Categorical Equivalence between PMV_f- product algebras and semi-low f_u-rings.

Lilian J. Cruz [*]
lilian.cruz@correounivalle.edu.co

Yuri A. Poveda [†]
yapoveda@utp.edu.co

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Abstract

An explicit categorical equivalence is defined between a proper subvariety of the class of PMV-algebras, as defined by Di Nola and Dvurečenskij, to be called PMV_f-algebras, and the category of semi-low f_u-rings. This categorical representation is done using the prime spectrum of the MV-algebras, through the equivalence between MV-algebras and l_u-groups established by Mundici, from the perspective of the Dubuc-Poveda approach, that extends the construction defined by Chang on chains. As a particular case, semi-low f_u-rings associated to Boolean algebras are characterized. Besides we show that class of PMV_f-algebras is coextensive.

Key words: PMV-algebra, PMV_f-algebra, l_u-ring, prime ideal, spectrum.

1 Introduction

In this paper the categorical equivalence is described, between a classical universal algebra variety, subvariety of the class of PMV-algebras, the PMV_f-algebras and the category of semi-low f_u-rings. This intermediate variety is a proper subvariety of the PMV-algebras defined by Di Nola y Dvurečenskij [5]. On the other hand, the variety of commutative unitary PMV-algebras studied by Montagna [11], to be called in this paper PMV_1-algebras, is a proper subvariety of the PMV_f. Estrada [9] defined the variety of MVW-rigs, and we defined the variety of PMV_f from it. The MVW-rigs contains strictly the variety of PMV-algebras. Every MV-algebra, with the infimum as product, is

[*]Universidad del Valle.
[†]Universidad Tecnológica de Pereira.

an MVW-rig (Proposition 3.1), it can happen that it is not a PMV-algebra; for example, the Łukasiewicz MV-algebras or the MV-algebra $[0, 1]$.

The equivalence between the category of PMV_f-algebras and the category of semi-low f_u-rings is established based of the equivalence proved by Mundici [12], but applying the construction introduced by Dubuc-Poveda [6], since it does not require the *good sequences*, and relies in the representation of any MV-algebra as a subdirect product of totally ordered MV-algebras, that will be called from here on chain MV-algebras or MV-chain. This representation only requires the prime spectrum of an MV-algebras and the equivalence between chain MV-algebras and the chain l_u-groups, established by Chang [3].

It is proved that for the representation established in this paper, it is enough with the prime spectrum of the subjacent MV-algebra, since every PMV_f-algebra A is a PMV-algebra that satisfies that $xy \leq x \wedge y$, y $x(y \ominus z) = xy \ominus xz$, for every $x, y, z \in A$, and every prime ideal of the subjacent MV-algebra is an ideal of the PMV_f-algebra A.

This construction finds explicit representations for the rings associated to notable examples of PMV_f-algebras. For example, the MV-algebra $[0, 1]$ with the usual product, the MV-algebra of the functions from $[0, 1]^n$ to $[0, 1]$ with the usual product, or the PMV_f-algebra of boolean algebras with product defined by the infimum. In this representation, the semi-low f_u-ring associated to the boolean algebra 2^n is precisely the ring $\mathbb{Z}^n$.

In section 2, the preliminary concepts about MV-algebras are presented. In section 3, the MV-algebras with product are defined, and in that context, the varieties of PMV_1, PMV_f, PMV-algebras and MVW-rigs. Some properties of the MVW-rigs are presented, with examples that illustrate the independence of the axioms chosen. Besides, it is shown that the inclusions between the categories are strict. In section 4 the semi-low l_u-rings are presented, (Definition 4.9) as well as one of the key results of this paper, Theorem 5.5, where the distributive property of the product for PMV_f-chains is proven. In section 5 we find the main result of this paper, the construction of the equivalence is extended to the category of PMV_f-algebras with product, and the category of semi-low f_u-rings. In section 6, some consequences of the equivalence are drawn, and in particular the construction of the ring associated to the boolean algebras is sketched. Finally, in section 7, we proof that the categories $\mathcal{PMV}_1$ and $\mathcal{PMV}_f$ are coextensive.

2 MV-algebras

Some properties of the theory of MV-algebras are presented, that are relevant to this work. The reader can find more complete information in [4].

Definition 2.1 (MV-algebra)**.** *An MV-algebra is a structure* $(A, \oplus, \neg, 0)$ *such that* $(A, \oplus, 0)$ *is a commutative monoid and the operation* $\neg$ *satisfies:*

i) $\neg(\neg x) = x$,

ii) $x \oplus \neg 0 = \neg 0$,

iii) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Because of properties of MV-algebras, $0 \leq a \leq u$ for all $a \in A$, with $u = \neg 0$. The operation $\neg$ is called **negation**, while the operation $\oplus$ is called **sum**.

Affirmation 2.1 (Order)**.** *Every MV-algebra A is ordered by the relation,*

$$x \leq y \text{ if and only if } x \ominus y = 0, \text{ for all } x, y \in A.$$

Definition 2.2 (Homomorphism)**.** *Given two MV-algebras A and B, a function $f : A \to B$ is a homomorphism of MV-algebras if for every x, y en A :*

i) $f(0) = 0$,

ii) $f(x \oplus y) = f(x) \oplus f(y)$,

iii) $f(\neg x) = \neg f(x)$.

Definition 2.3 (Ideal of an MV-algebra)**.** *A non-empty subset I of an MV-algebra A, is an ideal if and only if:*

i) If $a \leq b$ and $b \in I$, then $a \in I$.

ii) If $a, b \in I$, then $a \oplus b \in I$.

The set of all ideals of the MV-algebra A will be denoted by $Id(A)$.

Definition 2.4 (Prime ideal of an MV-algebra)**.** *An ideal P of an MV-algebra A, is prime if for all $a, b \in A$, $a \wedge b \in P$ implies $a \in P$ or $b \in P$.*

The set of all prime ideals of the MV-algebra A will be called $Spec(A)$, the spectrum of A.

Theorem 2.5 (Chang representation theorem [3])**.** *Every non trivial MV-algebra is isomorphic to a subdirect product of MV-chains.*

3 MV-algebras with product

Definition 3.1. *An MV-algebra with product is a structure $(A, \oplus, \cdot, \neg, 0)$ such that $(A, \oplus, \neg, 0)$ is an MV-algebra, and $(A, \cdot)$ is a semigroup.*

The operation $\cdot$ is called **product**, and the notation used is : $\underbrace{a \cdot a \cdot \ldots \cdot a}_{n-times} = a^n$.

Next, four varieties of MV-algebras with product are defined, namely the MVW-rigs, the PMV-algebras, the PMV_f-algebras and the unitary PMV_1-algebras. Some of their properties are proved and in particular, we show that each one is contained in the other.

From here on, all products are supposed to be commutative.

Definition 3.2. *(MVW-rig [[9],2.4]) An* ***MVW-rig*** $(A, \oplus, \cdot, \neg, 0)$ *is an MV-algebra with product such that*

i) $a0 = 0a = 0$,

ii) $\big(a(b \oplus c)\big) \ominus (ab \oplus ac) = 0$,

iii) $(ab \ominus ac) \ominus \big(a(b \ominus c)\big) = 0$.

Observation 3.1. *For every* $a, b, c \in A$, *axiom ii) is equivalent to*

$$a(b \oplus c) \leq ab \oplus ac \tag{1}$$

and axiom iii) is equivalent to

$$ab \ominus ac \leq a(b \ominus c). \tag{2}$$

Definition 3.3. *An MVW-rig A is called unitary if there exists an element s with the property that for every x in A $sx = xs = x$. It is follows that s is unique.*

Definition 3.4 (PMV [5])**.** *A PMV-algebra A, is an MV-algebra with product such that for every $a, b, c \in A$ i) $a \odot b = 0$ implies $ac \odot bc = 0$; ii) $a \odot b = 0$ implies $c(a \oplus b) = ca \oplus cb$.*

In Theorem 3.1 of [5], it is shown that the class PMV is equationally definible.

Definition 3.5 (PMV_f)**.** *A PMV_f-algebra is an MVW-rig such that for every $a, b, c \in A$, $ab \leq a \wedge b$, and $a(b \ominus c) = ab \ominus ac$.*

Definition 3.6 (PMV-Unitary algebra [10])**.** *A PMV- unitary algebra is an MV-algebra A with product such that for every $a, b, c \in A$, $au = a$, y $a(b \ominus c) = ab \ominus ac$.*

Theorem 3.7. *The following inclusions hold:*

$$PMV_1 \subset PMV_f \subseteq PMV \subset MVW\text{-}rig$$

Proof. The first inclusion, $PMV_1 \subset PMV_f$, follows from Lemma 2.9-iii on [10] and example 3.23.

For the second inclusion, given $a, b, c \in PMV_f$, if $a \odot b = 0$, since $ac \leq a$ and $bc \leq b$ then $ac \odot bc \leq a \odot b = 0$. On the other hand, $a \odot b = 0$ implies

$a \leq u \ominus b$ and therefore $ca \leq c(u \ominus b) \leq cu$, and this last inequality implies (see proposition 2.7-vii, [10]) $c(b \oplus a) = c(u \ominus ((u \ominus b) \ominus a)) = cu \ominus (c(u \ominus b) \ominus ca) = (cu \ominus c(u \ominus b)) \oplus ca = cb \oplus ca$.

