# CONVERGENCE OF MULTI-BLOCK BREGMAN ADMM FOR NONCONVEX COMPOSITE PROBLEMS

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ABSTRACT. The alternating direction method with multipliers (ADMM) has been one of most powerful and successful methods for solving various composite problems. The convergence of the conventional ADMM (i.e., 2-block) for convex objective functions has been justified for a long time, and its convergence for nonconvex objective functions has, however, been established very recently. The multi-block ADMM, a natural extension of ADMM, is a widely used scheme and has also been found very useful in solving various nonconvex optimization problems. It is thus expected to establish convergence theory of the multi-block ADMM under nonconvex frameworks. In this paper we present a Bregman modification of 3-block ADMM and establish its convergence for a large family of nonconvex functions. We further extend the convergence results to the N-block case ( $N \ge 3$ ), which underlines the feasibility of multi-block ADMM applications in nonconvex settings. Finally, we present a simulation study and a real-world application to support the correctness of the obtained theoretical assertions.

Keywords: nonconvex regularization, alternating direction method, subanalytic function, K-L inequality, Bregman distance.

#### 1. Introduction

Many problems arising in the fields of signal & image processing and machine learning [7, 34] involve finding a minimizer of the sum of N ( $N \ge 2$ ) functions with linear equality constraint. If N = 2, the problem then consists of solving

$$\min f(x) + g(y)$$
s.t.  $Ax + By = 0$  (1)

where  $A \in \mathbb{R}^{m \times n_1}$  and  $B \in \mathbb{R}^{m \times n_2}$  are given matrices,  $f : \mathbb{R}^{n_1} \to \mathbb{R}$  is a proper lower semicontinuous function, and  $g : \mathbb{R}^{n_2} \to \mathbb{R}$  is a smooth function. Because of its separable structure, problem (1) can be efficiently solved by ADMM, namely, through the procedure

$$\begin{cases} x^{k+1} = \arg\min_{x \in \mathbb{R}^{n_1}} L_{\alpha}(x, y^k, p^k) \\ y^{k+1} = \arg\min_{y \in \mathbb{R}^{n_2}} L_{\alpha}(x^{k+1}, y, p^k) \\ p^{k+1} = p^k + \alpha(Ax^{k+1} + By^{k+1}) \end{cases}$$
 (2)

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where  $\alpha$  is a penalty parameter and

$$L_{\alpha}(x, y, p) := f(x) + g(y) + \langle p, Ax + By \rangle + \frac{\alpha}{2} ||Ax + By||^2$$

is the associated augmented Lagrangian function with multiplier *p*. So far, various variants of the conventional ADMM have been suggested. Among such varieties, Bregman ADMM (BADMM) is the one designed to improve the performance of procedure (2) [20, 42, 43, 56]. More specifically, BADMM takes the following iterative form:

$$x^{k+1} = \arg\min_{x \in \mathbb{R}^{n_1}} L_{\alpha}(x, y^k, p^k) + \Delta_{\phi}(x, x^k)$$

$$y^{k+1} = \arg\min_{y \in \mathbb{R}^{n_2}} L_{\alpha}(x^{k+1}, y, p^k) + \Delta_{\psi}(y, y^k)$$

$$p^{k+1} = p^k + \alpha(Ax^{k+1} + By^{k+1}),$$
(3)

where  $\triangle_{\phi}$  and  $\triangle_{\psi}$  are the Bregman distance with respect to functions  $\phi$  and  $\psi$ , respectively.

ADMM was introduced in the early 1970s [21, 22], and its convergence properties for convex objective functions have been extensively studied. The first convergent result was established for strongly convex functions [21, 22], and then extended to general convex functions [17, 18]. It has been shown that ADMM can converge at a sublinear rate of O(1/k) [25, 36], and  $O(1/k^2)$  for the accelerated version [23]. The convergence of BADMM for convex objective functions has also been examined with the Euclidean distance [14], Mahalanobis distance [56], and the general Bregman distance [56].

Recently, there has been an increasing interest in the study of ADMM for nonconvex objective functions. On one hand, the ADMM algorithm is highly successful in solving various nonconvex examples ranging from nonnegative matrix factorization, distributed matrix factorization, distributed clustering, sparse zero variance discriminant analysis, polynomial optimization, tensor decomposition, to matrix completion (see e.g. [26, 33, 47, 53, 55]). On the other hand, the convergence analysis of nonconvex ADMM is generally very difficult, due to the failure of the Féjer monotonicity of iterates. In [27], the subsequential convergence of ADMM for general nonconvex functions has been proved. Furthermore, the global convergence of ADMM for certain type of nonconvex functions has been proved in [31, 44].

The purpose of the present study is to examine convergence of ADMM with 3 blocks (i.e., N = 3). The obtained results then can naturally be generalized to the case of ADMM with multiple blocks. Thus, in the present paper we first consider the following 3-block composite optimization problem:

$$\min f(x) + g(y) + h(z)$$
s.t.  $Ax + By + Cz = 0$  (4)

where  $A \in \mathbb{R}^{m \times n_1}$ ,  $B \in \mathbb{R}^{m \times n_2}$  and  $C \in \mathbb{R}^{m \times n_3}$  are given matrices,  $f : \mathbb{R}^{n_1} \to \mathbb{R}$ ,  $g : \mathbb{R}^{n_2} \to \mathbb{R}$  are proper lower semicontinuous functions, and  $h : \mathbb{R}^{n_3} \to \mathbb{R}$  is a smooth function. To solve such a

problem, it is natural to extend the ADMM to the following form:

$$\begin{cases} x^{k+1} = \arg\min_{x \in \mathbb{R}^{n_1}} L_{\alpha}(x, y^k, z^k, p^k) \\ y^{k+1} = \arg\min_{y \in \mathbb{R}^{n_2}} L_{\alpha}(x^{k+1}, y, z^k, p^k) \\ z^{k+1} = \arg\min_{z \in \mathbb{R}^{n_3}} L_{\alpha}(x^{k+1}, y^{k+1}, z, p^k) \\ p^{k+1} = p^k + \alpha(Ax^{k+1} + By^{k+1} + Cz^{k+1}) \end{cases}$$

$$(5)$$

where the augmented Lagrangian function  $L_{\alpha}: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \times \mathbb{R}^m \to \mathbb{R}$  is defined by

$$L_{\alpha}(x, y, z, p) := f(x) + g(y) + h(z) + \langle p, Ax + By + Cz \rangle + \frac{\alpha}{2} ||Ax + By + Cz||^{2}.$$
 (6)

Unlike the conventional ADMM with 2 blocks, the convergence of algorithm (5), called the 3-block ADMM henceforth, has remained unclear even for convex objective functions. Although it is not necessarily convergent in general [13], the 3-block ADMM does converge under some restrictive conditions; for example, under the strong convexity condition of all objective functions (see e.g. [25]). Recently, Li, Sun, and Toh [32] proposed a modification of algorithm (5), called the semi-proximal 3-block ADMM as follows

$$\begin{cases} x^{k+1} = \arg\min_{x \in \mathbb{R}^{n_1}} L_{\alpha}(x, y^k, z^k, p^k) + \frac{1}{2} ||x - x^k||_{T_1}^2 \\ y^{k+1} = \arg\min_{y \in \mathbb{R}^{n_2}} L_{\alpha}(x^{k+1}, y, z^k, p^k) + \frac{1}{2} ||y - y^k||_{T_2}^2 \\ z^{k+1} = \arg\min_{z \in \mathbb{R}^{n_3}} L_{\alpha}(x^{k+1}, y^{k+1}, z, p^k) + \frac{1}{2} ||z - z^k||_{T_3}^2 \\ p^{k+1} = p^k + \alpha (Ax^{k+1} + By^{k+1} + Cz^{k+1}) \end{cases}$$

$$(7)$$

where  $\|\cdot\|_{T_i}$  denotes ellipsoidal norms, i = 1, 2, 3. They proved the convergence of the algorithm when f, g, h are all convex and one of them is at least strongly convex.

Motivated by Bregman ADMM, we propose to use the following 3-block Bregman ADMM for solving the optimization problem (4):

$$\begin{cases} x^{k+1} = \arg\min_{x \in \mathbb{R}^{n_1}} L_{\alpha}(x, y^k, z^k, p^k) + \Delta_{\phi}(x, x^k) \\ y^{k+1} = \arg\min_{y \in \mathbb{R}^{n_2}} L_{\alpha}(x^{k+1}, y, z^k, p^k) + \Delta_{\psi}(y, y^k) \\ z^{k+1} = \arg\min_{y \in \mathbb{R}^{n_3}} L_{\alpha}(x^{k+1}, y^{k+1}, z, p^k) + \Delta_{\varphi}(z, z^k) \\ p^{k+1} = p^k + \alpha(Ax^{k+1} + By^{k+1} + Cz^{k+1}) \end{cases}$$
(8)

where, as mentioned before,  $\triangle_{\phi}$ ,  $\triangle_{\psi}$  and  $\triangle_{\varphi}$  are the Bregman distance associated with functions  $\phi$ ,  $\psi$ , and  $\varphi$ , respectively. In the present paper, our aim is to justify the convergence of 3-block BADMM under nonconvex frameworks. We will show that the 3-block BADMM can converge if the objective function is subanalytic and matrix C has full-row rank.

### 2. Preliminaries

In what follows,  $R^n$  will stand for the *n*-dimensional Euclidean space,

$$\langle x, y \rangle = x^{\mathsf{T}} y = \sum_{i=1}^{n} x_i y_i, ||x|| = \sqrt{\langle x, x \rangle},$$

where  $x, y \in \mathbb{R}^n$  and  $\top$  stands for the transpose operation.

2.1. **Subdifferentials.** Given a function  $f : \mathbb{R}^n \to \mathbb{R}$  we denote by dom f the domain of f, namely, dom  $f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ . A function f is said to be proper if dom  $f \neq \emptyset$ ; lower semicontinuous at the point  $x_0$  if

$$\liminf_{x \to x_0} f(x) \ge f(x_0).$$

If f is lower semicontinuous at every point of its domain of definition, then it is simply called a lower semicontinuous function.

**Definition 2.1.** Let  $f: \mathbb{R}^n \to \mathbb{R}$  be a proper lower semi-continuous function.

(i) Given  $x \in \text{dom } f$ , the Fréchet subdifferential of f at x, written by  $\widehat{\partial} f(x)$ , is the set of all elements  $u \in \mathbb{R}^n$  which satisfy

$$\lim_{y \neq x} \inf_{y \to x} \frac{f(y) - f(x) - \langle u, y - x \rangle}{\|x - y\|} \ge 0.$$

(ii) The limiting subdifferential, or simply subdifferential, of f at x, written by  $\partial f(x)$ , is defined as

$$\partial f(x) = \{ u \in \mathbb{R}^n : \exists x^k \to x, f(x^k) \to f(x),$$
  
$$u^k \in \widehat{\partial} f(x^k) \to u, k \to \infty \}.$$

(iii) A critical point or stationary point of f is a point  $x^*$  in the domain of f satisfying  $0 \in \partial f(x^*)$ .

