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Kurahashi, Taishi

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On inclusions between quantified provability logics

Taishi Kurahashi[*][†]

Abstract

We investigate several consequences of inclusion relations between quantified provability logics. Moreover, we give a necessary and sufficient condition for the inclusion relation between quantified provability logics with respect to Σ_1 arithmetical interpretations.

1 Introduction

The notion of provability is a kind of modality, and modal logical studies of formalized provability have been extensively proceeded by many authors. Such studies have had many successes, especially in the framework of propositional modal logic. Solovay's arithmetical completeness theorem [13] is one of them. For every recursively enumerable extension T of Peano Arithmetic **PA**, let $\mathrm{Pr}_T(x)$ be a usual provability predicate of T. A *T-arithmetical interpretation* is a mapping f_T from the set of all propositional modal formulas to the set of sentences of arithmetic such that f_T commutes with each propositional connective and f_T maps $\Box A$ to $\mathrm{Pr}_T(\ulcorner f_T(A) \urcorner)$. Let $\mathsf{PL}(T)$ be the set of all propositional modal formulas A such that $T \vdash f_T(A)$ for every T-arithmetical interpretation f_T. This set is called the *propositional provability logic* of T. Solovay's arithmetical completeness theorem states that if T is a Σ_1-sound recursively enumerable extension of **PA**, then $\mathsf{PL}(T)$ is exactly the propositional modal logic **GL**. Thus $\mathsf{PL}(T)$ is recursive, but does not contain any elements specific to the theory T.

Formalized provability is also studied in the framework of quantified modal logic. The main target of this study is the *quantified provability logic* $\mathsf{QPL}(T)$ of T, which consists of quantified modal sentences verifiable in T under any T-arithmetical interpretation. Boolos [3] asked if $\mathsf{QPL}(\mathbf{PA})$ is recursively enumerable or not, and in contrast to the propositional case, Vardanyan [14] proved that $\mathsf{QPL}(\mathbf{PA})$ is Π^0_2-complete. Hence the analogue of Solovay's arithmetical completeness theorem never holds in the case of quantified modal logic. Moreover,

[*]Email: kurahashi@people.kobe-u.ac.jp

[†]Graduate School of System Informatics, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501, Japan.

Montagna [12] showed that some results which hold in the case of propositional logic are not inherited in the quantified case. Among other things, he proved that $\mathsf{QPL}(\mathbf{PA})$ is not a subset of $\mathsf{QPL}(\mathbf{BG})$, where $\mathbf{BG}$ is the Bernays–Gödel set theory. Thus $\mathsf{QPL}(T)$ can vary depending on the theory T.

Artemov [1] showed that the quantified provability logic $\mathsf{QPL}(T)$ of T can be different depending on the choice of a formula defining T. More precisely, we say that a formula $\tau(v)$ is a *definition* of a theory T if for any natural number n, $\tau(\overline{n})$ is true if and only if n is the Gödel number of an axiom of T. For each Σ_1 definition $\tau(v)$ of T, we can construct a Σ_1 provability predicate $\mathrm{Pr}_\tau(x)$ of T saying that "x is (the Gödel number of a formula) provable in the theory defined by $\tau(v)$". The notion of τ-arithmetical interpretations is introduced as well by using $\mathrm{Pr}_\tau(x)$ instead of $\mathrm{Pr}_T(x)$. Then, the quantified provability logic $\mathsf{QPL}_\tau(T)$ of $\tau(v)$ is defined to be the set of all quantified modal sentences provable in T under all τ-arithmetical interpretations. Artemov proved that for any Σ_1-sound recursively enumerable extension T of $\mathbf{PA}$ and any Σ_1 definition $\tau_0(v)$ of T, there exists a Σ_1 definition $\tau_1(v)$ of T such that $\mathsf{QPL}_{\tau_0}(T) \nsubseteq \mathsf{QPL}_{\tau_1}(T)$.

The results of Montagna and Artemov seem to indicate that inclusion relations between quantified provability logics are rarely established. Indeed, Kurahashi [9] proved that for any natural numbers i and j with $0 < i < j$, there exists a Σ_1 definition $\sigma_i(v)$ of the theory $\mathbf{I\Sigma_i}$ such that for all Σ_1 definitions $\sigma_j(v)$ of $\mathbf{I\Sigma_j}$, $\mathsf{QPL}_{\sigma_i}(\mathbf{I\Sigma_i}) \nsubseteq \mathsf{QPL}_{\sigma_j}(\mathbf{I\Sigma_j})$ and $\mathsf{QPL}_{\sigma_j}(\mathbf{I\Sigma_j}) \nsubseteq \mathsf{QPL}_{\sigma_i}(\mathbf{I\Sigma_i})$. The situation of the inclusion relation between quantified provability logics is completely different from that of propositional case: it is known that for any theories T_0 and T_1, at least one of $\mathsf{PL}(T_0) \subseteq \mathsf{PL}(T_1)$ and $\mathsf{PL}(T_1) \subseteq \mathsf{PL}(T_0)$ holds (cf. Visser [15]).

From this point of view, in the present paper, we investigate several consequences of the inclusion $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$ between quantified provability logics. Among other things, we prove that if $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$, then

1. $T_0 + \mathrm{Con}_{\tau_0}$ is a subtheory of $T_1 + \mathrm{Con}_{\tau_1}$;
2. T_0 is Σ_1-conservative over T_1;
3. Con_{τ_0} and Con_{τ_1} are provably equivalent over T_1; and
4. For any formula $\varphi(\vec{x})$,

$$T_1 \vdash \forall \vec{x} \left(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\vec{\dot{x}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi(\vec{\dot{x}}) \urcorner) \right).$$

Thus from our results, we certify that the inclusion relation between quantified provability logics holds only under limited situations. Moreover, our results also show that the quantified provability logic $\mathsf{QPL}_\tau(T)$ is not only complex, but also possesses much information about the theory T and the provability predicate $\mathrm{Pr}_\tau(x)$.

We also investigate provability logics with respect to Σ_1 arithmetical interpretations. In the propositional case, a T-arithmetical interpretation f_T is called Σ_1 if for any propositional variable p, $f_T(p)$ is a Σ_1 sentence. Let $\mathsf{PL}^{\Sigma_1}(T)$ be the set of all propositional modal formulas A such that $T \vdash f_T(A)$ for every T-arithmetical interpretation f_T which is Σ_1. Visser proved that $\mathsf{PL}^{\Sigma_1}(\mathbf{PA})$ is also recursive and exactly the propositional modal logic **GLV** (see Boolos [4]). In the quantified case, Berarducci [2] also proved that $\mathsf{QPL}^{\Sigma_1}(\mathbf{PA})$ is Π^0_2-complete. Thus, the situations of Σ_1 provability logics do not seem to be different from those of usual provability logics.

On the other hand, there is an advantage to dealing with Σ_1 arithmetical interpretations for our purposes, which allows us to improve Artemov's Lemma used in the proof of Vardanyan's theorem. Then, we can give a necessary and sufficient condition for the inclusion relation between quantified provability logics with respect to Σ_1 arithmetical interpretations. Namely, we prove that $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$ if and only if T_0 is a subtheory of T_1 and for any formula $\varphi(\vec{x})$, $T_1 \vdash \forall \vec{x}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner))$.

2 Preliminaries

Let $\mathcal{L}_A = \{0, S, +, \times, <, =\}$ be the language of first-order arithmetic. We call a set of $\mathcal{L}_A$-sentences simply a *theory*. Peano Arithmetic **PA** is the theory consisting of basic axioms for $\mathcal{L}_A$ and induction axioms for $\mathcal{L}_A$-formulas. $\mathbf{I\Sigma_1}$ is the theory obtained from **PA** by restricting induction axioms to Σ_1 formulas. Throughout the present paper, T, T_0 and T_1 always denote recursively enumerable extensions of $\mathbf{I\Sigma_1}$[1]. In the present paper. Let $\mathsf{Th}(T)$ be the set of all $\mathcal{L}_A$-sentences provable in T. Also, for each class Γ of formulas, let $\mathsf{Th}_\Gamma(T) := \mathsf{Th}(T) \cap \Gamma$. The standard model of arithmetic is denoted by $\mathbb{N}$. We say that T is Σ_1-*sound* if every element of $\mathsf{Th}_{\Sigma_1}(T)$ is true in $\mathbb{N}$. Notice that Σ_1-soundness implies consistency.

For each natural number n, the numeral for n is denoted by $\overline{n}$. We fix some natural Gödel numbering, and for each $\mathcal{L}_A$-formula φ, let $\ulcorner \varphi \urcorner$ be the numeral for the Gödel number of φ. We say a formula $\tau(v)$ is a *definition* of a theory T if for any natural number n, $\mathbb{N} \models \tau(\overline{n})$ if and only if n is the Gödel number of some axiom of T. Hereafter, we assume that $\tau(v)$, $\tau_0(v)$ and $\tau_1(v)$ always denote Σ_1 definitions of T, T_0 and T_1, respectively. Then, we can construct a Σ_1 formula $\mathrm{Pr}_\tau(x)$ saying that "x is (the Gödel number of a formula) provable in the theory defined by $\tau(v)$". The following fact is well-known.

Fact 2.1 (Derivability conditions (see Boolos [4] and Lindström [11]))**.** *For any formulas $\varphi(\vec{x})$ and $\psi(\vec{x})$,*

1. *If $T \vdash \varphi(\vec{x})$, then $\mathbf{I\Sigma_1} \vdash \mathrm{Pr}_\tau(\ulcorner \varphi(\dot{\vec{x}}) \urcorner)$;*
2. *$\mathbf{I\Sigma_1} \vdash \mathrm{Pr}_\tau(\ulcorner \varphi(\dot{\vec{x}}) \to \psi(\dot{\vec{x}}) \urcorner) \to (\mathrm{Pr}_\tau(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \to \mathrm{Pr}_\tau(\ulcorner \psi(\dot{\vec{x}}) \urcorner))$;*
3. *If $\varphi(\vec{x})$ is a Σ_1 formula, then $\mathbf{I\Sigma_1} \vdash \varphi(\vec{x}) \to \mathrm{Pr}_\tau(\ulcorner \varphi(\dot{\vec{x}}) \urcorner)$.*

[1] Based on the result of de Jonge [5] that Artemov's Lemma (Fact 2.8) holds for the theory $\mathbf{I\Sigma_1}$, we adopted $\mathbf{I\Sigma_1}$ as the base theory in this paper. See the paragraph immediately following Fact 2.10.

□

Here $\ulcorner \varphi(\vec{\dot{x}}) \urcorner$ is an abbreviation for $\ulcorner \varphi(\dot{x_1}, \ldots, \dot{x_n}) \urcorner$ that is a primitive recursive term corresponding to a primitive recursive function calculating the Gödel number of $\varphi(\overline{k_1}, \ldots, \overline{k_n})$ from $k_1, \ldots, k_n$.