The inclusion $PMV \subset MVW$-rig is proven in proposition 6.3. To see that it is a strict inclusion, see example 3.13. $\square$

3.1 Examples and properties of the MVW-rigs

Example 3.8. *Every MV-algebra with the product defined by $ab = 0$, for all $a, b \in A$, is an MVW-rig.*

Example 3.9. *The MV-algebra $[0,1]$ with the usual multiplication inherited from $\mathbb{R}$ is a commutative MVW-rig with unitary element $u = 1$.*

Example 3.10. *The MV-algebra $[0,u]$ of real numbers with $0 \leq u < 1$ is a commutative MVW-rig, but it is not unitary.*

Example 3.11. *The MV-algebra of the continuous functions from $[0,1]^n$ to $[0,1]$ with the usual product for functions is an MVW-rig with the property: $xy \leq x \wedge y$.*

Example 3.12. *[[9],2.10]. Consider the algebra $\widetilde{Ł_n} = \{\frac{m}{n^k} \in \mathbb{Q} \cap [0,1] |\ k, m \in \mathbb{N}\}$ obtained by closing the Łukasiewicz algebra $Ł_n$ under products, where $Ł_n = \left\langle \{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}, \oplus, \neg \right\rangle$ with the usual product, is an MVW-rig .*

Example 3.13. *[[9],2.11]. $\mathbf{Z}_n = \{0, 1, \ldots, n\}$ with $n \in \mathbb{N}$, $u = n$ as strong unit, $x \oplus y = min\{n, x + y\}$, $\neg x = n - x$ y $xy = min\{n, x \cdot y\}$, is an MVW-rig where sum and product are the usual operations on the natural numbers.*

$\mathbf{Z}_n$ is unitary, and $u \neq 1$. The cancellation law does not hold, because the product of two elements can be larger than the supremum. In some cases, the strict inequality in (2) holds, even though the equality (1) is always true. For example, in $\mathbf{Z}_{10}$, $2(7 \ominus 6) = 2[\neg(\neg 7 \oplus 6)] = 2[\neg(3 \oplus 6)] = 2[\neg 9] = 2(1) = 2 > (2)(7) \ominus (2)(6) = 10 \ominus 10 = 0$.

$\mathbf{Z}_n$ is not always a PMV algebra either, because for example in $\mathbf{Z}_{10}$, $(3)(2) \odot (3)(2) = 6 \odot 6 = 2$, even though $2 \odot 2 = 0$.

Example 3.14. *$\widehat{Ł}_{n+1} = \left\langle \{0, \frac{1}{n}, \cdots \frac{n-1}{n}, 1\}, \oplus, \cdot, \neg, 0, 1 \right\rangle$, with product defined by $\dfrac{x}{n} \cdot \dfrac{y}{n} = \dfrac{min\{n, x \cdot y\}}{n}$, for $x, y \in \{0, 1, \cdots, n\}$, is an MVW-rig isomorphic to $\mathbf{Z}_n$, and the isomorphism is defined by $\varphi : Ł_{n+1} \longrightarrow \mathbf{Z}_n$, $\frac{x}{n} \longmapsto x$.*

Proposition 3.1. *Every MV-algebra A with product defined by the infimum $x \cdot y = x \wedge y$ is an MVW-rig.*

Proof. Because the product is defined in terms of the order, by the Chang representation theorem, it is enough to show that the result holds for every totally ordered PMV algebra. Since the product defined by the infimum is associative and commutative, it is enough to prove the inequalities (1) and (2). Consider $a, b, c \in A$ an MV-chain. If $b \oplus c \leq a$ then $b \oplus c = a \wedge (b \oplus c) \leq b \oplus c = (a \wedge b) \oplus (a \wedge c)$. If on the contrary $a \leq b \oplus c$, and $a \leq b, c$, then $a = a \wedge (b \oplus c) \leq a \oplus a = (a \wedge b) \oplus (a \wedge c)$. If $a \leq b \oplus c$ and $b \leq a \leq c$, then $a = a \wedge (b \oplus c) \leq a \oplus a = (a \wedge b) \oplus (a \wedge c)$. Similarly it can be proved that $a \wedge (b \ominus c) \geq a \wedge b \ominus a \wedge c$. □

Observation 3.2. *Even though every MV-algebra is an MVW-rig with the product given by the infimum, in general it is not a PMV-algebra, as is shown in the next example.*

The Łukasiewics MV-algebra Ł$_4$ with the product defined by the infimum is not a PMV algebra because $\frac{1}{3} \odot \frac{1}{3} = 0$ and $\frac{1}{3} = \frac{1}{3} \wedge \left(\frac{1}{3} \oplus \frac{1}{3}\right) < \frac{1}{3} \oplus \frac{1}{3} = \frac{2}{3}$.

Proposition 3.2. *Every MV-algebra A with product defined by the supremum for non-zero elements, namely, $ab = a \vee b$, si $a \neq 0$ y $b \neq 0$ and zero otherwise, is an MVW-rig.*

Proof. It is enough to show (1) and (2) for totally ordered MVW-rigs. □

Example 3.15. *An interesting and relevant particular case of the proposition is when A is a boolean algebra. A boolean algebra A can be considered as an MV-algebra, where the sum is given by the supremum and negation is the complement. If the product is defined as in the propositions 3.1 or 3.2, every boolean algebra is naturally an MVW-rig.*

Proposition 3.3. *Axiom iii) in the definition 3.2, is independent of the other axioms for MVW-rig. Similarly, axiom i) is independent of the others.*

Proof. Consider the Łukasiewicz MV-algebra Ł$_4$ with product defined by :

$$a \cdot b = \begin{cases} 0, & si \quad a = 0 \ o \ b = 0, \\ a \oplus b, & if \quad a \odot b = 0, \\ a \odot b, & if \quad a \odot b \neq 0. \end{cases}$$

In this structure axiom iii) does not hold, but the others do. The product is equivalent to the sum on the integers mod 3, $\mathbb{Z}_3$ for the elements of Ł$_4 - \{0\}$. Therefore, this product is associative and commutative.

On the other hand, every MV-algebra with the supremum as product is a model for all the axioms of MVW-rigs, except for axiom i). The proof is similar to the one given in proposition 3.2. □

Proposition 3.4. *[[9],2.5]. For every $a, b, c \in A$ a commutative MVW-rig the following properties hold:*

i) If $a \leq b$ then $ac \leq bc$,

ii) $u^2 \leq u$,

iii) $a \leq b$ *and* $c \leq d$ *then* $ac \leq bd$.

Proof. Property *iii)* follows from *i)*; in fact, $a \leq b$ and $c \leq d$ imply that $ac \leq bc$ and $cb \leq db$. $\square$

3.1.1 Ideals and Homomorphisms of MVW-rigs

Definition 3.16. *Given A, B MVW-rigs, a function $f : A \to B$ is a homomorphism of MVW-rigs in and only if*

i) *f is a homomorphism of MV-algebras and*

ii) $f(ab) = f(a)f(b)$.

Definition 3.17. *The kernel of a homomorphism $\varphi : A \to B$ of MVW-rigs is*

$$ker(\varphi) := \varphi^{-1}(0) = \{x \in A | \varphi(x) = 0\}.$$

Definition 3.18. *An ideal of an MVW-rig A is a subset I of A that has the following properties:*

i) *I is an ideal of the subjacent MV-algebra A.*

ii) *Given $a \in I$, and $b \in A$, $ab \in I$ (Absorbent Property).*

$Id_W(A)$ denotes the set of all ideals of the MVW-rig A.

Example 3.19 (Boolena Algebras)**.** *Every boolean algebra is an MVW-rig, taking the supremum as the sum and the infimum as the product. The ideals of this MVW-rig are the ideals of the MV-algebra, that are at the same time ideals for the lattice.*

Observation 3.3. *Note that in proposition 3.2, the MVW-rig has no proper non trivial ideals. Its only ideals are zero and the MVW-rig.*

Definition 3.20 (Prime ideal of an MVW-rig)**.** *An ideal P of an MVW-rig, is called prime if for every $a, b \in A$, $ab \in P$ implies $a \in P$ o $b \in P$.*

The set of all prime ideals of the MVW-rig A is denoted $Spec_W(A)$.

Proposition 3.5. *[4, 9]. There is a bijective correspondence between the set of all ideals of an MVW-rig A and the set of its congruences. Namely, given I an ideal of the MVW-rig A, the binary relation defined by $x \equiv_I y$ if and only if $(x \ominus y) \oplus (y \ominus x) \in I$ is a congruence relation, and given $\equiv$ any congruence relation in A the set $\{x \in A | x \equiv 0\}$ is an ideal of A.*

Because it is relevant, the proof of the compatibility of the product is reproduced. The full proof can be found on [[9],2.29].

Proof. Since A is an MV-algebra and I is an MV-ideal, $a \equiv_I b$ and $c \equiv_I d$ imply $a \oplus c \equiv_I b \oplus d$ and $\neg a \equiv_I \neg b$, (see [[4],1.2.6]). It is left then to prove that $a \equiv_I b$ and $c \equiv_I d$ imply $ac \equiv_I bd$.