**Definition 2.2.** An element  $w^* := (x^*, y^*, z^*, p^*)$  is called a critical point or stationary point of the Lagrangian function  $L_{\alpha}$  defined as in (6) if it satisfies:

$$\begin{cases} A^{\top} p^* \in -\partial f(x^*), \ B^{\top} p^* \in -\partial g(y^*), \\ C^{\top} p^* = -\nabla h(z^*), \ Ax^* + By^* + Cz^* = 0. \end{cases}$$
(9)

The existence of proper lower semicontinuous functions and properties of subdifferential can see [37]. We particularly collect the following basic properties of the subdifferential.

**Proposition 2.1.** Let  $f: \mathbb{R}^n \to \mathbb{R}$  and  $g: \mathbb{R}^n \to \mathbb{R}$  be proper lower semi-continuous functions. Then the following holds:

- (i)  $\partial f(x) \subset \partial f(x)$  for each  $x \in \mathbb{R}^n$ . Moreover, the first set is closed and convex, while the second is closed, and not necessarily convex.
- (ii) Let  $(u^k, x^k)$  be sequences such that  $x^k \to x, u^k \to u, f(x^k) \to f(x)$  and  $u^k \in \partial f(x^k)$ . Then  $u \in \partial f(x)$ .
- (iii) The Fermat's rule remains true: if  $x_0 \in \mathbb{R}^n$  is a local minimizer of f, then  $x_0$  is a critical point or stationary point of f, that is,  $0 \in \partial f(x_0)$ .
- (iv) If f is continuously differentiable function, then  $\partial(f+g)(x) = \nabla f(x) + \partial g(x)$ .

A function f is said to be  $\ell_f$ -Lipschitz continuous ( $\ell_f \ge 0$ ) if

$$||f(x) - f(y)|| \le \ell_f ||x - y||,$$

for any  $x, y \in \text{dom } f$ ;  $\mu$ -strongly convex  $(\mu > 0)$  if

$$f(y) \ge f(x) + \langle \xi(x), y - x \rangle + \frac{\mu}{2} ||y - x||^2,$$
 (10)

for any  $x, y \in \text{dom } f$  and  $\xi(x) \in \partial f(x)$ ; coercive if

$$\lim_{\|x\| \to \infty} f(x) = +\infty. \tag{11}$$

2.2. **Kurdyka-Łojasiewicz inequality.** The Kurdyka-Łojasiewicz (K-L) inequality was first introduced by Łojasiewicz [38] for real analytic functions, and then was extended by Kurdyka [29] to smooth functions whose graph belongs to an o-minimal structure. Recently, this notion was further extended for nonsmooth subanalytic functions [4].

**Definition 2.3** (K-L inequality). A function  $f: \mathbb{R}^n \to \mathbb{R}$  is said to satisfy the K-L inequality at  $x_0$  if there exists  $\eta > 0$ ,  $\delta > 0$ ,  $\varphi \in \mathscr{A}_{\eta}$ , such that for all  $x \in O(x_0, \delta) \cap \{x : f(x_0) < f(x) < f(x_0) + \eta\}$ 

$$\varphi'(f(x) - f(x_0))\operatorname{dist}(0, \partial f(x)) \ge 1,$$

where  $\operatorname{dist}(x_0, \partial f(x)) := \inf\{||x_0 - y|| : y \in \partial f(x)\}$ , and  $\mathcal{A}_{\eta}$  stand for the class of functions  $\varphi : [0, \eta) \to \mathbb{R}^+$  with the properties: (a)  $\varphi$  is continuous on  $[0, \eta)$ ; (b)  $\varphi$  is smooth concave on  $(0, \eta)$ ; (c)  $\varphi(0) = 0, \varphi'(x) > 0, \forall x \in (0, \eta)$ .

The following is an extension of the conventional K-L inequality [5].

**Lemma 2.2** (K-L inequality on compact subsets). Let  $f: \mathbb{R}^n \to \mathbb{R}$  be a proper lower semicontinuous function and let  $\Omega \subseteq \mathbb{R}^n$  be a compact set. If f is a constant on  $\Omega$  and f satisfies the K-L inequality at each point in  $\Omega$ , then there exists  $\eta > 0$ ,  $\delta > 0$ ,  $\varphi \in \mathcal{A}_{\eta}$ , such that for all  $x_0 \in \Omega$ and for all  $x \in \{x \in \mathbb{R}^n : \operatorname{dist}(x,\Omega) < \delta\} \cap \{x \in \mathbb{R}^n : f(x_0) < f(x_0) + \eta\}$ ,

$$\varphi'(f(x) - f(x_0)) \operatorname{dist}(0, \partial f(x)) \ge 1.$$

Typical functions satisfying the K-L inequality include strongly convex functions, real analytic functions, semi-algebraic functions and subanalytic functions.

A subset  $C \subset \mathbb{R}^n$  is said to be *semi-algebraic* if it can be written as

$$C = \bigcup_{j=1}^{r} \bigcap_{i=1}^{s} \{x \in \mathbb{R}^{n} : g_{i,j}(x) = 0, h_{i,j}(x) < 0\},$$

where  $g_{i,j}, h_{i,j}: \mathbb{R}^n \to \mathbb{R}$  are real polynomial functions. Then a function  $f: \mathbb{R}^n \to \mathbb{R}$  is called *semi-algebraic* if its graph

$$G(f) := \{(x, y) \in \mathbb{R}^{n+1} : f(x) = y\}$$

is a semi-algebraic subset in  $\mathbb{R}^{n+1}$ . For example, the  $\ell_q$  norm  $||x||_q := \sum_i |x_i|^q$  with  $0 < q \le 1$ , the sup-norm  $||x||_{\infty} := \max_i |x_i|$ , the Euclidean norm ||x||,  $||Ax - b||_q$ , ||Ax - b|| and  $||Ax - b||_{\infty}$  are all semi-algebraic functions for any matrix A [5, 48].

A real function on R is said to be *analytic* if it possesses derivatives of all orders and agrees with its Taylor series in a neighborhood of every point. For a real function f on  $\mathbb{R}^n$ , it is said to be analytic if the function of one variable g(t) := f(x + ty) is analytic for any  $x, y \in \mathbb{R}^n$ . It is readily seen that real polynomial functions such as quadratic functions  $||Ax - b||^2$  are analytic. Moreover, the  $\varepsilon$ -smoothed  $\ell_q$  norm  $||x||_{\varepsilon,q} := \sum_i (x_i^2 + \varepsilon)^{q/2}$  with  $0 < q \le 1$  and the logistic loss function  $\log(1 + e^{-t})$  are all examples for real analytic functions [48].

A subset  $C \subset \mathbb{R}^n$  is said to be *subanalytic* if it can be written as

$$C = \bigcup_{j=1}^{r} \bigcap_{i=1}^{s} \{x \in \mathbb{R}^{n} : g_{i,j}(x) = 0, h_{i,j}(x) < 0\},\$$

where  $g_{i,j}, h_{i,j}: \mathbb{R}^n \to \mathbb{R}$  are real analytic functions. Then a function  $f: \mathbb{R}^n \to \mathbb{R}$  is called subanalytic if its graph G(f) is a subanalytic subset in  $\mathbb{R}^{n+1}$ . It is clear that both real analytic and semi-algebraic functions are subanalytic. Generally speaking, the sum of two subanalytic functions is not necessarily subanalytic. It is known, however, that for two subanalytic functions, if at least one function maps bounded sets to bounded sets, then their sum is also subanalytic, as shown in [4, 48]. In particular, the sum of a subanalytic function and a analytic function is subanalytic. Some subanalytic functions that are widely used are as follows:

- (i)  $||Ax b||^2 + \lambda ||y||_q^q$ ;
- (ii)  $||Ax b||^2 + \lambda \sum_{i} (y_i^2 + \varepsilon)^{q/2}$ ;
- (iii)  $\frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(-c_i(a_i^{\top} x + b)) + \lambda ||y||_q^q;$ (iv)  $\frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(-c_i(a_i^{\top} x + b)) + \lambda \sum_i (y_i^2 + \varepsilon)^{q/2}.$

2.3. Bregman distance. The Bregman distance, first introduced in 1967 [8], plays an important role in various iterative algorithms. As a generalization of squared Euclidean distance, the Bregman distance share many similar nice properties of the Euclidean distance. However, the Bregman distance is not a real metric, since it does not satisfy the triangle inequality nor symmetry. For a convex differential function  $\phi$ , the associated Bregman distance is defined as

$$\triangle_{\phi}(x, y) = \phi(x) - \phi(y) - \langle \nabla \phi(y), x - y \rangle.$$

In particular, if we let  $\phi(x) := ||x||^2$  in the above, then it is reduced to  $||x - y||^2$ , namely, the classical Euclidean distance. Some nontrivial examples of Bregman distance include [2]:

- (i) Itakura-Saito distance:  $\sum_i x_i (\log x_i/y_i) \sum_i (x_i y_i)$ ;
- (ii) Kullback-Leibler divergence:  $\sum_i x_i (\log x_i/y_i)$ ;
- (iii) Mahalanobis distance:  $||x y||_Q^2 = \langle Qx, x \rangle$  with Q a symmetric positive definite matrix.

The following proposition collects some useful properties of Bregman distance.

**Proposition 2.3.** Let  $\phi$  be a convex differential function and  $\Delta_{\phi}(x,y)$  the associated Bregman distance.

- (i) Non-negativity:  $\triangle_{\phi}(x, y) \ge 0, \triangle_{\phi}(x, x) = 0$  for all x, y.
- (ii) Convexity:  $\triangle_{\phi}(x, y)$  is convex in x, but not necessarily in y.
- (iii) Strong Convexity: If  $\phi$  is  $\delta$ -strongly convex, then  $\Delta_{\phi}(x,y) \geq \frac{\delta}{2}||x-y||^2$  for all x, y.

2.4. **Basic assumptions.** In the research of present paper, we will make the following assumptions:

**Assumption 1.** We assume that functions  $f, g, h, C, \phi, \psi, \varphi$  in problem (4) have the following properties:

- (a1)  $\langle CC^{\intercal}x, x \rangle = ||x||_{C^{\intercal}}^2 \ge \sigma_C ||x||^2, \forall x \in \mathbb{R}^m$ , namely, C is full row rank;
- (a2)  $\nabla h, \nabla \phi, \nabla \psi, \nabla \varphi$  are Lipshitz continuous;
- (a3) either f or  $\phi$ , either g or  $\psi$ , and either h or  $\varphi$  are strongly convex;
- (a4) f + g + h is subanalytic,

where  $\sigma_C$  and  $\ell_h$  are both positive real numbers.