Let Con_τ be the Π_1 sentence $\neg \mathrm{Pr}_\tau(\ulcorner 0 = \overline{1} \urcorner)$ stating that the theory defined by $\tau(v)$ is consistent. For each sentence φ, let $(\tau + \varphi)(v)$ be the Σ_1 definition $\tau(v) \lor v = \ulcorner \varphi \urcorner$ of $T + \varphi$. Then it is known that the formalized version of the deduction theorem holds: $\mathbf{I\Sigma_1} \vdash \forall x(\mathrm{Pr}_{\tau+\varphi}(x) \leftrightarrow \mathrm{Pr}_\tau(\ulcorner \varphi \urcorner \dot{\to} x))$. Here $u \dot{\to} v$ is a primitive recursive term corresponding to a primitive recursive function calculating the Gödel number of $\varphi \to \psi$ from the Gödel numbers of φ and ψ.

The language of quantified modal logic is the language of first-order predicate logic without function and constant symbols equipped with the unary modal operators $\Box$ and $\Diamond$. We may assume that the languages of quantified modal logic and first-order arithmetic have the same variables.

Definition 2.2. A mapping f from the set of all atomic formulas of quantified modal logic to the set of $\mathcal{L}_A$-formulas satisfying the following condition is called an *arithmetical interpretation*: For each atomic formula $P(x_1, \ldots, x_n)$, $f(P(x_1, \ldots, x_n))$ is an $\mathcal{L}_A$-formula $\varphi(x_1, \ldots, x_n)$ with the same free variables, and moreover $f(P(y_1, \ldots, y_n))$ is $\varphi(y_1, \ldots, y_n)$ for any variables $y_1, \ldots, y_n$.

Definition 2.3. Each arithmetical interpretation f is uniquely extended to a mapping f_τ from the set of all quantified modal formulas to the set of $\mathcal{L}_A$-formulas inductively as follows:

1. $f_\tau(\bot)$ is $0 = \overline{1}$;
2. f_τ commutes with each propositional connective and quantifier;
3. $f_\tau(\Box A(x_1, \ldots, x_n))$ is the formula $\mathrm{Pr}_\tau(\ulcorner f_\tau(A(\dot{x_1}, \ldots, \dot{x_n})) \urcorner)$.

Notice that any quantified modal formula A has the same free variables as $f_\tau(A)$. We are ready to introduce the quantified provability logic of $\tau(v)$.

Definition 2.4. The *quantified provability logic* $\mathsf{QPL}_\tau(T)$ *of* $\tau(v)$ is the set

$$\{A \mid A \text{ is a sentence and for all arithmetical interpretations } f, T \vdash f_\tau(A)\}.$$

The main purpose of the present paper is to investigate the inclusion relation $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$ between quantified provability logics. For this purpose, we heavily use Artemov's Lemma (Fact 2.8) that is used in the proof of Vardanyan's theorem on the Π_2-completeness of the quantified provability logic of **PA**. To state Artemov's Lemma, we prepare some definitions.

Definition 2.5. We prepare predicate symbols $P_Z(x)$, $P_S(x, y)$, $P_A(x, y, z)$, $P_M(x, y, z)$, $P_L(x, y)$ and $P_E(x, y)$ corresponding to members 0, S, $+$, $\times$, $<$ and $=$ of $\mathcal{L}_A$, respectively. For each $\mathcal{L}_A$-formula φ, let φ^* be a logically equivalent $\mathcal{L}_A$-formula where each atomic formula is one of the forms $x = 0$, $S(x) = y$,

$x + y = z$, $x \times y = z$, $x < y$ and $x = y$. Let φ° be a relational formula obtained from φ^* by replacing each atomic formula with the corresponding relation symbol in $\{P_Z, P_S, P_A, P_M, P_L, P_E\}$ adequately. Then φ° is a quantified modal formula.

Let $\mathrm{Seq}(s)$ be the formula naturally expressing that "s is a finite sequence". Also let $lh(s)$ and $(s)_x$ be primitive recursive terms corresponding to primitive recursive functions calculating the length and x-th component of a finite sequence s, respectively.

Definition 2.6. For each arithmetical interpretation f, let $R_f(x, y)$ be the formula

$$\exists s(\mathrm{Seq}(s) \wedge lh(s) = x+1 \wedge (s)_x = y \wedge f(P_Z((s)_0)) \wedge \forall z < x\, f(P_S((s)_z, (s)_{z+1}))).$$

Let $R_f(\vec{x}, \vec{y})$ denote a conjunction $R_f(x_0, y_0) \wedge R_f(x_1, y_1) \wedge \cdots \wedge R_f(x_n, y_n)$.

The formula $R_f(x, y)$ means that y represents x under the interpretation that $f(P_Z(u))$ and $f(P_S(u, v))$ say "u represents 0" and "v represents the successor of a number represented by u", respectively.

We introduce the modal sentence D asserting the completeness of P_K and $\neg P_K$ for every newly introduced predicate symbol P_K.

Definition 2.7. Let D be the modal sentence

$$\bigwedge_{K \in \{Z,S,A,M,L,E\}} \Big(\forall \vec{x}(P_K(\vec{x}) \to \Box P_K(\vec{x})) \wedge \forall \vec{x}(\neg P_K(\vec{x}) \to \Box \neg P_K(\vec{x})) \Big).$$

We are ready to state Artemov's Lemma. In the statement of the lemma, the $\mathcal{L}_A$-sentence χ is a conjunction of several basic sentences of arithmetic such as $\forall x \exists y(S(x) = y)$ and $\forall x(x + 0 = x)$, which serves to incorporate a structure of arithmetic into a set.

Fact 2.8 (Artemov's Lemma (see [4, p.232])). *There exists an $\mathcal{L}_A$-sentence χ such that $\mathbf{I\Sigma_1} \vdash \chi$ and for any arithmetical interpretation f and $\mathcal{L}_A$-formula $\varphi(\vec{x})$,*

$$\mathbf{I\Sigma_1} \vdash \mathrm{Con}_\tau \wedge f_\tau(\mathrm{D}) \wedge f_\tau(\chi^\circ) \wedge R_f(\vec{x}, \vec{y}) \to \Big(\varphi(\vec{x}) \leftrightarrow f_\tau(\varphi^\circ(\vec{y})) \Big).$$

$\Box$

We give a short outline of a proof of Artemov's Lemma based on the presentation in [8]. Let M be a model of $\mathbf{I\Sigma_1} + \mathrm{Con}_\tau \wedge f_\tau(\mathrm{D}) \wedge f_\tau(\chi^\circ)$. By the aid of $f_\tau(\chi^\circ)$, $f_\tau(P_E(x, y))$ defines an equivalence relation $\sim$ on M. Let $[a]$ be the equivalence class of $a \in M$ with respect to $\sim$. Then, the relations on M defined by the formulas $P_K(\vec{x})$ for $K \in \{Z, S, A, M, L\}$ induce an $\mathcal{L}_A$-structure M_f with the domain $\{[a] \mid a \in M\}$. For instance, $M_f \models [a] + [b] = [c] \iff M \models f_\tau(P_A(a, b, c))$. The sentence $f_\tau(\chi^\circ)$ guarantees that M_f is well-defined and indeed an $\mathcal{L}_A$-structure satisfying a sufficiently strong fragment of $\mathbf{I\Sigma_1}$, and that for any $\vec{a} \in M$, $M_f \models \varphi(\overrightarrow{[a]}) \iff M \models f_\tau(\varphi^\circ(\vec{a}))$. Also M is isomorphic to an initial segment of M_f via an embedding defined by the formula $R_f(x, y)$. Moreover, from the sentence $\mathrm{Con}_\tau \wedge f_\tau(\mathrm{D})$, we obtain the equivalences

$$f_\tau(P_K(\vec{x})) \leftrightarrow \mathrm{Pr}_\tau(\ulcorner f_\tau(P_K(\dot{\vec{x}}))\urcorner) \text{ and } \neg f_\tau(P_K(\vec{x})) \leftrightarrow \mathrm{Pr}_\tau(\ulcorner \neg f_\tau(P_K(\dot{\vec{x}}))\urcorner)$$

in M for each $K \in \{Z, S, A, M, L, E\}$. Then both $f_\tau(P_K(\vec{x}))$ and $\neg f_\tau(P_K(\vec{x}))$ are equivalent to Σ_1 formulas in M. By applying a proof of Tennenbaum's theorem (see Kaye [7]), we obtain that M and M_f are in fact isomorphic, and hence are elementarily equivalent. Therefore, if $M \models R_f(\vec{a}, \vec{b})$, then $M \models \varphi(\vec{a})$ is equivalent to $M_f \models \varphi(\overrightarrow{[b]})$. Hence $M \models \varphi(\vec{a}) \leftrightarrow f_\tau(\varphi^\circ(\vec{b}))$.

In the proof of Artemov's Lemma, the following facts are also used.

Fact 2.9 (See Boolos [4, Lemma 17.6])**.** *For any Σ_1 formula $\varphi(\vec{x})$ and arithmetical interpretation f,*

$$\mathbf{I\Sigma_1} \vdash f_\tau(\chi^\circ) \wedge R_f(\vec{x}, \vec{y}) \to \Big(\varphi(\vec{x}) \to f_\tau(\varphi^\circ(\vec{y}))\Big).$$

□

Fact 2.10 (See Boolos [4, Lemma 17.8])**.** *For any arithmetical interpretation f,*

$$\mathbf{I\Sigma_1} \vdash \mathrm{Con}_\tau \wedge f_\tau(\mathrm{D}) \wedge f_\tau(\chi^\circ) \to \forall y \exists x R_f(x, y).$$

□

Facts 2.9 and 2.10 follow from the observations that M is isomorphic to an initial segment of M_f and $R_f(x, y)$ defines a surjection from M onto M_f, respectively. In Boolos [4], these facts including Artemov's Lemma are stated in the forms that the corresponding formulas are proved in **PA**, and de Jonge [5] proved that **PA** can be replaced by $\mathbf{I\Sigma_1}$ (see also [8]).

Definition 2.11. An arithmetical interpretation f is *natural* if for each $K \in \{Z, S, A, M, L, E\}$, f maps $P_K(\vec{x})$ to the intended atomic formula (for example, $f(P_A(x, y, z))$ is $x + y = z$).

For every quantified modal formula A, let $\boxdot A$ be an abbreviation for $A \wedge \Box A$.

Proposition 2.12. *Let f be any natural arithmetical interpretation.*

1. *For any $\mathcal{L}_A$-formula $\varphi(\vec{x})$, $\mathbf{I\Sigma_1} \vdash \forall \vec{x}(f_\tau(\varphi^\circ(\vec{x})) \leftrightarrow \varphi(\vec{x}))$;*
2. *$\mathbf{I\Sigma_1} \vdash f_\tau(\boxdot \mathrm{D}) \wedge f_\tau(\boxdot \chi^\circ)$.*

Proof. 1. By induction on the construction of $\varphi(\vec{x})$.