Given $a \equiv_I b$ and $c \equiv_I d$ then $a \ominus b \in I$ and $c \ominus d \in I$ respectively, so $ac \leq (a \vee b)(c \vee d) = \big((a \ominus b) \oplus b\big)\big((c \ominus d) \oplus d\big) \leq (a \ominus b)\big((c \ominus d) \oplus d\big) \oplus b\big((c \ominus d) \oplus d\big) \leq (a \ominus b)(c \ominus d) \oplus (a \ominus b)d \oplus b(c \ominus d) \oplus bd$. Equivalently

$$ac \ominus bd \leq (a \ominus b)(c \ominus d) \oplus (a \ominus b)d \oplus b(c \ominus d) \in I,$$

because $(a \ominus b)$ and $(c \ominus d) \in I$ and I is absorbent, $ac \ominus bd \in I$. Similarly $bd \ominus ac \in I$, then $(ac \ominus bd) \oplus (bd \ominus ac) \in I$, and therefore $ac \equiv_I bd$. □

Observation 3.4. *For $a \in A$, the equivalent class of a respect to $\equiv_I$ will be denoted by $[a]_I$ and the quotient set $A/\equiv_I$ by A/I.*

Since $\equiv_I$ is a congruence, the operations $\neg[a]_I = [\neg a]_I$, $[a]_I \oplus [b]_I = [a \oplus b]_I$ y $[a]_I[b]_I = [ab]_I$, are well defined over A/I.

Proposition 3.6. *[[9],2.31]. If $I \in Id_w(A)$, then A/I is an MVW-rig.*

Corolary 3.21. *Consider $I \in Spec(A)$ and A an MVW-rig. If I is absorbent, then A/I is a totally ordered MVW-rig.*

3.2 Examples and properties of the PMV_f-algebras

Example 3.22. *The MV-algebra $[0,1]$ with the usual multiplication inherited from $\mathbb{R}$ is a PMV_f.*

Example 3.23. *The MV-algebra $[0,u]$ with the usual multiplication , and $0 < u < 1$, is a non-unitary PMV_f.*

Example 3.24. *The set of all continuous functions from $[0,1]^n$ to $[0,1]$ with truncated sum and the usual multiplication is a PMV_f.*

Example 3.25. *$F[x_1, \cdots, x_n]$ the set of all continuous functions from $[0,1]^n$ to $[0,1]$, that are constituted by finite polynomials in $\mathbb{Z}[x_1, \cdots, x_n]$, namely, $f(x_1, \cdots, x_n) \in F[x_1, \cdots, x_n] \Leftrightarrow \exists p_1, \cdots, p_k \in \mathbb{Z}[x_1, \cdots, x_n]$ such that for all $z \in [0,1]^n, f(z) = p_i(z)$, for some $i \in \{1, \cdots, n\}$, is a PMV_f.*

Example 3.26. *Every boolean algebra, as in the example 3.19 is a PMV_f.*

Example 3.27. *Every MV-algebra with multiplication defined by the infimum is an MVW-rig, not necessarily PMV_f-algebra, as was established on the example 3.1.*

Proposition 3.7. *Given a PMV_f-algebra A, $Id_W(A) = Id(A)$ and furthermore $Spec_W(A) \subseteq Spec(A)$.*

Proof. By definition, $Id_W(A) \subseteq Id(A)$. Additionally, given $I \in Id(A)$ and $a \in I$, for all $c \in A$, $ac \leq a \wedge c \in I$, so $I \in Id_W(A)$. On the other hand, given $I \in Spec_w(A)$ and $a, b \in A$ such that $a \wedge b \in I$, then $ab \in I$ because $ab \leq a \wedge b$. Consequently $a \in I$ or $b \in I$, therefore $I \in Spec(A)$. □

Proposition 3.8. *If $I \in Id_w(A)$ and A is a PMV_f-algebra, then A/I is a PMV_f-algebra.*

Proof. It follows directly from proposition 3.5 and proposition 3.6. □

4 l_u-rings

Definition 4.1 (l-group [1, 4])**.** *A l-group G is a lattice abelian group $(G, +, -, 0)$, such that, the order $<$ is compatible with the sum.*

Definition 4.2. *For each x in an l-group G, its absolute value is defined by $|x| = x^+ + x^-$, where $x^+ = x \vee 0$ is the positive part of x and $x^- = -x \vee 0$ is the negative part.*

Definition 4.3. *A strong unit u of an l-group G is an element u such that $0 \leq u \in G$ and for all $x \in G$ there exists an integer $n \geq 0$ with $|x| \leq nu$.*

An l-group with strong unit u will be called an l_u-group.

Definition 4.4 (l-ideal)**.** *An l-ideal of an l-group G is a subgroup J of G that satisfies: if $x \in J$ and $|y| \leq |x|$ then $y \in J$.*

Definition 4.5 (l-prime ideal)**.** *An l-ideal P of an l_u-group G, is prime if and only if G/P is a chain.*

The set of all l-prime ideals of G is called the spectrum of G and denoted by $Spec_g(G)$.

Definition 4.6. *[[1],XVII.1]. An l_u-ring is a ring $R = (|R|, +, \cdot, \leq, u)$ such that $\langle R, +, \leq, u\rangle$ is an l_u-group and, $0 \leq x$, $0 \leq y$ implies $0 \leq xy$, where $|R|$ denotes the subjacent set.*

From this point on, all rings will be assumed to be commutative.

Definition 4.7. *(L-ideal [[1],XVII.3]). An L-ideal I from an l-ring R is an l-ideal such that for every $y \in I$ and $x \in R$, $xy \in I$. I is called irreducible if and only if R/I is totally ordered.*

The set of all L-ideals of R is called $Id(R)$ and the set of all l-ideals of the subjacent group is called $Id_g(R)$.

Definition 4.8 (Low l-ring [11])**.** *An l-ring is called low if and only if, for all $x, y \geq 0 \in R$ we have that $xy \leq x \wedge y$.*

Definition 4.9 (Semi-low l_u-ring)**.** *An l_u-ring R is semi-low if and only if, for all $a, b \in [0, u]$, $ab \leq a \wedge b$.*

Theorem 4.10. *Given R an l-ring and $u \in R$, $u > 0$, and the segment $[0, u] = \{a \in R \mid 0 \leq a \leq u\}$ with $[0, u]^\sharp \subset R$, the subring generated by $[0, u]$, then:*

a) For every $A \subset [0, u]$, a PMV_f, $A^\sharp \subset [0, u]^\sharp$, the subring generated by A, $A^\sharp$ is a semi-low l_u-ring with strong unit u and $A = \Gamma(A^\sharp, u)$.

b) Every semi-low l_u-ring is generated by its segments,

$$[0, u]^\sharp = \{x \in R \mid \exists n \geq 0, |x| \leq nu\}.$$

Proof. a) Given $A = \langle |A|, \oplus, \cdot, \neg, 0\rangle$ a PMV_f, then $A = \langle |A|, \oplus, \neg, 0\rangle$ is an MV-algebra; call A^* the associated l_u-group. Then the subjacent sets are equal $|A^\sharp| = |A^*|$, because for every $a \in A^\sharp$, $a = \sum \epsilon_i b_i c_i + \sum \delta_j d_j$, with $b_i, c_i, b_i c_i, d_j \in A$ and $\epsilon_i, \delta_j \in \{1, -1\}$, is a sum of elements of A. Therefore $A = \Gamma(A^\sharp, u)$ because of Theorem 1.2-a) of [6]. On the other hand, $x, y \in A^\sharp \cap [0, u]$ implies $x, y \in A$ y $xy \leq x \wedge y$.

b) From theorem 1.2-b) of [6], it follows that

$$[0, u]^* = \{x \in R \mid \exists n \geq 0, |x| \leq nu\}$$

is an l_u-group with strong unit u, and for the reasons exposed above, the subjacent sets are $|R| = |[0, u]^*| = |[0, u]^\sharp|$, so $R = [0, u]^\sharp$.

□

5 Equivalence between the categories $\mathcal{PMV}_f$ and $\mathcal{LR}_u$.

Definition 5.1. *We called $\mathcal{PMV}_f$ and $\mathcal{CPMV}_f$ to the categories whose objects are PMV_f-algebras and PMV_f-chains and homomorphisms between them respectively.*

Definition 5.2. *We called $\mathcal{LR}_u$, and $\mathcal{CLR}_u$ to the categories whose objects are semi-low l_u-rings and chain semi-low l_u-rings, and homomorphisms between them respectively.*

5.1 Categorical Equivalence between $\mathcal{CPMV}_f$ and $\mathcal{CLR}_u$

Theorem 5.3 (Chang's construction of the l_u-group A^* [3], Lemma 5)**.** *Given A an MV-chain, $A^* = \langle \mathbb{Z} \times A, +, \leq\rangle$ together with the operations $(m + 1, 0) =$*

(m,u), $(-m-1,\neg a)=-(m,a)$, and $(m,a)+(n,b)=(m+n,a\oplus b)$ if $a\oplus b\neq u$, or $(m,a)+(n,b)=(m+n+1,a\odot b)$ if $a\oplus b=u$, is a chain l_u-group with strong unit $\mathbf{u}=(1,0)=(0,u)$, where the order $\leq$ is given by $(m,a)\leq(n,b)$ if and only if $m<n$ or $m=n$ and $a\leq b$.

