A Correlation-Breaking Interleaving of Polar Codes in Concatenated Systems

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Abstract—It is known that the bit errors of polar codes with successive cancellation (SC) decoding are coupled. However, existing concatenation schemes of polar codes with other error correction codes rarely take this coupling effect into consideration. To achieve a better error performance of concatenated systems with polar codes as inner codes, one can divide all bits in an outer block into different polar blocks to completely decorrelate the possible coupled errors in the transmitter side. We call this interleaving a blind interleaving (BI) which serves as a benchmark. Two BI schemes, termed BI-DP and BI-CDP, are proposed in the paper. To better balance performance, memory size, and the decoding delay from the de-interleaving, a novel interleaving scheme, named the correlation-breaking interleaving (CBI), is proposed. The CBI breaks the correlated information bits based on the error correlation pattern proposed and proven in this paper. The proposed CBI scheme is general in the sense that any error correction code can serve as the outer code. In this paper, Low-Density Parity-Check (LDPC) codes and BCH codes are used as two examples of the outer codes of the interleaving scheme. The CBI scheme 1) can keep the simple SC polar decoding while achieving a better error performance than the state-of-the-art (SOA) direct concatenation of polar codes with LDPC codes and BCH codes; 2) achieves a comparable error performance as the BI-DP scheme with a smaller memory size and a shorter decoding delay. Numerical results are provided to verify the performance of the BI schemes and the CBI scheme.

Keywords—Polar codes, SC decoding, BP decoding, interleaving, code concatenation

I. INTRODUCTION

THE channel polarization and polar codes were discovered by Arıkan in [1] which made a great progress in coding theory. Polar codes provably achieve the capacity of symmetric binary-input discrete memoryless channels (B-DMCs) with a low encoding and decoding complexity. The encoding and decoding process (with successive cancellation, SC) can be implemented with a complexity of $\mathcal{O}(N\log N)$, where N is the block length. The idea of polar codes is to transmit information bits on noiseless bit channels while fixing the information bits on the completely noisy bit channels. The

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fixed bits (also called as the frozen bits) are made known to both the transmitter and receiver. The standard format of polar codes in [1] is non-systematic. Later, the systematic version of polar codes was proposed in [2]. The construction of polar codes is studied in [3–6] and the hardware implementation is presented in [7–9].

To improve the polar code performance with the finite block length, various decoding processes [10–13] and concatenation schemes [14–18] were proposed. The decoding processes in these works have higher complexity than the original SC decoding of [1]. The performance improvements in these decoding algorithms are at the cost of the decoding complexity. The introduction of the systematic polar codes [2] provides a new way to improve bit error rate (BER) performance while still maintaining almost the same decoding complexity as non-systematic polar codes.

The spreading effect of the error bit on the following decoding steps results in the known error propagation problem. The better BER performance of systematic polar codes can be thought of coming from the error-decoupling. The non-systematic encoding is $x_1^N = u_1^N G$, where the vector u_1^N contains the source bits and G is the generator matrix. From the two-step decoding of systematic polar codes (first estimating \hat{u}_1^N and then calculating \hat{x}_1^N from it), this decoupling must be accomplished through the re-encoding $\hat{x}_1^N = \hat{u}_1^N G$ after obtaining the estimate \hat{u}_1^N . From $\hat{x}_1^N = \hat{u}_1^N G$ and that the number of errors in \hat{x}_1^N is smaller than that of \hat{u}_1^N , it can be concluded that the coupling of the errors in \hat{u}_1^N is controlled by the columns of G. A proposition of this error correlation pattern is formally stated and proven in this paper.

Two blind interleaving (BI) schemes are presented to decorrelate the coupled errors. A concatenation scheme, which divides all bits in an outer code block into different polar blocks to completely de-correlate the possible coupled errors, is first introduced as a benchmark. Note that this BI scheme is also called a direct product of the inner and outer code, termed as BI-DP in the paper. The BI scheme can keep the simple SC polar decoding while achieving a better BER performance than the state-of-the-art concatenation of polar codes with outer codes. An improved BI scheme, called 'quasi' cyclicly shifted direct product BI (BI-CDP), is introduced to improve the BI-DP scheme. This BI-CDP scheme takes into consideration the different levels of protection experienced by the information bits in one polar block, and assigns the coded bits from the outer code into cyclicly shifted information positions of the inner code. This BI-CDP scheme is shown to yield a better error performance than the BI-DP scheme. Note that the BI-CDP is different from the Twill interleaving in [19] since it does not require the greatest common divisor (gcd) of the number of the encoder for the inner code and the outer code equal to 1. In this paper, the number of the encoder for the inner code and the outer code is the code length of the outer code and the number of the information bits of the inner code, respectively.

From the error correlation pattern presented in the paper, a novel interleaving scheme, named the correlation-breaking interleaving (CBI), is proposed to better balance among performance, memory size, and the decoding delay from the deinterleaving operation. The proposed CBI scheme divides the information bits into two groups: the group of the correlated bits A_c and the group of the uncorrelated bits \bar{A}_c . Theoretical foundation for procedures to assign elements into these two groups is provided. As in the BI scheme, the CBI scheme assigns $|A_c|$ encoded bits from $|A_c|$ different outer code blocks to the correlated information bits of one polar block. Different from the BI scheme, the CBI scheme assigns $|\bar{\mathcal{A}}_c|$ encoded bits from one outer code block to the uncorrelated information bits of one polar block, which saves the required number of inner polar code blocks. As a result, the memory size for the deinterleaver and the decoding delay of the outer code can be saved.

Although any outer code works in the CBI scheme, LDPC and BCH codes are chosen in this paper as examples: the former requiring an iterative soft decoding process while the latter only requiring a simpler syndrome decoder [20]. Note that the concatenation of polar codes with LDPC codes is studied in [14] and [15] where no interleaving is used and BP (belief-propagation) decoding is applied for polar codes. For the ease of description, let us denote polar codes applying SC decoding as POLAR(N,K)-SC, and polar codes applying BP decoding as POLAR(N,K)-BP, where K is the number of information bits of polar codes in one code block. Also let us denote the direct concatenation system with a LDPC code as the outer code and a polar code as the inner code as LDPC (N_l, K_l) +POLAR(N, K), where N_l and K_l are the code length and the number of information bits in one LDPC block, respectively. If a CBI scheme is used between the outer and the inner code, then we denote such a system as LDPC(N_l,K_l)+CBI+POLAR(N,K). Similarly, the blind interleaving systems, BI-DP and BI-CDP, are denoted as $LDPC(N_l, K_l) + BI-DP+POLAR(N, K)$ and $LDPC(N_l, K_l) + BI-DP+POLAR(N, K)$ CDP+POLAR(N,K), respectively.

Simulation results are provided to verify the BER performance of the interleaving schemes in this paper. At a BER = 10^{-4} , the LDPC(155,64)+CBI+POLAR(256,64)-SC system achieves 1.4 dB and 1.2 dB gains over the direct concatenation systems LDPC(155,64)+POLAR(256,64)and LDPC(155,64)+POLAR(256,64)-BP, respectively. SC The LDPC(155,64)+CBI+POLAR(256,64)-SC system also achieves a comparable performance as that of the LDPC(155,64)+BI-DP+POLAR(256,64)-SC system. proposed LDPC(155,64)+BI-CDP+POLAR(256,64)-SC outperforms all the concatenation systems reported. The CBI scheme also works for BCH codes. Here we take the BCH(127,57) with the code length 127 and the number of information bits 57 in one code block as an example. At a

BER = 10^{-4} , the BCH(127,57)+CBI+POLAR(256,64)-SC system has a 0.7 dB gain over the direct concatenation system BCH(127,57)+POLAR(256,64)-SC.