2. For each $K \in \{Z, S, A, M, L, E\}$, since $f_\tau(P_K(\vec{x}))$ is Δ_0, it follows from Fact 2.1.3 that $\mathbf{I\Sigma_1}$ proves $f_\tau(P_K(\vec{x})) \to \mathrm{Pr}_\tau(\ulcorner f_\tau(P_K(\dot{\vec{x}}))\urcorner)$ and $\neg f_\tau(P_K(\vec{x})) \to \mathrm{Pr}_\tau(\ulcorner \neg f_\tau(P_K(\dot{\vec{x}}))\urcorner)$. Thus $\mathbf{I\Sigma_1} \vdash f_\tau(\mathrm{D})$. By Fact 2.1.1, $\mathbf{I\Sigma_1} \vdash \mathrm{Pr}_\tau(\ulcorner f_\tau(\mathrm{D})\urcorner)$, and hence $\mathbf{I\Sigma_1} \vdash f_\tau(\boxdot \mathrm{D})$.

Also by Clause 1, $\mathbf{I\Sigma_1} \vdash f_\tau(\chi^\circ) \leftrightarrow \chi$. Since $\mathbf{I\Sigma_1}$ proves χ, $\mathbf{I\Sigma_1} \vdash f_\tau(\chi^\circ)$. As above, $\mathbf{I\Sigma_1} \vdash f_\tau(\boxdot \chi^\circ)$ also holds. □

Artemov's Lemma is used to prove Vardanyan's theorem, but what is important to us is the following observation by Visser and de Jonge.

Fact 2.13 (Visser and de Jonge [16, Theorem 3])**.** *For any $\mathcal{L}_A$-sentence φ, the following are equivalent:*

1. $T + \mathrm{Con}_\tau \vdash \varphi$.
2. $\Diamond\top \wedge \mathrm{D} \wedge \chi^\circ \to \varphi^\circ \in \mathsf{QPL}_\tau(T)$.

We give a proof of Visser and de Jonge's fact.

Proof. ($1 \Rightarrow 2$): Suppose $T + \mathrm{Con}_\tau \vdash \varphi$. By Artemov's Lemma, for any arithmetical interpretation f,

$$T \vdash \mathrm{Con}_\tau \wedge f_\tau(\mathrm{D}) \wedge f_\tau(\chi^\circ) \to f_\tau(\varphi^\circ).$$

Thus $T \vdash f_\tau(\Diamond\top \wedge \mathrm{D} \wedge \chi^\circ \to \varphi^\circ)$. Hence $\Diamond\top \wedge \mathrm{D} \wedge \chi^\circ \to \varphi^\circ \in \mathsf{QPL}_\tau(T)$.

($2 \Rightarrow 1$): Suppose $\Diamond\top \wedge \mathrm{D} \wedge \chi^\circ \to \varphi^\circ \in \mathsf{QPL}_\tau(T)$. For a natural arithmetical interpretation f,

$$T \vdash \mathrm{Con}_\tau \wedge f_\tau(\mathrm{D}) \wedge f_\tau(\chi^\circ) \to f_\tau(\varphi^\circ).$$

By Proposition 2.12, $T + \mathrm{Con}_\tau \vdash \varphi$. $\square$

Visser and de Jonge's fact states that $\mathsf{QPL}_\tau(T)$ has the complete information about $\mathsf{Th}(T + \mathrm{Con}_\tau)$. Then we obtain some corollaries concerning inclusions between quantified provability logics.

Corollary 2.14.

1. *If* $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$*, then* $\mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0}) \subseteq \mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1})$*;*
2. *If* $\mathsf{QPL}_{\tau_0}(T_0) = \mathsf{QPL}_{\tau_1}(T_1)$*, then* $\mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0}) = \mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1})$*.*

Proof. 1. Suppose $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$. Let φ be any $\mathcal{L}_A$-sentence with $T_0 + \mathrm{Con}_{\tau_0} \vdash \varphi$. Then from Fact 2.13, $\Diamond\top \wedge \mathrm{D} \wedge \chi^\circ \to \varphi^\circ \in \mathsf{QPL}_{\tau_0}(T_0)$. By the supposition, $\Diamond\top \wedge \mathrm{D} \wedge \chi^\circ \to \varphi^\circ \in \mathsf{QPL}_{\tau_1}(T_1)$. From Fact 2.13 again, $T_1 + \mathrm{Con}_{\tau_1} \vdash \varphi$. Therefore $\mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0}) \subseteq \mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1})$.

Clause 2 follows from Clause 1. $\square$

The following corollary is an immediate consequence of Corollary 2.14.2.

Corollary 2.15. *If* $\mathsf{QPL}_{\tau_0}(T_0) = \mathsf{QPL}_{\tau_1}(T_1)$ *and* $\mathsf{Th}(T_0) \subseteq \mathsf{Th}(T_1)$*, then* $T_1 \vdash \mathrm{Con}_{\tau_0} \leftrightarrow \mathrm{Con}_{\tau_1}$*.* $\square$

3 On inclusions between quantified provability logics

Inspired by Visser and de Jonge's fact, we explore further consequences of inclusion relationships between quantified provability logics that result from Artemov's Lemma.

3.1 Variations of Fact 2.13 and its consequences

In this subsection, we prove variations of Visser and de Jonge's Fact 2.13 and its consequences. The following proposition is a variation of Fact 2.13 with respect to Σ_1 sentences.

Proposition 3.1. *For any Σ_1 sentence φ, the following are equivalent:*

1. $T \vdash \varphi$.
2. $\chi^\circ \to \varphi^\circ \in \mathsf{QPL}_\tau(T)$.

Proof. $(1 \Rightarrow 2)$: Suppose $T \vdash \varphi$. By Fact 2.9, for any arithmetical interpretation f, $\mathbf{I\Sigma_1} \vdash f_\tau(\chi^\circ) \land \varphi \to f_\tau(\varphi^\circ)$. Hence $T \vdash f_\tau(\chi^\circ \to \varphi^\circ)$. We have $\chi^\circ \to \varphi^\circ \in \mathsf{QPL}_\tau(T)$.

$(2 \Rightarrow 1)$: This is trivial by considering a natural arithmetical interpretation. ☐

Then we obtain a variation of Corollary 2.14 by a similar proof.

Corollary 3.2.

1. *If $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$, then $\mathsf{Th}_{\Sigma_1}(T_0) \subseteq \mathsf{Th}_{\Sigma_1}(T_1)$;*
2. *If $\mathsf{QPL}_{\tau_0}(T_0) = \mathsf{QPL}_{\tau_1}(T_1)$, then $\mathsf{Th}_{\Sigma_1}(T_0) = \mathsf{Th}_{\Sigma_1}(T_1)$.*

☐

By applying Fact 2.9, Corollary 2.15 is strengthened as follows.

Proposition 3.3. *If $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$, then $T_1 \vdash \mathrm{Con}_{\tau_0} \leftrightarrow \mathrm{Con}_{\tau_1}$.*

Proof. Suppose $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$. Then, $T_1 \vdash \mathrm{Con}_{\tau_1} \to \mathrm{Con}_{\tau_0}$ by Corollary 2.14.1, and so it suffices to prove $T_1 \vdash \mathrm{Con}_{\tau_0} \to \mathrm{Con}_{\tau_1}$. Let f be any arithmetical interpretation. Since $\neg\mathrm{Con}_{\tau_0}$ is a Σ_1 sentence, by Fact 2.9,

$$\mathbf{I\Sigma_1} \vdash f_{\tau_0}(\chi^\circ) \to (\neg\mathrm{Con}_{\tau_0} \to f_{\tau_0}(\neg\mathrm{Con}_{\tau_0}^\circ)).$$

Hence $T_0 \vdash f_{\tau_0}(\chi^\circ \land \Box\bot \to \neg\mathrm{Con}_{\tau_0}^\circ)$, and thus $\chi^\circ \land \Box\bot \to \neg\mathrm{Con}_{\tau_0}^\circ$ is in $\mathsf{QPL}_{\tau_0}(T_0)$. From the supposition, $\chi^\circ \land \Box\bot \to \neg\mathrm{Con}_{\tau_0}^\circ \in \mathsf{QPL}_{\tau_1}(T_1)$. By considering a natural arithmetical interpretation, we obtain that T_1 proves $\neg\mathrm{Con}_{\tau_1} \to \neg\mathrm{Con}_{\tau_0}$. Therefore $T_1 \vdash \mathrm{Con}_{\tau_0} \to \mathrm{Con}_{\tau_1}$. ☐

Corollary 3.4. *If T_1 is consistent and $T_1 \vdash \mathrm{Con}_{\tau_0}$, then $\mathsf{QPL}_{\tau_0}(T_0) \nsubseteq \mathsf{QPL}_{\tau_1}(T_1)$.*

Proof. Assume that T_1 is consistent and $T_1 \vdash \mathrm{Con}_{\tau_0}$. If $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$, then by Proposition 3.3, $T_1 \vdash \mathrm{Con}_{\tau_0} \leftrightarrow \mathrm{Con}_{\tau_1}$. From the supposition, $T_1 \vdash \mathrm{Con}_{\tau_1}$ and this contradicts Gödel's second incompleteness theorem. Therefore we get $\mathsf{QPL}_{\tau_0}(T_0) \nsubseteq \mathsf{QPL}_{\tau_1}(T_1)$. ☐

The following corollary is a refinement of the result of Artemov [1].

Corollary 3.5. *Suppose that T is Σ_1-sound. Then, for any Σ_1 definition $\tau(v)$ of T, there exists a Σ_1 definition $\tau'(v)$ of T such that $\mathsf{QPL}_\tau(T) \nsubseteq \mathsf{QPL}_{\tau'}(T)$ and $\mathsf{QPL}_{\tau'}(T) \nsubseteq \mathsf{QPL}_\tau(T)$.*

Proof. Let $\tau(v)$ be any Σ_1 definition of T. Since $\neg\mathrm{Con}_\tau$ is Σ_1, by Fact 2.1.3, $T \vdash \neg\mathrm{Con}_\tau \to \mathrm{Pr}_\tau(\ulcorner\neg\mathrm{Con}_\tau\urcorner)$. Equivalently, $T \vdash \mathrm{Con}_{\tau+\mathrm{Con}_\tau} \to \mathrm{Con}_\tau$. Since T is Σ_1-sound, $\mathrm{Con}_{\tau+\mathrm{Con}_\tau}$ is a true Π_1 sentence. Then, it is known that there exists a Σ_1 definition $\tau'(v)$ of T such that $T \vdash \mathrm{Con}_{\tau'} \leftrightarrow \mathrm{Con}_{\tau+\mathrm{Con}_\tau}$ (cf. Lindström [11, Theorem 2.8.(b)]).

Suppose, towards a contradiction, $T \vdash \mathrm{Con}_\tau \to \mathrm{Con}_{\tau'}$. Then, T proves $\mathrm{Con}_\tau \to \mathrm{Con}_{\tau+\mathrm{Con}_\tau}$ and $\mathrm{Pr}_\tau(\ulcorner\neg\mathrm{Con}_\tau\urcorner) \to \neg\mathrm{Con}_\tau$. By Löb's theorem, T also proves $\neg\mathrm{Con}_\tau$. This contradicts the Σ_1-soundness of T. Thus $T \nvdash \mathrm{Con}_\tau \to \mathrm{Con}_{\tau'}$.