Observation 5.1. *Denote $|A^*|=\mathbb{Z}\times A$.*

5.1.1 The functor $(-)^\sharp\colon \mathcal{CPMV}_f\to\mathcal{CLR}_u$

Definition 5.4. *Given A a PMV_f-chain, the structure $A^\sharp$ is define following the Chang's construction [3] as follows. $A^\sharp=\langle|A^*|,+,\cdot,\leq\rangle$, with $\langle|A^*|,+,\mathbf{u},\leq\rangle$, its associated l_u-group where the product is defined by*

$$(m,a)\cdot(n,b):=mn(0,u^2)+m(0,bu)+n(0,au)+(0,ab).$$

with $n(0,x)=\underbrace{(0,x)+\ldots+(0,x)}_{n-times}$ for $n\geq 0$ and $n(0,x)=\underbrace{-(0,x)-\ldots-(0,x)}_{n-times}$ for $n<0$.

Proposition 5.1. *The product defined above is well defined.*

Proof. Note that in A^*, $(m+1,0)=(m,u)$; it is enough then to observe directly from the definition of the product that $(m,u)\cdot(n,b)=(m+1,0)\cdot(n,b)$. □

Affirmation 5.1. *For every $x,y,z\in A$,*

$$(0,x)(0,y\odot z)=(0,xy)+(0,xz)-(0,xu).$$

Proof. The equality follows directly from theorem 3.7 and definition 5.4, if $y\odot z=0$. On the other hand, $y\odot z\neq 0\iff\neg y\oplus\neg z\neq u,\iff\neg y\odot\neg z=0$, implies

$$\begin{aligned}
(0,x)(0,y\odot z) &= (0,x)(0,\neg(\neg y\oplus\neg z))\\
&= (0,x)\left[-(-1,\neg y\oplus\neg z)\right]\\
&= -(0,x)(-1,\neg y\oplus\neg z)\\
&= -(0,x)\left[(0,\neg y)+(-1,\neg z)\right]\\
&= -(0,x)(0,\neg y)-(0,x)(-1,\neg z)\\
&= (0,x)\left[-(0,\neg y)\right]+(0,x)\left[-(-1,\neg z)\right]\\
&= (0,x)(-1,y)+(0,x)(0,z)\\
&= (0,xy)-(0,xu)+(0,xz).
\end{aligned}$$

□

Theorem 5.5. *Given A a $CPMV_f$, $A^\sharp$ is a chain semi-low l_u-ring.*

Proof. It is clear that $A^\sharp$ with the sum operation and the associated order is a chain l_u-group. It is enough to show that with the product given by definition 5.4, it is a semi-low ring.

For every $(m,x),(n,y),(s,z) \in A^\sharp$, the following properties hold:

Distributivity

$$(m,x)[(n,y)+(s,z)] = (m,x)(n,y)+(m,x)(s,z).$$

Because of the theorems 3.7 and 5.3, $z \odot y = 0$ implies $xz \odot xy = 0$, and $x(z \oplus y) = xz \oplus xy$, so, $(m,z)+(n,y) = (m+n, z\oplus y)$ and $(m,xz)+(n,xy) = (m+n, xz \oplus xy)$.

This affirmation will be proved dividing the proof in two cases.

Case 1. $y \odot z = 0$.

$$\begin{aligned}
(m,x)[(n,y)+(s,z)] &= (m,x)[(n+s, y\oplus z)] \\
&= m(n+s)(0,u^2)+m(0,(y\oplus z)u)+(n+s)(0,xu) \\
&\quad +(0,x(y\oplus z)) \\
&= mn(0,u^2)+ms(0,u^2)+m(0,yu\oplus zu)+n(0,xu) \\
&\quad +s(0,xu)+(0,xy\oplus xz) \\
&= mn(0,u^2)+ms(0,u^2)+m(0,yu)+m(0,zu) \\
&\quad +n(0,xu)+s(0,xu)+(0,xy)+(0,xz) \\
&= \big[mn(0,u^2)+m(0,yu)+n(0,xu)+(0,xy)\big] \\
&\quad +\big[ms(0,u^2)+m(0,zu)+s(0,xu)+(0,xz)\big] \\
&= (m,x)(n,y)+(m,x)(s,z).
\end{aligned}$$

From theorem 5.3, $y \odot z \neq 0$ implies, $(n,y)+(s,z) = (n+s+1, y\odot z)$, and form affirmation 5.1, $(0,x)(0,y\odot z) = (0,x(y\odot z)) = (0,xy)+(0,xz)-(0,xu)$.

Case 2. $y \odot z \neq 0$

$$\begin{aligned}
(m,x)[(n,y)+(s,z)] &= (m,x)[(n+s+1, y\odot z)] \\
&= m(n+s+1)(0,u^2)+m(0,(y\odot z)u) \\
&\quad +(n+s+1)(0,xu)+(0,x(y\odot z)) \\
&= m(n+s+1)(0,u^2)+m\big[(0,yu)+(0,zu)-(0,u^2)\big] \\
&\quad +(n+s+1)(0,xu)+\big[(0,xy)+(0,xz)-(0,xu)\big] \\
&= mn(0,u^2)+ms(0,u^2)+m(0,u^2)+m(0,yu) \\
&\quad +m(0,zu)-m(0,u^2)+n(0,xu)+s(0,xu) \\
&\quad +(0,xu)+(0,xy)+(0,xz)-(0,xu) \\
&= \big[mn(0,u^2)+m(0,yu)+n(0,xu)+(0,xy)\big] \\
&\quad +\big[ms(0,u^2)+m(0,zu)+s(0,xu)+(0,xz)\big] \\
&= (m,x)(n,y)+(m,x)(s,z).
\end{aligned}$$

Associativity

$$\begin{aligned}
(m,x)\big[(n,y)(s,z)\big] &= (m,x)\big[ns(0,u^2)+n(0,zu)+s(0,yu)+(0,yz)\big]\\
&= mns(0,u^3)+ns(0,xu^2)+mn(0,zu^2)+n(0,xzu)\\
&\quad +ms(0,yu^2)+s(0,xyu)+m(0,yzu)+(0,xyz)\\
&= mns(0,u^3)+mn(0,zu^2)+ms(0,yu^2)+m(0,zyu)\\
&\quad +ns(0,xu^2)+n(0,xuz)+s(0,xyu)+(0,xyz)\\
&= \big[mn(0,u^2)+m(0,yu)+n(0,xu)+(0,xy)\big](s,z)\\
&= \big[(m,x)(n,y)\big](s,z).
\end{aligned}$$

Given $(0,0)\le(m,x)$ and $(0,0)\le(n,y)$ it is clear that $0\le m$ and $0\le n$, so

$$(0,0)\le mn(0,u^2)+m(0,yu)+n(0,xu)+(0,xy)=(m,x)(n,y).$$

Now it can be proved that $A^\sharp$ is semi-low.
Given $(0,0)\le(m,x),(n,y)\le(0,u)$ it must be that $m=n=0$, so

$$(0,x)(0,y)=(0,xy)\le(0,x\wedge y)=(0,x)\wedge(0,y).$$

□

Corolary 5.6. *For every PMV_f-chain A,*

$$\left(\sum_{i=1}^{n}(0,x_i)\right)\left(\sum_{i=1}^{n}(0,y_i)\right)=\sum_{i=1}^{n}(0,x_iy_i),$$

in $A^\sharp$.

Proposition 5.2. $(-)^\sharp$ *is functorial.*

For $h:A\to B$ in $\mathcal{CPMV_E}$, define $h^\sharp:A^\sharp\to B^\sharp$ en $\mathcal{CLR}_u$ as follows: $h^\sharp(m,a):=(m,h(a))$. By construction (see [[6], 2.2]), $h^\sharp$ is a homomorphism of l_u-groups, so it is enough to prove that $h^\sharp$ is a homomorphism of l_u-rings. This follows directly from definition 5.4. Namely, $h^\sharp[(m,a)(n,b)]=h^\sharp(m,a)h^\sharp(n,b)$.

Besides, given $(m,a)\in A^\sharp$, $(gh)^\sharp(m,a)=(m,gh(a))=g^\sharp(m,h(a))=g^\sharp\circ h^\sharp(m,a)$.

5.1.2 The Functor $\Gamma\colon\mathcal{CLR}_u\to\mathcal{CPMV}_f$

Definition 5.7. *For (R,u) a chain semi-low l_u-ring, define $\Gamma(R,u)=\{x\in R\mid 0\le x\le u\}$ together with the operations $x\oplus y=(x+y)\wedge u$, $\neg x=u-x$ and, $x\cdot y=xy$. The multiplication is well defined because $xy\le x\wedge y\le u$.*

Proposition 5.3 ([5],3.2)**.** *Given (R,u) an l_u-ring that satisfies $u^2\le u$, $\Gamma(R,u)$ is a PMV-algebra.*

Observation 5.2. *If R is a chain semi-low l_u-ring, then $x(y \vee z) = xy \vee xz$. In fact, it can be assume without loss of generality that $y \leq z$. Then $x(y \vee z) = xz = xy \vee xz$. A similar statement for the infimum is true. Consequently, in this case $x(y \ominus z) = xy \ominus xz$.*

Corolary 5.8. *For every R a chain semi-low l_u-ring, $\Gamma(R, u)$ is an $\mathcal{PMV}_f$.*

Affirmation 5.2. *Γ is functorial.*

Given $\alpha : (R, u) \to (H, v)$ in $\mathcal{CLR}_u$, define $\Gamma(\alpha) : \Gamma(R, u) \to \Gamma(H, v)$ in $\mathcal{CPMV}_f$ as follows: $\Gamma(\alpha) := \alpha|_{[0,u]}$. By construction $\Gamma(\alpha)$ is a homomorphism of totally ordered MV-algebras. Then it is enough to see that it respects products, that is,

$$\Gamma(\alpha)(a)\Gamma(\alpha)(b) = \alpha(a)\alpha(b) = \alpha(ab) = \Gamma(\alpha)(ab).$$

Therefore, $\Gamma(\alpha)$ is a morphism in $\mathcal{CLR}_u$, such that for all $x \in \Gamma(R, u)$, it holds that $\Gamma(\beta)\Gamma(\alpha)(x) = \Gamma(\beta)(\Gamma(\alpha)(x)) = \Gamma(\beta)(\alpha(x)) = \beta(\alpha(x)) = (\beta\alpha)(x) = \Gamma(\beta\alpha)(x)$.

