The category* $\mathfrak{BSL}$ *is equivalent to* Sem-dir-$\mathfrak{DL}$.

Using Priestley duality and Theorem 3.8 we have

Theorem 5.5. *The categories* Sem-inv-$\mathfrak{PS}$ *and* $\mathfrak{BSL}$ *are dually equivalent.*

As the category of GR spaces is the dual category of $\mathfrak{BSL}$ (see Theorem 2.6), this means that Sem-inv-$\mathfrak{PS}$ are equivalent to a single class of spaces, namely

Corollary 5.6. *The category* Sem-inv-$\mathfrak{PS}$ *is equivalent to the category of GR spaces.*

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