# Stochastic Evolution Equations of Jump Type with Random Coefficients: Existence, Uniqueness and Optimal Control * 

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#### Abstract

We study a class of stochastic evolution equations of jump type with random coefficients and its optimal control problem. There are three major ingredients. The first is to prove the existence and uniqueness of the solutions by continuous dependence theorem of solutions combining with the parameter extension method. The second is to establish the stochastic maximum principle and verification theorem for our optimal control problem by the classic convex variation method and dual technique. The third is to represent an example of a Cauchy problem for a controlled stochastic partial differential equation with jumps which our theoretical results can solve.


Keywords: Stochastic Evolution Equation; Poisson Random Martingale Measure; Stochastic Maximum Principle; Verification Theorem

## 1 Introduction

In this paper, we study the following stochastic evolution equation with jump

$$
\left\{\begin{align*}
d X(t) & =[A(t) X(t)+b(t, X(t))] d t+[B(t) X(t)+g(t, X(t))] d W(t)+\int_{E} \sigma(t, X(t), e) \tilde{\mu}(d e, d t),  \tag{1.1}\\
X(0) & =x, \quad t \in[0, T]
\end{align*}\right.
$$

in the framework of a Gelfand triple $V \subset H=H^{*} \subset V^{*}$, where $H, V$ are Hilbert spaces. Here $W$ is a one-dimensional Brownian motion and $\tilde{\mu}$ is a Poisson random martingale measure on a filtrated probability $\left(\Omega, \mathscr{F},\left\{\mathscr{F}_{t}\right\}_{0 \leq t \leq T}, P\right)$, $A:[0, T] \times \Omega \longrightarrow \mathscr{L}\left(V, V^{*}\right), B:[0, T] \times \Omega \longrightarrow \mathscr{L}(V, H), b:[0, T] \times \Omega \times H \longrightarrow H, g:[0, T] \times \Omega \times H \longrightarrow H$ and $\sigma:[0, T] \times \Omega \times E \times H \longrightarrow H$ are given random mappings. Here we denote by $\mathscr{L}\left(V, V^{*}\right)$ the space of bounded linear transformations of V into $V^{*}$, by $\mathscr{L}(V, H)$ the space of bounded linear transformations of $H$ into $V$. An adapted solution of (1.1) is a $V$-valued, $\left\{\mathscr{F}_{t}\right\}_{0 \leq t \leq T}$-adapted process $X(\cdot)$ under some appropriate sense.

Stochastic partial differential equations and stochastic evolution equations driven only by Wiener processes have been investigated in depth and a great deal of advances have been made by many authors, see, for example the monographs [5],[6],[23] for their general theory. Most recently, thanks to comprehensive practical applications, many attentions have been paid to stochastic partial differential equations driven by jump processes, (cf., for example,[1],[2],[4],[10],[21],[22],[25],[26],[28], [29] and the references therein). It is worth mentioning that Röcker and Zhang [23] established the existence and uniqueness theory for solutions of stochastic evolution equations of type (1.1) by a successive approximations, in which case the operator does not exist and all coefficients involved are deterministic mappings.

The purpose of this paper is to show the existence and uniqueness of solutions to the stochastic evolution equation (1.1) and establish the corresponding maximum principle and verification theorem for the optimal control

[^0]problem where the state is driven by a controlled stochastic evolution equation (1.1). The main feature of this paper is that all coefficients of (1.1) are allowed to be random and time varying. Moreover, different from Picard iteration approach adopted by Röcker and Zhang [25] to prove the existence and uniqueness results, our approach is that we first establish continuous dependence theorem of solutions and then combine the parameter extension method to construct a contractive mapping. For optimal control problems of stochastic evolution equation or stochastic partial differential equation, many works are concerned with systems driven only by Wiener process and the corresponding stochastic maximum principles are establish, see e.g. [3],[7],[9],[11],[12], [14],[15],[16],[17], [18], [20],[30]. In contrast, there have not been very much results on the optimal control for stochastic partial differential equations driven by jump processes. In 2005, Oksendal etl [23] studied the optimal control problem for a stochastic reaction diffusion equation driven by Poisson random measure in which case the operator $B$ is absence and all the coefficients are deterministic. Thus our system cover the model studied by Øksendal etl [23] and is more general in such a way that all coefficients are random and time-varying.

The rest of this paper is structured as follows. In Section 2, we provide the basic notations and recall Itô formula for jump diffusion in Hilbert space used frequently in this paper. Section 3 establishes the existence and uniqueness of solutions to the stochastic evolution equation (1.1). Section 4 formulates the optimal control problem specifying the hypotheses. In section 5 , adjoint equation is introduced which turns out to be a backward stochastic evolution equation with jumps. In Section 6, we establish the stochastic maximum principle by the classic convex variation method. In Sections 7, the verification theorem for optimal controls is obtained by dual technique. In section 8 , we present one example of application of our results.

## 2 Notations and Itô Formula for Jump Diffusion in Hilbert Space

In this section, we first introduce the notations which will be used in our paper. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space equipped with a one-dimensional standard Brownian motion $\{W(t), 0 \leq t \leq T\}$ and a stationary Poisson point process $\left\{\eta_{t}\right\}_{t \geq 0}$ defined on a fixed nonempty Borel measurable subset $E$ of $\mathbb{R}^{1}$. Denote by $\mathbb{E}[\cdot]$ the expectation under the probability $\mathbb{P}$. We denote by $\mu(d e, d t)$ the counting measure induced by $\left\{\eta_{t}\right\}_{t \geq 0}$ and by $\nu(d e)$ the corresponding characteristic measure. Then the compensatory random martingale measure is denoted by $\tilde{\mu}(d e, d t):=\mu(d e, d t)-\nu(d e) d t$ which is assumed to be independent of the Brownian motion. Furthermore, we assume that $\nu(E)<\infty$. Let $\left\{\mathscr{F}_{t}\right\}_{0 \leq t \leq T}$ be the P-augmentation of the natural filtration generated by $\left\{W_{t}\right\}_{t \geq 0}$ and $\left\{\eta_{t}\right\}_{t \geq 0}$. By $\mathscr{P}$ we denote the predictable $\sigma$ field on $\Omega \times[0, T]$ and by $\mathscr{B}(\Lambda)$ the Borel $\sigma$-algebra of any topological space $\Lambda$. Let $X$ be a separable Hilbert space with norm $\|\cdot\|_{X}$. Denote by $M^{\nu, 2}(E ; X)$ the set of all $X$-valued measurable functions $r=\{r(e), e \in E\}$ defined on the measure space $(E, \mathscr{B}(E) ; v)$ such that $\|r\|_{M^{\nu, 2}(E ; X)} \triangleq \sqrt{\int_{E}\|r(e)\|_{X}^{2} v(d e)}<\infty$, by $M_{\mathscr{F}}^{\nu, 2}([0, T] \times E ; X)$ the set of all $\mathscr{P} \times \mathscr{B}(E)$-measurable $X$-valued processes $r=\{r(t, \omega, e),(t, \omega, e) \in[0, T] \times \Omega \times E\}$ such that $\|r\|_{M_{\mathscr{F}}^{\nu, 2}([0, T] \times E ; X)} \triangleq \sqrt{\mathbb{E}\left[\int_{0}^{T} \int_{E}\|r(t, e)\|_{X}^{2} \nu(d e) d t\right]}<\infty$, by $M_{\mathscr{F}}^{2}(0, T ; X)$ the set of all $\mathscr{F}_{t}$-adapted $X$ valued processes $f=\{f(t, \omega),(t, \omega) \in[0, T] \times \Omega\}$ such that $\|f\|_{M_{\mathscr{F}}^{2}(0, T ; X)} \triangleq \sqrt{\mathbb{E}\left[\int_{0}^{T}\|f(t)\|_{X}^{2} d t\right]}<\infty$, by $S_{\mathscr{F}}^{2}(0, T ; X)$ the set of all $\mathscr{F}_{t}$-adapted $X$-valued càdlàg processes $f=\{f(t, \omega),(t, \omega) \in[0, T] \times \Omega\}$ such that $\|f\|_{S_{\mathscr{F}}^{2}(0, T ; X)} \triangleq \sqrt{\mathbb{E}\left[\sup _{0 \leq t \leq T}\|f(t)\|_{X}^{2}\right]}<+\infty$, by $L^{2}(\Omega, \mathscr{F}, P ; X)$ the set of all $X$-valued random variables $\xi$ on $(\Omega, \mathscr{F}, P)$ such that $\|\xi\|_{L^{2}(\Omega, \mathscr{F}, P ; X)} \triangleq \sqrt{\mathbb{E}\left[\|\xi\|_{X}^{2}\right]}<\infty$. Throughout this paper, we let $C$ and $K$ be two generic positive constants, which may be different from line to line.

Let $V$ and $H$ be two separable (real) Hilbert spaces such that $V$ is densely embedded in $H$. We identify $H$ with its dual space by the Riesz mapping. Then we can take $H$ as a pivot space and get a Gelfand triple $V \subset H=H^{*} \subset V^{*}$, where $H^{*}$ and $V^{*}$ denote the dual spaces of $H$ and $V$, respectively. Denote by $\|\cdot\|_{V},\|\cdot\|_{H}$ and $\|\cdot\|_{V^{*}}$ the norms of $V, H$ and $V^{*}$, respectively, by $(\cdot, \cdot)_{H}$ the inner product in $H$, by $\langle\cdot, \cdot\rangle$ the duality product between $V$ and $V^{*}$. Moreover we write $\mathscr{L}\left(V, V^{*}\right)$ the space of bounded linear transformations of V into $V^{*}$.

Now we recall an Itô's formula in Hilbert space which will be frequently used in this paper.