Theorem 5.9. *For every PMV_f-chain A and chain semi-low l_u-ring (R, u), the following are isomorphisms:*

$$A \cong \Gamma(A^\sharp, u) \text{ and } R \cong (\Gamma(R, u))^\sharp$$

Proof. The correspondences i and υ :

$$\begin{array}{rcl} i : A & \longrightarrow & \Gamma(A^\sharp, u) \\ a & \longmapsto & (0, a) \end{array} \qquad \begin{array}{rcl} \upsilon : (\Gamma(R, u))^\sharp & \longrightarrow & R \\ (m, x) & \longmapsto & mu + x \end{array}$$

are isomorphisms of MV-algebras and l_u-groups respectively ([3], Lemma 6). It is then enough to prove that they respect the product. For $a, b \in A$,

$$i(ab) = (0, ab) = (0, a)(0, b) = i(a)i(b).$$

On the other hand, given $(m, a), (n, b) \in A^\sharp$, it is true that:
$\upsilon[(m, a)(n, b)] = \upsilon\left[mn(0, u^2) + m(0, bu) + n(0, au) + (0, ab)\right] = mn\left[\upsilon(0, u^2)\right] + m[\upsilon(0, bu)] + n[\upsilon(0, bu)] + \upsilon(0, ab) = mnu^2 + mbu + nau + ab = mu(nu + b) + a(nu + b) = (mu + a)(nu + b) = \upsilon(m, a)\upsilon(n, b)$. ☐

It is now easy to prove that the isomorphisms defined based on Chang's construction given in theorem 5.3, i and υ, determine a categorical equivalence.

Theorem 5.10. *The isomorphisms i and υ defined above are natural transformations associated to the functors $\Gamma(-)^\sharp$ and $(-)^\sharp\Gamma$ respectively and establish an equivalence of categories*

$$\mathcal{CPMV}_f \xrightarrow{(-)^\sharp} \mathcal{CLR}_u \qquad \mathcal{CLR}_u \xrightarrow{\Gamma} \mathcal{CPMV}_f$$

Proof. The proof is analogous to theorem 2.2 on [6]. Given $A \xrightarrow{h} B$ in $\mathcal{CPMV}_f$, and with $(R,u) \xrightarrow{\varphi} (H,w)$ in $\mathcal{CLR}_u$, the naturality of i and v follows from the commutativity of the following diagrams:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "h"'] & \Gamma(A^\sharp,u) \arrow[r, hook] \arrow[d, "\Gamma(h^\sharp)"] & A^\sharp \arrow[d] \\ B \arrow[r, "i"] & \Gamma(B^\sharp,u) \arrow[r, hook] & B^\sharp \end{tikzcd} \qquad \begin{tikzcd} \Gamma(R,u)^\sharp \arrow[r, "v"] \arrow[d, "(\Gamma\varphi)^\sharp"'] & (R,u) \arrow[d, "\varphi"] \\ \Gamma(H,w)^\sharp \arrow[r, "v"] & (H,w) \end{tikzcd}$$

Given $a \in A$, it holds that $\Gamma(h^\sharp)i(a) = \Gamma(h^\sharp)(0,a) = h^\sharp(0,a) = (0,h(a)) = ih(a)$, and for $(n,x) \in \Gamma(R,w)^\sharp$, then $\varphi v(n,x) = \varphi(nu+x) = nw+\varphi(x) = v(n,\varphi(x)) = v(n,(\Gamma\varphi)(x)) = v(\Gamma\varphi)^\sharp(n,x)$. □

5.2 Categorical equivalence between the categories $\mathcal{PMV}_f$ and $\mathcal{LR}_u$.

5.2.1 Subdirect representation of $\mathcal{PMV}_f$-algebras by chains

Recall the partial order isomorphism between the ideals of an l_u-group G and the ideals of its MV-algebra $\Gamma(G,u)$, established on theorem 7.2.2 of [4].

Theorem 5.11. *Given G an l_u-group and $A = \Gamma(G,u)$, the correspondence*

$$\begin{array}{rcl} \phi : \mathcal{I}(A) & \longrightarrow & \mathcal{I}(G) \\ J & \longmapsto & \phi(J) = \{x \in G | \, |x| \wedge u \in J\} \end{array}$$

is a partial order isomorphism between the ideals of the MV-algebra $\Gamma(G,u)$ and the l-ideals of the l_u-group, and its inverse is given by $H \mapsto \psi(H) = H \cap [0,u]$.

Proposition 5.4. *For a semi-low l_u-ring R, an ideal J of the PMV_f-algebra $\Gamma(R,u)$ and $\phi(J)$ the ideal of the l_u-group $(R,+,u)$ as in theorem 5.11, it holds that $\phi(J) = J^\sharp$ with*

$$J^\sharp = \left\{ x \in R \mid x = \sum_{i=1}^{m} \epsilon_i c_i, \; c_i \in J, \epsilon_i \in \{-1,1\} \right\}.$$

Proof. $J^\sharp$ is an l-ideal of the l_u-group $(R,+,u)$. In fact, $J^\sharp$ is a subgroup of R by construction. Next it must be proven that given $x \in J^\sharp$ and $y \in R$ such that $|y| \leq |x|$, $y \in J^\sharp$. Suppose without loss of generality that $|x| = x^+ = x$ and $|y| = y^+ = y$.

Since $x = \sum\limits_{i=1}^{m} \epsilon_i c_i$ with $c_i \in J$,

$$x \wedge u = \left(\sum_{i=1}^{m} \epsilon_i c_i \right) \wedge u = \left| \sum_{i=1}^{m} \epsilon_i c_i \right| \wedge u \leq \left(\sum_{i=1}^{m} c_i \right) \wedge u = \oplus_{i=1}^{n} c_i \in J, \qquad (3)$$

therefore, $x \wedge u \in J$.

By theorem 1.5-c of [6], it is enough to consider $x = \sum_{k=0}^{n} a_k$, with $a_k = (x - ku) \wedge u \vee 0$, where $0 < x < nu$ for some $n \in \mathbb{N}$, since the elements are in an l_u-group.

If $x - ku > 0$, $a_k \in J$ because

$$(x - ku) \wedge u \vee 0 = (x - ku) \wedge u \leq x \wedge u \in J,$$

by inequality (3). If $(x - ku) < 0$ then $a_k = 0 \in J$.

Since $0 \leq y \leq x \leq nu$ implies $b_k = (y - ku) \wedge u \vee 0 \leq (x - ku) \wedge u \vee 0 = a_k \in J$, then $y = \sum_{k=0}^{n} b_k \in J^\sharp$.

The same proof can be used if $x = x^-$ and $y = y^-$. Because $|x| = x^+ + x^-$ and $|y| = y^+ + y^-$, both are sums of positive elements and $J^\sharp$ is a subgroup of R, $|y| \leq |x|$ and $x \in J^\sharp$ imply $y \in J^\sharp$.

By construction $J \subseteq J^\sharp$, and for inequality (3), $J^\sharp \cap [0, u] \subseteq J$ so

$$J^\sharp \cap [0, u] = J = \phi(J) \cap [0, u],$$

thus, by the isomorphism given in theorem 5.11, $J^\sharp = \phi(J)$. □

Corolary 5.12. *$\phi(J) = J^\sharp$ is an ideal of the l_u-ring.*

Proof. It is enough to show that $J^\sharp$ is absorbent. For any $r \in R$, $r = \sum_{j=1}^{m} \alpha_j d_j$ with $d_j \in [0, u]$ and $\alpha_j \in \{-1, 1\}$, because of theorem 4.10, and given $x \in J^\sharp$, $x = \sum_{i=1}^{n} \epsilon_i c_i$, with $c_i \in J$, then

$$rx = \sum_{j=1}^{m} \alpha_j d_j \sum_{i=1}^{n} \epsilon_i c_i = \sum_{i,j=1}^{mn} \alpha_j \epsilon_i d_j c_i,$$

where $d_j c_i \in J$, since this is absorbent, therefore $rx \in J^\sharp$. □

Corolary 5.13. *In a semi-low l_u- ring every l-ideal is an L-ideal.*

$$Id_g(R) = Id(R).$$

Proof. For any $J \in Id_g(R)$, because of theorem 5.11 it holds that $J \cap [0, u] \in Id(\Gamma(R, u))$. Because $\Gamma(R, u) \in \mathcal{PMV}_f$, $J \cap [0, u]$ absorbs, then $J \cap [0, u] \in Id_w(\Gamma(R, u))$ and consequently by proposition 5.4, $J = (J \cap [0, u])^\sharp \in Id(R)$. In particular, $Spec_g(R) \subset Id(R)$. □

Corolary 5.14. *For any $J \in Id_g(R)$ where R is a semi-low l_u-ring, R/J is a semi-low l_u-ring.*

Theorem 5.15. *For any $J \in Id_g(R)$ where R is a semi-low l_u-ring,*

$$\Theta\colon \Gamma(R/J, u_J) \to \Gamma(R,u)/(J \cap [0,u]);\ [x]_J \longmapsto [x]_{J\cap[0,u]}$$

is an isomorphism of PMV_f-algebras.