Moreover, $T \nvdash \mathrm{Con}_\tau \leftrightarrow \mathrm{Con}_{\tau'}$. It follows from Proposition 3.3 that $\mathsf{QPL}_\tau(T) \nsubseteq \mathsf{QPL}_{\tau'}(T)$ and $\mathsf{QPL}_{\tau'}(T) \nsubseteq \mathsf{QPL}_\tau(T)$. $\square$

3.2 On provable equivalences of provability predicates

In this subsection, we investigate further consequences of inclusions between quantified provability logics via Artemov's Lemma. In particular, we show that some provable equivalences of provability predicates are derived from inclusion. First, we prepare the following lemma.

Lemma 3.6. *Let f be any arithmetical interpretation.*

1. $\mathbf{PA} \vdash f_\tau(\mathrm{D}) \to (R_f(x,y) \to \mathrm{Pr}_\tau(\ulcorner R_f(\dot{x},\dot{y})\urcorner))$*;*
2. *If $f(P_Z(x))$ and $f(P_S(x,y))$ are Σ_1 formulas, then $\mathbf{I\Sigma_1} \vdash R_f(x,y) \to \mathrm{Pr}_\tau(\ulcorner R_f(\dot{x},\dot{y})\urcorner)$.*

Proof. 1. By the definition of D, $f_\tau(P_Z(x)) \to \mathrm{Pr}_\tau(\ulcorner f_\tau(P_Z(\dot{x}))\urcorner)$ and $f_\tau(P_S(x,y)) \to \mathrm{Pr}_\tau(\ulcorner f_\tau(P_S(\dot{x},\dot{y}))\urcorner)$ are provable in $\mathbf{PA} + f_\tau(\mathrm{D})$. Also if $\mathbf{PA} \vdash \varphi_0 \to \mathrm{Pr}_\tau(\ulcorner\varphi_0\urcorner)$ and $\mathbf{PA} \vdash \varphi_1 \to \mathrm{Pr}_\tau(\ulcorner\varphi_1\urcorner)$, then $\mathbf{PA} \vdash \varphi_0 \wedge \varphi_1 \to \mathrm{Pr}_\tau(\ulcorner\varphi_0 \wedge \varphi_1\urcorner)$ and $T \vdash \exists s\varphi_0 \to \mathrm{Pr}_\tau(\ulcorner\exists s\varphi_0\urcorner)$. Thus it suffices to show that $\mathbf{PA} + f_\tau(\mathrm{D})$ proves

$$\forall z < x\ f_\tau(P_S((s)_z,(s)_{z+1})) \to \mathrm{Pr}_\tau(\ulcorner\forall z < \dot{x}\ f_\tau(P_S((\dot{s})_z,(\dot{s})_{z+1}))\urcorner).$$

Let $\psi(x)$ denote this formula. Since $T \vdash \forall z < 0\ f_\tau(P_S((s)_z,(s)_{z+1}))$, by Fact 2.1.1, $\mathbf{PA} \vdash \mathrm{Pr}_\tau(\ulcorner\forall z < 0\ f_\tau(P_S((\dot{s})_z,(\dot{s})_{z+1}))\urcorner)$. Thus $\mathbf{PA} \vdash \psi(0)$. Also $\mathbf{PA} + f_\tau(\mathrm{D})$ proves

$$\begin{aligned}
&\psi(x) \wedge \forall z < S(x)\ f_\tau(P_S((s)_z,(s)_{z+1})) \\
&\qquad \to \forall z < x\ f_\tau(P_S((s)_z,(s)_{z+1})) \wedge f_\tau(P_S((s)_x,(s)_{x+1})), \\
&\qquad \to \mathrm{Pr}_\tau(\ulcorner\forall z < \dot{x}\ f_\tau(P_S((\dot{s})_z,(\dot{s})_{z+1})) \wedge f_\tau(P_S((\dot{s})_{\dot{x}},(\dot{s})_{\dot{x}+1}))\urcorner), \\
&\qquad \to \mathrm{Pr}_\tau(\ulcorner\forall z < S(\dot{x})\ f_\tau(P_S((\dot{s})_z,(\dot{s})_{z+1}))\urcorner).
\end{aligned}$$

Hence $\mathbf{PA} + f_\tau(\mathrm{D}) \vdash \psi(x) \to \psi(S(x))$, and by the induction axiom, we conclude $\mathbf{PA} + f_\tau(\mathrm{D}) \vdash \forall x\psi(x)$.

2. If $f(P_Z(x))$ and $f(P_S(x,y))$ are Σ_1 formulas, then $R_f(x,y)$ is also a Σ_1 formula. Then the statement follows from Fact 2.1.3. $\square$

We are ready to prove one of our main theorem of this subsection.

Theorem 3.7. *Suppose* $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$. *If* $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$, *then for any* $\mathcal{L}_A$*-formula* $\varphi(\vec{y})$,

$$T_1 \vdash \forall \vec{y} \left(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{y}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi(\dot{\vec{y}}) \urcorner) \right).$$

Proof. Suppose $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$ and $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$. Let f be any arithmetical interpretation. By Artemov's Lemma,

$$\mathbf{I\Sigma_1} \vdash \mathrm{Con}_{\tau_0} \land f_{\tau_0}(\mathrm{D}) \land f_{\tau_0}(\chi^\circ) \land R_f(\vec{x}, \vec{y}) \to (\varphi(\vec{x}) \leftrightarrow f_{\tau_0}(\varphi^\circ(\vec{y}))).$$

Then T_0 proves

$$f_{\tau_0}(\mathrm{D}) \land f_{\tau_0}(\chi^\circ) \land R_f(\vec{x}, \vec{y}) \to ((\mathrm{Con}_{\tau_0} \to \varphi(\vec{x})) \leftrightarrow (\mathrm{Con}_{\tau_0} \to f_{\tau_0}(\varphi^\circ(\vec{y})))).$$

By Fact 2.1, we have

$$\begin{aligned} \mathbf{I\Sigma_1} \vdash\ & f_{\tau_0}(\Box \mathrm{D}) \land f_{\tau_0}(\Box \chi^\circ) \land \mathrm{Pr}_{\tau_0}(\ulcorner R_f(\dot{\vec{x}}, \dot{\vec{y}}) \urcorner) \\ & \to \left(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow f_{\tau_0}(\Box(\Diamond \top \to \varphi^\circ(\vec{y}))) \right). \end{aligned} \tag{1}$$

By Artemov's Lemma again,

$$\begin{aligned} \mathbf{I\Sigma_1} \vdash\ & \mathrm{Con}_{\tau_0} \land f_{\tau_0}(\mathrm{D}) \land f_{\tau_0}(\chi^\circ) \land R_f(\vec{x}, \vec{y}) \\ & \to \left(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{y}}) \urcorner)^\circ) \right). \end{aligned} \tag{2}$$

From Lemma 3.6.1, $\mathbf{PA} + f_{\tau_0}(\mathrm{D}) \vdash R_f(\vec{x}, \vec{y}) \to \mathrm{Pr}_{\tau_0}(\ulcorner R_f(\dot{\vec{x}}, \dot{\vec{y}}) \urcorner)$. By combining this with (1) and (2), we obtain

$$\begin{aligned} \mathbf{PA} \vdash\ & \mathrm{Con}_{\tau_0} \land f_{\tau_0}(\boxdot \mathrm{D}) \land f_{\tau_0}(\boxdot \chi^\circ) \land R_f(\vec{x}, \vec{y}) \\ & \to \left(f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{y}}) \urcorner)^\circ) \leftrightarrow f_{\tau_0}(\Box(\Diamond \top \to \varphi^\circ(\vec{y}))) \right). \end{aligned}$$

Since $\vec{x}$ does not appear in the consequent of the formula,

$$\begin{aligned} \mathbf{PA} \vdash\ & \mathrm{Con}_{\tau_0} \land f_{\tau_0}(\boxdot \mathrm{D}) \land f_{\tau_0}(\boxdot \chi^\circ) \land \exists \vec{x} R_f(\vec{x}, \vec{y}) \\ & \to \left(f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{y}}) \urcorner)^\circ) \leftrightarrow f_{\tau_0}(\Box(\Diamond \top \to \varphi^\circ(\vec{y}))) \right). \end{aligned}$$

From Fact 2.10, $\mathbf{I\Sigma_1} \vdash \mathrm{Con}_{\tau_0} \land f_{\tau_0}(\mathrm{D}) \land f_{\tau_0}(\chi^\circ) \to \forall \vec{y} \exists \vec{x} R_f(\vec{x}, \vec{y})$. Hence

$$\begin{aligned} \mathbf{PA} \vdash\ & \mathrm{Con}_{\tau_0} \land f_{\tau_0}(\boxdot \mathrm{D}) \land f_{\tau_0}(\boxdot \chi^\circ) \\ & \to \left(f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{y}}) \urcorner)^\circ) \leftrightarrow f_{\tau_0}(\Box(\Diamond \top \to \varphi^\circ(\vec{y}))) \right). \end{aligned}$$

Since $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$, we obtain that the sentence

$$\forall \vec{y} \left(\Diamond \top \land \boxdot \mathrm{D} \land \boxdot \chi^\circ \to \left(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{y}}) \urcorner)^\circ \leftrightarrow \Box(\Diamond \top \to \varphi^\circ(\vec{y})) \right) \right)$$

is contained in $\mathsf{QPL}_{\tau_0}(T_0)$. By the supposition, this sentence is also in $\mathsf{QPL}_{\tau_1}(T_1)$. By considering a natural arithmetical interpretation and by Proposition 2.12,

$$T_1 + \mathrm{Con}_{\tau_1} \vdash \forall \vec{y} \left(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{y}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi(\dot{\vec{y}}) \urcorner) \right).$$

By Proposition 3.3, $T_1 \vdash \mathrm{Con}_{\tau_0} \to \mathrm{Con}_{\tau_1}$. Thus $T_1 + \neg\mathrm{Con}_{\tau_1} \vdash \neg\mathrm{Con}_{\tau_0}$, and hence

$$T_1 + \neg\mathrm{Con}_{\tau_1} \vdash \forall \vec{y} \left(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{y}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi(\dot{\vec{y}}) \urcorner) \right).$$

Therefore we conclude

$$T_1 \vdash \forall \vec{y} \left(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{y}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi(\dot{\vec{y}}) \urcorner) \right).$$

□

In our proof of Theorem 3.7, Lemma 3.6 is used to replace the formula $\mathrm{Pr}_{\tau_0}(\ulcorner R_f(\dot{\vec{x}}, \dot{\vec{y}}) \urcorner)$ with $R_f(\vec{x}, \vec{y})$ in the antecedent of a formula. If φ is a sentence, then this procedure is no longer needed, and so the proof proceeds without using Lemma 3.6. Then other parts of our proof of Theorem 3.7 work within $\mathbf{I\Sigma_1}$. Thus we also obtain the following theorem.

Theorem 3.8. *If* $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$*, then for any* $\mathcal{L}_A$*-sentence* φ*,*

$$T_1 \vdash \mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi \urcorner).$$

□

Using Fact 2.9, we prove a variation of Theorem 3.7 with respect to Π_1 formulas.