In implementation of BADMM (8), the parameter  $\alpha$ , and the smooth convex functions  $\phi$ ,  $\psi$ , and  $\varphi$  should be regularized. We further assume **Assumption 2**:

$$\alpha > \frac{4[(\ell_h + \ell_\varphi)^2 + \ell_\varphi^2]}{\mu_3 \sigma_C},\tag{12}$$

where  $\mu_3$  is the strong convexity coefficient of h or  $\varphi$ , and  $\ell_h$  and  $\ell_{\varphi}$  are respectively the Lipschitz coefficient of  $\nabla h$  and  $\nabla \varphi$ .

We remark that conditions (a1)-(a2) above are standard assumptions even for convex settings. Condition (a3) is used to guarantee the sufficient descent property of iterates, and condition (a4) is a basic assumption assuring that the function  $\hat{L}$ , to be defined in the next section, can satisfy the K-L inequality, which in turn will imply the global convergence of the proposed algorithm.

The smooth convex functions in the Bregman distance are very easily specified; for example, take  $\phi(\cdot) = \psi(\cdot) = \frac{1}{2}||\cdot||^2$ . Note that if  $\phi$  is  $\mu_1$ -strongly convex, then its Bregman distance satisfies

$$\Delta_{\phi}(x, y) \ge \frac{\mu_1}{2} ||x - y||^2,$$
 (13)

which follows from Proposition 2.3.

#### 3. Convergence Analysis

In this section, under the Assumptions 1 and 2 we firstly give a convergence result for the BADMM with 3-block procedure (8), and then extend this result to the N-block ( $N \ge 3$ ) case. The main results are presented in the subsection 3.4.

For convenience, we first fix the following notations:

$$\sigma_{0} = \frac{2\ell_{\varphi}^{2}}{\alpha\sigma_{C}}, \ \sigma_{1} = \frac{1}{2}\min\left(\mu_{1}, \mu_{2}, \mu_{3} - \frac{4(\ell_{h} + \ell_{\varphi})^{2}}{\alpha\sigma_{C}} - \frac{4\ell_{\varphi}^{2}}{\alpha\sigma_{C}}\right),$$

$$u = (x, y, z), w = (x, y, z, p), \hat{w} = (x, y, z, p, \hat{z}),$$

$$u^{k} = (x^{k}, y^{k}, z^{k}), w^{k} = (x^{k}, y^{k}, z^{k}, p^{k}), \hat{w}^{k} = (x^{k}, y^{k}, z^{k}, p^{k}, z^{k-1}),$$

$$||w|| = (||x||^{2} + ||y||^{2} + ||z||^{2})^{1/2}, ||w||_{1} = ||x|| + ||y|| + ||z||,$$

where  $\mu_1$  is the strong convexity coefficient of f or  $\phi$ , and  $\mu_2$  is the strong convexity coefficient of g or  $\psi$ . Clearly both  $\sigma_0$  and  $\sigma_1$  are positive by our assumptions. Also, we define a new function  $\hat{L}: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3} \times \mathbb{R}^m \times \mathbb{R}^{n_3} \to \mathbb{R}$  by

$$\hat{L}(\hat{w}) = L_{\alpha}(w) + \sigma_0 ||z - \hat{z}||^2. \tag{14}$$

3.1. **Some lemmas.** We establish a series of lemmas to support the proof of convergence of BADMM with 3-block procedure (8).

**Lemma 3.1.** For each  $k \in \mathbb{N}$ 

$$||p^{k+1} - p^k||^2 \le \frac{2(\ell_h + \ell_\varphi)^2}{\sigma_C} ||z^{k+1} - z^k||^2 + \frac{2\ell_\varphi^2}{\sigma_C} ||z^k - z^{k-1}||^2.$$
(15)

*Proof.* By our assumptions on C, we have

$$||C^{\mathsf{T}}(p^{k+1} - p^k)||^2 = \langle CC^{\mathsf{T}}(p^{k+1} - p^k), p^{k+1} - p^k \rangle \ge \sigma_C ||p^{k+1} - p^k||^2.$$
(16)

Applying Fermat's rule to z-subproblem in (8), we then get

$$\nabla h(z^{k+1}) + C^{\intercal}(p^k + \alpha(Ax^{k+1} + By^{k+1} + Cz^{k+1})) + \nabla \varphi(z^{k+1}) - \nabla \varphi(z^k) = 0.$$

Note that  $p^{k+1} = p^k + \alpha (Az^{k+1} + By^{k+1} + Cz^{k+1})$ . It then follows that

$$\nabla h(z^{k+1}) + C^{\mathsf{T}} p^{k+1} + \nabla \varphi(z^{k+1}) - \nabla \varphi(z^{k}) = 0, \tag{17}$$

so that

$$\begin{split} & \|C^{\intercal}(p^{k+1} - p^k)\|^2 \\ &= \|\nabla h(z^{k+1}) - \nabla h(z^k) + (\nabla \varphi(z^{k+1}) - \nabla \varphi(z^k)) + (\nabla \varphi(z^{k-1}) - \nabla \varphi(z^k))\|^2 \\ &\leq (\|\nabla h(z^{k+1}) - \nabla h(z^k)\| + \|\nabla \varphi(z^{k+1}) - \nabla \varphi(z^k)\| + \|\nabla \varphi(z^{k-1}) - \nabla \varphi(z^k)\|)^2 \\ &\leq (\ell_h \|z^{k+1} - z^k\| + \ell_{\varphi} \|z^k - z^{k+1}\| + \ell_{\varphi} \|z^k - z^{k-1}\|)^2 \\ &\leq 2(\ell_h + \ell_{\varphi})^2 \|z^{k+1} - z^k\|^2 + 2\ell_{\varphi}^2 \|z^k - z^{k-1}\|^2. \end{split}$$

This together with (16) at once yields inequality (15).

## **Lemma 3.2.** For each $k \in \mathbb{N}$

$$L_{\alpha}(w^{k+1}) \leq L_{\alpha}(w^{k}) + \left(\frac{2(\ell_{h} + \ell_{\varphi})^{2}}{\alpha\sigma_{C}} - \frac{\mu_{3}}{2}\right) \|z^{k+1} - z^{k}\|^{2} + \frac{2\ell_{\varphi}^{2}}{\alpha\sigma_{C}} \|z^{k} - z^{k-1}\|^{2} - \frac{\mu_{1}}{2} \|x^{k+1} - x^{k}\|^{2} - \frac{\mu_{2}}{2} \|y^{k+1} - y^{k}\|^{2}.$$

$$(18)$$

*Proof.* First we show that if either f or  $\phi$  is strongly convex, then it follows that

$$L_{\alpha}(x^{k+1}, y^k, z^k, p^k) \le L_{\alpha}(x^k, y^k, z^k, p^k) - \frac{1}{2\mu_1} ||x^{k+1} - x^k||^2.$$
(19)

In fact, if f is strongly convex, then  $L_{\alpha}(x, y^k, z^k, p^k) + \triangle_{\phi}(x, x^k)$  is strongly convex with modulus  $\mu_1$ , and thus inequality (19) follows from (10). Let us now justify the case whenever  $\phi$  is strongly convex. As  $y^{k+1}$  is a minimizer of  $L_{\alpha}(x, y^k, z^k, p^k) + \triangle_{\phi}(x, x^k)$ , we have

$$\begin{split} L_{\alpha}(x^{k+1}, y^k, z^k, p^k) &\leq L_{\alpha}(x, y^k, z^k, p^k) - \triangle_{\phi}(x^{k+1}, x^k) \\ &\leq L_{\alpha}(x, y^k, z^k, p^k) - \frac{1}{2\mu_1} \|x^{k+1} - x^k\|^2, \end{split}$$

where the last inequality follows from (13). Similarly, we have

$$L_{\alpha}(x^{k+1}, y^{k+1}, z^k, p^k) \le L_{\alpha}(x^{k+1}, y^k, z^k, p^k) - \frac{1}{2\mu_2} ||y^{k+1} - y^k||^2$$
  
$$L_{\alpha}(x^{k+1}, y^{k+1}, z^{k+1}, p^k) \le L_{\alpha}(x^{k+1}, y^{k+1}, z^k, p^k) - \frac{1}{2\mu_3} ||z^{k+1} - z^k||^2,$$

and from the last equality in (8) we have

$$L_{\alpha}(x^{k+1}, y^{k+1}, z^{k+1}, p^{k+1}) = L_{\alpha}(x^{k+1}, y^{k+1}, z^{k+1}, p^{k}) + \frac{1}{\alpha} ||p^{k+1} - p^{k}||^{2}.$$

Adding up the above formulas, we get

$$L_{\alpha}(w^{k+1}) \le L_{\alpha}(w^{k}) + \frac{1}{\alpha} ||p^{k+1} - p^{k}||^{2} - \frac{1}{2\mu_{1}} ||x^{k+1} - x^{k}||^{2} - \frac{1}{2\mu_{2}} ||y^{k+1} - y^{k}||^{2} - \frac{1}{2\mu_{3}} ||z^{k+1} - z^{k}||^{2}.$$

$$(20)$$

This together with (15) yields inequality (18) as desired.

### **Lemma 3.3.** For each $k \in \mathbb{N}$

$$\hat{L}(\hat{w}^{k+1}) \leq \hat{L}(\hat{w}^k) - \sigma_1(\|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2 + \|z^k - z^{k+1}\|^2).$$

Proof. It follows from lemmas 3.1 and 3.2 that

$$\begin{split} &L_{\alpha}(x^{k+1},y^{k+1},z^{k+1},p^{k+1}) - L_{\alpha}(x^{k},y^{k},z^{k},p^{k}) \\ &\leq \left(\frac{2(\ell_{h}+\ell_{\varphi})^{2}}{\alpha\sigma_{C}} - \frac{\mu_{3}}{2}\right) \|z^{k+1} - z^{k}\|^{2} + \frac{2\ell_{\varphi}^{2}}{\alpha\sigma_{C}} \|z^{k} - z^{k-1}\|^{2} \\ &- \frac{\mu_{1}}{2} \|x^{k+1} - x^{k}\|^{2} - \frac{\mu_{2}}{2} \|y^{k+1} - y^{k}\|^{2}, \end{split}$$

which implies

$$\begin{split} & L_{\alpha}(x^{k+1}, y^{k+1}, z^{k+1}, p^{k+1}) + \sigma_{0} \|z^{k+1} - z^{k}\|^{2} \\ & \leq L_{\alpha}(x^{k}, y^{k}, z^{k}, p^{k}) + \sigma_{0} \|z^{k} - z^{k-1}\|^{2} \\ & - \left(\frac{\mu_{3}}{2} - \frac{2(\ell_{h} + \ell_{\varphi})^{2}}{\alpha \sigma_{C}} - \frac{2\ell_{\varphi}^{2}}{\alpha \sigma_{C}}\right) \|z^{k} - z^{k+1}\|^{2} \\ & - \frac{\mu_{1}}{2} \|x^{k+1} - x^{k}\|^{2} - \frac{\mu_{2}}{2} \|y^{k+1} - y^{k}\|^{2} \\ & \leq L_{\alpha}(x^{k}, y^{k}, z^{k}, p^{k}) + \sigma_{0} \|z^{k} - z^{k-1}\|^{2} \end{split}$$

$$-\sigma_1(||x^{k+1}-x^k||^2+||y^{k+1}-y^k||^2+||z^k-z^{k+1}||^2).$$

Then lemma 3.3 follows from our notations.