Note that portions of this work are investigated in [21] where the theorems of the error correlation pattern are not proven and the BI scheme is only one of the two BI schemes in this paper. What's more, the CBI scheme in this paper has a different assignment of the $|\mathcal{A}_c|$ correlated information bits from that in [21]. In this paper, we provide the proofs of the theorems, improve the BI scheme and the CBI scheme, and provide examples of the CBI scheme. Specifically, the contribution of this paper can be summarized as: 1) Theoretically, we prove that the errors from the SC decoding are coupled. The error correlation pattern is found and proven from two perspectives; 2) Two BI schemes are introduced and a universal CBI scheme (based on the error correlation pattern) is proposed; 3) The CBI scheme is theoretically explained based on the cyclic arrangements of coded bits from the outer code to the inner code, and details and examples are provided to illustrate the key parameters.

In this paper, we use v_1^N to represent a row vector with elements $(v_1, v_2, ..., v_N)$. For a vector v_1^N , the vector v_i^j is a subvector $(v_i, ..., v_j)$ with $1 \le i, j \le N$. For a given set $A \in \{1, 2, ..., N\}$, v_A denotes a subvector with elements in $\{v_i, i \in A\}$.

The rest of the paper is organized as follows. Section II introduces the fundamentals of non-systematic and systematic polar codes. The error correlation pattern is raised and proven in section III. Section IV introduces the two BI schemes and proposes the novel CBI scheme. Section V presents the simulation results. The conclusion remarks are provided at the end

II. BACKGROUND OF POLAR CODES

In this section, the relevant theories on non-systematic polar codes [1] and systematic polar codes [2] are presented.

A. Preliminaries of Non-Systematic Polar Codes

Let $W: \mathcal{X} \to \mathcal{Y}$ denote a B-DMC where $\mathcal{X} = \{0, 1\}$ is the input and \mathcal{Y} is the output alphabet of the channel. The transition probability is denoted by W(y|x), $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

The generator matrix for polar codes is $G_N = BF^{\otimes n}$ where B is a bit-reversal matrix, $F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $n = \log_2 N$, N is the block length, and $F^{\otimes n}$ is the nth Kronecker power of the matrix F over the binary field \mathbb{F}_2 . In this paper, we consider an encoding matrix $G_N = F^{\otimes n}$ without the permutation matrix B, which only affects the decoding order [2]. For compactness, the subscript of G_N is sometimes omitted as G without causing confusion of the block length N.

The channel polarization process is performed as follows. The $N=2^n (n \geq 1)$ independent copies of W are first combined and then split into N bit channels $\{W_N^{(i)}\}_{i=1}^N$ with:

$$W_N^{(i)}(y_1^N, u_1^{i-1}|u_i) = \sum_{u_{i+1}^N \in \mathcal{X}^{N-i}} \frac{1}{2^{N-1}} W_N(y_1^N|u_1), \quad (1)$$

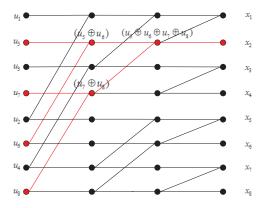


Fig. 1. An encoding circuit of the non-systematic polar codes with N=8. Signals flow from the left to the right. Each edge carries a signal of 0 or 1.

where

$$W_N(y_1^N|u_1^N) = W^N(y_1^N|u_1^NG_N) = \prod_{i=1}^N W(y_i|x_i).$$
 (2)

Mathematically, the encoding is a process to obtain the codeword x_1^N through $x_1^N = u_1^N G$ for given source bits u_1^N . The source bits u_1^N consists of the information bits and the frozen bits, denoted by $u_{\mathcal{A}}$ and $u_{\bar{\mathcal{A}}}$, respectively. Frozen bits refer to the fixed transmission bits which are known to both the transmitter and the receiver. The set \mathcal{A} includes the indices for the information bits and $\bar{\mathcal{A}}$ is the complementary set, which can be constructed as in [1, 3–6]. Both sets \mathcal{A} and $\bar{\mathcal{A}}$ are in $\{1,2,...,N\}$ for polar codes of length N. The source bits u_1^N can be split as $u_1^N = (u_{\mathcal{A}},u_{\bar{\mathcal{A}}})$. The codeword can then be expressed as

$$x_1^N = u_{\mathcal{A}} G_{\mathcal{A}} + u_{\bar{\mathcal{A}}} G_{\bar{\mathcal{A}}},\tag{3}$$

where G_A is the submatrix of G_N with rows specified by the set A.

An encoding diagram is shown in Fig. 1. Each node adds the signals on all incoming edges from the left and sends the result out on all edges to the right. The operations are done in the binary field \mathbb{F}_2 . One such encoding process is highlighted in Fig. 1 for $x_2=u_5\oplus u_6\oplus u_7\oplus u_8$. If the nodes in Fig. 1 are viewed as memory elements, the encoding process is to calculate the corresponding binary values to fill all the memory elements from the left to the right. This view is helpful when it comes to systematic polar codes in the following section.

B. Systematic Polar Codes

The systematic polar code is constructed by specifying a set of indices of the codeword x_1^N as the indices to convey the information bits. Denote this set as \mathcal{B} ($|\mathcal{B}|=K$) and the complementary set as $\bar{\mathcal{B}}$. The codeword x_1^N is thus split as $(x_{\mathcal{B}}, x_{\bar{\mathcal{B}}})$. Define a matrix $G_{\mathcal{A}\mathcal{B}}$ that is a submatrix of the generator matrix with elements $\{G_{i,j}\}_{i\in\mathcal{A},j\in\mathcal{B}}$. Splitting x_1^N in (3) into $(x_{\mathcal{B}}, x_{\bar{\mathcal{B}}})$ requires splitting the matrices $G_{\mathcal{A}}$ and $G_{\bar{\mathcal{A}}}$ as:

$$G_{\mathcal{A}} = (G_{\mathcal{A}\mathcal{B}}, G_{\mathcal{A}\bar{\mathcal{B}}}), \tag{4}$$

$$G_{\bar{A}} = (G_{\bar{A}B}, G_{\bar{A}\bar{B}}). \tag{5}$$

Then x_1^N can be split as the following:

$$\begin{cases} x_{\mathcal{B}} = u_{\mathcal{A}} G_{\mathcal{A}\mathcal{B}} + u_{\bar{\mathcal{A}}} G_{\bar{\mathcal{A}}\mathcal{B}}, \\ x_{\bar{\mathcal{B}}} = u_{\mathcal{A}} G_{\mathcal{A}\bar{\mathcal{B}}} + u_{\bar{\mathcal{A}}} G_{\bar{\mathcal{A}}\bar{\mathcal{B}}}. \end{cases}$$
(6)