Proof. Because the MV-algebras are isomorphic, due to theorem 7.2.4 of [4], it is enough to see that the isomorphism respects products. Using corollary 5.14, proposition 5.3 and the definition of Θ it follows that

$$\Theta([a]_J[b]_J) = \Theta([ab]_J) = [ab]_{J\cap[0,u]} = \left([a]_{J\cap[0,u]}\right)\left([b]_{J\cap[0,u]}\right).$$

□

Corolary 5.16. *If $J \in Spec_g(R)$ then Θ is an isomorphism of PMV_f-chains.*

Proof. It follows from the last theorem and the corollary 5.8. □

Theorem 5.17. *Every PMV_f-algebra is isomorphic to a subdirect product of PMV_f-chains.*

Proof. For any PMV_f-algebra A there is an injective homomorphism of MV-algebras,

$$\widehat{(\,)} : A \to \prod_{P\in Spec(A)} A/P,$$

mapping $a \mapsto \widehat{a}$ where $\widehat{a}\colon Spec\,A \to \bigsqcup_{P\in Spec\,A} A/P$ with $\widehat{a}(P) = [a]_P$. It is a homomorphism of PMV_f-algebras, and $\pi_P \circ \widehat{(\,)}\colon A \to A/P$ is a surjective homomorphism for each prime ideal $P \in Spec(A)$. In fact, every prime ideal P in the MV-algebra is an ideal in the PMV_f-algebra, as proven in proposition 3.7, where $\widehat{ab} = \widehat{a}\cdot\widehat{b}$, due to the correspondence between ideals and congruences in any PMV_f-algebra. □

Corolary 5.18. *Every PMV_f-equation (see [4], section 1.4) that holds in any PMV_f-chain holds in every PMV_f-algebra.*

Corolary 5.19. *In every PMV_f-algebra it holds that $a(b\wedge c) = ab\wedge ac,\ a(b\vee c) = ab \vee ac$.*

Corolary 5.20. *If (R,u) is a semi-low l_u-ring, $\Gamma((R,u))$ is a PMV_f-algebra.*

Proof. From the corollary 5.8 it follows that every $\Gamma(R,u)/P$ is a PMV_f-chain for every P, and the other hand, $\Gamma((R,u))$ is isomorphic to a subdirect product of $\prod_{P\in Spec(\Gamma(R,u))} \Gamma(R,u)/P$ and therefore $\Gamma((R,u))$ is a PMV_f-algebra. □

Theorem 5.21. *Every semi-low l_u-ring R is isomorphic to a subdirect product of chains.*

Proof. It is enough to show that the injective homomorphism of l_u-groups given by

$$\widehat{(-)}^{g}: R \to \prod_{P \in Spec_g(R)} R/P; \quad x \longmapsto [x]_P,$$

is an l_u-ring homomorphism. In fact, from theorem 7.2.2 of [4] and corollary 5.12 it follows directly that R/P is a semi-low l_u-ring for every $P \in Spec_g(R)$. $\square$

5.2.2 Extension of the functors $(-)^\sharp$ y Γ

The diagram on the left will be completed to extend the construction of Chang to the functor $\mathcal{PMV}_f \xrightarrow{(-)^\sharp} \mathcal{LR}_u$, and then it will be proven that this extends the equivalence from the first row to an equivalence in the second.

$$\begin{tikzcd} \mathcal{CPMV}_f \arrow[r, "(-)^\sharp"] \arrow[d, hook, "i_{\mathcal{M}}"'] & \mathcal{CR}_u\mathcal{E} \arrow[d, hook, "i_{\mathcal{R}}"'] \\ \mathcal{PMV}_f & \mathcal{LR}_u \end{tikzcd} \qquad \begin{tikzcd} \mathcal{CLR}_u \arrow[r, "\Gamma"] \arrow[d, hook, "i_{\mathcal{R}}"'] & \mathcal{CPMV}_f \arrow[d, hook, "i_{\mathcal{M}}"'] \\ \mathcal{LR}_u \arrow[r, "\Gamma"] & \mathcal{PMV}_f \end{tikzcd}$$

Definition 5.22. *For any PMV_f-algebra A, we define*

$$A^\circ = \{(0, \widehat{a}) : a \in A\} \subseteq \prod_{P \in Spec\, A} (A/P)^\sharp.$$

Definition 5.23 (Associate l_u-ring)**.** *For any PMV_f-algebra A we define $A^\sharp = gen(A^\circ)$ as the l-ring generated in the l-ring $\prod_{P \in Spec\, A} (A/P)^\sharp$.*

Notation. $|A^*| = \left\{ x \in \prod_{P \in Spec\, A} (A/P)^* \mid x = \sum_{i=1}^{n} \epsilon_i (0, \widehat{a_i}), a_i \in A, n \in \mathbb{N} \right\}.$

Affirmation 5.3. *$A^\sharp$ is a semi-low l_u-ring and $A^\sharp = \langle |A^*|, +, \cdot, u, \leq \rangle$ where $A^* = \langle |A^*|, +, u, \leq \rangle$ is the l_u-group associated to the subjacent MV-algebra A, and the product is defined as follows:*

$$\begin{array}{rcl} \varphi : |A^*|^2 & \longrightarrow & |A^*| \\ (x, y) & \longmapsto & \varphi(x, y) := x \cdot y. \end{array}$$

with $x = \sum_{i=1}^{n} \epsilon_i (0, \widehat{a_i})$, $y = \sum_{j=1}^{m} \delta_j (0, \widehat{b_j})$, $x \cdot y = \sum_{i,j=1}^{nm} \epsilon_i \delta_k (0, \widehat{a_i b_j})$, $\epsilon_i, \delta_j \in \{-1, 1\}$ *y*, $a_i, b_j \in A$,.

Proof. φ is well defined because for each $P \in Spec(A)$ the product $(x \cdot y)(P) = x(P) \cdot y(P)$ coincides with the product given in definition 5.4 and described in corollary 5.6. From theorem 5.5 it follows that the operation is associative and distributive.

On the other hand, $\langle |A^*|, +, \cdot, \leq \rangle$ is a semi-low l_u-ring because for every $\mathbf{0} \leq x\,, y \leq \mathbf{u}$,

$$x=\sum_{i=1}^{n}(0,\widehat{a_i})=(0,\oplus_{i=1}^{n}\widehat{a_i}) \text{ e } y=\sum_{j=1}^{m}\left(0,\widehat{b_j}\right)=\left(0,\oplus_{j=1}^{m}\widehat{b_j}\right),$$

$(x\cdot y)(P) = x(P)\cdot y(P) = (0,[\oplus a_i]_P)\cdot\left(0,[\oplus b_j]_P\right) = \left(0,[\oplus a_i]_P\cdot[\oplus b_j]_P\right) \leq \left(0,[\oplus a_i]_P\wedge[\oplus b_j]_P\right) = (0,[\oplus a_i]_P)\wedge\left(0,[\oplus b_j]_P\right) = x(P)\wedge y(P)$.

Since $A^\circ\subseteq|A^*|$, and every l_u-ring H that contains A°, must contain all finite sums and products of elements of A°, $\langle|A^*|,+,\cdot,u,\leq\rangle\subseteq H$. $\square$

Definition 5.24. *For any $h: A\to B$ in the category $\mathcal{PMV}_f$, we define $h^\sharp: A^\sharp\to B^\sharp$ en $\mathcal{LR}_u$ by* $h^\sharp\left(\sum_{i=1}^{n}\epsilon_i(0,\widehat{a_i})\right):=\sum_{i=1}^{n}\epsilon_i\left(0,\widehat{h(a_i)}\right)$.

Theorem 5.25. *The application $(-)^\sharp\colon\mathcal{PMV}_f\to\mathcal{LR}_u$ that assigns to each PMV_f-algebra A the l_u-ring $A^\sharp$, is functorial.*

Proof. Since for every $h: A\to B$ in the category $\mathcal{PMV}_f$ $h^\sharp$ is a homomorphism of l_u-rings and $\mathbf{h}^\sharp$ is a homomorphism of l-rings such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "\cong"] \arrow[d, "h"'] & A^\circ \arrow[r, hook, "="] \arrow[d, dashed, "h^\sharp"] & \Gamma(A^\sharp,u) \arrow[r, hook] \arrow[d, dashed, "\Gamma(h^\sharp)"] & A^\sharp \arrow[r, hook] \arrow[d, dashed, "h^\sharp"] & \prod_{P\in Spec\,A}(A/P)^\sharp \arrow[d, "\mathbf{h}^\sharp"] \\
B \arrow[r, "\cong"] & B^\circ \arrow[r, hook, "="] & \Gamma(B^\sharp,u) \arrow[r, hook] & B^\sharp \arrow[r, hook] & \prod_{Q\in Spec\,B}(B/Q)^\sharp
\end{tikzcd}$$

According to theorem 3.3 on [6], $h^\sharp$ is a homomorphism of l_u-groups and $\mathbf{h}^\sharp$ is a homorphims of l-groups. Recall that on the proof of theorem 3.3 on [6], for any $Q\in Spec(B)$ the well defined morphism $h|_Q\colon A/h^{-1}Q\to B/Q$; $h|_Q\left([a]_{h^{-1}Q}\right)=[h(a)]_Q$, makes the following diagram commute

$$\begin{tikzcd}
A/h^{-1}Q \arrow[r, "i"] \arrow[d, "h|Q"'] & (A/h^{-1}Q)^\sharp \arrow[d, "(h|Q)^\sharp"] \\
B/Q \arrow[r, "i"] & (B/Q)^\sharp
\end{tikzcd}$$

and therefore the group homomorphims $\mathbf{h}^\sharp$ can be defined as follows:
given $\sigma\in\prod_{P\in Spec\,A}(A/P)^\sharp$, $\mathbf{h}^\sharp(\sigma)(Q)=(h|_Q)^\sharp(\sigma(h^{-1}Q))$, $(\mathbf{h_1h_2})^\sharp=\mathbf{h_1}^\sharp\mathbf{h_2}^\sharp$, and $\mathbf{h}^\sharp|_{A^\sharp}=h^\sharp$.