Lemma 2.1. Let $\varphi \in L^{2}\left(\Omega, \mathscr{F}_{0}, P ; H\right)$. Let $Y, Z$ and $\Gamma$ be three progressively measurable stochastic processes defined on $[0, T] \times \Omega$ with values in $V, H$ and $V^{*}$ such that $Y \in M_{\mathscr{F}}^{2}(0, T ; V), Z \in M_{\mathscr{F}}^{2}(0, T ; H)$ and $\Gamma \in M_{\mathscr{F}}^{2}\left(0, T ; V^{*}\right)$, respectively. Let $R$ be a $\mathscr{P} \otimes \mathscr{B}(E)$-measurable stochastic process defined on $[0, T] \times \Omega \times E$ with values in $H$ such that $R \in M_{\mathscr{F}}^{\nu, 2}([0, T] \times E ; H)$. Suppose that for every $\eta \in V$ and almost every $(\omega, t) \in \Omega \times[0, T]$, it holds that

$$
(\eta, Y)_{H}=(\eta, \varphi)_{H}+\int_{0}^{t}\langle\eta, \Gamma(s)\rangle d s+\int_{0}^{t}(\eta, Z)_{H} d W(s)+\int_{0}^{t} \int_{E}(\eta, R(s, e))_{H} \tilde{\mu}(d e, d s)
$$

Then $Y$ is a $H$-valued strongly cadlag $\mathscr{F}_{t}$-adapted process such that

$$
\begin{equation*}
\mathbb{E}\left[\sup _{0 \leq t \leq T}\|Y\|_{H}^{2}\right]<\infty \tag{2.1}
\end{equation*}
$$

and the following Itô formula holds

$$
\begin{align*}
\|Y(t)\|_{H}^{2}= & \|\varphi\|^{2}+2 \int_{0}^{t}\langle\Gamma(s), Y(s)\rangle d s+2 \int_{0}^{t}(Z(s), Y(s))_{H} d W(s)+\int_{0}^{t}\|Z(s)\|_{H}^{2} d s  \tag{2.2}\\
& +\int_{0}^{t} \int_{E}\left[\|R(s, e)\|_{H}^{2}+2(Y(s), R(s, e))_{H}\right] \tilde{\mu}(d e, d s)+\int_{0}^{t} \int_{E}\|R(s, e)\|_{H}^{2} \nu(d e) d s
\end{align*}
$$

Proof. The proof follows that of Theorem 1 in Gyöngy and Krylov [13].

## 3 Stochastic Evolution Equation with Jumps

In this section, we study the existence and uniqueness of the solution of the following stochastic evolution equation (SEE, for short) with jumps:

$$
\left\{\begin{align*}
d X(t)= & {[A(t) X(t)+b(t, X(t))] d t+[B(t) X(t)+g(t, X(t))] d W(t) }  \tag{3.1}\\
& +\int_{E} \sigma(t, e, X(t)) \tilde{\mu}(d e, d t) \\
X(0)= & x \in H, \quad t \in[0, T]
\end{align*}\right.
$$

where $A, B, b, g$ and $\sigma$ are given random mappings which satisfying the following standing assumptions.
Assumption 3.1. (i) The operator processes $A:[0, T] \times \Omega \longrightarrow \mathscr{L}\left(V, V^{*}\right)$ and $B:[0, T] \times \Omega \longrightarrow \mathscr{L}(V, H)$ are weakly predictable; i.e., $\langle A(\cdot) x, y\rangle$ and $(B(\cdot) x, y)_{H}$ are both predictable process for every $x, y \in V$, and satisfy the coercive condition, i.e., there exists some constants $C, \alpha>0$ and $\lambda$ such that a.s. $(t, \omega) \in[0, T] \times \Omega$ for all $x \in V$,

$$
\begin{equation*}
-\langle A(t) x, x\rangle+\lambda\|x\|_{H} \geq \alpha\|x\|_{V}+\|B x\|_{H} \tag{3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\sup _{(t, \omega) \in[0, T] \times \Omega}\|A(t, \omega)\|_{\mathscr{L}\left(V, V^{*}\right)}+\sup _{(t, \omega) \in[0, T] \times \Omega}\|B(t, \omega)\|_{\mathscr{L}(V, H)} \leq C \tag{3.3}
\end{equation*}
$$

Assumption 3.2. The mappings $b:[0, T] \times \Omega \times H \longrightarrow H$ and $g:[0, T] \times \Omega \times H \longrightarrow H$ are both $\mathscr{P} \times \mathscr{B}(H) / \mathscr{B}(H)$ measurable; the mapping $\sigma:[0, T] \times \Omega \times E \times H \longrightarrow H$ is $\mathscr{P} \times \mathscr{B}(E) \times \mathscr{B}(H) / \mathscr{B}(H)$-measurable. And there exists a constant $C$ such that for all $x, \bar{x} \in V$ and a.s. $(t, \omega) \in[0, T] \times \Omega$

$$
\begin{equation*}
\|b(t, x)-b(t, x)\|_{H}+\|g(t, x)-g(t, x)\|_{H}+\|\sigma(t, x, \cdot)-\sigma(t, x \cdot \cdot)\|_{M^{\nu, 2}(Z ; H)} \leq C\|x-\bar{x}\|_{H} \tag{3.4}
\end{equation*}
$$

Definition 3.1. A $V$-valued, $\left\{\mathscr{F}_{t}\right\}_{0 \leq t \leq T}$-adapted process $X(\cdot)$ is said to be a solution to the SEE (3.1), if $X(\cdot) \in$ $M_{\mathscr{F}}^{2}(0, T ; V)$, such that for every $\phi \in \bar{V}$ and a.e. $(t, \omega) \in[0, T] \times \Omega$, it holds that

$$
\left\{\begin{align*}
(X(t), \phi)_{H}= & (x, \phi)_{H}+\int_{0}^{t}\langle A(s) X(s), \phi\rangle d s+\int_{0}^{t}\left(b(s, X(s), \phi)_{H} d s\right.  \tag{3.5}\\
& +\int_{0}^{t}(B(s) X(s)+g(s, X(s)), \phi)_{H} d W(s) \\
& +\int_{0}^{t} \int_{E}(\sigma(s, e, X(s)), \phi)_{H} d \tilde{\mu}(d e, d s), \quad t \in[0, T] \\
X(0)= & x \in H
\end{align*}\right.
$$

or alternatively, $X(\cdot)$ satisfies the following Itô's equation in $V^{*}$ :

$$
\left\{\begin{align*}
X(t)= & x+\int_{0}^{t} A(s) X(s) d s+\int_{0}^{t} b\left(s, X(s) d s+\int_{0}^{t}[B(s) X(s)+g(s, X(s))] d W(s)\right.  \tag{3.6}\\
& +\int_{0}^{t} \int_{E} \sigma(s, e, X(s)) d \tilde{\mu}(d e, d s), \quad t \in[0, T] \\
X(t)= & x \in H
\end{align*}\right.
$$

Now we state our main result.
Theorem 3.1. Let Assumptions 3.1 3.2 be satisfied by given coefficients $(A, B, b, g, \sigma)$. Then SEE (3.1) has a unique solution $X(\cdot) \in M_{\mathscr{F}}^{2}(0, T ; V) \bigcap S_{\mathscr{F}}^{2}(0, T ; H)$.

To prove this theorem, we need the following result on the a prior estimate for the solution to SEE (3.1).
Theorem 3.2. Let $X(\cdot)$ be the solution to the $S E E$ (3.1) with the coefficients $(A, B, b, g, \sigma)$ satisfying Assumptions 3.1 3.2. Then the following estimate holds:

$$
\begin{align*}
& \mathbb{E}\left[\sup _{0 \leq t \leq T}\|X(t)\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\|X(t)\|_{V}^{2} d t\right] \\
& \leq K\left\{\|x\|_{H}+\mathbb{E}\left[\int_{0}^{T}\|b(t, 0)\|_{H}^{2} d t\right]+\mathbb{E}\left[\int_{0}^{T}\|g(t, 0)\|_{H}^{2} d t\right]+\mathbb{E}\left[\int_{0}^{T} \int_{E}\|\sigma(t, e, 0)\|_{H}^{2} \nu(d e) d t\right]\right\} . \tag{3.7}
\end{align*}
$$

Furthermore, suppose that $\bar{X}(\cdot)$ is the solution to the SEE (3.1) with the initial value $\bar{X}(0)=\bar{x}$ and coefficients ( $A, B, \bar{b}, \bar{g}, \bar{\sigma}$ ) satisfying Assumptions 3.1 3.2, then we have

$$
\begin{align*}
& \mathbb{E}\left[\sup _{0 \leq t \leq T}\|X(t)-\bar{X}(t)\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\|X(t)-\bar{X}(t)\|_{V}^{2} d t\right] \\
& \leq K\left\{\|x-\bar{x}\|_{H}^{2}+\mathbb{E}\left[\int_{0}^{T}\|b(t, \bar{X}(t))-\bar{b}(t, \bar{X}(t))\|_{H}^{2} d t\right]\right.  \tag{3.8}\\
& \left.+\mathbb{E}\left[\int_{0}^{T}\|g(t, \bar{X}(t))-\bar{g}(t, \bar{X}(t))\|_{H}^{2} d t\right]+\mathbb{E}\left[\int_{0}^{T} \int_{E}\|\sigma(t, e, \bar{X}(t))-\bar{\sigma}(t, e, \bar{X}(t))\|_{H}^{2} \nu(d e) d t\right]\right\}
\end{align*}
$$