We can see from (6) that, in systematic polar codes, $x_{\mathcal{B}}$ plays the role that $u_{\mathcal{A}}$ plays in non-systematic polar codes. Given a non-systematic encoder $(\mathcal{A}, u_{\bar{\mathcal{A}}})$, there exists a systematic encoder $(\mathcal{B}, u_{\bar{\mathcal{A}}})$ if \mathcal{A} and \mathcal{B} have the same number of elements and the matrix $G_{\mathcal{A}\mathcal{B}}$ is invertible [2]. Then a systematic encoder can perform the mapping $x_{\mathcal{B}} \mapsto x_1^N = (x_{\mathcal{B}}, x_{\bar{\mathcal{B}}})$. To realize this systematic mapping, $x_{\bar{\mathcal{B}}}$ needs to be computed for any given information bits $x_{\mathcal{B}}$. To this end, we see from (6) that $x_{\bar{\mathcal{B}}}$ can be computed if $u_{\mathcal{A}}$ is known. The vector $u_{\mathcal{A}}$ can be obtained as the following

$$u_{\mathcal{A}} = (x_{\mathcal{B}} - u_{\bar{\mathcal{A}}} G_{\bar{\mathcal{A}}\mathcal{B}}) (G_{\mathcal{A}\mathcal{B}})^{-1}.$$
 (7)

In [2], it is shown that $\mathcal{B} = \mathcal{A}$ satisfies all these conditions in order to establish the one-to-one mapping $x_{\mathcal{B}} \mapsto u_{\mathcal{A}}$. In the rest of the paper, the systematic encoding of polar codes adopts this selection of \mathcal{B} : $\mathcal{B} = \mathcal{A}$. Therefore we can rewrite (6) as

$$\begin{cases} x_{\mathcal{A}} = u_{\mathcal{A}}G_{\mathcal{A}\mathcal{A}} + u_{\bar{\mathcal{A}}}G_{\bar{\mathcal{A}}\mathcal{A}}, \\ x_{\bar{\mathcal{A}}} = u_{\mathcal{A}}G_{\mathcal{A}\bar{\mathcal{A}}} + u_{\bar{\mathcal{A}}}G_{\bar{\mathcal{A}}\bar{\mathcal{A}}}. \end{cases}$$
(8)

Note that the submatrix $G_{\mathcal{A}\mathcal{A}}$ is a lower triangular matrix with ones at the diagonal. The entries above the diagonal are all zeros.

Let us go back to the diagram in Fig. 1. For systematic polar codes, the information bits are now conveyed in the right-hand side in x_A . To calculate $x_{\bar{A}}$, u_A in the left-hand side needs to be calculated first. Once u_A is obtained, systematic encoding can be performed in the same way as the non-systematic encoding: performing binary additions from the left to the right. Therefore, compared with non-systematic encoding, systematic encoding has an additional round of binary additions from the right to the left. The detailed analysis of systematic encoding can be found in [22, 23].

C. SC Decoding

The SC decoding of polar codes follows the same graph as shown in Fig. 1. The likelihood ratio (LR) of bit channel i is defined as:

$$L_N^{(i)} = \frac{W_N^{(i)}(y_1^N, u_1^{i-1}|0)}{W_N^{(i)}(y_1^N, u_1^{i-1}|1)}.$$
 (9)

From [1], it is shown that the transition probability of bit channel i can be recursively calculated, which results in a recursive calculation of the LRs as:

$$\begin{split} &L_{N}^{(2i-1)}(y_{1}^{N},\hat{u}_{1}^{2i-2}) = \\ &\frac{L_{N/2}^{(i)}(y_{1}^{N/2},\hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2})L_{N/2}^{(i)}(y_{N/2+1}^{N},\hat{u}_{1,e}^{2i-2}) + 1}{L_{N/2}^{(i)}(y_{1}^{N/2},\hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2}) + L_{N/2}^{(i)}(y_{N/2+1}^{N},\hat{u}_{1,e}^{2i-2})},\\ &L_{N}^{(2i)}(y_{1}^{N},\hat{u}_{1}^{2i-1}) = [L_{N/2}^{(i)}(y_{1}^{N/2},\hat{u}_{1,o}^{2i-2} \oplus \hat{u}_{1,e}^{2i-2})]^{(1-2\hat{u}_{2i-1})} \end{split}$$

$$L_{N/2}^{(i)}(y_{N/2+1}^N, \hat{u}_{1,e}^{2i-2}).$$

$$(11)$$

III. ERROR CORRELATION PATTERN

In [2][24], it is shown that the re-encoding process of $\hat{x}_1^N =$ \hat{u}_1^NG after decoding \hat{u}_1^N does not amplify the number of errors in \hat{u}_1^N . Instead, there are less errors in \hat{x}_1^N than in \hat{u}_1^N . In this section, we state a corollary proven in [25] and then provide a proposition to show the error correlation pattern of the errors in \hat{u}_1^N . This pattern is used in Section IV to design the CBI scheme.

Corollary 1. The matrix $G_{\bar{A}A} = 0$.

The proof of this corollary can be found in [25].

Now let us define the set A_i containing the non-zero positions of column j of G as:

$$A_j = \{i \mid 1 \le i \le N \text{ and } G_{i,j} = 1\}.$$
 (12)

Assume the entries of the set A_i are arranged in the ascending order. Define $A_i(a:b)$ as a vector containing element a to element b of the set A_i . The following lemma can be deduced directly from the construction and the SC decoding of polar codes.

Lemma 1. Let A_i be as defined in (12) and j = i - N/2 $(N/2+1 \le i \le N)$. Then the LR of $\sum_{k \in A_i} u_k$ is directly affected by the decision of $\sum_{l \in A_i(1:N/2)} u_l$.

Proof: To understand the decoding process, let us first look closely at the encoding process of polar codes. Fig. 2 shows the structure of the generator matrix $G = G_N$ and the corresponding details of the matrix, with respect to the matrix $G_{N/2}$. Two basic facts of the generator matrix G_N are that:

- Fact One. Rows N/2 + 1 to N of $G = G_N$ contain two
- copies of $G_{N/2}$ as: $(G_{N/2} \ G_{N/2})$.

 Fact Two. Colums 1 to N/2 of $G = G_N$ contain two copies of $G_{N/2}$ as: $\begin{pmatrix} G_{N/2} \\ G_{N/2} \end{pmatrix}$.

In the encoding process, the following two coded bits are achieved:

$$x_i = \sum_{k \in A_i} u_k \tag{13}$$

$$x_{i} = \sum_{k \in \mathcal{A}_{i}} u_{k}$$

$$x_{j} = \sum_{l \in \mathcal{A}_{j}(1:N/2)} u_{l} + \sum_{l' \in \mathcal{A}_{j}(N/2+1:N)} u_{l'}$$

$$(13)$$

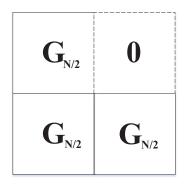
Because of Fact One of the generator matrix G_N , the set $A_j(N/2+1:N)$ (j=i-N/2) is the same as the set A_i . Therefore the coded bit x_i is:

$$x_j = \sum_{l \in \mathcal{A}_j(1:N/2)} u_l + \sum_{k \in \mathcal{A}_i} u_k \tag{15}$$

The coded bits x_1^N are transmitted over N independent underlying channels W, producing corresponding y_1^N observations at the receiver side.

In the decoding process, when estimation of $u_1^{N/2}$ is done, denoted as $\hat{u}_1^{N/2}$, then Fact Two can be employed to provide the other N/2 observations of the coded bits $x_{N/2+1}^N$. For example, the coded bit $x_i = \sum_{k \in \mathcal{A}_i} u_k$ is observed from the corresponding received sample y_i .