Theorem 3.9. *Suppose* $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$*. If* $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$*, then for any* Π_1 *formula* $\varphi(\vec{y})$*,*

$$T_1 \vdash \forall \vec{y}(\mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner) \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner)).$$

Proof. Suppose $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$ and $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$. Let f be any arithmetical interpretation and let $\varphi(\vec{y})$ be any Π_1 formula. Since $\neg\varphi(\vec{y})$ is Σ_1, by Fact 2.9, $\mathbf{I\Sigma_1} \vdash f_{\tau_0}(\chi^\circ) \wedge R_f(\vec{x}, \vec{y}) \wedge \neg\varphi(\vec{x}) \to f_{\tau_0}(\neg\varphi^\circ(\vec{y}))$. Then, $T_0 \vdash f_{\tau_0}(\chi^\circ) \wedge R_f(\vec{x}, \vec{y}) \wedge f_{\tau_0}(\varphi^\circ(\vec{y})) \to \varphi(\vec{x})$. By Fact 2.1,

$$\mathbf{I\Sigma_1} \vdash f_{\tau_0}(\Box\chi^\circ) \wedge \mathrm{Pr}_{\tau_0}(\ulcorner R_f(\dot{\vec{x}}, \dot{\vec{y}}) \urcorner) \wedge f_{\tau_0}(\Box\varphi^\circ(\vec{y})) \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner). \quad (3)$$

By Artemov's Lemma, $\mathbf{I\Sigma_1}$ proves

$$\mathrm{Con}_{\tau_0} \wedge f_{\tau_0}(\mathrm{D}) \wedge f_{\tau_0}(\chi^\circ) \wedge R_f(\vec{x}, \vec{y}) \wedge \mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \to f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner)^\circ). \quad (4)$$

By combining Lemma 3.6 with (3) and (4), $\mathbf{PA}$ proves

$$\mathrm{Con}_{\tau_0} \wedge f_{\tau_0}(\mathrm{D}) \wedge f_{\tau_0}(\boxdot\chi^\circ) \wedge R_f(\vec{x}, \vec{y}) \wedge f_{\tau_0}(\Box\varphi^\circ(\vec{y})) \to f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner)^\circ).$$

As in the proof of Theorem 3.7, $R_f(\vec{x}, \vec{y})$ is removed from the antecedent of the formula, that is,

$$\mathbf{PA} \vdash \mathrm{Con}_{\tau_0} \land f_{\tau_0}(\mathrm{D}) \land f_{\tau_0}(\boxdot\chi^\circ) \land f_{\tau_0}(\Box\varphi^\circ(\vec{y})) \to f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}})\urcorner)^\circ).$$

Since $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$,

$$\forall\vec{y}\left(\Diamond\top \land \mathrm{D} \land \boxdot\chi^\circ \land \Box\varphi^\circ(\vec{y}) \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}})\urcorner)^\circ\right) \in \mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1).$$

By considering a natural arithmetical interpretation, we obtain

$$T_1 + \mathrm{Con}_{\tau_1} \vdash \forall\vec{y}(\mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{y}})\urcorner) \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}})\urcorner)).$$

By Proposition 3.3, $T_1 + \neg\mathrm{Con}_{\tau_1} \vdash \neg\mathrm{Con}_{\tau_0}$, and in particular, $T_1 + \neg\mathrm{Con}_{\tau_1}$ proves $\forall\vec{y}\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}})\urcorner)$. Therefore we conclude

$$T_1 \vdash \forall\vec{y}(\mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{y}})\urcorner) \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}})\urcorner)).$$

□

As above, we also obtain the following theorem.

Theorem 3.10. *If $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$, then for any Π_1 sentence φ,*

$$T_1 \vdash \mathrm{Pr}_{\tau_1}(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi \urcorner).$$

□

As consequences of theorems proved in this subsection, we obtain several corollaries.

Corollary 3.11. *If $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$ and T_1 is Σ_1-sound, then*

1. $\mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0}) = \mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1})$*; and*
2. $\mathsf{Th}_{\Pi_1}(T_1) \subseteq \mathsf{Th}_{\Pi_1}(T_0)$.

Proof. Suppose $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$ and T_1 is Σ_1-sound.

1. By Corollary 2.14.1, $\mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0}) \subseteq \mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1})$. On the other hand, let φ be any $\mathcal{L}_A$-sentence φ with $T_1 + \mathrm{Con}_{\tau_1} \vdash \varphi$. Then, $T_1 \vdash \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi \urcorner)$ by Fact 2.1.1. By Theorem 3.8,

$$T_1 \vdash \mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi \urcorner),$$

and hence $T_1 \vdash \mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi \urcorner)$. Then, $\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi \urcorner)$ is true in $\mathbb{N}$ because T_1 is Σ_1-sound. This means $T_0 \vdash \mathrm{Con}_{\tau_0} \to \varphi$. Therefore we conclude $\mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1}) \subseteq \mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0})$.

2. Let φ be any Π_1 sentence such that $T_1 \vdash \varphi$. Then $T_1 \vdash \mathrm{Pr}_{\tau_1}(\ulcorner \varphi \urcorner)$ by Fact 2.1.1. By Theorem 3.10, $T_1 \vdash \mathrm{Pr}_{\tau_1}(\ulcorner \varphi \urcorner) \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi \urcorner)$, and hence $T_1 \vdash \mathrm{Pr}_{\tau_0}(\ulcorner \varphi \urcorner)$. Since T_1 is Σ_1-sound, $T_0 \vdash \varphi$. Thus $\mathsf{Th}_{\Pi_1}(T_1) \subseteq \mathsf{Th}_{\Pi_1}(T_0)$.

□

In the next subsection, we will prove that the assumption of the Σ_1-soundness of T_1 in the statement of Corollary 3.11 cannot be removed (see Propositions 3.21 and 3.22).

Remark 3.12. We say that a theory T_1 is *faithfully interpretable* in a theory T_0 if there exists an interpretation I of T_1 in T_0 such that for any $\mathcal{L}_A$-sentence φ, $T_1 \vdash \varphi$ if and only if $T_0 \vdash I(\varphi)$. Lindström [10] proved that if T_0 and T_1 are consistent recursively enumerable extensions of $\mathbf{PA}$, then T_1 is faithfully interpretable in T_0 if and only if $\mathsf{Th}_{\Pi_1}(T_1) \subseteq \mathsf{Th}_{\Pi_1}(T_0)$ and $\mathsf{Th}_{\Sigma_1}(T_0) \subseteq \mathsf{Th}_{\Sigma_1}(T_1)$. Therefore from Corollaries 3.11 and 3.2.1, we obtain that if $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0) \cap \mathsf{Th}(T_1)$, $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$ and T_1 is Σ_1-sound, then T_1 is faithfully interpretable in T_0.

We show that if T_1 is Σ_1-sound and proves the Σ_1-soundness of T_0, then $\mathsf{QPL}_{\tau_0}(T_0)$ and $\mathsf{QPL}_{\tau_1}(T_1)$ are incomparable in the following strong sense.

Corollary 3.13. *Suppose that T_0 is consistent, T_1 is Σ_1-sound and for some Σ_1 definition $\sigma_0(v)$ of T_0, for all Σ_1 sentences φ, $T_1 \vdash \mathrm{Pr}_{\sigma_0}(\ulcorner \varphi \urcorner) \to \varphi$. Then, for any respective Σ_1 definitions $\tau_0(v)$ and $\tau_1(v)$ of T_0 and T_1, $\mathsf{QPL}_{\tau_0}(T_0) \nsubseteq \mathsf{QPL}_{\tau_1}(T_1)$ and $\mathsf{QPL}_{\tau_1}(T_1) \nsubseteq \mathsf{QPL}_{\tau_0}(T_0)$.*

Proof. First, we show $\mathsf{QPL}_{\tau_0}(T_0) \nsubseteq \mathsf{QPL}_{\tau_1}(T_1)$. By the supposition, T_1 proves $\mathrm{Pr}_{\sigma_0}(\ulcorner 0 = \overline{1} \urcorner) \to 0 = \overline{1}$ which is equivalent to Con_{σ_0}. On the other hand, $T_0 \nvdash \mathrm{Con}_{\sigma_0}$ by the second incompleteness theorem. Since Con_{σ_0} is a Π_1 sentence, $\mathsf{Th}_{\Pi_1}(T_1) \nsubseteq \mathsf{Th}_{\Pi_1}(T_0)$. Therefore $\mathsf{QPL}_{\tau_0}(T_0) \nsubseteq \mathsf{QPL}_{\tau_1}(T_1)$ by Corollary 3.11 because T_1 is Σ_1-sound.

Secondly, we show $\mathsf{QPL}_{\tau_1}(T_1) \nsubseteq \mathsf{QPL}_{\tau_0}(T_0)$. Since $\neg\mathrm{Con}_{\tau_0}$ is Σ_1, T_1 proves $\mathrm{Pr}_{\sigma_0}(\ulcorner \neg\mathrm{Con}_{\tau_0} \urcorner) \to \neg\mathrm{Con}_{\tau_0}$, and also proves $\mathrm{Con}_{\tau_0} \to \mathrm{Con}_{\sigma_0 + \mathrm{Con}_{\tau_0}}$. On the other hand, assume, towards a contradiction, that $T_0 + \mathrm{Con}_{\tau_0}$ proves the sentence $\mathrm{Con}_{\tau_0} \to \mathrm{Con}_{\sigma_0 + \mathrm{Con}_{\tau_0}}$. Then, $T_0 + \mathrm{Con}_{\tau_0}$ proves its own consistency, and hence it is inconsistent by the second incompleteness theorem. We have $T_0 \vdash \neg\mathrm{Con}_{\tau_0}$. By Fact 2.1.1, $T_1 \vdash \mathrm{Pr}_{\sigma_0}(\ulcorner \neg\mathrm{Con}_{\tau_0} \urcorner)$. Hence $T_1 \vdash \neg\mathrm{Con}_{\tau_0}$, and this contradicts the Σ_1-soundness of T_1. We obtain $T_0 + \mathrm{Con}_{\tau_0} \nvdash \mathrm{Con}_{\tau_0} \to \mathrm{Con}_{\sigma_0 + \mathrm{Con}_{\tau_0}}$. Therefore $\mathsf{Th}(T_1) \nsubseteq \mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0})$. By Corollary 2.14.1, we conclude $\mathsf{QPL}_{\tau_1}(T_1) \nsubseteq \mathsf{QPL}_{\tau_0}(T_0)$. ☐

Remark 3.14. Let i and j be any natural numbers with $0 < i < j$. Then, the theory $\mathbf{I\Sigma_j}$ is Σ_1-sound and proves $\mathrm{Pr}_{\mathbf{I\Sigma_i}}(\ulcorner \varphi \urcorner) \to \varphi$ for all Σ_1 sentences φ (cf. Hájek and Pudlák [6, Corollary I.4.34]). From Corollary 3.13, for any respective Σ_1 definitions $\sigma_i(v)$ and $\sigma_j(v)$ of $\mathbf{I\Sigma_i}$ and $\mathbf{I\Sigma_j}$, $\mathsf{QPL}_{\sigma_i}(\mathbf{I\Sigma_i}) \nsubseteq \mathsf{QPL}_{\sigma_j}(\mathbf{I\Sigma_j})$ and $\mathsf{QPL}_{\sigma_j}(\mathbf{I\Sigma_j}) \nsubseteq \mathsf{QPL}_{\sigma_i}(\mathbf{I\Sigma_i})$. This is a refinement of a result of Kurahashi [9].