Proof. The estimate (3.7) can be directly obtained by the estimate (3.8) by taking the initial value $\bar{X}(0)=0$ and coefficients $(A, B, \bar{b}, \bar{g}, \bar{\sigma})=(A, B, 0,0,0)$ which imply that $\bar{X}(\cdot) \equiv 0$. Therefore, it suffices to prove the estimate (3.8). For the sake of simplicity, in the following discussion, we will use the following shorthand notation:

$$
\hat{X}(t) \triangleq X(t)-\bar{X}(t), \quad \hat{x} \triangleq x-\bar{x}
$$

$$
\begin{aligned}
& \Lambda \triangleq\|x-\bar{x}\|_{H}^{2}+\mathbb{E}\left[\int_{0}^{T}\|b(t, \bar{X}(t))-\bar{b}(t, \bar{X}(t))\|_{H}^{2} d t\right]+\mathbb{E}\left[\int_{0}^{T}\|g(t, \bar{X}(t))-\bar{g}(t, \bar{X}(t))\|_{H}^{2} d t\right] \\
& \quad+\mathbb{E}\left[\int_{0}^{T} \int_{E}\|\sigma(t, \bar{X}(t), e)-\bar{\sigma}(t, \bar{X}(t), e)\|_{H}^{2} \nu(d e) d t\right] \\
& \tilde{\phi}(t) \triangleq \phi(t, X(t))-\bar{\phi}(t, \bar{X}(t)), \hat{\phi}(t) \triangleq \phi(t, \bar{X}(t))-\bar{\phi}(t, \bar{X}(t)), \Delta \phi(t) \triangleq \phi(t, X(t))-\phi(t, \bar{X}(t)),
\end{aligned}
$$

where $\phi=b, g, \sigma$.
Applying Itô formula in Lemma 2.1 to $\|\hat{X}(t)\|_{H}^{2}$ and using Assumption 3.1]3.2 and the elementary inequalities $|a+b|^{2} \leq 2 a^{2}+2 b^{2}$ and $2 a b \leq \frac{1}{\epsilon} a^{2}+\epsilon b^{2}, \forall a, b>0, \epsilon>0$ and, we get that

$$
\begin{align*}
\|\hat{X}(t)\|_{H}^{2}=\quad & \|\hat{x}\|_{H}^{2}+2 \int_{0}^{t}\langle A(s) \hat{X}(s), \hat{X}(s)\rangle d s+2 \int_{0}^{t}(\hat{X}(s), \tilde{b}(s))_{H} d s+\int_{0}^{t}\|B(s) \hat{X}(s)+\tilde{g}(s)\|_{H}^{2} d s \\
& +\int_{0}^{t} \int_{E}\|\tilde{\sigma}(s, e)\|_{H}^{2} \nu(d e) d s+2 \int_{0}^{t}(\hat{X}(s), B(s) \hat{X}(s)+\tilde{g}(s))_{H} d W(s) \\
& +\int_{0}^{t} \int_{E}\left[\|\tilde{\sigma}(s, e)\|_{H}^{2}+2(\hat{X}(s), \tilde{\sigma}(s, e))\right] \tilde{\mu}(d e, d s) \\
=\quad & \|\hat{x}\|_{H}^{2}+2 \int_{0}^{t}\left[\langle A(s) \hat{X}(s), \hat{X}(s)\rangle+\|B(s) \hat{X}(s)\|_{H}^{2}\right] d s+\int_{0}^{t}\|\Delta g(s)+\hat{g}(s)\|_{H}^{2} \\
& +2 \int_{0}^{t}(B(s) \hat{X}(s), \Delta g(s)+\hat{g}(s))_{H} d s+2 \int_{0}^{t}(\hat{X}(s), \Delta b(s)+\hat{b}(s))_{H} d s \\
& +\int_{0}^{t} \int_{E}\|\Delta \sigma(s, e)+\hat{\sigma}(s, e)\|_{H}^{2} \nu(d e) d s+2 \int_{0}^{t}(\hat{X}(s), B(s) \hat{X}(s)+\tilde{g}(s))_{H} d W(s) \\
& +\int_{0}^{t} \int_{E}\left[\|\tilde{\sigma}(s, e)\|_{H}^{2}+2(\hat{X}(s), \tilde{\sigma}(s, e))\right] \tilde{\mu}(d e, d s) \\
& K \Lambda+(-2 \alpha+\varepsilon) \mathbb{E}\left[\int_{0}^{t}\|\hat{x}(s)\|_{V}^{2} d s\right]+K \mathbb{E}\left[\int_{0}^{t}\|\hat{X}(s)\|_{H}^{2} d t\right] \\
& +2 \int_{0}^{t}(\hat{X}(s), B(s) \hat{X}(s)+\tilde{g}(s))_{H} d W(s)  \tag{3.9}\\
& +\int_{0}^{t} \int_{E}\left[\|\tilde{\sigma}(s, e)\|_{H}^{2}+2(\hat{X}(s), \tilde{\sigma}(s, e))\right] \tilde{\mu}(d e, d s) . \tag{3.10}
\end{align*}
$$

Taking expectation on both sides of the above and taking $\varepsilon$ small enough such that $-2 \alpha+\varepsilon<0$, we conclude that

$$
\begin{equation*}
\mathbb{E}\left[\|\hat{X}(t)\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{t}\|\hat{X}(s)\|_{V}^{2} d s\right] \leq K \Lambda+K \mathbb{E}\left[\int_{0}^{t}\|\hat{X}(s)\|_{H}^{2} d t\right] \tag{3.11}
\end{equation*}
$$

Then by virtue of Grönwall's inequality to $\mathbb{E}\left[\|X(t)\|_{H}^{2}\right]$, we obtain

$$
\begin{equation*}
\sup _{0 \leq t \leq T} \mathbb{E}\left[\|\hat{X}(t)\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\|\hat{X}(s)\|_{V}^{2} d s\right] \leq K \Lambda \tag{3.12}
\end{equation*}
$$

Furthermore, applying Burkholder-Davis-Gundy inequality in (3.9) and using the estimate (3.11), we get that

$$
\begin{align*}
\mathbb{E}\left[\sup _{0 \leq t \leq T}\|\hat{X}(t)\|_{H}^{2}\right] \leq & K \Lambda+2 \mathbb{E}\left\{\sup _{0 \leq t \leq T}\left|\int_{0}^{t}(\hat{X}(s), B(s) \hat{X}(s)+\tilde{g}(s))_{H} d W(s)\right|\right\} \\
& +2 \mathbb{E}\left\{\sup _{0 \leq t \leq T}\left|\int_{0}^{t} \int_{E}\left[\|\tilde{\sigma}(s, e)\|_{H}^{2}+2(\hat{X}(s), \tilde{\sigma}(s, e))\right] \tilde{\mu}(d e, d t)\right|\right\} \\
\leq & K \Lambda+K \mathbb{E}\left\{\int_{0}^{T}\left|(\hat{X}(s), B(s) \hat{X}(s)+\tilde{g}(s))_{H}\right|^{2} d s\right\}^{\frac{1}{2}}  \tag{3.13}\\
& +K \mathbb{E}\left\{\int_{0}^{T} \int_{E}\left|\|\tilde{\sigma}(s, e)\|_{H}^{2}+2(\hat{X}(s), \tilde{\sigma}(s, e))\right| \nu(d e) d t\right\} \\
& \leq \frac{1}{2} \mathbb{E}\left[\sup _{0 \leq t \leq T}\|\hat{X}(t)\|_{H}^{2}\right]+K \Lambda
\end{align*}
$$

which implies that

$$
\begin{equation*}
\mathbb{E}\left[\sup _{0 \leq t \leq T}\|\hat{X}(t)\|_{H}^{2}\right] \leq K \Lambda \tag{3.14}
\end{equation*}
$$

Combining (3.12) and (3.14), we get the desired result. The proof is complete.
Now we give the existence and uniqueness of the solution of (3.1) for a simple case where the coefficients $(b, g, \sigma)$ is independent of the variable $x$.

Lemma 3.3. Given three stochastic processes $b, g$ and $\sigma$ such that $b \in M_{\mathscr{F}}^{2}(0, T ; H), g \in M_{\mathscr{F}}^{2}(0, T ; H)$ and $\sigma \in$ $M_{\mathscr{F}}^{\nu, 2}(0, T ; H)$. Suppose that the operators $A$ and $B$ satisfy Assumption 3.1. There exists a unique solution to the following SEE:

$$
\left\{\begin{align*}
d X(t) & =[A(t) X(t)+b(t)] d t+[B(t) X(t)+g(t)] d W(t)+\int_{E} \sigma(t, e) \tilde{\mu}(d e, d t)  \tag{3.15}\\
X(0) & =x, \quad t \in[0, T]
\end{align*}\right.
$$

Proof. The proof can be obtained by Galerkin approximations in the same way as the proof of Theorem 3.2 in [11] with minor change.