With the estimated $\hat{u}_1^{N/2}$, another observation of $x_i =$ $\sum_{k \in \mathcal{A}_i} u_k$ is readily calculated as: $y_j - \sum_{l \in \mathcal{A}_i (1:N/2)} \hat{u}_l$. This



(a) The structure of the generator matrix G

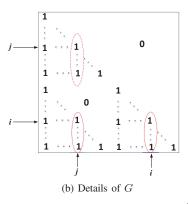


Fig. 2. The structure of the generior matrix $G = G_N = F^{\otimes n}$ and the details of it. The variable j is spaced by N/2 from i.

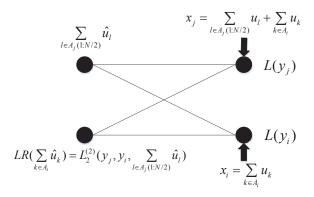


Fig. 3. The LR calculation of one stage involving x_j and x_i where j =i - N/2.

process is captured by the recursive LR calculation in (11) where $\sum_{l \in A_j(1:N/2)} \hat{u}_l$ is the estimated decision of the upper left node and the LR of $\sum_{k \in A_i} u_k$ (at the lower left node) is to be calculated at that specific connection. Fig. 3 shows the connection of that stage. Therefore, the LR of $\sum_{k \in \mathcal{A}_i} u_k$ is affected by the decision of $\sum_{l \in \mathcal{A}_j(1:N/2)} u_l$ for j = i - N/2: if the decision of $\sum_{l \in \mathcal{A}_j(1:N/2)} u_l$ is incorrect, then the incorrect decision can cause the LR value of $\sum_{k \in \mathcal{A}_i} u_k$ incorrect.

Proposition 1. Let A_i be defined as in (12). Then, the errors of \hat{u}_{A_i} are dependent (or coupled).

Before going into the proof of this Proposition 1, we provide an example to explain the meaning of it. As noted in Section

I, the notation $v_{\mathcal{A}}$ is a subvector of v_1^N with elements specified by the set \mathcal{A} . Here is an example to show what exactly $\hat{u}_{\mathcal{A}_i}$ is. Let the block length be N=16, the code rate of the polar code be R=0.5, and the underlying channel is the BEC channel with an erasure probability 0.2. The set \mathcal{A} is calculated to be $\mathcal{A}=\{8,10,11,12,13,14,15,16\}$. Let i=10, then we take the indices of non-zero entries of column 10 of G as \mathcal{A}_{10} , which is a collecting set of indices 10, 12, 14, 16. Therefore, $\hat{u}_{\mathcal{A}_{10}}$ is a subvector of \hat{u}_1^N which contains elements of \hat{u}_{10} , \hat{u}_{12} , \hat{u}_{14} , \hat{u}_{16} .

Proof: We provide proofs of this proposition from two perspectives: 1) From the SC decoding process; 2) From a contradiction perspective with respect to the performance of systematic polar codes.

First, let us prove this proposition from the SC decoding process. The same reasoning in the proof of Lemma 1 can be applied here: the LR of $\sum_{k\in\mathcal{A}_i}u_k$ is directly affected by the decision of $\sum_{l\in\mathcal{A}_j(1:N/2)}u_l$ for j=i-N/2. With $\mathcal{A}_i=\mathcal{A}_j(N/2+1:N)$, it is exactly saying that the decision of $\sum_{l\in\mathcal{A}_j(1:N/2)}u_l$ (from the first half of column j) affects the decoding of the $\sum_{l'\in\mathcal{A}_j(N/2+1:N/2)}u_{l'}$ (from another half of column j). Since the recursive LR calculation of $u_{j'}$ ($j'\in\mathcal{A}_j(N/2+1:N)$) involves the LR of $\sum_{l\in\mathcal{A}_j(1:N/2)}u_l$ (from the nature of the polar encoding graph), the decision of bit $u_{j'}$ is therefore affected by the LR of $\sum_{l\in\mathcal{A}_j(1:N/2)}u_l$. In other words, any error u_j ($j\in\mathcal{A}_j(1:N/2)$) affects the decision of the subsequent bit $u_{j'}$ with $j'\in\mathcal{A}_j(N/2+1:N)$. Therefore the errors in $\hat{u}_{\mathcal{A}_i}$ are correlated.

Now let us prove the proposition from a contradiction. Assume the errors in \hat{u}_{A_i} are independent. For non-systematic polar codes, we define a set $A_t \subset A$ containing the indices of the incorrect information bits in an error event. In the same way, we define a set $A_{sys,t} \subset A$ containing the corresponding indices of the information bits in error for systematic polar codes. Let v_1^N be an error indicator vector: a N-element vector with 1s in the positions specified by the error event A_t and 0s elsewhere. Let the error probability being: $Pr(v_m = 1) = p_m$. From the independence assumption of errors, it is known that $0 \le p_m \le 0.5$ for information bits. Correspondingly, we set a vector q_1^N with 1s in the positions specified by $\mathcal{A}_{sys,t}$ and 0s elsewhere. From the systematic encoding process, we have $q_1^N = v_1^N G$. Correspondingly, $q_A = v_1^N G(:, A)$ where G(:, A)denotes the submatirx of G composed of the columns specified by A. Since the frozen bits are always correctly determined, $v_{\bar{A}} = 0_1^{N-K}$ (note that 0_1^{N-K} is a zero vector with N-Kelements all being zeros). This leads to $q_A = v_A G_{AA}$. In this way, we convert the errors of non-systematic polar codes and systematic polar codes to the weight of the vectors v_1^N and q_1^N .

Denote the Hamming weight of the vector v_1^N as $w_H(v_1^N)$. Specifically, the element q_i $(i \in \mathcal{A})$ is one if $v_{\mathcal{A}_i}$ has an odd number of ones. With the independent assumption of errors in $\hat{u}_{\mathcal{A}_i}$, the probability that the *i*th information bit \hat{x}_i is in error is

$$\tilde{p}_i = \frac{1}{2} - \frac{1}{2} \prod_{m=1}^{K_i} (1 - 2p_m)$$
 (16)

where $K_i = |A_i|$. The proof of (16) is given in Appendix. In

TABLE I COUPLING EFFECT FOR N=16 and R=0.5 in a BEC Channel with an Erasure Probability of 0.2

Column Index	Coupling coefficient
10	76%
11	74%
13	74%

(16), we can order the probabilities $\{p_m\}_{m=1}^{K_i}$ $(0 \le p_m \le 0.5)$ in the ascending order. Applying the Monotone Convergence Theorem to real numbers [26], we have:

$$\lim_{K_i \to \infty} \tilde{p}_i = \lim_{K_i \to \infty} \left[\frac{1}{2} - \frac{1}{2} \prod_{m=1}^{K_i} (1 - 2p_m) \right] = \frac{1}{2}$$
 (17)

Thus, the mean Hamming weight of $q_1^N \colon w_H(q_1^N) = \frac{K}{2} \ge w_H(v_1^N)$, meaning the average number of errors of the systematic polar codes is larger than the average number of errors of non-systematic polar codes. This contradicts with the existing results that systematic polar codes outperform non-systematic polar codes. Thus, we can conclude the errors of $u_{\mathcal{A}_i}$ are dependent.