Lemma 3.15. *Let $\sigma(v)$ be any Σ_1 definition of some theory. Suppose that for all $\mathcal{L}_A$-formulas $\varphi(\vec{x})$, $T \vdash \forall \vec{x}(\mathrm{Pr}_\sigma(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow \mathrm{Pr}_\tau(\ulcorner \varphi(\dot{\vec{x}}) \urcorner))$. Then, for any quantified modal formula A and any arithmetical interpretation f, $T \vdash f_\sigma(A) \leftrightarrow f_\tau(A)$.*

Proof. We prove the lemma by induction on the construction of A. We only give a proof of the case that A is of the form $\Box B$. Assume that T proves $f_\sigma(B) \leftrightarrow f_\tau(B)$. Then, by Fact 2.1, $\mathbf{I\Sigma_1} \vdash \mathrm{Pr}_\tau(\ulcorner f_\sigma(B) \urcorner) \leftrightarrow f_\tau(\Box B)$. Since $T \vdash f_\sigma(\Box B) \leftrightarrow \mathrm{Pr}_\tau(\ulcorner f_\sigma(B) \urcorner)$ by the supposition, we obtain that $f_\sigma(\Box B) \leftrightarrow f_\tau(\Box B)$ is provable in T. $\Box$

Corollary 3.16. *If* $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$ *and* $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$, *then* $\mathsf{QPL}_{\tau_0 + \mathrm{Con}_{\tau_0}}(T_0 + \mathrm{Con}_{\tau_0}) \subseteq \mathsf{QPL}_{\tau_1 + \mathrm{Con}_{\tau_1}}(T_1 + \mathrm{Con}_{\tau_1})$.

Proof. Suppose $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$ and $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$. Let A be any element of $\mathsf{QPL}_{\tau_0 + \mathrm{Con}_{\tau_0}}(T_0 + \mathrm{Con}_{\tau_0})$ and f be an arbitrary arithmetical interpretation. Then, $T_0 + \mathrm{Con}_{\tau_0} \vdash f_{\tau_0 + \mathrm{Con}_{\tau_0}}(A)$. Since $\mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0}) \subseteq \mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1})$ by Corollary 2.14.1, $T_1 + \mathrm{Con}_{\tau_1} \vdash f_{\tau_0 + \mathrm{Con}_{\tau_0}}(A)$. By Theorem 3.7, for any $\mathcal{L}_A$-formula $\varphi(\vec{x})$,

$$T_1 \vdash \forall \vec{x} \left(\mathrm{Pr}_{\tau_0 + \mathrm{Con}_{\tau_0}}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1 + \mathrm{Con}_{\tau_1}}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \right).$$

Thus by Lemma 3.15, $T_1 + \mathrm{Con}_{\tau_1} \vdash f_{\tau_0 + \mathrm{Con}_{\tau_0}}(A) \leftrightarrow f_{\tau_1 + \mathrm{Con}_{\tau_1}}(A)$, and hence $T_1 + \mathrm{Con}_{\tau_1} \vdash f_{\tau_1 + \mathrm{Con}_{\tau_1}}(A)$. Since f is arbitrary, A is contained in $\mathsf{QPL}_{\tau_1 + \mathrm{Con}_{\tau_1}}(T_1 + \mathrm{Con}_{\tau_1})$. $\Box$

Moreover, we strengthen Proposition 3.3 and Corollary 3.16.

Definition 3.17. We define a sequence $(\mathrm{Con}^n_\tau)_{n \in \mathbb{N}}$ of Π_1 consistency statements of T inductively as follows:

1. $\mathrm{Con}^0_\tau :\equiv 0 = 0$; and
2. $\mathrm{Con}^{n+1}_\tau :\equiv \mathrm{Con}_{\tau + \mathrm{Con}^n_\tau}$.

Since $\neg \mathrm{Con}^n_\tau$ is a Σ_1 sentence, $\mathbf{I\Sigma_1} \vdash \neg \mathrm{Con}^n_\tau \to \mathrm{Pr}_\tau(\ulcorner \neg \mathrm{Con}^n_\tau \urcorner)$ by Fact 2.1.3. Equivalently, $\mathbf{I\Sigma_1} \vdash \mathrm{Con}^{n+1}_\tau \to \mathrm{Con}^n_\tau$. Thus $\mathrm{Con}^n_\tau \land \mathrm{Con}_{\tau + \mathrm{Con}^n_\tau}$ is provably equivalent to Con^{n+1}_τ over $\mathbf{I\Sigma_1}$.

Corollary 3.18. *If* $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$ *and* $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$, *then for any natural number* $n \geq 1$,

1. $\mathsf{QPL}_{\tau_0 + \mathrm{Con}^n_{\tau_0}}(T_0 + \mathrm{Con}^n_{\tau_0}) \subseteq \mathsf{QPL}_{\tau_1 + \mathrm{Con}^n_{\tau_1}}(T_1 + \mathrm{Con}^n_{\tau_1})$; *and*
2. $T_1 \vdash \mathrm{Con}^n_{\tau_0} \leftrightarrow \mathrm{Con}^n_{\tau_1}$.

Proof. Suppose $\mathsf{Th}(\mathbf{PA}) \subseteq \mathsf{Th}(T_0)$ and $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$.

1. By induction on $n \geq 1$. For $n = 1$, the statement is exactly Corollary 3.16. Suppose $\mathsf{QPL}_{\tau_0 + \mathrm{Con}^n_{\tau_0}}(T_0 + \mathrm{Con}^n_{\tau_0}) \subseteq \mathsf{QPL}_{\tau_1 + \mathrm{Con}^n_{\tau_1}}(T_1 + \mathrm{Con}^n_{\tau_1})$. As commented above, $\mathrm{Con}^n_{\tau_i} \land \mathrm{Con}_{\tau_i + \mathrm{Con}^n_{\tau_i}}$ is equivalent to $\mathrm{Con}^{n+1}_{\tau_i}$ for $i \in \{0, 1\}$, and hence by Corollary 3.16,

$$\mathsf{QPL}_{\tau_0 + \mathrm{Con}^{n+1}_{\tau_0}}(T_0 + \mathrm{Con}^{n+1}_{\tau_0}) \subseteq \mathsf{QPL}_{\tau_1 + \mathrm{Con}^{n+1}_{\tau_1}}(T_1 + \mathrm{Con}^{n+1}_{\tau_1}).$$

2. By induction on $n \geq 1$. For $n = 1$, the statement is exactly Proposition 3.3. Suppose $T_1 \vdash \mathrm{Con}^n_{\tau_0} \leftrightarrow \mathrm{Con}^n_{\tau_1}$. By Clause 1,

$$\mathsf{QPL}_{\tau_0+\mathrm{Con}^n_{\tau_0}}(T_0 + \mathrm{Con}^n_{\tau_0}) \subseteq \mathsf{QPL}_{\tau_1+\mathrm{Con}^n_{\tau_1}}(T_1 + \mathrm{Con}^n_{\tau_1}).$$

Then by Proposition 3.3, $T_1 + \mathrm{Con}^n_{\tau_1}$ proves $\mathrm{Con}_{\tau_0+\mathrm{Con}^n_{\tau_0}} \leftrightarrow \mathrm{Con}_{\tau_1+\mathrm{Con}^n_{\tau_1}}$. This means

$$T_1 + \mathrm{Con}^n_{\tau_1} \vdash \mathrm{Con}^{n+1}_{\tau_0} \leftrightarrow \mathrm{Con}^{n+1}_{\tau_1}. \tag{5}$$

We prove $T_1 \vdash \mathrm{Con}^{n+1}_{\tau_0} \leftrightarrow \mathrm{Con}^{n+1}_{\tau_1}$. Since $T_1 + \mathrm{Con}^{n+1}_{\tau_1} \vdash \mathrm{Con}^n_{\tau_1}$, it follows from (5) that $T_1 + \mathrm{Con}^{n+1}_{\tau_1} \vdash \mathrm{Con}^{n+1}_{\tau_0}$. Conversely, since $T_1 + \mathrm{Con}^{n+1}_{\tau_0} \vdash \mathrm{Con}^n_{\tau_0}$, $T_1 + \mathrm{Con}^{n+1}_{\tau_0} \vdash \mathrm{Con}^n_{\tau_1}$ by induction hypothesis. Then, $T_1 + \mathrm{Con}^{n+1}_{\tau_0} \vdash \mathrm{Con}^{n+1}_{\tau_1}$ from (5). $\square$

Under certain suppositions, we give the following necessary and sufficient condition for $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$.

Corollary 3.19. *Suppose that $\mathsf{Th}(T_0) \subseteq \mathsf{Th}(T_1)$ and there exists a Π_1 sentence π satisfying the following two conditions:*

- $T_0 \vdash \mathrm{Con}_{\tau_0} \to \neg \mathrm{Pr}_{\tau_0}(\ulcorner \pi \urcorner)$;
- $T_1 \vdash \mathrm{Pr}_{\tau_1}(\ulcorner \pi \urcorner)$.

Then, $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$ if and only if $T_1 \vdash \neg \mathrm{Con}_{\tau_0} \land \neg \mathrm{Con}_{\tau_1}$.

Proof. ($\Rightarrow$): Suppose $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$. Let π be a Π_1 sentence satisfying the two conditions stated above. By Theorem 3.10, $\mathrm{Pr}_{\tau_1}(\ulcorner \pi \urcorner) \to \mathrm{Pr}_{\tau_0}(\ulcorner \pi \urcorner)$ is provable in T_1, and hence $T_1 \vdash \mathrm{Pr}_{\tau_0}(\ulcorner \pi \urcorner)$ by the choice of π. On the other hand, by Corollary 2.14.1, $\mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0}) \subseteq \mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1})$, and thus $T_1 + \mathrm{Con}_{\tau_1} \vdash \neg \mathrm{Pr}_{\tau_0}(\ulcorner \pi \urcorner)$. Therefore $T_1 + \mathrm{Con}_{\tau_1}$ is inconsistent, and we obtain $T_1 \vdash \neg \mathrm{Con}_{\tau_1}$. By Proposition 3.3, $T_1 \vdash \mathrm{Con}_{\tau_0} \to \mathrm{Con}_{\tau_1}$. Hence $T_1 \vdash \neg \mathrm{Con}_{\tau_0}$.