**Lemma 3.4.** If the sequence  $\{u^k\}$  is bounded, then we have

$$\sum_{k=0}^{\infty} ||w^k - w^{k+1}||^2 < \infty.$$

In particular, the sequence  $||w^k - w^{k+1}||$  is asymptotically regular, namely,  $||w^k - w^{k+1}|| \to 0$  as  $k \to \infty$ . Moreover, any cluster point of  $w^k$  is a stationary point of the augmented Lagrangian function  $L_\alpha$  defined as in (6).

*Proof.* We first show that the sequence  $\{w^k\}$  is bounded. Indeed we deduce from Eq. (17) that

$$\begin{split} \|C^{\intercal}p^k\|^2 &= \|\nabla h(z^k) + \nabla \varphi(z^k) - \nabla \varphi(z^{k-1})\|^2 \\ &\leq (\|\nabla h(z^k)\| + \ell_{\varphi}\|z^k - z^{k-1}\|)^2 \\ &\leq 2(\|\nabla h(z^k)\|^2 + \ell_{\varphi}^2\|z^k - z^{k-1}\|^2). \end{split}$$

Since C has full row rank, we have

$$\sigma_C \|p^k\|^2 \le 2(\|\nabla h(z^k)\|^2 + \ell_{\varphi}^2 \|z^k - z^{k-1}\|^2). \tag{21}$$

Note that  $\{u^k\}$  is bounded. This implies that the sequence  $\{p^k\}$  is bounded and so are the sequences  $\{w^k\}$  and  $\{\hat{w}^k\}$ .

Since  $\hat{w}^k$  is bounded, there exists a subsequence  $\hat{w}^{k_j}$  so that it is convergent to some element  $\hat{w}^*$ . By our hypothesis, the function  $\hat{L}$  is lower semicontinuous, which leads to

$$\liminf_{j\to\infty} \hat{L}(\hat{w}^{k_j}) \ge \hat{L}(\hat{w}^*),$$

so that  $\hat{L}(\hat{w}^{k_j})$  is bounded from below. By Lemma 3.3,  $\hat{L}(\hat{w}^k)$  is nonincreasing, so that  $\hat{L}(\hat{w}^{k_j})$  is a convergent sequence. Moreover  $\hat{L}(\hat{w}^k)$  is also convergent and  $\hat{L}(\hat{w}^k) \ge \hat{L}(\hat{w}^*)$  for each k.

Now fix  $k \in \mathbb{N}$ . By Lemma 3.3, we have

$$\sigma_{1} \sum_{i=1}^{k} (\|x^{k+1} - x^{k}\|^{2} + \|y^{k+1} - y^{k}\|^{2} + \|z^{k} - z^{k+1}\|^{2})$$

$$\leq \sum_{i=1}^{k} \hat{L}(\hat{w}^{i}) - \hat{L}(\hat{w}^{i+1}) = \hat{L}(\hat{w}^{1}) - \hat{L}(\hat{w}^{k+1})$$

$$\leq \hat{L}(\hat{w}^{1}) - \hat{L}(\hat{w}^{*}) < \infty.$$

Moreover, by inequality (15), we see that  $\sum_{k=0}^{\infty} \|p^k - p^{k+1}\|^2 < \infty$ . This implies  $\sum_{k=0}^{\infty} \|w^k - w^{k+1}\|^2 < \infty$ , and hence  $\|w^k - w^{k+1}\| \to 0$ .

Let  $w^* = (x^*, y^*, z^*, p^*)$  be any cluster point of  $w^k$  and let  $w^{kj}$  be a subsequence of  $w^k$  converging to  $w^*$ . It then follows from algorithm (8) that

$$p^{k+1} = p^k + \alpha (Ax^{k+1} + By^{k+1} + Cz^{k+1}),$$

$$\begin{split} -\partial f(x^{k+1}) \ni A^{\intercal} p^k + \alpha A^{\intercal} (Ax^{k+1} + By^k + Cz^k) + \nabla \phi(x^{k+1}) - \nabla \phi(x^k) \\ &= A^{\intercal} p^{k+1} + \alpha A^{\intercal} B(y^k - y^{k+1}) + \alpha A^{\intercal} C(z^k - z^{k+1}) + \nabla \phi(x^{k+1}) - \nabla \phi(x^k), \\ -\partial g(y^{k+1}) \ni B^{\intercal} p^k + \alpha B^{\intercal} (Ax^{k+1} + By^{k+1} + Cz^k) + \nabla \psi(y^{k+1}) - \nabla \psi(y^k) \\ &= B^{\intercal} p^{k+1} + \alpha B^{\intercal} C(z^k - z^{k+1}) + \nabla \psi(y^{k+1}) - \nabla \psi(y^k), \\ -\nabla h(z^{k+1}) = C^{\intercal} p^k + \alpha C^{\intercal} (Ax^{k+1} + By^{k+1} + Cz^{k+1}) + \nabla \varphi(z^k) - \nabla \varphi(z^{k+1}) \\ &= C^{\intercal} p^{k+1} + \nabla \varphi(z^k) - \nabla \varphi(z^{k+1}). \end{split}$$

Since  $||w^k - w^{k+1}||$  tends to zero, letting  $j \to \infty$  in the above formulas yields

$$A^{\mathsf{T}}p^* \in -\partial f(x^*), \ B^{\mathsf{T}}p^* \in -\partial g(y^*),$$
  
 $C^{\mathsf{T}}p^* = -\nabla h(z^*), \ Ax^* + By^* + Cz^* = 0,$ 

which implies that  $w^*$  is a stationary point of  $L_{\alpha}$ .

**Lemma 3.5.** There exists  $\kappa > 0$  such that for each k

$$\operatorname{dist}(0,\partial \hat{L}(\hat{w}^{k+1})) \leq \kappa(\|x^k - x^{k+1}\| + \|y^k - y^{k+1}\| + \|z^k - z^{k+1}\| + \|z^k - x^{k-1}\|).$$

Proof. First, we deduce from algorithm (8) that

$$\partial \hat{L}_{x}(\hat{w}^{k+1}) = \partial f(x^{k+1}) + A^{\mathsf{T}} p^{k+1} + \alpha A^{\mathsf{T}} (A x^{k+1} + B y^{k+1} + C z^{k+1}), \tag{22}$$

$$\partial \hat{L}_{v}(\hat{w}^{k+1}) = \partial g(y^{k+1}) + B^{\mathsf{T}} p^{k+1} + \alpha B^{\mathsf{T}} (A x^{k+1} + B y^{k+1} + C z^{k+1}), \tag{23}$$

$$\partial \hat{L}_z(\hat{w}^{k+1}) = \nabla h(z^{k+1}) + C^{\intercal} p^{k+1} + \alpha C^{\intercal} (Ax^{k+1} + By^{k+1} + Cz^{k+1})$$

$$+2\sigma_0(z^{k+1}-z^k),$$
 (24)

$$\partial \hat{L}_{\hat{z}}(\hat{z}^{k+1}) = -\sigma_0(z^{k+1} - z^k), \, \partial \hat{L}_p(\hat{z}^{k+1}) = \frac{1}{\alpha}(p^{k+1} - p^k). \tag{25}$$

Second, we apply Fermat's rule to algorithm (8) to get

$$\begin{split} 0 &\in \partial f(x^{k+1}) + A^{\mathsf{\scriptscriptstyle T}} p^k + \alpha A^{\mathsf{\scriptscriptstyle T}} (A x^{k+1} + B y^k + C z^k) + \nabla \phi(x^{k+1}) - \nabla \phi(x^k), \\ 0 &\in \partial g(y^{k+1}) + B^{\mathsf{\scriptscriptstyle T}} p^k + \alpha B^{\mathsf{\scriptscriptstyle T}} (A x^{k+1} + B y^{k+1} + C z^k) + \nabla \psi(y^{k+1}) - \nabla \psi(y^k), \end{split}$$

Substituting this into (22) and (23), we obtain

$$\begin{split} \partial \hat{L}_{x}(\hat{w}^{k+1}) &\ni \alpha A^{\intercal} B(y^{k+1} - y^{k}) + \alpha A^{\intercal} C(z^{k+1} - z^{k}) \\ &+ \nabla \phi(x^{k}) - \nabla \phi(x^{k+1}) + A^{\intercal} (p^{k+1} - p^{k}), \\ \partial \hat{L}_{y}(\hat{w}^{k+1}) &\ni \alpha B^{\intercal} C(z^{k+1} - z^{k}) + B^{\intercal} (p^{k+1} - p^{k}) \\ &+ \nabla \psi(y^{k}) - \nabla \psi(y^{k+1}). \end{split}$$

We also substitute (17) into (24) to get

$$\partial \hat{L}_{z}(\hat{w}^{k+1}) = \nabla \varphi(z^{k}) - \nabla \varphi(z^{k+1}) + C^{\mathsf{T}}(p^{k+1} - p^{k}) + 2\sigma_{0}(z^{k+1} - z^{k}),$$

where the last equality follows from (8).

As  $\nabla \phi$ ,  $\nabla \psi$ ,  $\nabla \varphi$  are all Lipshitz continuous and matrices A, B, C are all bounded, the above series of estimations show that there exists  $\kappa_0 > 0$  such that

$$\operatorname{dist}(0, \partial \hat{L}(\hat{w}^{k+1})) \le \kappa_0(\|x^k - x^{k+1}\| + \|y^{k+1} - y^k\| + \|z^{k+1} - z^k\| + \|p^{k+1} - p^k\|). \tag{26}$$

On the other hand, it follows from Lemma 3.1 that

$$||p^{k+1} - p^k|| \le \frac{\sqrt{2}(\ell_h + \ell_\varphi)}{\sqrt{\sigma_C}} ||z^{k+1} - z^k|| + \frac{\sqrt{2}\ell_\varphi}{\sqrt{\sigma_C}} ||z^k - z^{k-1}||$$
 (27)

$$\leq \frac{\sqrt{2}(\ell_h + \ell_{\varphi})}{\sqrt{\sigma_C}}(\|z^{k+1} - z^k\| + \|z^k - z^{k-1}\|). \tag{28}$$

Letting  $\kappa_1 := \sqrt{2}(\ell_h + \ell_\varphi)/\sqrt{\sigma_C}$ , we then have

$$||p^{k+1} - p^k|| \le \kappa_1(||z^{k+1} - z^k|| + ||z^k - z^{k-1}||).$$
(29)

Let  $\kappa := (\kappa_1 + 1)(\kappa_0 + 1)$ . Hence Lemma 3.5 follows immediately.