Proof of Theorem 3.1. The uniqueness of the solution to the SEE (3.1) can be got by the a priori estimate (3.8) directly. For $\rho \in[0,1]$ and three any given stochastic processes $b_{0}(\cdot) \in M_{\mathscr{F}}^{2}(0, T ; H), g_{0}(\cdot) \in M_{\mathscr{F}}^{2}(0, T ; H)$, and $\sigma_{0}(\cdot) \in M_{\mathscr{F}}^{\nu, 2}(0, T ; H)$, we introduce a family of parameterized SEEs as follows:

$$
\begin{align*}
X(t)= & \left.x+\int_{0}^{t} A(s) X(s) d s+\int_{0}^{t}[\rho b(s, X(s))]+b_{0}(t)\right] d s+\int_{0}^{t}\left[B(s) X(s)+\rho g(s, X(s))+g_{0}(t)\right] d W(s)  \tag{3.16}\\
& +\int_{E}\left[\rho \sigma(t, e, X(t))+\sigma_{0}(t, e)\right] \tilde{\mu}(d e, d t)
\end{align*}
$$

It is easy to see that when we take the parameter $\rho=1$ and $b_{0}(\cdot) \equiv 0, g_{0}(\cdot) \equiv 0, \sigma_{0}(\cdot, \cdot) \equiv 0$, the SEE (3.16) is reduced to the original SEE (3.1). Obviously, the coefficients of the SEE (3.16) satisfy Assumption 3.1 and 3.2 with $(A, B, b, g, \sigma)$ replaced by $\left(A, B, \rho b+b_{0}, \rho g+g_{0}, \rho \sigma+\sigma_{0}\right)$. Suppose for any $b_{0}(\cdot) \in M_{\mathscr{F}}^{2}(0, T ; H), g_{0}(\cdot) \in M_{\mathscr{F}}^{2}(0, T ; H)$, $\sigma_{0}(\cdot) \in M_{\mathscr{F}}^{\nu, 2}(0, T ; H)$, and some parameter $\rho=\rho_{0}$, there exists a unique solution $X(\cdot) \in M_{\mathscr{F}}^{2}(0, T ; V)$ to the SEE
(3.16). For any parameter $\rho$, the SEE (3.16) can be rewritten as

$$
\begin{align*}
X(t)= & x+\int_{0}^{t} A(s) X(s) d s+\int_{0}^{t}\left[\rho_{0} b(s, X(s))+b_{0}(t)+\left(\rho-\rho_{0}\right) b(s, X(s))\right] d s \\
& +\int_{0}^{t}\left[B(s) X(s)+\rho_{0} g(s, X(s))+g_{0}(t)+\left(\rho-\rho_{0}\right) g(s, X(s))\right] d W(s)  \tag{3.17}\\
& +\int_{0}^{t} \int_{E}\left[\rho_{0} \sigma(s, e, X(s))+\sigma_{0}(t, e)+\left(\rho-\rho_{0}\right) \sigma(s, e, X(s))\right] d \tilde{\mu}(d e, d s) .
\end{align*}
$$

Therefore, by the above assumption, for any $x(\cdot) \in M_{\mathscr{F}}^{2}(0, T ; V)$, the SEE

$$
\begin{align*}
X(t)= & x+\int_{0}^{t} A(s) X(s) d s+\int_{0}^{t}\left[\rho_{0} b(s, X(s))+b_{0}(t)+\left(\rho-\rho_{0}\right) b(s, x(s))\right] d s \\
& +\int_{0}^{t}\left[B(s) X(s)+\rho_{0} g(s, X(s))+g_{0}(t)+\left(\rho-\rho_{0}\right) g(s, x(s))\right] d W(s)  \tag{3.18}\\
& +\int_{0}^{t} \int_{E}\left[\rho_{0} \sigma(s, e, X(s))+\sigma_{0}(t, e)+\left(\rho-\rho_{0}\right) \sigma(s, e, x(s))\right] d \tilde{\mu}(d e, d s)
\end{align*}
$$

admits a unique solution $X(\cdot) \in M_{\mathscr{F}}^{2}(0, T ; V)$. Now define a mapping from $M_{\mathscr{F}}^{2}(0, T ; V)$ onto itself denoted by

$$
X(\cdot)=\Gamma(x(\cdot))
$$

Then for any $x_{i}(\cdot) \in M_{\mathscr{F}}^{2}(0, T ; V), i=1,2$, from the Lipschitz continuity of $b, g, \sigma$ and a priori estimate (3.7), it follows that

$$
\begin{aligned}
\left\|\Gamma\left(x_{1}(\cdot)\right)-\Gamma\left(x_{2}(\cdot)\right)\right\|_{M_{\mathscr{F}}^{2}(0, T ; V)}^{2} & =\left\|X_{1}(\cdot)-X_{2}(\cdot)\right\|_{M_{\mathscr{F}}^{2}(0, T ; V)}^{2} \\
& \leq K\left|\rho-\rho_{0}\right|^{2} \cdot\left\|x_{1}(\cdot)-x_{2}(\cdot)\right\|_{M_{\mathscr{F}}^{2}(0, T ; V)}^{2}
\end{aligned}
$$

Here $K$ is a positive constant independent of $\rho$. If $\left|\rho-\rho_{0}\right|<\frac{1}{2 \sqrt{K}}$, the mapping $\Gamma$ is strictly contractive in $M_{\mathcal{F}}^{2}(0, T ; V)$. Hence it implies that the SEE (3.16) with the coefficients $\left(A, B, \rho b+b_{0}, \rho g+g_{0}, \rho \sigma+\sigma_{0},\right)$ admits a unique solution $X(\cdot) \in M_{\mathcal{F}}^{2}(0, T ; V)$. From Lemma 3.3, the uniqueness and existence of a solution to the SEE (3.16) is true for $\rho=0$. Then starting from $\rho=0$, one can reach $\rho=1$ in finite steps and this finishes the proof of solvability of the SEE (3.1). Moreover, from Lemma 2.1 and the estimate (2.1), we obtain $X(\cdot) \in \mathcal{S}_{\mathscr{F}}^{2}(0, T ; H)$. This completes the proof.

## 4 Formulation of Optimal Control Problem

Let $U$ be a real-valued Hilbert space standing for the control space. Let $\mathscr{U}$ be a nonempty convex closed subset of $U$. An admissible control process $u(\cdot) \triangleq\{u(t), 0 \leq t \leq T\}$ is defined as follows.

Definition 4.1. A stochastic process $u(\cdot)$ defined on $[0, T] \times \Omega$ is called an admissible control process if it is a predictable process such that $u(\cdot) \in M^{2}(0, T ; U)$. The set of all admissible controls is denoted by $\mathcal{A}$.

We make the following basic assumptions.

## Assumption 4.1.

(i) $A:[0, T] \times \Omega \longrightarrow \mathscr{L}\left(V, V^{*}\right)$ and $B:[0, T] \times \Omega \longrightarrow \mathscr{L}(V, H)$ are operator-valued stochastic process satisfying (i) in Assumption 3.1
(iii) $b, g:[0, T] \times \Omega \times H \times \mathscr{U} \rightarrow H$ are $\mathscr{P} \times \mathscr{B}(H) \times \mathscr{B}(\mathscr{U}) / \mathscr{B}(H)$ measurable mappings and $\sigma:[0, T] \times \Omega \times$ $E \times H \times \mathscr{U} \longrightarrow H$ is a $\mathscr{P} \times \mathscr{B}(E) \times \mathscr{B}(H) \times \mathscr{B}(U) / \mathscr{B}(H)$-measurable mapping such that $b(\cdot, 0,0), g(\cdot, 0,0) \in$ $M_{\mathscr{F}}^{2}(0, T ; H), \sigma(\cdot, \cdot, 0,0) \in M_{\mathscr{F}}^{\nu, 2}([0, T] \times E ; H)$. Moreover, for almost all $(t, \omega, e) \in[0, T] \times \Omega \times E, h, g$ and $\sigma$ are Gâteaux differentiable in $(x, u)$ with continuous bounded Gâteaux derivatives $b_{x}, g_{x}, \sigma_{x}, b_{u}, g_{u}$ and $\sigma_{u}$;
(iv) $l:[0, T] \times \Omega \times H \times \mathscr{U} \rightarrow H$ is a $\mathscr{P} \otimes \mathscr{B}(H) \otimes \mathscr{B}(\mathscr{U}) / \mathscr{B}(\mathbb{R})$-measurable mapping and $\Phi: \Omega \times H \rightarrow \mathbb{R}$ is a $\mathscr{F}_{T} \otimes \mathscr{B}(H) / \mathscr{B}(\mathbb{R})$-measurable mapping. For almost all $(t, \omega) \in[0, T] \times \Omega, l$ is continuous Gâteaux differentiable in $(x, u)$ with continuous Gâteaux derivatives $l_{x}$ and $l_{u}, \Phi$ is Gâteaux differentiable with continuous Gâteaux derivative $\Phi_{x}$. Moreover, for almost all $(t, \omega) \in[0, T] \times \Omega$, there exists a constant $C>0$ such that for all $(x, u) \in H \times \mathscr{U}$

$$
\begin{aligned}
|l(t, x, u)| & \leq C\left(1+\|x\|_{H}^{2}++\|u\|_{U}^{2}\right) \\
\left\|l_{x}(t, x, u)\right\|_{H}+ & +\left\|l_{u}(t, x, u)\right\|_{U} \leq C\left(1+\|x\|_{H}+\|u\|_{U}\right)
\end{aligned}
$$

and

$$
\begin{gathered}
|\Phi(x)| \leq C\left(1+\|x\|_{H}^{2}\right) \\
\left\|\Phi_{x}(x)\right\|_{H} \leq C\left(1+\|x\|_{H}\right)
\end{gathered}
$$

In the Gelfand triple $\left(V, H, V^{*}\right)$, for any admissible control $u(\cdot) \in \mathcal{A}$, we consider the following SEE

$$
\left\{\begin{align*}
& d X(t)= {[A(t) X(t)+b(t, X(t), u(t))] d t+[B(t) X(t)+g(t, X(t), u(t))] d W(t) }  \tag{4.1}\\
&+\int_{E} \sigma(t, e, X(t), u(t)) \tilde{\mu}(d e, d t) \\
& X(0)=x, \quad t \in[0, T]
\end{align*}\right.
$$

with the cost functional

$$
\begin{equation*}
J(u(\cdot))=\mathbb{E}\left[\int_{0}^{T} l(t, x(t), u(t)) d t+\Phi(x(T))\right] \tag{4.2}
\end{equation*}
$$

For any admissible control $u(\cdot)$, the solution of the system (4.1), denoted by $X^{u}(\cdot)$ or $X(\cdot)$, if its dependence on admissible control $u(\cdot)$ is clear from the context, is called the state process corresponding to the control process $u(\cdot)$, and $(u(\cdot), X(\cdot))$ is called an admissible pair.

