Every maximally monotone operator of Fitzpatrick-Phelps type is actually of dense type

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Abstract

We show that every maximally monotone operator of Fitzpatrick-Phelps type defined on a real Banach space must be of dense type. This provides an affirmative answer to a question posed by Stephen Simons in 2001 and implies that various important notions of monotonicity coincide.

1   Introduction

Throughout this note, we assume that X is a real Banach space with norm $\|\cdot\|$, that X^* is the continuous dual of X, and that X and X^* are paired by $\langle\cdot,\cdot\rangle$. Let $A\colon X \rightrightarrows X^*$ be

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a *set-valued operator* (also known as multifunction) from X to X^*, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A = \left\{(x, x^*) \in X \times X^* \mid x^* \in Ax\right\}$ denote the *graph* of A. The *domain* of A is $\operatorname{dom} A = \left\{x \in X \mid Ax \neq \varnothing\right\}$, while $\operatorname{ran} A = A(X)$ is the *range* of A. Recall that A is *monotone* if

$$(1) \qquad \langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*) \in \operatorname{gra} A \; \forall (y, y^*) \in \operatorname{gra} A,$$

and *maximally monotone* if A is monotone and A admits no proper monotone extension. It will be convenient to also say that $\operatorname{gra} A$ is monotone or maximally monotone respectively in this case. We can then simply say that $(x, x^*) \in X \times X^*$ is monotonically related to $\operatorname{gra} A$ if $\{(x, x^*)\} \cup \operatorname{gra} A$ is monotone.

We now recall the three fundamental types of monotonicity.

Definition 1.1 *Let $A : X \rightrightarrows X^*$ be maximally monotone. Then three key types of monotone operators are defined as follows.*

(i) *A is of dense type or type (D) (1971, [11]) if for every $(x^{**}, x^*) \in X^{**} \times X^*$ with*

$$\inf_{(a, a^*) \in \operatorname{gra} A} \langle a - x^{**}, a^* - x^* \rangle \geq 0,$$

there exist a bounded net $(a_\alpha, a_\alpha^*)_{\alpha \in \Gamma}$ in $\operatorname{gra} A$ such that $(a_\alpha, a_\alpha^*)_{\alpha \in \Gamma}$ weak*$\times$strong converges to (x^{**}, x^*).

(ii) *A is of type negative infimum (NI) (1996, [16]) if*

$$\sup_{(a, a^*) \in \operatorname{gra} A} \left(\langle a, x^* \rangle + \langle a^*, x^{**} \rangle - \langle a, a^* \rangle \right) \geq \langle x^{**}, x^* \rangle, \quad \forall (x^{**}, x^*) \in X^{**} \times X^*.$$

(iii) *A is of type Fitzpatrick-Phelps (FP) (1992, [10]) if whenever U is an open convex subset of X^* such that $U \cap \operatorname{ran} A \neq \varnothing$, $x^* \in U$, and $(x, x^*) \in X \times X^*$ is monotonically related to $\operatorname{gra} A \cap (X \times U)$ it must follow that $(x, x^*) \in \operatorname{gra} A$.*

All three of these properties are known to hold for the subgradient of a closed convex function and for every maximally monotone operator on a reflexive space. These and other relationships known amongst these and other monotonicity notions are described in [6, Chapter 8]. Monotone operators are fundamental objects in modern Optimization and Variational Analysis; see, e.g., [3, 4, 5], the books [2, 6, 7, 14, 17, 19, 15, 21] and the references therein.

In Theorem 3.1 of this paper, we provide an affirmative to the following question, posed by S. Simons [18, Problem 18, page 406]:

> *Let $A : X \rightrightarrows X^*$ be maximally monotone such that A is of type (FP).*
> *Is A necessarily of type (D)?*

In consequence, in Corollary 3.2 we record that the three notions in Definition 1.1 actually coincide.

We shall utilize the following notation, in addition to standard notions from convex analysis: The *open unit ball* in X is $U_X = \{x \in X \mid \|x\| < 1\}$, and the *closed unit ball* is $\{x \in X \mid \|x\| \leq 1\}$. It is very convenient to identify X with its canonical image in the bidual space X^{**}. Moreover, $X \times X^*$ and $(X \times X^*)^* = X^* \times X^{**}$ are paired via

$$\langle (x, x^*), (y^*, y^{**}) \rangle = \langle x, y^* \rangle + \langle x^*, y^{**} \rangle,$$

where $(x, x^*) \in X \times X^*$ and $(y^*, y^{**}) \in X^* \times X^{**}$. We recall the following basic fact regarding the second dual ball:

Fact 1.2 (Goldstine) (See [13, Theorem 2.6.26] or [8, Theorem 3.27].) *The weak*-closure of B_X in X^{**} is $B_{X^{**}}$.*

The remainder of this paper is organized as follows. In Section 2, we record auxiliary results for subsequent use. The main result (Theorem 3.1) and the promised corollary (Corollary 3.2) are provided in Section 3.