### 3.2. Convergence analysis.

**Theorem 3.6.** Under the Assumptions 1 and 2, if the sequence  $\{u^k\}$  is bounded, then

$$\sum_{k=0}^{\infty} \| w^k - w^{k+1} \|_1 < \infty.$$

In particular, the sequence  $\{w^k\}$  converges to a stationary point of  $L_\alpha$  defined as in (6).

*Proof.* From the proof of Lemma 3.4, we see that the sequence  $\{\hat{w}^k\}$  is bounded. Let  $\Omega$  stand for the cluster point set of  $\hat{w}^k$ . Take any  $\hat{w}^* \in \Omega$  and let  $\hat{w}^{k_j}$  be a subsequence of  $\hat{w}^k$  converging to  $\hat{w}^*$ . Since by Lemma 3.3 the sequence  $\hat{L}(\hat{w}^k)$  is convergent, it follows that

$$\hat{L}(\hat{w}^*) = \lim_{j \to \infty} \hat{L}(\hat{w}^{k_j}) = \lim_{k \to \infty} \hat{L}(\hat{w}^k),$$

so that the function  $\hat{L}(\cdot)$  is a constant on  $\Omega$ .

Let us now consider two possible cases on  $\hat{L}(\hat{w}^k)$ . First assume that there exists  $k_0 \in \mathbb{N}$  such that  $\hat{L}(\hat{w}^{k_0}) = \hat{L}(\hat{w}^*)$ . Then we deduce from Lemma 3.3 that for any  $k > k_0$ 

$$\sigma_1 \| \boldsymbol{w}^{k+1} - \boldsymbol{w}^k \|^2 \leq \hat{L}(\hat{\boldsymbol{w}}^k) - \hat{L}(\hat{\boldsymbol{w}}^{k+1}) \leq \hat{L}(\hat{\boldsymbol{w}}^{k_0}) - \hat{L}(\hat{\boldsymbol{w}}^*) = 0,$$

where we have used the fact that  $\hat{L}(\hat{w}^k)$  is nonincreasing. This together with (26) implies that  $(w^k)$  is a constant sequence except for finite terms, and thus the proof is finished in this case.

Let us now assume that  $\hat{L}(\hat{w}^k) > \hat{L}(\hat{w}^*)$  for each  $k \in \mathbb{N}$ . By Assumption 1, It is easy to know that  $\hat{L}(\cdot)$  is a subanalytic function and thus satisfies the K-L inequality. Then by Lemma 2.2 there exists  $\eta > 0, \delta > 0, \varphi \in \mathscr{A}_{\eta}$ , such that

$$\varphi'(\hat{L}(\hat{w}) - \hat{L}(\hat{w}^*)) \operatorname{dist}(0, \partial \hat{L}(\hat{w})) \ge 1.$$

for all  $\hat{w}$  satisfying  $\operatorname{dist}(\hat{w}, \Omega) < \delta$  and  $\hat{L}(\hat{w}^*) < \hat{L}(\hat{w}) < \hat{L}(\hat{w}^*) + \eta$ . By definition of  $\Omega$  we have  $\lim_k \operatorname{dist}(\hat{w}^k, \Omega) = 0$ . This together with the fact  $\hat{L}(\hat{w}^k) \to \hat{L}(\hat{w}^*)$  implies that there exists  $k_1 \in \mathbb{N}$  such that  $\operatorname{dist}(\hat{w}^k, \Omega) < \delta$  and  $\hat{L}(\hat{w}^k) < \hat{L}(\hat{w}^*) + \eta$  for all  $k \geq k_1$ .

Let us fix  $k > k_1$  in the following. Then the K-L inequality

$$\operatorname{dist}(0, \partial \hat{L}(\hat{w}^k))\varphi \prime (\hat{L}(\hat{w}^k) - \hat{L}(\hat{w}^*)) \ge 1$$

holds, which along with Lemma 3.5 then yields

$$\frac{1}{\varphi \prime (\hat{L}(\hat{w}^k) - \hat{L}(\hat{w}^*))} \le \operatorname{dist}(0, \partial \hat{L}(\hat{w}^{k+1})) 
\le \kappa (\|x^k - x^{k-1}\| + \|y^k - y^{k-1}\| + \|z^k - z^{k-1}\| + \|z^{k-2} - z^{k-1}\|).$$

By Lemma 3.2, the last inequality and the concavity of  $\varphi$  show

$$\begin{split} &\sigma_{1}||w^{k+1}-w^{k}||^{2} \leq \hat{L}(\hat{w}^{k}) - \hat{L}(\hat{w}^{k+1}) \\ &= (\hat{L}(\hat{w}^{k}) - \hat{L}(\hat{w}^{*})) - (\hat{L}(\hat{w}^{k+1}) - \hat{L}(\hat{w}^{*})) \\ &\leq \frac{\varphi(\hat{L}(\hat{w}^{k}) - \hat{L}(\hat{w}^{*})) - \varphi(\hat{L}(\hat{w}^{k+1}) - \hat{L}(\hat{w}^{*}))}{\varphi'(\hat{L}(\hat{w}^{k}) - \hat{L}(\hat{w}^{*}))} \\ &\leq \kappa(||x^{k} - x^{k-1}|| + ||y^{k} - y^{k-1}|| + ||z^{k} - z^{k-1}|| + ||z^{k-2} - z^{k-1}||) \\ &\times [\varphi(\hat{L}(\hat{w}^{k}) - \hat{L}(\hat{w}^{*})) - \varphi(\hat{L}(\hat{w}^{k+1}) - \hat{L}(\hat{w}^{*}))], \end{split}$$

or, equivalently,

$$\begin{split} & \|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2 + \|z^{k+1} - z^k\|^2 \\ & \leq \frac{\kappa}{\sigma_1} (\|x^k - x^{k-1}\| + \|y^k - y^{k-1}\| + \|z^k - z^{k-1}\| + \|z^{k-2} - z^{k-1}\|) \\ & \times [\varphi(\hat{L}(\hat{w}^k) - \hat{L}(\hat{w}^*)) - \varphi(\hat{L}(\hat{w}^{k+1}) - \hat{L}(\hat{w}^*))]. \end{split}$$

We thus have

$$3(\|x^{k} - x^{k+1}\| + \|y^{k} - y^{k+1}\| + \|z^{k+1} - z^{k}\|)$$

$$\leq 3\sqrt{3}(\|x^{k+1} - x^{k}\|^{2} + \|y^{k+1} - y^{k}\|^{2} + \|z^{k+1} - z^{k}\|^{2})^{1/2}$$

$$\leq 2(\|x^{k} - x^{k-1}\| + \|y^{k} - y^{k-1}\| + \|z^{k} - z^{k-1}\| + \|z^{k-2} - z^{k-1}\|)^{1/2}$$

$$\times \sqrt{\frac{27\kappa}{4\sigma_{1}}} [\varphi(\hat{L}(\hat{w}^{k}) - \hat{L}(\hat{w}^{*})) - \varphi(\hat{L}(\hat{w}^{k+1}) - \hat{L}(\hat{w}^{*}))]^{1/2}.$$
(30)

On the other hand, we observe that

$$\begin{split} &2(\|\boldsymbol{x}^{k}-\boldsymbol{x}^{k-1}\|+\|\boldsymbol{y}^{k}-\boldsymbol{y}^{k-1}\|+\|\boldsymbol{z}^{k}-\boldsymbol{z}^{k-1}\|+\|\boldsymbol{z}^{k-2}-\boldsymbol{z}^{k-1}\|)^{1/2}\\ &\times\sqrt{\frac{27\kappa}{4\sigma_{1}}}[\varphi(\hat{L}(\hat{w}^{k})-\hat{L}(\hat{w}^{*}))-\varphi(\hat{L}(\hat{w}^{k+1})-\hat{L}(\hat{w}^{*}))]^{1/2}\\ &\leq\|\boldsymbol{x}^{k}-\boldsymbol{x}^{k-1}\|+\|\boldsymbol{y}^{k}-\boldsymbol{y}^{k-1}\|+\|\boldsymbol{z}^{k}-\boldsymbol{z}^{k-1}\|+\|\boldsymbol{z}^{k-2}-\boldsymbol{z}^{k-1}\|\\ &+\frac{27\kappa}{4\sigma_{1}}[\varphi(\hat{L}(\hat{w}^{k})-\hat{L}(\hat{w}^{*}))-\varphi(\hat{L}(\hat{w}^{k+1})-\hat{L}(\hat{w}^{*}))], \end{split}$$

which along with (30) yields

$$3(||x^k - x^{k+1}|| + ||y^k - y^{k+1}|| + ||z^{k+1} - z^k||)$$

$$\leq \|x^{k} - x^{k-1}\| + \|y^{k} - y^{k-1}\| + \|z^{k} - z^{k-1}\| + \|z^{k-2} - z^{k-1}\| + \frac{27\kappa}{4\sigma_{1}} [\varphi(\hat{L}(\hat{w}^{k}) - \hat{L}(\hat{w}^{*})) - \varphi(\hat{L}(\hat{w}^{k+1}) - \hat{L}(\hat{w}^{*}))].$$

Hence we have

$$\begin{split} & \sum_{i=k_1}^k 3(\|x^i-x^{i+1}\|+\|y^i-y^{i+1}\|+\|z^i-z^{i+1}\|) \\ & \leq \sum_{i=k_1}^k (\|x^i-x^{i-1}\|+\|y^i-y^{i-1}\|+\|z^i-z^{i-1}\|+\|z^{i-1}-z^{i-2}\|) \\ & + \frac{27\kappa}{4\sigma_1} \sum_{i=k_1}^k [\varphi(\hat{L}(\hat{w}^i)-\hat{L}(\hat{w}^*))-\varphi(\hat{L}(\hat{w}^{i+1})-\hat{L}(\hat{w}^*))]. \end{split}$$