From Proposition 1, an error correlation pattern among the errors in \hat{u}_1^N can be deduced. We call bits $\hat{u}_{\mathcal{A}_i}$ the correlated estimated bits. This says that statistically, the errors of bits $\hat{u}_{\mathcal{A}_i}$ are coupled. To show this coupling, we use the same example as the one after Proposition 1. The number of times the errors of $\hat{u}_{\mathcal{A}_i}$ ($i\in\mathcal{A}$) happening simultaneously (denoted by N_s) over the number of times any of the bits $\hat{u}_{\mathcal{A}_i}$ in error (denoted by N_e) is called the coupling coefficient, which is equal to N_s/N_e . The coupling coefficients (similar to the correlation coefficient) of bits indicated by non-zero positions of column 10, 11, and 13 is shown in Table I. It can be seen from Table I that if there are errors in $\hat{u}_{\mathcal{A}_{10}}=\{\hat{u}_{10},\hat{u}_{12},\hat{u}_{14},\hat{u}_{16}\}$, then 76% of times these bits errors happen simultaneously, resulting in a coupling coefficient 0.76 for errors in $\hat{u}_{\mathcal{A}_{10}}$. The coupling coefficients for $\hat{u}_{\mathcal{A}_{11}}$ and $\hat{u}_{\mathcal{A}_{13}}$ are 0.74 in Table I.

To the authors' knowledge, there is no attempt yet to utilize the error correlation pattern to improve the performance of polar codes. In the next section of this paper, we propose novel interleaving schemes to break the coupling of errors to improve the BER performance of polar codes in concatenation systems while still maintaining the low complexity of the SC decoding.

IV. THE CORRELATION-BREAKING INTERLEAVING SCHEMES

In this section we consider interleaving schemes of polar codes (the inner code) with an outer LDPC code as an example. The introduced schemes work for all types of outer codes. From Proposition 1, we know the exact correlated information bits of polar codes. The interleaving scheme is thus to make sure that the correlated bits of the inner polar codes come from differen LDPC blocks in the transmitter side. In this way, the de-interleaved LDPC blocks have independent errors. A blind interleaving (BI) (also known as direct product) is first introduced, which breaks all bits in one LDPC block into different polar code blocks in the interleaver. Then an

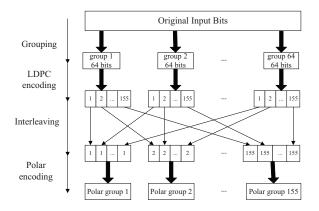


Fig. 4. A blind interleaving scheme with direct product (BI-DP). The block length of the LDPC code is $N_l=155$, and the code rate is 64/155. The block length of the polar code is N=256, and the code rate is R=1/4.

improved BI scheme and a correlation breaking interleaving (CBI) scheme, that only breaks the correlated bits, are presented.

In this section, we also compare the time complexity and the required memory size of the CBI and the BI schemes. The time complexity is in terms of the decoding delay from the de-interleaving operation: the time from transmitting the first outer code block to decoding the first outer code block in each round of transmission.

A. The Blind Interleaving Schemes

In this section, the scheme of scattering all bits in a LDPC block into different polar code blocks is introduced. The N_l bits of one LDPC block are divided into N_l polar code blocks, which guarantees that the received error information bits in each LDPC block are independent as they come from different polar code blocks during de-interleaving.

1) Direct Product Blind Interleaving: Denote $c_i^{(j)}$ $(1 \le i \le N_l, 1 \le j \le K)$ as the ith coded bit of the jth LDPC block. Also denote $u_k^{(d)}$ as the kth information bit of the dth polar block. Bits i $(c_i^{(j)})$ of all LDPC code blocks form the input vector to the ith polar code encoder. The input bits of the ith polar block are arranged in the order of the LDPC blocks: $u_j^{(i)} = c_i^{(j)}$. For example, $\{c_1^{(j)}, 1 \le j \le K\}$ of all LDPC blocks produce the input for the first polar block, and $u_j^{(1)} = c_1^{(j)}$, meaning that bit one of the jth LDPC block is set as the jth input bit of polar block one. This interleaving is called the blind interleaving with direct product (BI-DP).

We give an example in Fig. 4 where $K_l=64$ and $N_l=155$. Polar code in this example has N=256, K=64 and a code rate R=1/4. Fig. 4 is an exact illustration of the BI-DP scheme: bits one of all LDPC blocks serve as the input to polar block one, bits two of all LDPC blocks serve as the input to polar block two, and so on.

To compare with the subsequent improved blind interleaving, define a $N_l \times K$ matrix C, that contains the elements of the input of polar blocks. The entry of the ith row and jth column is $C_{i,j} = u_i^{(i)}$. For the BI-DP scheme, $C_{i,j} = u_i^{(i)} = c_i^{(j)}$.

2) Cyclic Direct Product Blind Interleaving: One problem with the BI-DP scheme is that for LDPC block j, all the coded bits of it are placed as the jth input bits of all polar blocks. For example, all the bits $c_i^{(1)}$ of LDPC block one are the first information bits of all polar blocks in the receiver side. Given that information bits of polar codes are not equally protected, it can happen that LDPC block j is exposed to a large amount of errors if bit j of the polar code is a poorly protected bit in the decoding process. An improved BI, termed cyclic DP (BI-CDP), is thus introduced below to overcome this problem.

Denote $N_l = n_u K + k_l$, where n_u and k_l are the quotient and the reminder of N_l divided by K, respectively. Define a basic polynomial $p(x) = j'x^{j'}$ $(0 \le j' \le K - 1)$. For the ith polar code block $(1 \le i \le n_u K)$, the assignments of the LDPC coded bits to this polar block can be obtained from the i'th (i' = i - 1) quasi cyclic shift $(0 \le i' \le n_u K - 1)$:

$$p^{(i')}(x) = ((j' + |i'/K|K)x^{(i'+j')}) \pmod{K}$$
 (18)

where 'mod' is the modulo operator. Here the word 'quasi' means that it is not the traditional cyclic shift operation of $x^{i'}p(x)$ because of the jump of the coefficients every K shifts.

Let $m=(j'+\lfloor i'/K\rfloor K)+1$, $q=((i'+j') \bmod K)+1$ and $l=((m-1) \bmod K)+1$. Then the i'th cyclic shifted polynomial $p^{(i')}(x)$ carries the mth bit of the qth LDPC block $c_m^{(q)}$, which is applied to the lth bit of polar block i=i'+1, namely $u_l^{(i)}=c_m^{(q)}$.

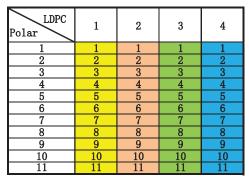
This quasi-cyclic arrangement of LDPC coded bits to the corresponding input bits of polar blocks works for the first $n_u K$ polar blocks. However it does not work for the last k_l polar blocks because $m = (j' + n_u K) + 1 > N_l$ when $k_l \leq j' \leq K - 1$.

There are many ways to arrange the input for the last k_l polar blocks. In the following, we propose one possible solution. Let $i'=i-1=n_uK+i_r$ $(n_uK< i\le N_l$ and $0\le i_r\le k_l-1)$. For the original polynomial $p(x)=j'x^{j'}$, when $j'=j-1=i_r$, the i'th cyclic shift is defined as $p^{(i')}(x)=i'x^{j'}$. When $j'=j-1\ne i_r$, define a new parameter j'' $(0\le j''\le K-2)$ for the other K-1 elements of the i'th shift of p(x) $(i'=i-1=n_uK+i_r)$:

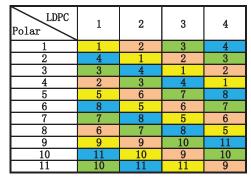
$$p^{(i')}(x) = \begin{cases} i'x^{j'}, & \text{if } j' = i_r, \\ (j^{''} \mod k_l + n_u K)x^{(i'+j^{''}+1) \mod K}, & \text{otherwise.} \end{cases}$$
(19)

It can be verified that the proposed arrangements assign the remaining LDPC coded bits to the last k_l polar blocks.