($\Leftarrow$): Assume that T_1 proves $\neg \mathrm{Con}_{\tau_0}$ and $\neg \mathrm{Con}_{\tau_1}$. Then, for any $\mathcal{L}_A$-formula $\varphi(\vec{x})$, $T_1 \vdash \forall \vec{x}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner))$. Let A be any element of $\mathsf{QPL}_{\tau_0}(T_0)$ and f be any arithmetical interpretation. Then, $T_0 \vdash f_{\tau_0}(A)$. Since $\mathsf{Th}(T_0) \subseteq \mathsf{Th}(T_1)$, $T_1 \vdash f_{\tau_0}(A)$. By Lemma 3.15, $f_{\tau_0}(A) \leftrightarrow f_{\tau_1}(A)$ is provable in T_1, and hence $T_1 \vdash f_{\tau_1}(A)$. Therefore $A \in \mathsf{QPL}_{\tau_1}(T_1)$. We have proved $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$. $\square$

For example, for any Π_1 sentence π satisfying $T_0 \vdash \mathrm{Con}_{\tau_0} \to \neg \mathrm{Pr}_{\tau_0}(\ulcorner \pi \urcorner)$, the theories T_0 and $T_1 := T_0 + \pi$ satisfy the assumption of Corollary 3.19. Corollary 3.19 is used in the proof of Proposition 3.23 below.

3.3 Some counterexamples

In this subsection, we give some counterexamples to several statements. Before giving them, we prepare a lemma.

Lemma 3.20. *For any $\mathcal{L}_A$-sentence φ with $T \vdash \varphi \to \mathrm{Pr}_\tau(\ulcorner \varphi \urcorner)$,*

$$\mathsf{QPL}_\tau(T) \subseteq \mathsf{QPL}_{\tau+\varphi}(T+\varphi).$$

Proof. Suppose $T \vdash \varphi \to \mathrm{Pr}_\tau(\ulcorner \varphi \urcorner)$. Let A be any element of $\mathsf{QPL}_\tau(T)$ and f be any arithmetical interpretation. Then, $T \vdash f_\tau(A)$. Since $T+\varphi$ proves $\mathrm{Pr}_\tau(\ulcorner \varphi \urcorner)$, for any $\mathcal{L}_A$-formula $\psi(\vec{x})$, it follows from Fact 2.1.2 that

$$\begin{aligned} T+\varphi \vdash \mathrm{Pr}_{\tau+\varphi}(\ulcorner \psi(\dot{\vec{x}}) \urcorner) &\leftrightarrow \mathrm{Pr}_\tau(\ulcorner \varphi \to \psi(\dot{\vec{x}}) \urcorner), \\ &\leftrightarrow \mathrm{Pr}_\tau(\ulcorner \psi(\dot{\vec{x}}) \urcorner). \end{aligned}$$

Then by Lemma 3.15, $T+\varphi \vdash f_\tau(A) \leftrightarrow f_{\tau+\varphi}(A)$. Hence $T+\varphi \vdash f_{\tau+\varphi}(A)$. We conclude $\mathsf{QPL}_\tau(T) \subseteq \mathsf{QPL}_{\tau+\varphi}(T+\varphi)$. $\square$

The following two propositions show that in the statement of Corollary 3.11, the assumption of the Σ_1-soundness of T_1 cannot be omitted.

Proposition 3.21. *There exist consistent recursively enumerable extensions T_0 and T_1 of $\mathbf{I\Sigma_1}$ and respective Σ_1 definitions $\tau_0(v)$ and $\tau_1(v)$ of T_0 and T_1 satisfying the following conditions:*

1. $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$*;*
2. $T_0 + \mathrm{Con}_{\tau_0}$ *and* $T_1 + \mathrm{Con}_{\tau_1}$ *are consistent; and*
3. $\mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1}) \nsubseteq \mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0})$.

Proof. Let T_0 be any Σ_1-sound recursively enumerable extension of $\mathbf{I\Sigma_1}$ and $\tau_0(v)$ be any Σ_1 definition of T_0. Also let φ be the Σ_1 sentence $\neg\mathrm{Con}^2_{\tau_0}$. Then $\mathbb{N} \models \neg\varphi$. Let $T_1 := T_0 + \varphi$ and $\tau_1(v)$ be $(\tau_0+\varphi)(v)$.

1. Since φ is a Σ_1 sentence, $T_0 \vdash \varphi \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi \urcorner)$ by Fact 2.1.3. Then by Lemma 3.20, $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$.

2. Since T_0 is Σ_1-sound, $T_0 + \mathrm{Con}_{\tau_0}$ is consistent. Suppose, towards a contradiction, that $T_1 + \mathrm{Con}_{\tau_1}$ is inconsistent. Then $T_0 + \varphi \vdash \neg\mathrm{Con}_{\tau_0+\varphi}$, and hence $T_0 \vdash \varphi \to \mathrm{Pr}_{\tau_0}(\ulcorner \neg\varphi \urcorner)$. Since $T_0 \vdash \varphi \to \mathrm{Pr}_{\tau_0}(\ulcorner \varphi \urcorner)$, we have $T_0 \vdash \varphi \to \neg\mathrm{Con}_{\tau_0}$. It follows $T_0 \vdash \mathrm{Pr}_{\tau_0}(\ulcorner \neg\mathrm{Con}_{\tau_0} \urcorner) \to \neg\mathrm{Con}_{\tau_0}$. By Löb's theorem, $T_0 \vdash \neg\mathrm{Con}_{\tau_0}$. This contradicts the Σ_1-soundness of T_0. Therefore $T_1 + \mathrm{Con}_{\tau_1}$ is consistent.

3. Since $T_0 + \mathrm{Con}_{\tau_0}$ is also Σ_1-sound, $T_0 + \mathrm{Con}_{\tau_0} \nvdash \varphi$. On the other hand, $T_1 + \mathrm{Con}_{\tau_1} \vdash \varphi$, and hence $\mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1}) \nsubseteq \mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0})$. $\square$

Proposition 3.22. *There exist consistent recursively enumerable extensions T_0 and T_1 of $\mathbf{I\Sigma_1}$ and respective Σ_1 definitions $\tau_0(v)$ and $\tau_1(v)$ of T_0 and T_1 satisfying the following conditions:*

1. $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$*; and*
2. $\mathsf{Th}_{\Pi_1}(T_1) \nsubseteq \mathsf{Th}_{\Pi_1}(T_0)$.

Proof. Let T_0 be an arbitrary consistent recursively enumerable extension of $\mathbf{I\Sigma_1}$ and $\tau_0(v)$ be any Σ_1 definition of T_0. Let ρ be a Π_1 Rosser sentence of T_0 defined by using $\tau_0(v)$, and let $T_1 := T_0 + \neg\rho$ and $\tau_1(v)$ be $(\tau_0 + \neg\rho)(v)$. By Rosser's theorem, T_1 is consistent. Since $\neg\rho$ is Σ_1, by Lemma 3.20, $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$. It is easily shown that there exists a Π_1 sentence π such that $\mathbf{I\Sigma_1} \vdash \rho \vee \pi$ and $\mathbf{I\Sigma_1} \vdash \rho \wedge \pi \to \mathrm{Con}_{\tau_0}$. Then $T_1 \vdash \pi$ and $T_0 \nvdash \pi$ because $T_0 \nvdash \rho \to \mathrm{Con}_{\tau_0}$. Therefore $\mathsf{Th}_{\Pi_1}(T_1) \nsubseteq \mathsf{Th}_{\Pi_1}(T_0)$ (see also Lindström [11, Chapter 5 Exercise 1]). $\square$

The following proposition shows that the converse implications of Proposition 3.3, Theorem 3.7 and Corollary 3.11 do not hold.

Proposition 3.23. *There exist consistent recursively enumerable extensions T_0 and T_1 of $\mathbf{I\Sigma_1}$ and respective Σ_1 definitions $\tau_0(v)$ and $\tau_1(v)$ of T_0 and T_1 satisfying the following conditions:*

1. $\mathbf{I\Sigma_1} \vdash \mathrm{Con}_{\tau_0} \leftrightarrow \mathrm{Con}_{\tau_1}$*;*
2. T_1 *is Σ_1-sound and* $\mathsf{Th}(T_0 + \mathrm{Con}_{\tau_0}) = \mathsf{Th}(T_1 + \mathrm{Con}_{\tau_1})$*;*
3. *For any $\mathcal{L}_A$-formula $\varphi(\vec{x})$,*

$$\mathbf{I\Sigma_1} \vdash \forall \vec{x} \left(\mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{x}})\urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi(\dot{\vec{x}})\urcorner)\right);$$

4. $\mathsf{QPL}_{\tau_0}(T_0) \nsubseteq \mathsf{QPL}_{\tau_1}(T_1)$.

Proof. Let T_0 be any Σ_1-sound recursively enumerable extension of $\mathbf{I\Sigma_1}$ and $\tau_0(v)$ be any Σ_1 definition of T_0. Let ρ be a Π_1 Rosser sentence of T_0 defined by using $\tau_0(v)$. Also let $T_1 := T_0 + \rho$ and $\tau_1(v)$ be $(\tau_0 + \rho)(v)$.

1. Since $\mathbf{I\Sigma_1} \vdash \mathrm{Con}_{\tau_0} \leftrightarrow \neg\mathrm{Pr}_{\tau_0}(\ulcorner \neg\rho \urcorner)$, $\mathbf{I\Sigma_1} \vdash \mathrm{Con}_{\tau_0} \leftrightarrow \mathrm{Con}_{\tau_1}$.

2. Let ψ be any Σ_1 sentence with $T_1 \vdash \psi$. Then $T_0 \vdash \neg\rho \vee \psi$. Since T_0 is Σ_1-sound, $\mathbb{N} \models \neg\rho \vee \psi$. Since $\mathbb{N} \models \rho$, $\mathbb{N} \models \psi$. Hence T_1 is Σ_1-sound.

Moreover, since $\mathbf{I\Sigma_1} \vdash \mathrm{Con}_{\tau_0} \to \rho$, $T_0 + \mathrm{Con}_{\tau_0}$ is deductively equivalent to $T_0 + \rho + \mathrm{Con}_{\tau_0}$, and to $T_1 + \mathrm{Con}_{\tau_1}$.

3. For any $\mathcal{L}_A$-formula $\varphi(\vec{x})$,

$$\begin{aligned}\mathbf{I\Sigma_1} \vdash \mathrm{Pr}_{\tau_0}(\ulcorner \mathrm{Con}_{\tau_0} \to \varphi(\dot{\vec{x}})\urcorner) &\leftrightarrow \mathrm{Pr}_{\tau_0 + \mathrm{Con}_{\tau_0}}(\ulcorner \varphi(\dot{\vec{x}})\urcorner), \\ &\leftrightarrow \mathrm{Pr}_{\tau_0 + \rho + \mathrm{Con}_{\tau_0 + \rho}}(\ulcorner \varphi(\dot{\vec{x}})\urcorner), \\ &\leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \mathrm{Con}_{\tau_1} \to \varphi(\dot{\vec{x}})\urcorner).\end{aligned}$$

4. Since T_1 is Σ_1-sound and T_0 is consistent, $T_1 \nvdash \neg\mathrm{Con}_{\tau_0}$. It follows from Corollary 3.19 that $\mathsf{QPL}_{\tau_0}(T_0) \nsubseteq \mathsf{QPL}_{\tau_1}(T_1)$. $\square$

4 Σ_1 arithmetical interpretations

In this section, we investigate inclusions between quantified provability logics with respect to Σ_1 arithmetical interpretations. The main goal of this section is to give a necessary and sufficient condition for the inclusion relation between quantified provability logics with respect to Σ_1 arithmetical interpretations.