The following result gives the well-posedness of the state equation as well as some useful estimates.
Lemma 4.1. Let Assumption 4.1 be satisfied. Then for any admissible control $u(\cdot)$, the state equation (4.1) has a unique solution $X^{u}(\cdot) \in M_{\mathscr{F}}^{2}(0, T ; V) \bigcap S_{\mathscr{F}}^{2}(0, T ; H)$. Moreover, the following estimate holds

$$
\begin{equation*}
\mathbb{E}\left[\sup _{0 \leq t \leq T}\left\|X^{u}(t)\right\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\left\|X^{u}(t)\right\|_{V}^{2} d t\right] \leq K\left\{1+\|x\|_{H}+\mathbb{E}\left[\int_{0}^{T}\|u(t)\|_{U}^{2} d t\right]\right\} \tag{4.3}
\end{equation*}
$$

and

$$
\begin{equation*}
|J(u(\cdot))|<\infty \tag{4.4}
\end{equation*}
$$

Furthermore, let $X^{v}(\cdot)$ be the state process corresponding to another admissible control $v(\cdot)$, then

$$
\begin{equation*}
\mathbb{E}\left[\sup _{0 \leq t \leq T}\left\|X^{u}(t)-X^{v}(t)\right\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\left\|X^{u}(t)-X^{v}(t)\right\|_{V}^{2} d t\right] \leq K \mathbb{E}\left[\int_{0}^{T}\|u(t)-v(t)\|_{U}^{2} d t\right] \tag{4.5}
\end{equation*}
$$

Proof. Under Assumption 4.1 by Theorem 3.1, we can get directly the existence and uniqueness of the solution of the state equation (3.1), and the estimates (4.3) and (4.5) can be obtained by the estimates (3.8) and (3.9), respectively. Furthermore, from Assumption 4.1 and the estimate (4.3), it follows that

$$
\left.\left.|J(x, u(\cdot))| \leq K\left\{\mathbb{E}\left[\sup _{0 \leq t \leq T}\|X(t)\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\|u(t)\|_{U}^{2} d t\right]+1\right]\right\} \leq K\left\{\mathbb{E}\left[\int_{0}^{T}\|u(t)\|_{U}^{2} d t\right]+\|x\|_{H}+1\right]\right\}<\infty .(4.6)
$$

The proof is complete.

Therefore, by Lemma 4.1 we claim that the cost functional (4.2) is well-defined. Now we put forward our optimal control problem as follows.
Problem 4.2. Find an admissible control process $\bar{u}(\cdot) \in \mathcal{A}$ such that

$$
\begin{equation*}
J(\bar{u}(\cdot))=\inf _{u(\cdot) \in \mathcal{A}} J(u(\cdot)) \tag{4.7}
\end{equation*}
$$

The admissible control $\bar{u}(\cdot)$ satisfying (4.7) is called an optimal control process of Problem 4.2, Correspondingly, the state process $\bar{X}(\cdot)$ associated with $\bar{u}(\cdot)$ is called an optimal state process. Then $(\bar{u}(\cdot) ; \bar{X}(\cdot))$ is called an optimal pair of Problem 4.2

## 5 Regularity Result for the Adjoint Equation

For any admissible pair $(\bar{u}(\cdot) ; \bar{X}(\cdot))$, the corresponding adjoint processes is defined as the triple of processes $(p(\cdot), q(\cdot), r(\cdot, \cdot))$, which is the solution to the following backward stochastic equation with jump, called adjoint equation,

$$
\left\{\begin{align*}
d p(t)= & -\left[A^{*}(t) p(t)+b_{x}^{*}(t, \bar{X}(t), \bar{u}(t)) p(t)+B^{*}(t) q(t)+g_{x}^{*}(t, \bar{X}(t), \bar{u}(t)) q(t)\right.  \tag{5.1}\\
& \left.\quad+\int_{E} \sigma_{x}^{*}(t, e, \bar{X}(t), \bar{u}(t)) r(t, e) \nu(d e) d t+l_{x}(t, \bar{X}(t), \bar{u}(t))\right] d t \\
& \quad+q(t) d W(t)+\int_{E} r(t, e) \tilde{\mu}(d e, d t), \quad 0 \leqslant t \leqslant T \\
p(T)= & \Phi_{x}(T)
\end{align*}\right.
$$

Here $A^{*}$ denotes the adjoint operator of the operator $A$. Similarly, we can define the corresponding adjoint operator for other operators.
Under Assumptions 4.1, we have the following basic result on the adjoint processes.
Lemma 5.1. Let Assumptions 4.1 be satisfied. Then for any admissible pair $(\bar{u}(\cdot) ; \bar{X}(\cdot))$, there exists a unique adjoint processes $(p(\cdot), q(\cdot), r(\cdot)) \in M_{\mathscr{F}}^{2}(0, T ; H) \times M_{\mathscr{F}}^{2}(0, T ; H) \times M_{\mathscr{F}}^{\nu, 2}([0, T] \times E ; H)$. Moreover, the following estimate holds:

$$
\begin{align*}
& \mathbb{E}\left[\sup _{0 \leq t \leq T}\|p(t)\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\|p(t)\|_{V}^{2} d t\right]+\mathbb{E}\left[\int_{0}^{T}\|q(t)\|_{H}^{2} d t\right]+\mathbb{E}\left[\int_{0}^{T} \int_{E}\|r(t, e)\|_{H}^{2} \nu(d e) d t\right]  \tag{5.2}\\
& \leq K\left\{\mathbb{E}\left[\int_{0}^{T}\left\|l_{x}(t)\right\|_{H}^{2} d t\right]+\mathbb{E}\left[\left\|\Phi_{x}(T)\right\|_{H}^{2}\right]\right\}
\end{align*}
$$

Proof. From the property of adjoint operator, the adjoint operator $A^{*}$ of $A$ and the adjoint operator $B^{*}$ of $B$ also satisfies (i) in Assumption [3.1. Therefore, the desired result can be obtained by the existence and uniqueness theorem of solution of BSEE with jumps established in [19].

Define the Hamiltonian $\mathcal{H}:[0, T] \times \Omega \times H \times \mathscr{U} \times V \times H \times M^{\nu, 2}(E ; H) \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
\mathcal{H}(t, x, u, p, q, r):=(b(t, x, u), p)_{H}+(g(t, x, u), q)_{H}+\int_{E}(\sigma(t, e, x, u), r(t, e))_{H} \nu(d e)+l(t, x, u) \tag{5.3}
\end{equation*}
$$

Using Hamiltonian $\mathcal{H}$, the adjoint equation (5.1) can be written in the following form:

$$
\left\{\begin{array}{l}
d p(t)=-\left[A^{*}(t) p(t)+B(t)^{*} q(t)+\overline{\mathcal{H}}_{x}(t)\right] d t+q(t) d W(t)+\int_{E} r(t, e) \tilde{\mu}(d e, d t), \quad 0 \leqslant t \leqslant T  \tag{5.4}\\
p(T)=\Phi_{x}(\bar{X}(T))
\end{array}\right.
$$

where we write

$$
\begin{equation*}
\overline{\mathcal{H}}(t) \triangleq \mathcal{H}(t, \bar{x}(t), \bar{u}(t), p(t), q(t), r(t, \cdot)) \tag{5.5}
\end{equation*}
$$

## 6 Stochastic Maximum Principle

### 6.1 Variation of the State and Cost Functional

Let $(\bar{u}(\cdot) ; \bar{X}(\cdot))$ be an optimal pair. Define a convex perturbation of $\bar{u}(\cdot)$ as follows:

$$
u^{\epsilon}(\cdot) \triangleq \bar{u}(\cdot)+\epsilon(v(\cdot)-\bar{u}(\cdot)), \quad 0 \leq \epsilon \leq 1
$$

where $v(\cdot)$ is an arbitrarily admissible control. Since the control domain $\mathscr{U}$ is convex, $u^{\varepsilon}(\cdot)$ is also an element of $\mathcal{A}$. We denote by $X^{\varepsilon}(\cdot)$ the state process corresponding to the control $u^{\varepsilon}(\cdot)$. Now we introduce the following first order variation equation:

$$
\left\{\begin{align*}
d Y(t)= & {\left[A(t) Y(t)+b_{x}(t, \bar{X}(t), \bar{u}(t)) Y(t)+b_{u}(t, \bar{X}(t), \bar{u}(t))(v(t)-\bar{u}(t))\right] d t }  \tag{6.1}\\
& +\left[B(t) Y(t)+g_{x}(t, \bar{X}(t), \bar{u}(t)) Y(t)+g_{u}(t, \bar{X}(t), \bar{u}(t))(v(t)-\bar{u}(t))\right] d W(t) \\
& +\int_{E}\left[\sigma_{x}(t, e, \bar{X}(t), \bar{u}(t)) Y(t)+\sigma_{u}(t, e, \bar{X}(t), \bar{u}(t))(v(t)-\bar{u}(t))\right] \tilde{\mu}(d e, d t) \\
Y(0)= & 0
\end{align*}\right.
$$

Under Assumption 4.1, by Theorem 3.1, we see that the variation equation (6.1) has a unique solution $Y(\cdot) \in$ $M_{\mathscr{F}}^{2}(0, T ; V) \bigcap S_{\mathscr{F}}^{2}(0, T ; H)$.