This arrangement can be viewed from the matrix C defined in Sec IV-A1. Fig. 5 shows the assignments of LDPC coded bits to polar blocks, stored by this matrix C. In this example, $N_l=11$ and K=4. For LDPC block j $(1 \le j \le 4)$, the subscript of the coded bits $c_i^{(j)}$ $(1 \le i \le 11)$ are stored in column j of the two tables. For polar block i, the input information bits $u_j^{(i)}$ are stored in the ith row of the tables. Since the entries of the tables in Fig. 5 are the subscripts of $c_i^{(j)}$, the subscripts of the information bits $u_j^{(i)}$ are represented by different colors: yellow is j=1 $(u_1^{(i)})$, orange is j=2, green is j=3, and blue is j=4.



(a) BI-DP



(b) BI-CDP

Fig. 5. An example of the matrix C for $N_l=11$ and K=4. The row and column indices are the indices of polar and LDPC blocks. The entries of one column are the indices of LDPC coded bits of that specific LDPC block. The four colors of the background corresponding to the four positions of each polar block.

For BI-DP, the assignments of each LDPC coded bits are designed according to Section IV-A1. Clearly it can be seen from the same color of columns of Fig. 5-(a) that the coded bits of LDPC block j are assigned to the same bits (the jth bits) of all polar blocks. The assignments of LDPC coded bits for BI-CDP are done according to equations (18) and (19). Take column 1 (LDPC block 1) of Fig. 5-(b) as an example. It is shown that three coded bits (three colored yellow of the column 1) of LDPC block one are put as the first information bits for three polar blocks (polar block 1, 5, and 9), two coded bits (two colored orange) are the second information bits of polar blocks 4 and 8, three coded bits (three colored green) are the third information bits of polar blocks 3, 7 and 11, and three coded bits (three colored blue) are the fourth information bits of polar blocks 2, 6 and 10. On the other hand, all eleven coded bits of LDPC block one are the first information bits of eleven polar blocks for the BI-DP scheme.

Overall, the improved BI-CDP can scatter the LDPC coded bits evenly to the input of polar blocks to reduce the chance of simultaneous errors. It is expected that the BI-CDP scheme performing better than the BI-DP scheme.

B. The CBI Scheme

The two BI schemes in Section IV-A occupies a memory of $[N_l,K]$ received samples. The decoding delay of the BI scheme is $N_l \times N \times T_s$ (T_s is the symbol duration). From Section III, we know that it is not necessary to scatter all

bits in a LDPC block into different polar blocks, since not all bits in a polar block are correlated. The interleaving scheme in this section is to make the correlated information bits $u_{\mathcal{A}_i}$ ($1 \leq i \leq K$) of one polar block come from different LDPC blocks and the remaining uncorrelated information bits come from one LDPC block in the encoding process. Or in other words, the interleaving scheme is to scatter only the correlated information bits $u_{\mathcal{A}_i}$ ($1 \leq i \leq K$) of each polar block into different LDPC blocks and the uncorrelated information bits of each polar block are scattered into one LDPC block in the receiver side.

The difficulty in designing a CBI scheme is that the sets $\{\mathcal{A}_i\}_{i=1}^K$ are different for different block lengths and code rates. They are also different for different underlying channels for which polar codes are designed. A CBI scheme is dependent on three parameters: the block length N, the code rate R, and the underlying channel W. Let us denote a CBI scheme as $\mathrm{CBI}(N,R,W)$ to show this dependence. A $\mathrm{CBI}(N,R,W)$ optimized for one set of (N,R,W) is not necessarily optimized for another set (N',R',W'). It may not even work for the set (N',R',W') if $N'R'\neq NR$. In the following, we provide a CBI scheme which works for any sets of (N,R,W), but not necessarily optimal for one specific set of (N,R,W).

The set \mathcal{A}_i contains the indices of the non-zero entries of column $i \in \mathcal{A}$. First, the $K = |\mathcal{A}|$ columns of G are extracted, forming a submatrix $G(:,\mathcal{A})$. Divide this submatrix further as: $G(:,\mathcal{A}) = [G_{\bar{\mathcal{A}}\mathcal{A}} \ G_{\mathcal{A}\mathcal{A}}]$. Since the submatrix $G_{\bar{\mathcal{A}}\mathcal{A}} = \mathbf{0}$ from Corollary 1, it is only necessary to analyze the submatrix $G_{\mathcal{A}\mathcal{A}}$. If a CBI needs to look at each individual set \mathcal{A}_i , then a general CBI is beyond reach. However, we can simplify this problem by dividing the indices of information bits only into two groups: the correlated bits indices \mathcal{A}_c and the uncorrelated bits indices $\bar{\mathcal{A}}_c$.

Let ω_i denote the Hamming weight of row i of $G_{\mathcal{A}\mathcal{A}}$. The following proposition can be used to find the sets \mathcal{A}_c and $\bar{\mathcal{A}}_c$.

Proposition 2. For the submatrix G_{AA} , define $A_{cs} = \{i \mid 1 \le i \le K \text{ and } \omega_i > 1\}$, and $\bar{A}_{cs} = \{j \mid 1 \le j \le K \text{ and } \omega_j = 1\}$. The corresponding sets of A_{cs} and \bar{A}_{cs} with respect to the matrix G are the sets A_c and \bar{A}_c , respectively.

Proof: First, let us bear in mind that the submatrix $G_{\mathcal{A}\mathcal{A}}$ is a lower triangular matrix as discussed in Section II-B. This proposition is equivalent to the following assignment:

$$\begin{cases} i \in \bar{\mathcal{A}}_{cs}, & \text{if } \omega_i = 1, \\ i \in \mathcal{A}_{cs}, & \text{if } \omega_i > 1. \end{cases}$$
 (20)

For $\omega_i=1$, there is only one non-zero entry $G_{i,i}=1$ for row i. Let $K_c=|\mathcal{A}_{cs}|$ and $K_{uc}=|\bar{\mathcal{A}}_{cs}|$. Denote the submatrix formed by the rows of $G_{\mathcal{A}\mathcal{A}}$ indicated by $\bar{\mathcal{A}}_{cs}$ as $G_{\mathcal{A}\mathcal{A}}(\bar{\mathcal{A}}_{cs},:)$. Then each row of the submatrix $G_{\mathcal{A}\mathcal{A}}(\bar{\mathcal{A}}_{cs},:)$ has Hamming weight one. Extract the columns specified by $\bar{\mathcal{A}}_{cs}$ of $G_{\mathcal{A}\mathcal{A}}(\bar{\mathcal{A}}_{cs},:)$ to obtain a matrix denoted as G_{uc} . Similar to the process of extracting $G_{\mathcal{A}\mathcal{A}}$ from G, the extraction of rows and columns (indicated by $\bar{\mathcal{A}}_{cs}$) from $G_{\mathcal{A}\mathcal{A}}$ results in a final $K_{uc} \times K_{uc}$ identity matrix $G_{uc} = I_{K_{uc}}$.