To finish the proof, it is enough to show that $h^\sharp$ respects products in the generators of A°.
Given $P\in Spec\,A$, it follows from proposition 5.2 that
$h^\sharp\left[(0,\widehat{a})\left(0,\widehat{b}\right)\right](P)=h^\sharp\left[(0,[a]_P)(0,[b]_P)\right]=h^\sharp\left[(0,[a]_P)\right]h^\sharp\left[(0,[b]_P)\right]$.

As seen in the affirmation 5.3, $A^\sharp$ is a semi-low l_u-ring, and if $h\colon A\to B$ is a homomorphism of PMV_f-algebras, $h^\sharp\colon A^\sharp \to B^\sharp$ is a homomorphism of semi-low l_u-rings.

On the other hand, given $A \xrightarrow{h} B \xrightarrow{g} C \in \mathcal{PMV}_f$, $(gh)^\sharp = g^\sharp h^\sharp$ follows directly from definition 5.24. $\square$

Theorem 5.26. *$\Gamma\colon \mathcal{LR}_u \to \mathcal{PMV}_f$ is a functor, where $\Gamma(R,u) = [0,u]$ and $\Gamma(h) = h|_{[0,u]}$ for every homomorphism of l_u-rings $h\colon R \to R'$.*

Proof. It is follows directly from corollary 5.20 and the affirmation 5.2. $\square$

5.2.3 The equivalence

Theorem 5.27. *Given any PMV_f-algebra A and semi-low l_u-ring (R,u), the following homomorphisms are isomorphisms of PMV_f-algebras and semi-low l_u-rings.*

$$A \cong \Gamma(A^\sharp, u) \text{ and } (R,u) \cong (\Gamma(R,u))^\sharp.$$

Proof. For the first isomorphism $A \cong A^\circ$ as PMV_f-algebras, with $A^\circ \subset A^\sharp$ as shown in the following commutative diagram

$$\begin{tikzcd} A \arrow[r, tail, "\widehat{(-)}_A"] \arrow[dr, "\cong"'] & \prod_{P \in Spec\, A} (A/P) \arrow[r, tail, "i"] & \prod_{P \in Spec\, A} (A/P)^\sharp \\ & A^\circ \subset A^\sharp \arrow[ur, hook] & \end{tikzcd}$$

where i is built using the universal property as follows:

$$\begin{tikzcd} A/P \arrow[r, "i_P"] & (A/P)^\sharp \\ \prod_{P \in Spec\, A} A/P \arrow[u, "\pi_P"] \arrow[r, dashed, "\exists !i"] & \prod_{P \in Spec\, A} (A/P)^\sharp \arrow[u, "\pi_P^\sharp"'] \end{tikzcd}$$

with i_P the application defined for each P as

$$\begin{array}{rccl} i_P: & A/P & \longrightarrow & (A/P)^\sharp \\ & [a]_P & \longmapsto & (0, [a]_P). \end{array}$$

Because of theorem 4.10 a), $A^\circ = \Gamma(A^\sharp, u)$, and so $A \cong \Gamma(A^\sharp, u)$. $\square$

On the other hand, the isomorphisms of the chain semi-low l_u-rings, obtained from the Chang's construction in 5.3, in the theorem 5.9 on the fibers of $\Gamma(R,u)^\sharp$ determine an isomorphism of semi-low l_u-rings $\tau_R : \Gamma(R,u)^\sharp \longrightarrow (R,u)$, as follows:

$$
\begin{tikzcd}
\Gamma(R,u)^{\sharp} \arrow[r, hook, "\subset", "id"'] \arrow[d, dashed, "\tau_R"'] & \prod\limits_{P\in Spec\, R}(\Gamma(R,u)/P\cap[0,u])^{\sharp} \arrow[r, tail, "\Theta", "\cong"'] & \prod\limits_{P\in Spec\, R}\Gamma(R/P,u_P)^{\sharp} \arrow[d, "v", "\cong"'] \\
(R,u) \arrow[rr, tail, "\widehat{(-)}^{g}"] & & \prod\limits_{P\in Spec\, R}(R/P,u_P)
\end{tikzcd}
$$

where $\tau_R(0,\hat{x}) = \left[\widehat{(-)}^{g}\right]^{-1}(v\Theta(0,\hat{x})) = \left[\widehat{(-)}^{g}\right]^{-1}(\hat{x}^{g})$.

τ_R is well defined because the homomorphism $\widehat{(-)}^{g}$ is injective. The fact that τ_R is injective follows from the fact that for every $x,y\in\Gamma(R,u)$, $\widehat{x}^{g}=\widehat{y}^{g}$ implies $x=y$, and so $\widehat{x}=\widehat{y}$, since $\widehat{(-)}$ is a homomorphism. τ_R is surjective because for every $x\in(R,u)$ it holds that $x=\sum_{i=1}^{n}\epsilon_i x_i$ for some $x_i\in[0,u]$ by the theorem 4.10, b). Consequently $\tau_R(\sum_{i=1}^{n}\epsilon_i(0,\widehat{x_i}))=\sum_{i=1}^{n}\epsilon_i\tau_R(0,\widehat{x_i})=x$.

Theorem 5.28. *For every $A\in\mathcal{PMV}_f$ and $R\in\mathcal{LR}_u$ the isomorphisms*

$$
A \xrightarrow{i\widehat{(-)}_A} \Gamma(A^{\sharp},u) \qquad\qquad \Gamma(R,u)^{\sharp} \xrightarrow{\tau_R} (R,u)
$$

are natural transformations.

Proof. It follows directly from theorem 3.3 of [6]. □

6 PMV_f vs f-rings

Definition 6.1. *(f-rings [[1],XVII.5]). A function ring or f-ring is an l-ring that satisfies*

$$
a\wedge b=0 \textit{ and } c\geq 0 \textit{ implies } ac\wedge b=a\wedge cb=0.
$$

Proposition 6.1. *[[1],XVII.5]. In every f-ring it holds that*

$$
a\wedge b=0 \implies ab=0.
$$

Theorem 6.2. *(Fuchs [[1],XVII.5]). An l-ring is an f-ring if and only if all its l-closed ideals are L-ideals.*

Proposition 6.2. *For any PMV_f A, $A^{\sharp}$ is a semi-low f_u-ring.*

Proof. From affirmation 5.3, $A^{\sharp}$ is a semi-low l_u-ring. From corollary 5.13 and theorem 6.2, it is an f-ring, since all its l-ideals are L-ideals. □

Example 6.3. $[0,1]^{\sharp}=\mathbb{R}$.

Proposition 6.3. *$PMV\subset MVW$-rig.*

Proof. From theorem 4.2 of [5], it follows that for any PMV-algebra A there exists an l_u-ring R such that $\Gamma(R,u) \cong A$ and because of proposition 5.3, A is an MVW-rig. The inclusion is strict because of remark 3.2. $\square$

Affirmation 6.1. *For any set X the semi-low f_u-ring associated to the PMV_f-boolean algebra 2^X is isomorphic to the ring of bounded functions of $\mathbb{Z}^X$, $B(\mathbb{Z}^X)$.*

Proof. It is enough to see that $\left(2^X\right)^\sharp \cong B(\mathbb{Z}^X)$. The application Θ defined on the generators, for all $f \in 2^X$,

$$\begin{array}{rcl} \left(2^X\right)^\sharp & \xrightarrow{\Theta} & B(\mathbb{Z}^X) \\ i\widehat{f} & \longmapsto & \tilde{f}\colon X \longrightarrow \mathbb{Z} \\ & & \quad x \longmapsto f(x) \end{array}$$

with $\Theta\left(\sum_{j=1}^k i\widehat{f_j}\right) = \sum_{j=1}^k \Theta(i\widehat{f_j})$ and $\Theta\left((i\widehat{f})(i\widehat{g})\right) = \Theta((i\widehat{f}))\Theta\left((i\widehat{g})\right)$, is a ring isomorphism.