Lemma 6.1. Let Assumption 4.1 be satisfied. Then we have the following estimates:

$$
\begin{gather*}
\mathbb{E}\left[\sup _{0 \leq t \leq T}\left\|X^{\epsilon}(t)-\bar{X}(t)\right\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\left\|X^{\epsilon}(t)-\bar{X}(t)\right\|_{V}^{2} d t\right]=O\left(\epsilon^{2}\right),  \tag{6.2}\\
\mathbb{E}\left[\sup _{0 \leq t \leq T}\left\|X^{\epsilon}(t)-\bar{X}(t)-\varepsilon Y(t)\right\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\left\|X^{\epsilon}(t)-\bar{X}(t)-\varepsilon Y(t)\right\|_{V}^{2} d t\right]=o\left(\epsilon^{2}\right) . \tag{6.3}
\end{gather*}
$$

Proof. From the estimate (4.5), we have

$$
\begin{align*}
\mathbb{E}\left[\sup _{0 \leq t \leq T}\left\|X^{\varepsilon}(t)-\bar{X}(t)\right\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\left\|X^{\varepsilon}(t)-\bar{X}(t)\right\|_{V}^{2} d t\right] & \leq K \mathbb{E}\left[\int_{0}^{T}\left\|u^{\varepsilon}(t)-\bar{u}(t)\right\|_{U}^{2} d t\right] \\
& =K \varepsilon^{2} \mathbb{E}\left[\int_{0}^{T}\|v(t)-\bar{u}(t)\|_{U}^{2} d t\right]  \tag{6.4}\\
& =O\left(\varepsilon^{2}\right)
\end{align*}
$$

Set $\Xi^{\varepsilon}(t)=X^{\varepsilon}(t)-\bar{X}(t)-\varepsilon Y(t)$. From Taylor expanding, we have

$$
\left\{\begin{align*}
d \Xi^{\varepsilon}(t)= & {\left[A(t) \Xi^{\varepsilon}(t)+b_{x}(t, \bar{X}(t), \bar{u}(t)) \Xi^{\varepsilon}(t)+\alpha^{\varepsilon}(t)\right] d t+\left[B(t) \Xi^{\varepsilon}(t)+g_{x}(t, \bar{X}(t), \bar{u}(t)) \Xi^{\varepsilon}(t)+\beta^{\varepsilon}(t)\right] d W(t) }  \tag{6.5}\\
& +\int_{E}\left[\sigma_{x}(t, e, \bar{X}(t), \bar{u}(t)) \Xi^{\varepsilon}(t)+\gamma^{\varepsilon}(t)\right] d \tilde{\mu}(d e, d t) \\
X(0)= & x, \quad t \in[0, T]
\end{align*}\right.
$$

where

$$
\left\{\begin{align*}
\alpha^{\varepsilon}(t)= & \int_{0}^{1}\left[\left(b_{x}\left(t, \bar{X}(t)+\lambda\left(X^{\varepsilon}(t)-\bar{X}(t)\right), \bar{u}(t)+\lambda\left(u^{\varepsilon}(t)-\bar{u}(t)\right)\right)-b_{x}(t, \bar{X}(t), \bar{u}(t))\right)\left(X^{\varepsilon}(t)-\bar{X}(t)\right)\right. \\
& \left.+\left(b_{u}\left(t, \bar{X}(t)+\lambda\left(X^{\varepsilon}(t)-\bar{X}(t)\right), \bar{u}(t)+\lambda\left(u^{\varepsilon}(t)-\bar{u}(t)\right)\right)-b_{u}(t, \bar{X}(t), \bar{u}(t))\right)\left(u^{\varepsilon}(t)-\bar{u}(t)\right)\right] d \lambda \\
\beta^{\varepsilon}(t)= & \int_{0}^{1}\left[\left(g_{x}\left(t, \bar{X}(t)+\lambda\left(X^{\varepsilon}(t)-\bar{X}(t)\right), \bar{u}(t)+\lambda\left(u^{\varepsilon}(t)-\bar{u}\right)\right)-g_{x}(t, \bar{X}(t), \bar{u}(t))\right)\left(X^{\varepsilon}(t)-\bar{X}(t)\right)\right.  \tag{6.6}\\
& \left.+\left(g_{u}\left(t, \bar{X}(t)+\lambda\left(X^{\varepsilon}(t)-\bar{X}(t)\right), \bar{u}(t)+\lambda\left(u^{\varepsilon}(t)-\bar{X}\right)\right)-g_{u}(t, \bar{X}(t), \bar{u}(t))\right)\left(u^{\varepsilon}(t)-\bar{u}(t)\right)\right] d \lambda \\
\gamma^{\varepsilon}(t, e)= & \int_{0}^{1}\left[\left(\sigma_{x}\left(t, e, \bar{X}(t)+\lambda\left(X^{\varepsilon}(t)-\bar{X}(t)\right), \bar{u}(t)+\lambda\left(u^{\varepsilon}(t)-\bar{u}(t)\right)-\sigma_{x}(t, e, \bar{X}(t), \bar{u}(t))\right)\left(X^{\varepsilon}(t)-\bar{X}(t)\right)\right.\right. \\
& \left.+\left(\sigma_{u}\left(t, e, \bar{X}(t)+\lambda\left(X^{\varepsilon}(t)-\bar{X}\right), \bar{u}(t)+\lambda\left(u^{\varepsilon}-\bar{u}(t)\right)\right)-\sigma_{x}(t, e, \bar{X}(t), \bar{u}(t))\right)\left(u^{\varepsilon}(t)-\bar{u}(t)\right)\right] d \lambda
\end{align*}\right.
$$

From the estimates (3.7), (6.2) and Lebesgue dominated convergence theorem, we get that

$$
\begin{align*}
& \mathbb{E}\left[\sup _{0 \leq t \leq T}\|\Xi(t)\|_{H}^{2}\right]+\mathbb{E}\left[\int_{0}^{T}\|\Xi(t)\|_{V}^{2} d t\right]  \tag{6.7}\\
\leq & \mathbb{E}\left[\int_{0}^{T}\left\|\alpha^{\varepsilon}(t)\right\|_{H}^{2} d t\right]+\mathbb{E}\left[\int_{0}^{T}\left\|\beta^{\varepsilon}(t)\right\|_{H}^{2} d t\right]+\mathbb{E}\left[\int_{0}^{T} \int_{E}\left\|\gamma^{\varepsilon}(t, e)\right\|_{H}^{2} \nu(d e) d t\right] \\
= & o(\varepsilon) . \tag{6.8}
\end{align*}
$$

Lemma 6.2. Let Assumption 4.1 be satisfied. Let $(\bar{u}(\cdot) ; \bar{X}(\cdot))$ be an optimal pair of Problem 4.2 associated with the first order variation process $Y(\cdot)$. Then,

$$
\begin{align*}
J\left(u^{\varepsilon}(\cdot)\right)-J(\bar{u}(\cdot))= & \varepsilon \mathbb{E}\left[\left(\Phi_{x}(\bar{X}(T)), Y(T)\right)_{H}\right]+\varepsilon \mathbb{E}\left[\int_{0}^{T}\left(l_{x}(t, \bar{X}(t), \bar{u}(t)), Y(t)\right)_{H} d t\right] \\
& +\varepsilon \mathbb{E}\left[\int_{0}^{T}\left(l_{u}(t, \bar{X}(t), \bar{u}(t)), v(t)-u(t)\right)_{U} d t\right]+o(\varepsilon) \tag{6.9}
\end{align*}
$$

Proof. From the definition of the cost functional, we have

$$
\begin{equation*}
J\left(u^{\varepsilon}(\cdot)\right)-J(\bar{u}(\cdot))=\mathbb{E}\left[\int_{0}^{T}\left(l\left(t, X^{\varepsilon}(t), u^{\varepsilon}(t)\right)-l(t, \bar{X}(t), \bar{u}(t)) d t\right]+\mathbb{E}\left[\Phi\left(X^{\varepsilon}(T)\right)-\Phi(\bar{X}(T))\right]=I_{1}+I_{2}\right. \tag{6.10}
\end{equation*}
$$

Let us concentrate on $I_{1}$, in terms of Lemma 6.1 and the control convergence theorem, we have

$$
\begin{align*}
I_{1}= & \mathbb{E}\left[\int_{0}^{T} \int_{0}^{1}\left(b_{x}\left(t, \bar{X}(t)+\lambda\left(X^{\varepsilon}(t)-\bar{X}(t)\right), \bar{u}(t)+\lambda\left(u^{\varepsilon}(t)-\bar{u}(t)\right)\right)-b_{x}(t, \bar{X}(t), \bar{u}(t))\right)\left(X^{\varepsilon}(t)-\bar{X}(t)\right) d \lambda d t\right] \\
& +E\left[\int_{0}^{T} \int_{0}^{1}\left(b_{u}\left(t, \bar{X}(t)+\lambda\left(X^{\varepsilon}(t)-\bar{X}(t)\right), \bar{u}(t)+\lambda\left(u^{\varepsilon}(t)-\bar{u}(t)\right)\right)-b_{u}(t, \bar{X}(t), \bar{u}(t))\right)\left(u^{\varepsilon}(t)-\bar{u}(t)\right) d t\right] \\
& \left.\left.+\mathbb{E}\left[\int_{0}^{T} b_{x}(t, \bar{X}(t), \bar{u}(t))\right) \Xi^{\varepsilon}(t) d t\right]+\varepsilon \mathbb{E}\left[\int_{0}^{T} b_{x}(t, \bar{X}(t), \bar{u}(t))\right) Y(t) d \lambda d t\right]  \tag{6.11}\\
& \left.+\varepsilon E\left[\int_{0}^{T} b_{u}(t, \bar{X}(t), \bar{u}(t))\right)(u(t)-\bar{u}(t)) d t\right] \\
= & \varepsilon \mathbb{E}\left[\int_{0}^{T} b_{x}(t, \bar{X}(t), \bar{u}(t)) Y(t) d \lambda d t\right]+\varepsilon E\left[\int_{0}^{T} b_{u}(t, \bar{X}(t), \bar{u}(t))(u(t)-\bar{u}(t)) d t\right] \\
& +o(\varepsilon)
\end{align*}
$$

Similarly, we have

$$
\begin{equation*}
I_{1}=\varepsilon \mathbb{E}\left[\Phi_{x}(\bar{X}(T)) Y(T)\right]+o(\varepsilon) \tag{6.12}
\end{equation*}
$$

Then putting (6.11) and (6.12) into (6.10), we get (6.9). The proof is complete.