According to Proposition 1, the errors in \hat{u}_{A_i} (\hat{u}_{A_c}) are coupled. Now that each column of $G_{uc} = I_{K_{uc}}$ has Hamming

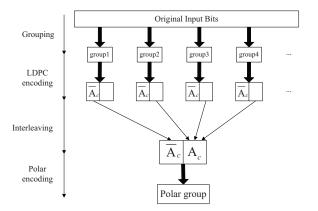


Fig. 6. A general correlation-breaking interleaving scheme. Here the set A_c consists of the indices of the correlated bits and the set \bar{A}_c is the complementary set of A_c .

weight one, the errors contained in $\hat{u}_{\bar{A}_c}$ are not coupled as indicated by Proposition 1.

We use the same example as before (the one after Proposition 1) to show how to use Proposition 2 to find the sets \mathcal{A}_c and $\bar{\mathcal{A}}_c$. With Proposition 2, we can easily find that $\mathcal{A}_{cs} = \{4,6,7,8\}$ for the submatrix $G_{\mathcal{A}\mathcal{A}}$. Relative to the matrix G_{16} , this set is $\mathcal{A}_c = \{12,14,15,16\}$. The uncorrelated set is thus $\bar{\mathcal{A}}_c = \{8,10,11,13\}$.

With the sets A_c and \bar{A}_c obtained for any (N,R,W), we can devise a CBI scheme. Fig. 6 is a general CBI scheme. As in Section IV-A2, let $N_l = n_u K + k_l$ and $K_n = K_c + 1$. For the general CBI scheme, the number of polar blocks, n_p , to transmit K_n LDPC blocks, is expressed as:

$$n_p = \begin{cases} (n_u + 1)K_n, & \text{if } k_l = 0 \text{ or } k_l > K_n, \\ n_u K_n + k_l, & \text{otherwise.} \end{cases}$$
 (21)

The assignment of LDPC coded bits to the polar blocks are similarly done as the BI-CDP scheme, except that there are coded bits which are put into the uncorrelated positions of the same polar block. Let $0 \le i' \le n_p - 1$ and $0 \le j' \le K_n - 1$. The general rules to determine the elements of the matrix C are the following:

- Consider elements of C within the first $n_u K_n$ rows. When $j'=i' \mod K_n$, $C_{i,j}$ contains K_{uc} bits from LDPC block j=(j'+1). These bits are put into positions $\bar{\mathcal{A}}_c$ of polar block i=i'+1. For the remaining K_c correlated information bits of polar block i, it takes coded bits from other different K_c LDPC blocks to put into correlated positions \mathcal{A}_c in the same fashion as the BI-CDP scheme
- Consider the rest of the rows (for the remaining $n_p n_u K_n$ polar blocks). When $j' = i' \mod K_n$, $C_{i,j}$ contains the remaining bits (smaller than K_{uc}) from LDPC block j = (j'+1). These bits are also put into positions $\bar{\mathcal{A}}_c$ of polar block i = i'+1. Polar block i' takes coded bits from other LDPC blocks for its correlated information bits, similarly to the arrangement of the BI-CDP scheme.

Two examples are given in Table II and Table III to explain the assignments for the two cases of (21): Table II is an example of the second case of (21) and Table III is an example of the first case of (21).

A polar code (32,16) concatenated with a LDPC code (21,8)shown in Table II is the example when $k_l < K_n$. The correlated set $A_c = \{16, 24, 26, 27, 28, 29, 30, 31, 32\}$. Therefore $K_c = |A_c| = 9, K_n = K_c + 1 = 10, n_u = \lfloor N_l/K \rfloor = 1,$ and $k_l = 5 < K_n$. To transmit $K_n = 10$ LDPC blocks, $n_p = n_u K_n + k_l = 15$ polar blocks are required. In Table II, the top row contains indices of the LDPC blocks, the first column is the indices of the polar blocks, and the entries of this table represent the indices of encoded bits of LDPC blocks. From Table II, for polar block one, bit 1 to bit 7 are taken from LDPC block one, and the other 9 bits are bits 8, 9, ..., 16 from LDPC blocks two to ten, respectively. The 7 bits from LDPC block one are placed at the uncorrelated positions $\bar{\mathcal{A}}_c$ of polar block one, and the other 9 bits from nine LDPC blocks are arranged at the correlated positions A_c of polar block one. The other polar blocks (polar block two to polar block ten) follow the same fashion in collecting the input bits. These first $n_u K_n$ rows follow the same cyclic assignments of LDPC coded bits to the inputs of polar blocks as the BI-CDP scheme. The remaining polar blocks (from polar block eleven to polar block fifteen) collect the remaining bits of LDPC blocks. For example, although polar block eleven can take $K_{uc} = 7$ uncorrelated bits from LDPC block one, there are not enough bits left from LDPC block one: only bits c_{17} to c_{21} are left. The assignments of the correlated positions of polar block eleven follows exactly that of the BI-CDP scheme.

Table III shows another example when $k_l > K_n$. In this example, the polar code (32,8) has an $\mathcal{A}_c = \{28,30,31,32\}$ with $K_c = 4$ and the LDPC is a (22,8) code. The parameters are $k_l = 6$ and $K_n = K_c + 1 = 5$. The total polar blocks $n_p = n_u \times K_n = 3 \times 5$ are used to transmit $K_n = 5$ LDPC blocks. For both examples, there are 0s at the left low corner, which means that there are polar positions which are not used. These positions are wasted which are the cost of the universal CBI design.

C. Complexity Analysis

For the CBI scheme, the interleaving requires a memory to store the decoded LR values from n_p polar blocks in order to do the de-interleaving. The memory size is therefore $[n_p, K]$. The decoder needs to wait n_p polar blocks to decode K_n LDPC blocks. The decoding delay of the outer code is therefore $n_p \times N \times T_s$, where T_s is the symbol duration in seconds. For the BI scheme, the memory size is $[N_l, K]$ and the decoding delay is $N_l \times N \times T_s$.

V. SIMULATION RESULTS

In this section, simulation results are provided to verify the performance of BI-DP, BI-CDP, and the CBI scheme. The first example we take is the same as the BI scheme in Fig. 4. The LDPC codes used in this section is the (155,64) MacKay code [27], where the code length is $N_l=155$ and the information bit length is $K_l=64$. The polar code is (256,64). The

TABLE II
THE CBI SCHEME FOR LDPC (21,8) and Polar (32,16). The Top Row Contains Indices of LDPC Blocks and the First Column Is the Indices of Polar Blocks. The entries of the table are bit indices of LDPC blocks.

LDPC Polar	1	2	3	4	5	6	7	8	9	10
1	1:7	8	9	10	11	12	13	14	15	16
2	16	1:7	8	9	10	11	12	13	14	15
3	15	16	1:7	8	9	10	11	12	13	14
4	14	15	16	1:7	8	9	10	11	12	13
5	13	14	15	16	1:7	8	9	10	11	12
6	12	13	14	15	16	1:7	8	9	10	11
7	11	12	13	14	15	16	1:7	8	9	10
8	10	11	12	13	14	15	16	1:7	8	9
9	9	10	11	12	13	14	15	16	1:7	8
10	8	9	10	11	12	13	14	15	16	1:7
11	17:21	17	18	19	20	21	17	18	19	20
12	0	18:21	17	18	19	20	21	17	18	19
13	0	0	19:21	17	18	19	20	21	17	18
14	0	0	0	20:21	17	18	19	20	21	17
15	0	0	0	0	21:21	17	18	19	20	21

TABLE III

THE CBI SCHEME FOR LDPC (22,8) AND POLAR (32,8). THE TOP ROW CONTAINS INDICES OF LDPC BLOCKS AND THE FIRST COLUMN IS THE INDICES OF POLAR BLOCKS. THE ENTRIES OF THE TABLE ARE BIT INDICES OF LDPC BLOCKS.