Definition 4.1. An arithmetical interpretation f is Σ_n if for any atomic formula $P(\vec{x})$ of quantified modal logic, $f(P(\vec{x}))$ is a Σ_n formula.

Notice that there are natural Σ_1 arithmetical interpretations. We introduce the quantified provability logics with respect to Σ_n arithmetical interpretations.

Definition 4.2. $\mathsf{QPL}_\tau^{\Sigma_n}(T) := \{\varphi \mid \varphi$ is a sentence and for all Σ_n arithmetical interpretations f, $T \vdash f_\tau(\varphi)\}$.

Berarducci [2] proved that restricting arithmetical interpretations to Σ_n does not change the complexity of quantified provability logics, that is, for each $n \geq 1$, the complexity of the quantified provability logic of **PA** with respect to Σ_n arithmetical interpretations is also Π^0_2-complete.

On the other hand, it is beneficial to deal with Σ_1 arithmetical interpretations in our study. In the proof of Artemov's Lemma, the assumption $\mathrm{Con}_\tau \wedge f_\tau(\mathrm{D})$ is prepared to make the formulas $f(P_K(x))$ and $\neg f(P_K(x,y))$ equivalent to Σ_1 formulas for each $K \in \{Z, S, A, M, L, E\}$. In the case that f is a Σ_1 arithmetical interpretation, the same result holds without the assumption $\mathrm{Con}_\tau \wedge f_\tau(\mathrm{D})$ by adding sufficiently many theorems of $\mathbf{I\Sigma_1}$ to the sentence χ as conjuncts. This is guaranteed by the following equivalences:

- $\neg P_Z(x) \leftrightarrow \exists y P_S(y, x)$;
- $\neg P_S(x, y) \leftrightarrow \exists z(P_S(x, z) \wedge (P_L(z, y) \vee P_L(y, z)))$;
- $\neg P_A(x, y, z) \leftrightarrow \exists w(P_A(x, y, w) \wedge (P_L(w, z) \vee P_L(z, w)))$;
- $\neg P_M(x, y, z) \leftrightarrow \exists w(P_M(x, y, w) \wedge (P_L(w, z) \vee P_L(z, w)))$;
- $\neg P_L(x, y) \leftrightarrow P_E(x, y) \vee P_L(y, x)$;
- $\neg P_E(x, y) \leftrightarrow P_L(x, y) \vee P_L(y, x)$.

Thus we obtain the following variation of Artemov's Lemma with respect to Σ_1 arithmetical interpretations.

Theorem 4.3 (Σ_1-Artemov's Lemma)**.** *There exists an $\mathcal{L}_A$-sentence χ such that $\mathbf{I\Sigma_1} \vdash \chi$ and for any Σ_1 arithmetical interpretation f and any $\mathcal{L}_A$-formula $\varphi(\vec{x})$,*

$$\mathbf{I\Sigma_1} \vdash f_\tau(\chi^\circ) \wedge R_f(\vec{x}, \vec{y}) \to (\varphi(\vec{x}) \leftrightarrow f_\tau(\varphi^\circ(\vec{y}))).$$

□

We also obtain a variation of Fact 2.10 with respect to Σ_1 arithmetical interpretations.

Proposition 4.4. *For any Σ_1 arithmetical interpretation f,*

$$\mathbf{I\Sigma_1} \vdash f_\tau(\chi^\circ) \to \forall y \exists x R_f(x, y).$$

$\square$

The following proposition is a variation of Fact 2.13 with respect to Σ_1 arithmetical interpretations.

Proposition 4.5. *For any $\mathcal{L}_A$-sentence φ, the following are equivalent:*

1. $T \vdash \varphi$.
2. $\chi^\circ \to \varphi^\circ \in \mathsf{QPL}^{\Sigma_1}_\tau(T)$.

Proof. ($1 \Rightarrow 2$): Suppose $T \vdash \varphi$. By Σ_1-Artemov's Lemma, for any Σ_1 arithmetical interpretation f, $\mathbf{I\Sigma_1} \vdash f_\tau(\chi^\circ) \to (\varphi \leftrightarrow f_\tau(\varphi^\circ))$. Then T proves $f_\tau(\chi^\circ \to \varphi^\circ)$. Thus $\chi^\circ \to \varphi^\circ \in \mathsf{QPL}^{\Sigma_1}_\tau(T)$.

($2 \Rightarrow 1$): Suppose $\chi^\circ \to \varphi^\circ \in \mathsf{QPL}^{\Sigma_1}_\tau(T)$. By considering a natural Σ_1 arithmetical interpretation, we obtain $T \vdash \varphi$. $\square$

We prove the following main theorem of this section.

Theorem 4.6. *The following are equivalent:*

1. $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$.
2. $\mathsf{Th}(T_0) \subseteq \mathsf{Th}(T_1)$ *and for any $\mathcal{L}_A$-formula $\varphi(\vec{x})$,*

$$T_1 \vdash \forall \vec{x}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner)).$$

Proof. ($1 \Rightarrow 2$): Suppose $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$.

First, we prove $\mathsf{Th}(T_0) \subseteq \mathsf{Th}(T_1)$. Let φ be any sentence with $T_0 \vdash \varphi$. Then by Proposition 4.5, $\chi^\circ \to \varphi^\circ \in \mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0)$. By the supposition, this sentence is also in $\mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$. Then by Proposition 4.5 again, we obtain $T_1 \vdash \varphi$. Therefore $\mathsf{Th}(T_0) \subseteq \mathsf{Th}(T_1)$.

Secondly, we prove the T_1-provable equivalence of the two provability predicates. Let $\varphi(\vec{y})$ be any $\mathcal{L}_A$-formula. By Σ_1-Artemov's Lemma, for any Σ_1 arithmetical interpretation f,

$$\mathbf{I\Sigma_1} \vdash f_{\tau_0}(\chi^\circ) \wedge R_f(\vec{x}, \vec{y}) \to (\varphi(\vec{x}) \leftrightarrow f_{\tau_0}(\varphi^\circ(\vec{y}))).$$

By Fact 2.1,

$$\mathbf{I\Sigma_1} \vdash f_{\tau_0}(\Box\chi^\circ) \wedge \mathrm{Pr}_{\tau_0}(\ulcorner R_f(\dot{\vec{x}}, \dot{\vec{y}}) \urcorner) \to \Big(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow f_{\tau_0}(\Box\varphi^\circ(\vec{y}))\Big). \quad (6)$$

By Σ_1-Artemov's Lemma again,

$$\mathbf{I\Sigma_1} \vdash f_{\tau_0}(\chi^\circ) \wedge R_f(\vec{x}, \vec{y}) \to \Big(f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner)^\circ) \leftrightarrow \mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner)\Big). \quad (7)$$

By Lemma 3.6.2, $\mathbf{I\Sigma_1} \vdash R_f(\vec{x}, \vec{y}) \to \mathrm{Pr}_{\tau_0}(\ulcorner R_f(\dot{\vec{x}}, \dot{\vec{y}}) \urcorner)$. By combining this with (6) and (7),

$$\mathbf{I\Sigma_1} \vdash f_{\tau_0}(\boxdot \chi^\circ) \wedge R_f(\vec{x}, \vec{y}) \to \Big(f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner)^\circ) \leftrightarrow f_{\tau_0}(\Box \varphi^\circ(\vec{y}))\Big).$$

Since $\vec{x}$ does not appear in the consequent of the formula,

$$\mathbf{I\Sigma_1} \vdash f_{\tau_0}(\boxdot \chi^\circ) \wedge \exists \vec{x} R_f(\vec{x}, \vec{y}) \to \Big(f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner)^\circ) \leftrightarrow f_{\tau_0}(\Box \varphi^\circ(\vec{y}))\Big).$$

By Proposition 4.4, $\mathbf{I\Sigma_1} \vdash f_{\tau_0}(\chi^\circ) \to \forall \vec{y} \exists \vec{x} R_f(\vec{x}, \vec{y})$. Then,

$$\mathbf{I\Sigma_1} \vdash f_{\tau_0}(\boxdot \chi^\circ) \to \Big(f_{\tau_0}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner)^\circ) \leftrightarrow f_{\tau_0}(\Box \varphi^\circ(\vec{y}))\Big).$$

We obtain

$$\forall \vec{y} \Big(\boxdot \chi^\circ \to \Big(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner)^\circ \leftrightarrow \Box \varphi^\circ(\vec{y})\Big)\Big) \in \mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1).$$

By considering a natural Σ_1 arithmetical interpretation, we conclude

$$T_1 \vdash \forall \vec{y} \Big(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{y}}) \urcorner)\Big).$$

($2 \Rightarrow 1$): Assume Clause 2 of the statement. Let A be any element of $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0)$ and f be any Σ_1 arithmetical interpretation. Then, $T_0 \vdash f_{\tau_0}(A)$. Since $\mathsf{Th}(T_0) \subseteq \mathsf{Th}(T_1)$, $T_1 \vdash f_{\tau_0}(A)$. By the assumption and Lemma 3.15, we have $T_1 \vdash f_{\tau_0}(A) \leftrightarrow f_{\tau_1}(A)$, and thus $T_1 \vdash f_{\tau_1}(A)$. Therefore A is in $\mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$. We have proved $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$. ☐

Similar to the proof of ($2 \Rightarrow 1$) of Theorem 4.6, it can be proved that Clause 2 in the statement of Theorem 4.6 implies $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$.

Corollary 4.7. *If* $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$, *then* $\mathsf{QPL}_{\tau_0}(T_0) \subseteq \mathsf{QPL}_{\tau_1}(T_1)$. ☐

We propose the following question.

Problem 4.8. *Does the converse implication of Corollary 4.7 hold?*

We close this section with the following corollary.

Corollary 4.9. *If* $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$ *and* T_1 *is* Σ_1*-sound, then* $\mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0) = \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$.

Proof. Suppose $\mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_0) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1)$ and T_1 is Σ_1-sound. By Theorem 4.6, $T_1 \vdash \forall \vec{x}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner))$ for any $\mathcal{L}_A$-formula $\varphi(\vec{x})$. Let ψ be any $\mathcal{L}_A$-sentence with $T_1 \vdash \psi$. Since T_1 proves $\mathrm{Pr}_{\tau_1}(\ulcorner \psi \urcorner)$ by Fact 2.1.1, we have $T_1 \vdash \mathrm{Pr}_{\tau_0}(\ulcorner \psi \urcorner)$. Since T_1 is Σ_1-sound, $T_0 \vdash \psi$. We have shown $\mathsf{Th}(T_1) \subseteq \mathsf{Th}(T_0)$. Then, for any $\mathcal{L}_A$-formula $\varphi(\vec{x})$, $T_0 \vdash \forall \vec{x}(\mathrm{Pr}_{\tau_0}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner) \leftrightarrow \mathrm{Pr}_{\tau_1}(\ulcorner \varphi(\dot{\vec{x}}) \urcorner))$. By Theorem 4.6, we conclude $\mathsf{QPL}^{\Sigma_1}_{\tau_1}(T_1) \subseteq \mathsf{QPL}^{\Sigma_1}_{\tau_0}(T_0)$. ☐

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