### 6.2 Main Results

Now we are in position to state and prove the maximum principle for our optimal control problem.
Theorem 6.3 (Maximum Principle). Let Assumption 4.1 be satisfied. Let $(\bar{u}(\cdot) ; \bar{X}(\cdot))$ be an optimal pair of Problem 4.2 associated with the adjoint processes $(p(\cdot), q(\cdot), r(\cdot, \cdot))$. Then the following minimum condition holds:

$$
\begin{equation*}
\left(\mathcal{H}_{u}(t, \bar{X}(t-), \bar{u}(t), p(t-), q(t), r(t, \cdot)), v-\bar{u}(t)\right)_{U} \geq 0, \quad \forall v \in \mathscr{U}, \text { for a.e. } t \in[0, T], \mathbb{P}-\text { a.s. } \tag{6.13}
\end{equation*}
$$

Proof. Applying Itô formula to $(p(t), Y(t))_{H}$ leads to

$$
\begin{align*}
& \mathbb{E}\left[\left(\Phi_{x}(\bar{X}(T)), Y(T)\right)_{H}\right]+\mathbb{E}\left[\int_{0}^{T}\left(l_{x}(t, \bar{X}(t), \bar{u}(t)), Y(t)\right)_{H} d t\right] \\
= & \mathbb{E}\left[\int_{0}^{T}\left(v(t)-\bar{u}(t), b_{u}^{*}(t, \bar{X}(t), \bar{u}(t)) p(t)+g_{u}^{*}(t, \bar{X}(t), \bar{u}(t)) q(t)+\int_{E} \sigma_{u}^{*}(t, e, \bar{X}(t), \bar{u}(t)) r(t, e) \nu(d e)\right)_{U} d t\right] \tag{6.14}
\end{align*}
$$

Since $\bar{u}(\cdot)$ is the optimal control, from (6.9) and the duality relation (5.3), we have

$$
\begin{align*}
0 \leq & \lim _{\varepsilon \longrightarrow 0} \frac{J\left(u^{\varepsilon}(\cdot)\right)-J(\bar{u}(\cdot))}{\varepsilon} \\
= & \mathbb{E}\left[\left(\Phi_{x}(\bar{X}(T)), Y(T)\right)_{H}\right]+\mathbb{E}\left[\int_{0}^{T}\left(l_{x}(t, \bar{X}(t), \bar{u}(t)), Y(t)\right)_{H} d t\right]+\mathbb{E}\left[\int_{0}^{T}\left(l_{u}(t, \bar{X}(t), \bar{u}(t)), v(t)-u(t)\right)_{U} d t\right] \\
= & \mathbb{E}\left[\int_{0}^{T}\left(v(t)-\bar{u}(t), b_{u}^{*}(t, \bar{X}(t), \bar{u}(t)) p(t)+g_{u}^{*}(t, \bar{X}(t), \bar{u}(t)) q(t)+\int_{E} \sigma_{u}^{*}(t, e, \bar{X}(t), \bar{u}(t)) r(t, e) \nu(d e)\right)_{U} d t\right](t  \tag{6.15}\\
& +\mathbb{E}\left[\int_{0}^{T}\left(l_{u}(t, \bar{X}(t), \bar{u}(t)), v(t)-\bar{u}(t)\right)_{U} d t\right] \\
= & E\left[\int_{0}^{T}\left(v(t)-\bar{u}(t), \mathcal{H}_{u}(t, \bar{X}(t), \bar{u}(t), p(t), q(t), r(t, \cdot))\right)_{U} d t\right]
\end{align*}
$$

This imply the minimum condition (6.13) holds since $v(\cdot)$ is arbitrary given admissible control.

## 7 Verification Theorem

In the following, we give the sufficient condition of optimality for the existence of an optimal control of Problem 4.2 .

Theorem 7.1 (Verification Theorem). Let Assumption 4.1 be satisfied. Let $(\bar{u}(\cdot) ; \bar{X}(\cdot))$ be an admissible pair of Problem 4.2 associated with the adjoint processes $(p(\cdot), q(\cdot), r(\cdot, \cdot))$. Suppose that $\mathcal{H}(t, x, u, p(t), q(t), r(t, \cdot))$ is convex in $(x, u)$, and $\Phi(x)$ is convex in $x$, moreover assume that the following optimality condition holds for almost all $(t, \omega) \in[0, T] \times \Omega:$

$$
\begin{equation*}
\mathcal{H}(t, \bar{X}(t), \bar{u}(t), p(t), q(t), r(t, \cdot))=\min _{u \in \mathscr{U}} \mathcal{H}(t, \bar{x}(t), u, p(t), q(t), r(t, \cdot)) . \tag{7.1}
\end{equation*}
$$

Then $(\bar{u}(\cdot) ; \bar{X}(\cdot))$ is an optimal pair of Problem 4.2.

Proof. Let $(u(\cdot) ; X(\cdot))$ be an any given admissible pairs. To simplify our notation, we define the following shorthand notations:

$$
\begin{align*}
& b(t) \triangleq b(t, X(t), u(t)), \bar{b}(t) \triangleq b(t, \bar{X}(t), \bar{u}(t)), \\
& g(t) \triangleq g(t, X(t), u(t)), \bar{g}(t) \triangleq g(t, \bar{X}(t), \bar{u}(t)), \\
& \sigma(t, e) \triangleq \sigma(t, e, X(t), u(t)), \bar{\sigma}(t) \triangleq \sigma(t, e, \bar{X}(t), \bar{u}(t)),  \tag{7.2}\\
& \mathcal{H}(t) \triangleq \mathcal{H}(t, X(t), u(t), p(t), q(t), r(t, \cdot)), \\
& \overline{\mathcal{H}}(t) \triangleq \mathcal{H}(t, \bar{X}(t), \bar{u}(t), p(t), q(t), r(t, \cdot)) .
\end{align*}
$$

From the definitions of the cost functional $J(u(\cdot))$ and the Hamiltonian $\mathcal{H}$ (see (4.2) and (5.3)), we can represent $J(u(\cdot))-J(\bar{u}(\cdot))$ as follows:

$$
\begin{align*}
J(u(\cdot))-J(\bar{u}(\cdot))= & \mathbb{E}\left[\int _ { 0 } ^ { T } \left\{\mathcal{H}(t)-\overline{\mathcal{H}}(t)-(\bar{p}(t), b(t)-\bar{b}(t))_{H}-(\bar{q}(t), g(t)-\bar{g}(t))_{H}\right.\right. \\
& \left.\left.-\int_{E}(\bar{r}(t, e), \sigma(t, e)-\bar{\sigma}(t, e))_{H} \nu(d e)\right\} d t\right]+\mathbb{E}[\Phi(X(T))-\Phi(\bar{X}(T))] \tag{7.3}
\end{align*}
$$

In terms of the state equation (3.1), we can check that $X(\cdot)-\bar{X}(\cdot)$ satisfies the following SEE:

$$
\left\{\begin{align*}
d(X(t)-\bar{X}(t))= & {[A(t)(X(t)-\bar{X}(t))+b(s)-\bar{b}(s)] d t+[B(t)(X(t)-\bar{X}(t))+g(s)-\bar{g}(s))] d W(t) }  \tag{7.4}\\
& \left.+\int_{E}[\sigma(s, e)-\bar{\sigma}(s, e))\right] d \tilde{\mu}(d e, t), \quad t \in[0, T] \\
X(0)-\bar{X}(0)= & 0
\end{align*}\right.
$$

Then recalling the adjoint equation (5.1) and applying Itô's formula to $(p(t), X(t)-\bar{X}(t))_{H}$, we get that

$$
\begin{align*}
& \mathbb{E}\left[\int_{0}^{T}\left\{(p(t), b(t)-\bar{b}(t))_{H}+(q(t), g(t)-\bar{g}(t))_{H}+\int_{E}(r(t, e), \sigma(t, e)-\bar{\sigma}(t, e))_{H} \nu(d e)\right\} d t\right] \\
= & \mathbb{E}\left[\int_{0}^{T}\left(\overline{\mathcal{H}}_{x}(t), X(t)-\bar{X}(t)\right)_{H} d t\right]+\mathbb{E}\left[\left(\Phi_{x}(\bar{X}(T)), X(T)-\bar{X}(T)\right)_{H}\right] . \tag{7.5}
\end{align*}
$$

Then substituting (7.5) into (7.3) leads to

$$
\begin{align*}
J(u(\cdot))-J(\bar{u}(\cdot))= & \mathbb{E}\left[\int_{0}^{T}\left\{\mathcal{H}(t)-\overline{\mathcal{H}}(t)-\left(\overline{\mathcal{H}}_{x}(t), X(t)-\bar{X}(t)\right)_{H}\right\} d t\right] \\
& +\mathbb{E}\left[\Phi(X(T))-\Phi(\bar{X}(T))-\left(\Phi_{x}(\bar{X}(T)), X(T)-\bar{x}(T)\right)_{H}\right] \tag{7.6}
\end{align*}
$$

On the other hand, the convexity of $\mathcal{H}(t)$ and $\Phi(x)$ leads to

$$
\begin{equation*}
\mathcal{H}(t)-\overline{\mathcal{H}}(t) \geq\left(\overline{\mathcal{H}}_{x}(t), X(t)-\bar{X}(t)\right)_{H}+\left(\overline{\mathcal{H}}_{u}(t), u(t)-\bar{u}(t)\right)_{U} \tag{7.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\Phi(X(T))-\Phi(\bar{X}(T)) \geq\left(\Phi_{x}(\bar{X}(T)), x(T)-\bar{x}(T)\right)_{H} \tag{7.8}
\end{equation*}
$$

In addition, the optimality condition (7.3) and the convex optimization principle (see Proposition 2.21 of [8]) yield that for almost all $(t, \omega) \in[0, T] \times \Omega$,

$$
\begin{equation*}
\left(\overline{\mathcal{H}}_{u}(t), u(t)-\bar{u}(t)\right)_{U} \geq 0 \tag{7.9}
\end{equation*}
$$

Then putting (7.7), (7.8) and (7.9) into (7.6), we get that

$$
\begin{equation*}
J(u(\cdot))-J(\bar{u}(\cdot)) \geq 0 \tag{7.10}
\end{equation*}
$$

Therefore, since $u(\cdot)$ is arbitrary, $\bar{u}(\cdot)$ is an optimal control process and $(\bar{u}(\cdot) ; \bar{x}(\cdot))$ is an optimal pair. The proof is complete.