Rearranging terms in the above inequality, we obtain

$$\begin{split} &2\sum_{i=k_1}^k \|x^i-x^{i+1}\| + 2\sum_{i=k_1}^k \|y^i-y^{i+1}\| + \sum_{i=k_1}^k \|z^i-z^{i+1}\| \\ &\leq \sum_{i=k_1}^k (\|x^i-x^{i-1}\| - \|x^i-x^{i+1}\|) \\ &+ \sum_{i=k_1}^k (\|y^i-y^{i-1}\| - \|y^i-y^{i+1}\|) \\ &+ \sum_{i=k_1}^k (\|z^i-z^{i-1}\| - \|z^i-z^{i+1}\|) \\ &+ \sum_{i=k_1}^k (\|z^{i-1}-z^{i-2}\| - \|z^i-z^{i+1}\|) \\ &+ \frac{27\kappa}{4\sigma_1} \sum_{i=k_1}^k [\varphi(\hat{L}(\hat{w}^i) - \hat{L}(\hat{w}^*)) - \varphi(\hat{L}(\hat{w}^{i+1}) - \hat{L}(\hat{w}^*))] \\ &= \|x^{k_1-1}-x^{k_1}\| - \|x^k-x^{k+1}\| + \|y^{k_1-1}-y^{k_1}\| - \|y^k-y^{k+1}\| \\ &+ \|z^{k_1-1}-z^{k_1-2}\| + 2\|z^{k_1}-z^{k_1-1}\| - \|z^k-z^{k-1}\| - 2\|z^k-z^{k+1}\| \\ &+ \frac{27\kappa}{4\sigma_1} [\varphi(\hat{L}(\hat{w}^{k_1}) - \hat{L}(\hat{w}^*)) - \varphi(\hat{L}(\hat{w}^{k+1}) - \hat{L}(\hat{w}^*))] \\ &\leq \|x^{k_1-1}-x^{k_1}\| + \|y^{k_1-1}-y^{k_1}\| + \|z^{k_1-1}-z^{k_1-2}\| \\ &+ 2\|z^{k_1}-z^{k_1-1}\| + \frac{27\kappa}{4\sigma_1} \varphi(\hat{L}(\hat{w}^0) - \hat{L}(\hat{w}^*)) \end{split}$$

where the last inequality follows from the fact that  $\varphi(\hat{L}(\hat{w}^{k+1}) - \hat{L}(\hat{w}^*)) \ge 0$ . Since k is chosen arbitrarily, we deduce that  $\sum_{k=0}^{\infty} (\|x^k - x^{k+1}\| + \|y^k - y^{k+1}\| + \|z^k - z^{k+1}\|) < \infty$ . By inequality

- (29), it then implies that  $\sum_{k=0}^{\infty} \|p^k p^{k+1}\| < \infty$ , from which  $\sum_{k=0}^{\infty} \|w^k w^{k+1}\| < \infty$  follows. Consequently  $\{w^k\}$  is a convergent sequence. This completes the proof of Theorem 3.6.
- 3.3. **Boundedness.** In the previous theorem, we have assumed the boundedness of the sequence  $\{u_k\}$ . This assumption is not restrictive in general. There are actually various sufficient conditions ensuring the boundedness of the sequence  $\{u_k\}$ . We present such a sufficient condition below.

**Theorem 3.7.** If (a1)-(a3) in Assumption 1 hold and the following (b1)-(b4) are satisfied:

- (b1) inf  $f(x) = f^* > -\infty$ , inf  $g(y) = g^* > -\infty$  and there exists  $\beta_0 > 0$  such that  $\inf\{h(z) \beta_0 ||\nabla h(z)||^2\} = h^* > -\infty$ ;
- (b2) f(x) + g(y) is coercive, namely,  $\lim_{\min(||x||, ||y||) \to \infty} f(x) + g(y) = +\infty$ ;
- (b3) either  $h(z) \beta_0 ||\nabla h(z)||^2$  is coercive or C is square;
- (b4)  $\alpha > \alpha_0$  where,

$$\alpha_0 = \left\{ \begin{array}{ll} \max\left(\frac{2}{\beta_0\sigma_C},\frac{4[(\ell_h + \ell_\varphi)^2 + \ell_\varphi^2]}{\mu_\varphi\sigma_C}\right), & if\ h(z) - \beta_0 \|\nabla h(z)\|^2\ is\ coercive\\ \|C^{-1}\|^2\max\left(\ell_h,\frac{4[(\ell_h + \ell_\varphi)^2 + \ell_\varphi^2]}{\mu_\varphi}\right), & if\ C\ is\ square; \end{array} \right.$$

then the sequence  $\{u^k\}$  is bounded.

*Proof.* First we deduce from Eq. (21) that

$$\frac{1}{\alpha} ||p^k||^2 \le \frac{2}{\alpha \sigma_C} ||\nabla h(z^k)||^2 + \sigma_0 ||z^k - z^{k-1}||^2,$$

which together with the definition of  $\hat{L}$  gets

$$\hat{L}(\hat{w}^{k}) = f(x^{k}) + g(y^{k}) + h(z^{k}) - \frac{1}{\alpha} \|p^{k}\|^{2} + \sigma_{0} \|z^{k} - z^{k-1}\|^{2} + \frac{\alpha}{2} \|Ax^{k} + By^{k} + Cz^{k} + \frac{p^{k}}{\alpha} \|^{2}$$

$$\geq f(x^{k}) + g(y^{k}) + h(z^{k}) - \frac{2}{\alpha \sigma_{C}} \|\nabla h(z^{k})\|^{2} + \frac{\alpha}{2} \|Ax^{k} + By^{k} + Cz^{k} + \frac{p^{k}}{\alpha} \|^{2}$$

$$\geq f(x^{k}) + g(y^{k}) + h(z^{k}) - \beta_{0} \|\nabla h(z^{k})\|^{2} + \frac{\alpha}{2} \|Ax^{k} + By^{k} + Cz^{k} + \frac{p^{k}}{\alpha} \|^{2}$$

where  $\beta_0$  is any constant such that  $\inf\{h(z) - \beta_0 || \nabla h(z)||^2\} > -\infty$  and  $h(z) - \beta_0 || \nabla h(z)||^2$  keeps coercive no matter whether C is regular or not. Thus from the monotonically decreasing property of  $\{\hat{L}(\hat{w}^k)\}$ , we obtain

$$\hat{L}(\hat{w}^1) \ge f(x^k) + g(y^k) + h(z^k) - \beta_0 ||\nabla h(z^k)||^2,$$

which then implies

$$f(x^k) + g(y^k) \le \hat{L}(\hat{w}^1) - h^* < \infty$$

and

$$h(z^k) - \beta_0 ||\nabla h(z^k)||^2 \le \hat{L}(\hat{w}^1) - f^* - g^* < \infty.$$

By condition (b2), this yields the boundedness of  $\{x^k\}$  and  $\{y^k\}$ , and the boundedness of  $\{z^k\}$  as well whenever  $h(z) - \beta_0 ||\nabla h(z)||^2$  is coercive.

Similarly, from Lemma 3.3, we can obtain

$$\sigma_1 \| z^k - z^{k-1} \|^2 \le \hat{L}(\hat{w}^1) - (f^* + g^* + h^*) := M_1 < \infty, \tag{31}$$

which shows the boundedness of  $\{||z^k - z^{k-1}||\}$ . Now, let us assume that the function  $h(z) - \beta_0 ||\nabla h(z)||^2$  is not coercive but the matrix C keeps nonsingular. We then justify the boundedness of  $\{z^k\}$  in this case. In effect, by using again Lemma 3.3 and inequality (31), we get

$$||Ax^k + By^k + Cz^k + \frac{p^k}{\alpha}|| \le \sqrt{\frac{M_1\alpha}{2}},$$

and using the inequality

$$||Ax^k + By^k + Cz^k + \frac{p^k}{\alpha}|| \ge ||Cz^k|| - ||Ax^k + By^k|| - \frac{1}{\alpha}||p^k||,$$

we then have

$$||Cz^k|| - \frac{1}{\alpha}||p^k|| \le \sqrt{\frac{M_1\alpha}{2}} + M_2,$$
 (32)

where  $M_2 := \sup ||Ax^k + By^k||$ . It thus follows from Eq. (17) and condition (c3) that

$$\begin{split} \|p^k\| &\leq \|(C^{\mathsf{T}})^{-1}\| \|C^{\mathsf{T}}p^k\| = \|C^{-1}\| \|C^{\mathsf{T}}p^k\| \\ &\leq \|C^{-1}\| \|\nabla h(z^k) + \nabla \varphi(z^k) - \nabla \varphi(z^{k-1})\| \\ &\leq \|C^{-1}\| (\|\nabla h(z^k)\| + \ell_{\varphi}\|z^k - z^{k-1}\|). \end{split}$$

With any fixed  $z^*$ , we clearly have

$$\|\nabla h(z^k)\| = \|\nabla h(z^k) - \nabla h(z^*)\| + \|\nabla h(z^*)\|$$

$$\leq \ell_h \|z^k - z^*\| + \|\nabla h(z^*)\|$$

$$\leq \ell_h (\|z^k\| + \|z^*\|) + \|\nabla h(z^*)\|,$$

and furthermore,

$$||p^k|| \le ||C^{-1}|| \{ \ell_h(||z^k|| + ||z^*||) + ||\nabla h(z^*)|| + \ell_{\varphi}||z^k - z^{k-1}|| \}.$$

Hence we have

$$\begin{split} & \|Cz^k\| - \frac{1}{\alpha} \|p^k\| \ge \frac{1}{\|C^{-1}\|} \|z^k\| - \frac{1}{\alpha} \|p^k\| \\ & \ge \frac{1}{\|C^{-1}\|} \|z^k\| - \frac{\|C^{-1}\|}{\alpha} \left\{ \ell_h(\|z^k\| + \|z^*\|) + \|\nabla h(z^*)\| + \ell_\varphi \|z^k - z^{k-1}\| \right\} \\ & \ge \left( \frac{1}{\|C^{-1}\|} - \frac{\|C^{-1}\|\ell_h}{\alpha} \right) \|z^k\| - \frac{\|C^{-1}\|}{\alpha} (\ell_h \|z^*\| + \|\nabla h(z^*)\|) - \frac{\|C^{-1}\|\ell_\varphi}{\alpha} \|z^k - z^{k-1}\|, \end{split}$$

which together with (32) implies

$$\left(\frac{1}{\|C^{-1}\|} - \frac{\|C^{-1}\|\ell_h}{\alpha}\right) \|z^k\| \le \sqrt{\frac{M_1\alpha}{2}} + M_2 + \frac{\|C^{-1}\|}{\alpha} (\ell_h \|z^*\| + \|\nabla h(z^*)\|) + \frac{\|C^{-1}\|\ell_{\varphi}}{\alpha} \|z^k - z^{k-1}\| \\
\le \sqrt{\frac{M_1\alpha}{2}} + M_2 + \frac{\|C^{-1}\|}{\alpha} \left(\ell_h \|z^*\| + \|\nabla h(z^*)\| + \ell_{\varphi} \sqrt{\frac{M_1}{\sigma_1}}\right),$$

where the last inequality follows from (31). By condition (b4), the sequence  $\{z^k\}$  is then bounded, and so is the sequence  $\{u^k\}$ .