LDPC Polar	1	2	3	4	5
1	1:4	5	6	7	8
2	8	1:4	5	6	7
3	7	8	1:4	5	6
4	6	7	8	1:4	5
5	5	6	7	8	1:4
6	9:12	13	14	15	16
7	16	9:12	13	14	15
8	15	16	9:12	13	14
9	14	15	16	9:12	13
10	13	14	15	16	9:12
11	17:20	17	18	19	20
12	21	18:21	17	18	19
13	22	22	19:22	17	18
14	0	0	0	20:22	17
15	0	0	0	0	21:22

overall code rate of the LDPC(N_l,K_l)+CBI+POLAR(N,K) concatenation system is $K_l/N_l \times R = 0.1$. The underlying channel is the AWGN channel. The construction of polar code is based on [3], which produces the set \mathcal{A} . Then the submatrix $G_{\mathcal{A}\mathcal{A}}$ is formed from the generator matrix G. Based on the submatrix $G_{\mathcal{A}\mathcal{A}}$ and Proposition 2, for polar code (256,64), the correlated bits indices is calculated to be \mathcal{A}_c ($K_c = 38$) and the uncorrelated bits indices $\bar{\mathcal{A}}_c$ ($K_{uc} = 26$) are also obtained.

In this example, the occupied memory size of the CBI scheme is [105,64], smaller than [155,64] of the two BI schemes. The decoding delay of the CBI scheme is 105×256 symbols, still smaller than 155×256 symbols of the BI schemes.

The performance of the BI-DP (dashed line with squares) and BI-CDP (solid line with squares) is shown in Fig. 7. At a BER = 10^{-5} , the improved BI-CDP scheme has a 0.4 dB advantage over the BI-DP scheme. To compare with the CBI scheme (the solid line with circles), two other schemes are also shown in Fig. 7: 1) the performance of the polar code (SC decoding) directly concatenated with the LDPC code (no interleaving being performed, denoted by the solid line with

triangles), with a legend of LDPC(155,64)+POLAR(256,64)-SC; 2) the performance of the direct concatenation but with the polar code employing the belief propagation (BP) decoding (denoted by the solid line with asterisks), with a legend of LDPC(155,64)+POLAR(256,64)-BP. At a BER = 10^{-4} , the LDPC(155,64)+CBI+POLAR(256,64)-SC system achieves 1.4 dB and 1.2 dB gains over the direct concatenation systems LDPC(155,64)+POLAR(256,64)-SC and LDPC(155,64)+POLAR(256,64)-BP, respectively.

Compared with the BI-DP scheme, the CBI scheme requires only an additional 0.05 dB of E_b/N_0 to achieve the BER at 10^{-5} . Also, the CBI scheme requires a memory size $N_l/n_p=1.5$ times smaller than that of the BI-DP scheme. At the same BER level, the BI-CDP scheme outperforms both the BI-DP and the CBI scheme, requiring 0.4 dB less to achieve this BER.

The proposed CBI scheme can also work with other outer codes, such as BCH codes. Fig. 8 shows the result of the polar code (256,64) with a BCH code (127,57) where the 127 and 57 are the code length and the number of information bits of BCH codes in one code block, respectively. It can be seen from Fig. 8 that the CBI scheme employing BCH code as an

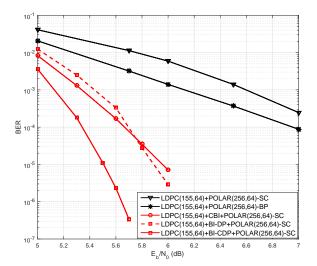


Fig. 7. The BER performance of polar code (256,64) concatenated with a LDPC code in AWGN channels. The LDPC code is the (155,64) MacKay code.

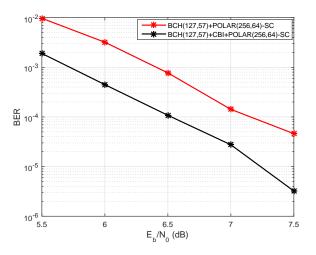


Fig. 8. The BER performance of the concatenation scheme in AWGN channels. The BCH code is (127, 57) and the polar code is (256, 64).

outer code has a 0.7 dB gain over the direct concatenation scheme at a BER = 10^{-4} .

VI. CONCLUSION

In this paper, a correlation pattern of bit errors of polar codes with the SC decoding are studied. Based on the studies, BI-DP, BI-CDP, and CBI schemes are proposed to decorrelate the coupled bit errors, while still maintaining the low complexity of the SC decoding of polar codes. The BI-CDP scheme cyclicly assigns the encoded bits from the outer code to the input of the inner encoder. As a result, the BI-CDP scheme enjoys a 0.4 dB gain over the BI-DP scheme for the presented results in the paper. The proposed novel CBI scheme has a much better performance than the direct concatenation schemes. Compared with the BI-DP scheme, the CBI scheme also achieves a comparable BER performance while requiring a smaller memory size and a shorter decoding delay. Simulation results verify the theories and the proposed schemes in the paper.

APPENDIX PROOF OF EQUATION(16)

Proof: Given a sequence of M independent binary digits v_1^M where the probability $\Pr(v_m=1)=p_m$, then the probability that v_1^M contains an odd number of 1's (denoted by P_M) is

$$P_M = \frac{1}{2} - \frac{1}{2} \prod_{m=1}^{M} (1 - 2p_m).$$
 (22)

We use induction to prove it. First, let M=1, then $P_1=p_m=\frac{1}{2}-\frac{1}{2}\prod_{m=1}^1(1-2p_m)$. Next assume when $M=k_m$, (22) holds. That is: $P_{k_m}=\frac{1}{2}-\frac{1}{2}\prod_{m=1}^{k_m}(1-2p_m)$. Now let us prove that when $M=k_m+1$, (22) still holds:

$$P_{k_m+1} = \frac{1}{2} - \frac{1}{2} \prod_{m=1}^{k_m+1} (1 - 2p_m).$$
 (23)

Starting from P_{k_m} , P_{k_m+1} can be derived as the following:

$$\begin{split} P_{k_m+1} &= p_{k_m+1} \times (1 - P_{k_m}) + P_{k_m} \times (1 - p_{k_m+1}) \\ &= p_{k_m+1} - 2 \times P_{k_m} \times p_{k_m+1} + P_{k_m} \\ &= p_{k_m+1} \times \prod_{m=1}^{k_m} (1 - 2p_m) + \frac{1}{2} - \frac{1}{2} \prod_{m=1}^{k_m} (1 - 2p_m). \end{split}$$

Let us extend the right-hand side of (23) as the following:

$$\frac{1}{2} - \frac{1}{2} \prod_{m=1}^{k_m+1} (1 - 2p_m)$$

$$= \frac{1}{2} - \frac{1}{2} \prod_{m=1}^{k_m} (1 - 2p_m) \times (1 - 2p_{k_m+1})$$

$$= p_{k_m+1} \times \prod_{m=1}^{k_m} (1 - 2p_m) + \frac{1}{2} - \frac{1}{2} \prod_{m=1}^{k_m} (1 - 2p_m).$$

which is equal to the one derived from P_{k_m} . Therefore, equation (16) is proven from the induction.

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