## 8 Application

We provide an example to which our results solve. We consider a controlled Cauchy problem, where the system is given by a stochastic partial differential equation driven by Brownian motion $W$ and Poisson random martingale $\tilde{\mu}$ in divergence form:

$$
\left\{\begin{align*}
d y(t, z)= & \left\{\partial_{z^{i}}\left[a^{i j}(t, z) \partial_{z^{j}} y(t, z)\right]+b^{i}(t, z) \partial_{z^{i}} y(t, z)+c(t, z) y(t, z)+u(t, z)\right\} d t  \tag{8.1}\\
& +\left\{\partial_{z^{i}}\left[\eta^{i}(t, z) y(t, z)\right]+\rho(t, z) y(t, z)+u(t, z)\right\} d W(t)+\int_{E}[\Gamma(t, e, z) y(t, z)+u(t, z)] \tilde{\mu}(d e, d t) \\
y(0, z)= & \xi(z) \in \mathbb{R}^{d} \quad(t, z) \in[0, T] \times \mathbb{R}^{d}
\end{align*}\right.
$$

with a quadratic cost functional

$$
\begin{equation*}
\mathbb{E}\left[\int_{\mathbb{R}^{d}} y^{2}(T, z) d z+\iint_{[0, T] \times \mathbb{R}^{d}} y^{2}(s, z) d z d s+\iint_{[0, T] \times \mathbb{R}^{d}} u^{2}(s, z) d z d s\right] . \tag{8.2}
\end{equation*}
$$

Here the unknown $y(t, z, \omega)$, representing the state of the system, is a real-valued process, the control is a predictable real-valued process $u(t, z, \omega)$. The coefficients $a, b, c, \eta, \rho, \Gamma$ are given random functions satisfying the following assumptions, for some fixed constants $K \in(1, \infty)$ and $\kappa \in(0,1)$ :

Assumption 8.1. The functions $a, b, c, \eta$, and $\rho$ are $\mathscr{P} \times \mathscr{B}\left(\mathbb{R}^{d}\right)$-measurable with values in the set of real symmetric $d \times d$ matrices $\mathbb{R}^{d}, \mathbb{R}, \mathbb{R}^{d}$ and $\mathbb{R}$, respectively, and are bounded by $K$. The function $\Gamma$ are $\mathscr{P} \times \mathscr{B}(E) \times \mathscr{B}\left(\mathbb{R}^{d}\right)$ measurable with value $\mathbb{R}$ and is bounded by $K . \xi \in L^{2}\left(\mathbb{R}^{d}\right)$.

Assumption 8.2. The super-parabolic condition holds, i.e.,

$$
\kappa I+\eta(t, z)(\eta(t, z))^{*} \leq 2 a(t, \omega, z) \leq K I, \quad \forall(t, \omega, z) \in[0, T] \times \Omega \times \mathbb{R}^{d}
$$

where $I$ is the $(d \times d)$-identity matrix.
Now we begin to transform (8.1) into a SEE with jump in the form of (3.1). To this end, let us recall some preliminaries of Sobolev spaces. For $m=0,1$, we define the space $H^{m} \triangleq\left\{\phi: \partial_{z}^{\alpha} \phi \in L^{2}\left(\mathbb{R}^{d}\right)\right.$, for any $\alpha:=$ $\left(\alpha_{1}, \cdots, \alpha_{d}\right)$ with $\left.|\alpha|:=\left|\alpha_{1}\right|+\cdots+\left|\alpha_{d}\right| \leq m\right\}$ with the norm

$$
\|\phi\|_{m} \triangleq\left\{\sum_{|\alpha| \leq m} \int_{\mathbb{R}^{d}}\left|\partial_{z}^{\alpha} \phi(z)\right|^{2} d z\right\}^{\frac{1}{2}}
$$

We denote by $H^{-1}$ the dual space of $H^{1}$. We set $V=H^{1}, H=H^{0}, V^{*}=H^{-1}$. Then $\left(V, H, V^{*}\right)$ is a Gelfand triple.

In our case, we assume control domain $\mathscr{U}=U=H$. The admissible control set $\mathcal{A}$ is defined as $M_{\mathscr{F}}^{2}(0, T ; U)$. Set

$$
\begin{aligned}
& X(t) \triangleq y(t, \cdot) \\
& (A(t) \phi)(z) \triangleq \partial_{z^{i}}\left[a^{i j}(t, z) \partial_{z^{j}} \phi(z)\right]+b^{i}(t, z) \partial_{z^{i}} \phi(z)+c(t, z) \phi(z), \quad \forall \phi \in V, \\
& (B(t) \phi)(z) \triangleq \partial_{z^{i}}\left[\eta^{i}(t, z) \phi(z)\right]+\rho(t, z) \phi(z), \quad \forall \phi \in V, \\
& b(t, \phi, u) \triangleq u, \quad \forall \phi \in H, u \in \mathcal{U} \\
& g(t, \phi, u) \triangleq u, \quad \forall \phi \in H, u \in \mathcal{U} \\
& \sigma(t, e, \phi, u) \triangleq \Gamma(t, e, \cdot) \phi+u, \quad \forall \phi \in H, u \in \mathcal{U} \\
& l(t, \phi, u) \triangleq(\phi, \phi)_{H}+(u, u)_{U}, \quad \forall \phi \in H, u \in \mathcal{U} \\
& \Phi(\phi) \triangleq(\phi, \phi)_{H}, \quad \forall \phi \in H .
\end{aligned}
$$

In the Gelfand triple $\left(V, H, V^{*}\right)$, using the above notations, we can rewrite the state equation (8.1) as follows:

$$
\left\{\begin{align*}
& d X(t)=[A(t) X(t)+b(t, X(t), u(t))] d t+[B(t) X(t)+g(t, X(t), u(t))] d W(t)  \tag{8.3}\\
&+\int_{E} \sigma(t, e, X(t), u(t)) \tilde{\mu}(d e, d t) \\
& X(0)=x, \quad t \in[0, T]
\end{align*}\right.
$$

and the cost functional (8.2) can be rewritten as

$$
\begin{equation*}
J(u(\cdot))=\mathbb{E}\left[\int_{0}^{T} l(t, x(t), u(t)) d t+\Phi(x(T))\right] \tag{8.4}
\end{equation*}
$$

where we set

$$
\begin{align*}
& l(t, x, u) \triangleq(x, x)_{H}+(u, u)_{H}, \forall x \in H, u \in U  \tag{8.5}\\
& \Phi(x) \triangleq(x, x)_{H}, \forall x \in H
\end{align*}
$$

Thus this optimal control problem is transformed into Problem4.2 as a special case. Under Assumptions $8.1 / 8.2$ it is easy to check that the coefficients of this optimal control problem satisfy Assumptions 4.1. So in this case, Theorem 6.3 and 7.1 hold. More precisely, the corresponding Hamiltonian $\mathcal{H}$ becomes

$$
\begin{equation*}
\mathcal{H}(t, x, u, p, q, r):=(u, p)_{H}+(u, q)_{H}+\int_{E}(\Gamma(t, e, \cdot) x+u, r(t, e))_{H} \nu(d e)+(x, x)_{H}+(u, u)_{H} \tag{8.6}
\end{equation*}
$$

The adjoint equation associated with the optimal pair $(\bar{u}(\cdot) ; \bar{X}(\cdot))$ becomes

$$
\left\{\begin{align*}
d p(t)= & {\left[A^{*}(t) p(t)+B^{*}(t) q(t)+\int_{E} \Gamma^{*}(t, e) r(t, e) \nu(d e) d t+2 X(t)\right] d t }  \tag{8.7}\\
& \quad+q(t) d W(t)+\int_{E} r(t, e) \tilde{\mu}(d e, d t), \quad 0 \leqslant t \leqslant T \\
p(T)= & \Phi_{x}(T)
\end{align*}\right.
$$

where

$$
\begin{aligned}
& A^{*}(t) \phi(z) \triangleq-\partial_{z^{i}}\left[a^{i j}(t, z) \partial_{z^{j}} \phi(z)\right]+\partial_{z^{i}}\left[b^{i}(t, z) \phi(z)\right]+c(t, z) \phi(z), \quad \forall \phi \in V, \\
& B^{*}(t) \phi(z) \triangleq-\eta^{i}(t, z) \partial_{z^{i}} \phi(z), \quad \forall \phi \in H \\
& \Gamma^{*}(t, e) \phi(z) \triangleq \Gamma(t, e, z) \phi(z), \quad \forall \phi \in H .
\end{aligned}
$$

Since $\mathscr{U}=U$, there is no constraint on the control and the minimum condition (6.13)

$$
\begin{equation*}
\mathcal{H}_{u}(t, \bar{X}(t-), \bar{u}(t), p(t-), q(t), r(t, \cdot))=0 \tag{8.8}
\end{equation*}
$$

which imply that

$$
\begin{equation*}
2 \bar{u}(t)+p(t)+q(t)+\int_{E} r(t, e) \nu(d e)=0 \tag{8.9}
\end{equation*}
$$

a.e. $t \in[0, T], \mathbb{P}$-a.s.. Thus the optimal control $\bar{u}(\cdot)$ is given by

$$
\bar{u}(t)=-\frac{1}{2}\left[\bar{p}(t)+\bar{q}(t)+\int_{E} r(t, e) \nu(d e)\right] .
$$

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