Because 2^X is a hyper-archimedean MV-algebra, every prime ideal is maximal and of the form $P_x = \{f \in 2^X : f(x) = 0\}$, for all $x \in X$ and $[f]_{P_x} = f(x)$. Then, if $x \in X$, $\tilde{f}(x) \neq \tilde{g}(x) \Leftrightarrow f(x) \neq g(x) \Leftrightarrow f \neq g \Leftrightarrow [f]_{P_x} \neq [g]_{P_x} \Leftrightarrow \widehat{f} \neq \widehat{g}$, implies that Θ is well defined and injective, with $P_x \in Spec(2^X)$.

On the other hand, for $h \in B(\mathbb{Z}^X)$ it holds that

$$h = \sum_{k=-n}^{n} k\lambda_k$$

with $|h| \leq n$, $\lambda_k \in 2^X$ such that $\lambda_k(x) = 1$ if $h(x) = k$ and zero elsewhere. Therefore Θ is surjective. By construction Θ is a homomorphism of l-rings. $\square$

Example 6.4. *The semi-low f_u-ring $(2^n)^\sharp$ is isomorphic to the ring $\mathbb{Z}^n$.*

Corolary 6.5. *Every boolean algebra seen as a PMV_f-algebra is a subalgebra of 2^X for some set X. Since the functor $(-)^\sharp$ preserves subalgebras, the semi-low f_u-ring associated to a boolean algebra is a subring of the semi-low f_u-ring $B(\mathbb{Z}^X)$.*

Example 6.6. $F[x] \subset \mathcal{C}\left([0,1]^{[0,1]}\right)$, *the semi-low f_u-ring of continuous functions defined as follows:*

$$f \in F[x] \Leftrightarrow \exists P_1, \cdots, P_k \in \mathbb{Z}[x], \text{ such that } \forall x \in [0,1]\, f(x) = P_i(x),$$

for some $1 \leq i \leq k$, is the semi-low f_u-ring associated to the PMV_f-algebra $\Gamma(F[x])$. This algebra is the minimum PMV_f-algebra that contain the MV-algebra $Free_1$.

7 The category $\mathcal{PMV}_f$ is coextensive

A category $\mathcal{C}$ is coextensive if only if $\mathcal{C}^{op}$ is extensive.

Definition 7.1. *[2] A category with finite products is coextensive if only if the projections of product is the terminal object and for all $g\colon A \times B \to C$, the following pushout exists and $C \cong C_1 \times C_2$.*

$$\begin{tikzcd} A \arrow[d, dashed] & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] \arrow[d, "g"] & B \arrow[d, dashed] \\ C_1 & C \arrow[l] \arrow[r] & C_2 \end{tikzcd}$$

Observation 7.1. *The terminal object of $\mathcal{PMV}_f$ category, is the PMV_f-algebra $\{0\} = \mathbf{1}$.*

Proposition 7.1. *The category $\mathcal{PMV}_f$ is coextensive.*

Proof. The pushout of projections π_A and π_B of PMV_f-algebras A, B, is the terminal object because of for all λ_A, λ_B, $\lambda_A \pi_A = \lambda_B \pi_B$ implied $\lambda_A = \lambda_B = 0$. It is enough to see that $(0,1) \in A \times B$ implies $\lambda_A \pi_A(0,1) = 0 = \lambda_B \pi_B(0,1) = 1$.

$$\begin{tikzcd} A \times B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \arrow[d] \arrow[dd, bend left, "\lambda_B"] \\ A \arrow[r] \arrow[dr, bend right, "\lambda_A"'] & \mathbf{1} \arrow[d, dotted, "\lambda"] \\ & C \end{tikzcd}$$

Let $g : A \times B \to C$, an homomorphism the PMV_f-algebras, we named $g(0,1) = e$, to idempotent element of C, and thus $g(1,0) = \neg e$ is idempotent too. We show that $\theta : C \to C/\langle e \rangle \times C/\langle \neg e \rangle\, ; c \mapsto ([c]_{\langle e \rangle}, [c]_{\langle \neg e \rangle})$, is an isomorphism of PMV_f-algebras, with $\langle e \rangle$ and $\langle \neg e \rangle$, the generated ideals of the subjacent MV-algebra of C. These ideals are ideals of the PMV_f-algebra C, proposition 3.7.

θ is well defined and is an homomorphism of PMV_f-algebras, proposition 3.8. It is injective because of exists $c \in C$ such that, $\theta(c) = ([c]_{\langle e \rangle}, [c]_{\langle \neg e \rangle}) = ([0]_{\langle e \rangle}, [0]_{\langle \neg e \rangle})$, then $c \in \langle e \rangle$ and $c \in \langle \neg e \rangle$, thus, $c \leq e$ and $c \leq \neg e$ because of e and $\neg e$ are idempotent elements, thus $c \leq e \wedge \neg e = g[(0,1) \wedge (1,0)] = 0$.

The other hand, for all $x, y \in C$, $x \equiv y \bmod \langle \langle e \rangle, \langle \neg e \rangle \rangle$, because of $\langle \langle e \rangle, \langle \neg e \rangle \rangle = C$. From the Chinese Remainder Theorem, Lemma 2 [8], exists $c \in C$ such that $c \equiv x \bmod \langle e \rangle$ y $c \equiv y \bmod \langle \neg e \rangle$, then $\theta(c) = ([c]_{\langle e \rangle}, [c]_{\langle \neg e \rangle}) = ([x]_{\langle e \rangle}, [y]_{\langle \neg e \rangle})$, for some $c \in C$, thus θ is surjective.

Now we defined $q_e = \pi_e \theta$ con $\pi_e \colon C/\langle e \rangle \times C/\langle \neg e \rangle \to C/\langle e \rangle$ and q_A as follows: for all $a \in A$, $q_A(a) = q_e(g(a,b))$. q_A is well defined because of $q_e(g(a,b)) = q_e(g(a,b'))$. If $\theta(g(a,b)) = ([x]_{\langle e \rangle}, [x]_{\langle \neg e \rangle})$ and $\theta(g(a,b')) = ([y]_{\langle e \rangle}, [y]_{\langle \neg e \rangle})$, it is enough to see that $[x]_{\langle e \rangle} = [y]_{\langle e \rangle}$.

$\theta(g(0,b)) = ([z]_{\langle e\rangle}, [z]_{\langle\neg e\rangle})$ with $z \leq e$. In effect, $([e]_{\langle e\rangle}, [e]_{\langle\neg e\rangle}) = \theta(g(0,1)) = \theta(g(0, b \oplus \neg b)) = \theta(g(0,b)) \oplus \theta(g(0,\neg b)) = ([z]_{\langle e\rangle}, [z]_{\langle\neg e\rangle}) \oplus ([w]_{\langle e\rangle}, [w]_{\langle\neg e\rangle}) = ([z \oplus w]_{\langle e\rangle}, [z \oplus w]_{\langle\neg e\rangle})$, with $z, w \in C$. Thus, $z \leq z \oplus w \leq e$.

Besides, $\theta(g(a,b)) \ominus \theta(g(a,b')) = \theta(g(0, b \ominus b')) = ([x \ominus y]_{\langle e\rangle}, [x \ominus y]_{\langle\neg e\rangle})$, and for previous affirmation, $x \ominus y \leq e$, and $[x]_{\langle e\rangle} = [y]_{\langle e\rangle} = q_A(a)$.

q_A is an homomorphism of PMV_f-algebras by construction.

Finally we show that the following diagrams are pushout.

$$\begin{tikzcd}
A \arrow[d, dashed, "q_A"] & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] \arrow[d, "g"] & B \arrow[d, dashed, "q_B"] \\
C/\langle e\rangle & C \arrow[l, "q_e"'] \arrow[r, "q_{\neg e}"] & C/\langle\neg e\rangle
\end{tikzcd}$$

Let λ_A and λ_g such that $\lambda_A \pi_A = \lambda_g g$, exists a unique λ such that the following diagram is commutative,

$$\begin{tikzcd}
 & A \arrow[d, "q_A"] \arrow[ddl, bend right, "\lambda_A"'] & A \times B \arrow[l, "\pi_A"'] \arrow[d, "g"] \\
 & C/\langle e\rangle \arrow[dl, dotted, "\lambda"] & C \arrow[l, "q_e"'] \arrow[dll, bend left, "\lambda_g"] \\
P & &
\end{tikzcd}$$

We defined $\lambda([c]_{\langle e\rangle}) = \lambda_g(c)$. λ is well defined, because of $[c]_{\langle e\rangle} = [c']_{\langle e\rangle} \leftrightarrow c \ominus c' \leq e$, and $\lambda_g(e) = \lambda_g(g(0,1)) = \lambda_A \pi_A((0,1)) = 0$. $\lambda q_e = \lambda_g$ by construction, and $\lambda q_A(a) = \lambda q_e(g(a,b)) = \lambda_g g(a,b) = \lambda_A \pi_A(a,b) = \lambda_A(a)$, with $b \in B$. λ is unique by construction.

The similar form we show that $q_{\neg e} g = q_B \pi_B$, is a pushout. □

Corolary 7.2. *The category $\mathcal{PMV}_1$ defined by Montagna [11], is coextensive.*

8 Conclusions

The construction of Dubuc-Poveda [6] lets you visualize the associate ring of each PMV_f-algebra, because this do not use the *good sequences*, and used the easy construction by Chang [3] for chains. The explicit construction of this equivalence permit us to study some properties of commutative algebra for the class of semi-low f_u-rings, in relationship with PMV_f-algebras. We know about the problem to study the free algebras of f_u-rings, however, its relationship with the PMV_f-algebras will let to see this from the other perspective.

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