Remark 1. It is easy to see that function  $h(x) = ||Ax - b||^2$  for any matrix A and b satisfies conditions (b1) and (b3) with  $\beta_0 = \frac{\|A\|^2}{4}$ .

#### 3.4. Main results.

Combining theorems 3.6 and 3.7, we present the following convergence theorem for the BADMM with 3-block procedure (8).

**Theorem 3.8.** If Assumption 1 and conditions (b1)-(b4) in Theorem 3.7 are satisfied, then the sequence  $\{w^k\}$  generated by procedure (8) converges to a stationary point of  $L_\alpha$  defined as in (6).

We now extend this result to the N-block case. Thus, let us consider the following composite optimization problem:

$$\min f_1(x_1) + f_2(x_2) + \dots + f_N(x_N)$$
s.t.  $A_1x_1 + A_2x_2 + \dots + A_Nx_N = 0$ , (33)

where  $A_i \in \mathbb{R}^{m \times n_i}$ ,  $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$ ,  $i = 1, 2, \dots, N-1$  are proper lower semicontinuous functions, and  $f_N: \mathbb{R}^{n_N} \to \mathbb{R}$  is a smooth function. The associated BADMM algorithm takes the form:

$$\begin{cases} x_1^{k+1} &= \arg\min_{x_1 \in \mathbb{R}^{n_1}} L_{\alpha}(x_1, x_2^k, \dots, x_N^k, p^k) + \triangle_{\phi_1}(x_1, x_1^k) \\ \vdots &= &\vdots & \vdots \\ x_N^{k+1} &= \arg\min_{x_N \in \mathbb{R}^{n_N}} L_{\alpha}(x_1^{k+1}, \dots, x_{N-1}^{k+1}, x_N, p^k) + \triangle_{\phi_N}(x_N, x_N^k) \\ p^{k+1} &= p^k + \alpha(A_1 x_1^{k+1} + A_2 x_2^{k+1} + \dots + A_N x_N^{k+1}) \end{cases}$$

$$(34)$$

where  $\triangle_{\phi_i}$ ,  $i = 1, 2, \dots, N$  are the Bregman distances associated with functions  $\phi_i$  and the corresponding Lagrangian function  $L_{\alpha}: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_N} \times \mathbb{R}^m \to \mathbb{R}$  is defined by

$$L_{\alpha}(x_1, x_2 \cdots, x_N, p) := \sum_{i=1}^{N} f_i(x_i) + \sum_{i=1}^{N} \langle p, A_i x_i \rangle + \frac{\alpha}{2} \| \sum_{i=1}^{N} A_i x_i \|^2.$$
 (35)

It is then straightforward to establish a similar convergence result with Theorem 3.8.

**Theorem 3.9.** If the following (d1)-(d7) are satisfied:

- (d1)  $\langle A_N A_N^\intercal x, x \rangle = ||x||_{A_N^\intercal}^2 \geq \sigma_{A_N} ||x||^2, \forall x \in \mathbb{R}^{n_N}, namely, A_N \text{ is full row rank;}$ (d2)  $\nabla f_N, \nabla \phi_i, i = 1, 2, \cdots, N \text{ are Lipschitz continuous;}$
- (d3) either  $f_i$  or  $\phi_i$ ,  $i = 1, 2, \dots, N$  is strongly convex;
- (d4)  $f_1 + f_2 + \cdots + f_N$  is subanalytic and coercive;
- (d5) inf  $f_i = f_i^* > -\infty$ ,  $i = 1, 2, \dots, N-1$ , and there exists  $\beta_0 > 0$  such that  $\inf\{f_N(x_N) 1\}$  $\beta_0 ||\nabla f_N(x_N)||^2 \} = f_N^* > -\infty;$
- (d6) either  $f_N \beta_0 ||\nabla f_N||^2$  is coercive, or  $A_N$  is square;

(d7)  $\alpha > \alpha_0$  where,

$$\alpha_0 = \left\{ \begin{array}{ll} \max\left(\frac{2}{\beta_0\sigma_{A_N}},\frac{4[(\ell_{f_N}+\ell_{\phi_N})^2+\ell_{\phi_N}^2]}{\mu_N\sigma_{A_N}}\right), & if \ f_N - \beta_0 ||\nabla f_N||^2 \ is \ coercive, \\ ||A_N^{-1}||^2 \max\left(\ell_{f_N},\frac{4[(\ell_{f_N}+\ell_{\phi_N})^2+\ell_{\phi_N}^2]}{\mu_N}\right), & if \ A_N \ is \ square; \end{array} \right.$$

where  $\mu_N$  is the strong convexity coefficient of  $f_N$  or  $\varphi_N$ , and  $\ell_{f_N}$  and  $\ell_{\phi_N}$  are respectively the Lipschitz coefficient of  $\nabla f_N$  and  $\nabla \phi_N$ ,

then the sequence  $\{x_1^k, x_2^k, \dots, x_N^k, p^k\}$  converges to a stationary point of  $L_\alpha$  defined as in (35).

Remark 2. We notice that whenever any  $f_i$  is strongly convex, the function  $\phi_i$  in the Bregman distance can be taken as zero in the *i*-th update of procedure (34).

*Remark* 3. For convenience of applications, we list some specifications of Theorem 3.9 as follows.

(i) Underdetermined linear system of equations: In this case,  $f_i \equiv 0, i = 1, 2, \dots, N$ , and  $m < \sum_{i=1}^{N} n_i$ . The problem (33) is degenerated to

min 0  
s.t. 
$$A_1x_1 + A_2x_2 + \dots + A_Nx_N = 0$$
 (36)

which amounts to solving the underdetermined linear system of equations:

$$Ax = 0 (37)$$

where  $A = [A_1, A_2, \cdots, A_N]$  and  $x = [x_1^\mathsf{T}, x_2^\mathsf{T}, \cdots, x_N^\mathsf{T}]^\mathsf{T}$ . In this case, the BADMM algorithm takes the form:

$$\begin{cases} x_1^{k+1} &= \arg\min_{x_1 \in \mathbb{R}^{n_1}} \frac{\alpha}{2} ||A_1 x_1 + A_2 x_2^k + \dots + A_N x_N^k + \frac{p^k}{\alpha}||^2 + \Delta_{\phi_1}(x_1, x_1^k) \\ \vdots &= &\vdots & \vdots \\ x_N^{k+1} &= \arg\min_{x_N \in \mathbb{R}^{n_N}} \frac{\alpha}{2} ||A_1 x_1^{k+1} + \dots + A_{N-1} x_{N-1}^{k+1} + A_N x_N + \frac{p^k}{\alpha} ||^2 + \Delta_{\phi_N}(x_N, x_N^k) \\ p^{k+1} &= p^k + \alpha (A_1 x_1^{k+1} + A_2 x_2^{k+1} + \dots + A_N x_N^{k+1}). \end{cases}$$

$$(38)$$

We easily check that in this special case all the assumptions in Theorem 3.9 are met whenever the matrix  $A_N$  is nonsingular. So, by Theorem 3.9, the procedure (38) can converge to a point  $(x_1^*, x_2^*, \dots, x_N^*, p^*)$ . The point  $(x_1^*, x_2^*, \dots, x_N^*)$  is clearly a solution of (37) by the last equation in (38). We notice that the same problem was studied by Sun, Luo and Ye [41], and they considered the case that A is a square nonsingular matrix. To solve the linear system of equations, they suggested a novel randomly permuted ADMM and proved its expected convergence.

(ii) Two blocks case: N = 2. It is easily seen that Theorem 3.9 in this case is degenerated to convergence of the conventional BADMM procedure:

$$\begin{cases} x_1^{k+1} &= \arg\min_{x_1 \in \mathbb{R}^{n_1}} L_{\alpha}(x_1, x_2^k, p^k) + \triangle_{\phi_1}(x_1, x_1^k) \\ x_2^{k+1} &= \arg\min_{x_2 \in \mathbb{R}^{n_2}} L_{\alpha}(x_1^{k+1}, x_2, p^k) + \triangle_{\phi_2}(x_2, x_2^k) \\ p^{k+1} &= p^k + \alpha(A_1 x_1^{k+1} + A_2 x_2^{k+1}) \end{cases}$$
(39)

for the problem:

$$\min f_1(x_1) + f_2(x_2)$$
s.t.  $A_1x_1 + A_2x_2 = 0$ . (40)

Thus, Theorem 3.9 includes the results established in [31, 44] as special cases.

(iii) The unconstrained minimization case:

$$\min f_1(x_1) + f_2(x_2) + \dots + f_N(x_N) \tag{41}$$

where  $f_i: \mathbb{R}^{n_i} \to \mathbb{R}, i = 1, 2, \dots, N-1$  are proper lower semicontinuous functions, and  $f_N: \mathbb{R}^{n_N} \to \mathbb{R}$  is a smooth function. Even no constraint exists in this case, a similar Bregman alternative direction method (BADM) can be defined as follows:

$$\begin{cases} x_1^{k+1} &= \arg\min_{x_1 \in \mathbb{R}^{n_1}} f_1(x_1) + \triangle_{\phi_1}(x_1, x_1^k) \\ \vdots &= &\vdots & \vdots \\ x_N^{k+1} &= \arg\min_{x_N \in \mathbb{R}^{n_N}} f_N(x_N) + \triangle_{\phi_N}(x_N, x_N^k). \end{cases}$$
(42)

Following exactly the procedure of proof of Theorems 3.6 and 3.7, we can immediately obtain the following convergence of (42) in the setting that:

- (e1) inf  $f_i = f_i^* > -\infty, i = 1, 2, \dots, N$ ;
- (e2)  $\nabla f_N$ ,  $\nabla \phi_i$ ,  $i = 1, 2, \dots, N$  are Lipschitz continuous;
- (e3) either  $f_i$  or  $\phi_i$ ,  $i = 1, 2, \dots, N$  is strongly convex;
- (e4)  $f_1 + f_2 + \cdots + f_N$  is subanalytic and coercive.

#### 4. Demonstration examples

In this section, a simulated example and a real-world application are provided to support the correctness of convergence of the proposed 3-block Bregman ADMM for solving non-convex composite problems.

Consider the non-convex optimization problem with 3-block variables deduced from matrix decomposition applications (see e.g. [3, 46, 57]):

$$\min_{\mathbf{L}, \mathbf{S}, \mathbf{T}} \|\mathbf{L}\|_* + \lambda \|\mathbf{S}\|_{1/2}^{1/2} + \frac{\mu}{2} \|\mathbf{T} - \mathbf{M}\|_F^2$$

$$s.t. \mathbf{T} = \mathbf{L} + \mathbf{S}, \tag{43}$$

where **M**, **T**, **L** and **S** are all  $m \times n$  matrices, **M** is a given observation, **T** is an ideal observation,  $\|\mathbf{L}\|_* := \sum_{i=1}^{\min(m,n)} \sigma_i(\mathbf{L})$  is the nuclear norm of **L**,  $\|\mathbf{S}\|_{1/2}^{1/2} := \sum_{i=1}^m \sum_{j=1}^n |\mathbf{S}_{ij}|^{1/2}$  is the  $\ell_{1/2}$  quasinorm of **S**,  $\lambda$  is a trade-off parameter between the spectral sparsity term  $\|\mathbf{L}\|_*$  and the elementwise sparsity term  $\|\mathbf{S}\|_{1/2}^{1/2}$ , and  $\mu$  is a parameter associated with the noise level. The augmented Lagrange function of this optimization problem is given by

$$L_{\alpha}(\mathbf{L}, \mathbf{S}, \mathbf{T}, \boldsymbol{\Lambda}) = \|\mathbf{L}\|_{*} + \lambda \|\mathbf{S}\|_{1/2}^{1/2} + \frac{\mu}{2} \|\mathbf{T} - \mathbf{M}\|_{F}^{2} + \langle \mathbf{p}, \mathbf{T} - (\mathbf{L} + \mathbf{S}) \rangle + \frac{\alpha}{2} \|\mathbf{T} - (\mathbf{L} + \mathbf{S})\|_{F}^{2}. \tag{44}$$

According to the 3-block BADMM (8), the optimization problem (43) can be solved by the following procedure

$$\begin{cases} \mathbf{L}^{k+1} = \arg\min_{\mathbf{L}} L_{\alpha}(\mathbf{L}, \mathbf{S}^{k}, \mathbf{T}^{k}, \boldsymbol{\Lambda}^{k}) + \triangle_{\phi}(\mathbf{L}, \mathbf{L}^{k}) \\ \mathbf{S}^{k+1} = \arg\min_{\mathbf{S}} L_{\alpha}(\mathbf{L}^{k+1}, \mathbf{S}, \mathbf{T}^{k}, \boldsymbol{\Lambda}^{k}) + \triangle_{\psi}(\mathbf{S}, \mathbf{S}^{k}) \\ \mathbf{T}^{k+1} = \arg\min_{\mathbf{T}} L_{\alpha}(\mathbf{L}^{k+1}, \mathbf{S}^{k+1}, \mathbf{T}, \boldsymbol{\Lambda}^{k}) + \triangle_{\varphi}(\mathbf{T}, \mathbf{T}^{k}) \\ \mathbf{p}^{k+1} = \mathbf{p}^{k} + \alpha(\mathbf{T}^{k+1} - (\mathbf{L}^{k+1} + \mathbf{S}^{k+1})). \end{cases}$$

$$(45)$$

Specifying  $\phi(\cdot) = \psi(\cdot) = \frac{\gamma_1}{2} ||\cdot||^2$ ,  $\varphi(\cdot) = \frac{\gamma_2}{2} ||\cdot||^2$  and substituting these formulations into the procedure (45), we then obtain the following closed-form iterative formulas of (45):

$$\begin{cases}
\mathbf{L}^{k+1} = \mathcal{S}_{M}(\frac{\alpha(\mathbf{T}^{k} - \mathbf{S}^{k} + \frac{\mathbf{p}^{k}}{\alpha \gamma_{1}}) + \gamma_{1} \mathbf{L}^{k}}{\alpha + \gamma_{1}}, \frac{\gamma_{1}}{\alpha + \gamma_{1}}) \\
\mathbf{S}^{k+1} = \mathcal{H}_{E}(\frac{\alpha(\mathbf{T}^{k} - \mathbf{L}^{k+1} + \frac{\mathbf{p}^{k}}{\alpha}) + \gamma_{1} \mathbf{S}^{k}}{\alpha + \gamma_{1}}, \frac{\lambda}{\alpha + \gamma_{1}}) \\
\mathbf{T}^{k+1} = \frac{\mu \mathbf{M} + \alpha(\mathbf{L}^{k+1} + \mathbf{S}^{k+1} - \frac{\mathbf{p}^{k}}{\alpha}) + \gamma_{2} \mathbf{T}^{k}}{\mu + \alpha + \gamma_{2}} \\
\mathbf{p}^{k+1} = \mathbf{p}^{k} + \alpha(\mathbf{T}^{k+1} - (\mathbf{L}^{k+1} + \mathbf{S}^{k+1}))
\end{cases} (46)$$

where  $S_M(\mathbf{A}, \cdot)$  indicates the operation of thresholding the singular values of matrix  $\mathbf{A}$  using the well-known soft shrinkage operator, and  $\mathcal{H}_E(\mathbf{A}, \cdot)$  the operation of thresholding the entries of matrix  $\mathbf{A}$  using the half shrinkage operator [49, 50, 51, 52]. The procedure (46) is the specification of BADMM (8) for the solution of problem (43) with functions f(x), g(y), h(z) defined by  $f(\mathbf{L}) = ||\mathbf{L}||_*$ ,  $g(\mathbf{S}) = \lambda ||\mathbf{S}||_{1/2}^{1/2}$ ,  $h(\mathbf{T}) = \frac{\mu}{2} ||\mathbf{T} - \mathbf{M}||^2$  and matrices A, B, C defined by A = I, B = -I, C = -I where I is the identity matrix. It is direct to see that all the assumptions of Theorem 3.8 are satisfied. Consequently, Theorem 3.8 can be applied to predict convergence of (46) in theory. We conduct a simulation study and an application example below for support of such theoretical assertion.

We first expatiate some implementation issues. We set  $\gamma_1 = \alpha$  and  $\gamma_2 = \alpha + \mu$  in (46). In order to avoid the tediousness of tuning the parameter  $\alpha$ , we exploit a dynamic updating scheme, e.g.,  $\alpha = \min(\alpha * 1.1, \alpha_{max})$ , where  $\alpha_{max}$  is a very large constant. Due to the nonconvexity of this optimization problem it is very important to choose a suitable initialization. In the following experiments, we initialized matrix **L** by the best rank r approximation of matrix **M**, i.e.,  $\mathbf{L} = \text{SVD}(\mathbf{M}, r)$ , where r was empirically set as  $\text{ceil}(0.01 \cdot \min(m, n))$ ; initialized matrix **S** as one zero matrix of size  $m \times n$ ; and then initialized matrix  $\mathbf{T} = \mathbf{L} + \mathbf{S}$ . Finally, we terminated the algorithm by the criterion relChg < 1e-8, where relChg is defined as

$$\text{relChg} := \frac{\|[\mathbf{L}^{k+1} - \mathbf{L}^k, \mathbf{S}^{k+1} - \mathbf{S}^k, \mathbf{T}^{k+1} - \mathbf{T}^k]\|_F}{\|[\mathbf{L}^k, \mathbf{S}^k, \mathbf{T}^k]\|_F + 1}.$$

(a) **Simulation study.** To check the validity of model (43) and the convergence of procedure (46), we generated an observation matrix  $\mathbf{M}$  from given  $\mathbf{L}$  and  $\mathbf{S}$  (namely, the true solution) with Gaussian random disturbance  $\mathbf{N}$ , and then we applied procedure (46) to recover  $\mathbf{L}$  and  $\mathbf{S}$ . The square matrices of size  $m \times m$  are randomly generated for our simulations. The matrix  $\mathbf{L}$  was taken as  $\mathbf{U}\mathbf{V}^T$ , where  $\mathbf{U}$  and  $\mathbf{V}$  are independent  $m \times r$  matrices whose elements are i.i.d.

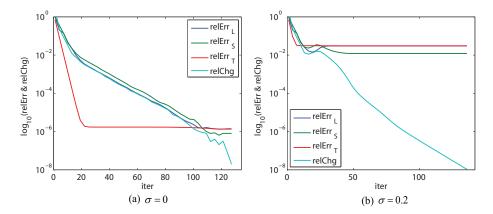


Figure 1: Separation results in simulated data.

Gaussian random variables with zero mean and unit variance, and **S** taken as a sparse matrix whose support was chosen uniformly at random with the entries uniformly specified in the interval [-50, 50]. Then, the measurement **M** was generated as  $\mathbf{M} = \mathbf{L} + \mathbf{S} + \mathbf{N}$ , where matrix **N** is Gaussian noise with mean zero and variance  $\sigma^2$ . Thus,  $\sigma = 0$  corresponds to the no noise case and  $\sigma \neq 0$  corresponds to the noisy case. In simulations, the parameter  $\mu$  in model (43) was set as a large value 1e+4 in the no noise setting, and a value in the noisy setting from a candidate set such that the proposed algorithm has the best performance. The parameter  $\lambda$  was empirically set as the value  $\frac{60}{\max(m,n)}$ . The performance of the algorithm is then measured in terms of the relative error defined by

$$relErr_{A} := \frac{\|\hat{\mathbf{A}} - \mathbf{A}^*\|_F}{\|\mathbf{A}^*\|_F},$$

where  $\hat{\mathbf{A}}$  indicates the recovery result of the algorithm, and  $\mathbf{A}^*$  indicates the true result.

With the above settings and measure, our simulation results are then shown in Figure 1. In Figure 1(a), they are exhibited the curves of the relative error relErr<sub>A</sub> (A := L, S, T) and the relative change relChg with respect to the iterative steps when no Gaussian noise is added, and in Figure 1(b) the curves when Gaussian noise is added with mean 0 and variance  $\sigma^2 = 0.2^2$ . From these curves, it can be seen that under the initialization in terms of the relative error and the relative change the procedure (46) does converge, as predicted.

(b) **An application example.** We further applied the model (43) with BADMM (46) to the background subtraction application. Background subtraction [6] is a fundamental task in the field of video surveillance. Its aim is to subtract the background from a video clip and meanwhile detect the anomalies (i.e., moving objects). From the webpage  $^2$ , we first download four video clips: Lobby, Bootstrap, Hall, and ShoppingMall. Then we chose 600 frames from each video clip and input these 600 frames into our algorithm. The parameter  $\lambda$  was set as the value  $\frac{50}{\max(m,n)}$ . In Figure 2, we exhibit the separation results of some frames in four video clips. From Figure 2, it can be seen that our algorithm can produce a clean video background and meanwhile detect

<sup>&</sup>lt;sup>2</sup>http://perception.i2r.a-star.edu.sg/bk\_model/bk\_index

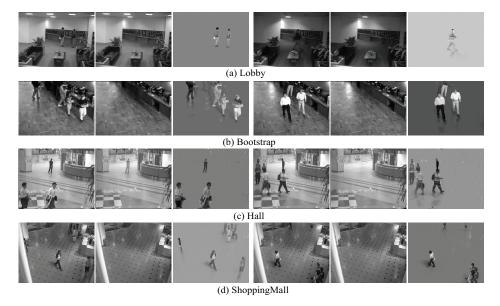


Figure 2: Separation results in real-world video clips.

a satisfactory video foreground, which supports the validity and convergence of the proposed BADMM.

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