2 Preliminary monotonicity results

A now fundamental tool of modern monotone operator theory originated with Simon Fitzpatrick in 1988. It is reprised next:

Fact 2.1 (Fitzpatrick) (See [9, Corollary 3.9].) *Let $A\colon X \rightrightarrows X^*$ be maximally monotone, and let us set*

$$(2) \qquad F_A\colon X^{**} \times X^* \to \,]{-\infty}, +\infty] : (x^{**}, x^*) \mapsto \sup_{(a, a^*) \in \operatorname{gra} A} \big(\langle x^{**}, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle \big).$$

Then for every $(x, x^*) \in X \times X^*$, the inequality

$$\langle x, x^* \rangle \leq F_A(x, x^*)$$

is true, and equality holds if and only if $(x, x^*) \in \operatorname{gra} A$.
The function $F_A|_{X \times X^*}$ is the classical Fitzpatrick function *associated with A.*

The first relevant relationship established for (FP) operators is due to Stephen Simons:

Fact 2.2 (Simons) (See [18, Theorem 17] or [19, Theorem 37.1].) *Let $A : X \rightrightarrows X^*$ be maximally monotone and of type (D). Then A is of type (FP).*

The most powerful current information is captured in the following result.

Fact 2.3 (Simons / Marques Alves and Svaiter) (See [16, Lemma 15] or [19, Theorem 36.3(a)], and [12, Theorem 4.4].) *Let $A: X \rightrightarrows X^*$ be maximally monotone. Then A is of type (D) if and only if it is of type (NI).*

The implication type (NI) implies type (D) — which we exploit — is very recently due to Marques Alves and Svaiter [12].

3   Main result

The next theorem is our main result. In conjunction with the corollary that follows, it provides the affirmative answer promised to Simons's problem posed in [18, Problem 18].

Theorem 3.1 *Let $A: X \rightrightarrows X^*$ be maximally monotone such that A is of type (FP). Then A is of type (NI).*

Proof. After translating the graph if necessary, we can and do suppose that $(0,0) \in \operatorname{gra} A$. Let $(x_0^{**}, x_0^*) \in X^{**} \times X^*$. We must show that

$$(3) \qquad\qquad F_A(x_0^{**}, x_0^*) \geq \langle x_0^{**}, x_0^* \rangle$$

and we consider two cases.

Case 1: $x_0^{**} \in X$.
Then (3) follows directly from Fact 2.1.

Case 2: $x_0^{**} \in X^{**} \smallsetminus X$.
By Fact 1.2, there exists a bounded net $(x_\alpha)_{\alpha \in I}$ in X that weak* converges to x_0^{**}. Thus, we have

$$(4) \qquad\qquad M = \sup_{\alpha \in I} \|x_\alpha\| < +\infty$$

and

$$(5) \qquad\qquad \langle x_\alpha, x_0^* \rangle \to \langle x_0^{**}, x_0^* \rangle.$$

Now we consider two subcases.

Subcase 2.1: There exists $\alpha \in I$, such that $(x_\alpha, x_0^*) \in \operatorname{gra} A$.
By definition,

$$F_A(x_0^{**}, x_0^*) \geq \langle x_\alpha, x_0^* \rangle + \langle x_0^{**}, x_0^* \rangle - \langle x_\alpha, x_0^* \rangle = \langle x_0^{**}, x_0^* \rangle.$$

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Hence (3) holds.

Subcase 2.2: We have

(6)
$$(x_\alpha, x_0^*) \notin \operatorname{gra} A, \quad \forall \alpha \in I.$$

Set

(7)
$$U_\varepsilon = [0, x_0^*] + \varepsilon U_{X^*},$$

where $\varepsilon > 0$. Observe that U_ε is open and convex. Since $(0,0) \in \operatorname{gra} A$, we have, by definition of U_ε, $0 \in \operatorname{ran} A \cap U_\varepsilon$ and $x_0^* \in U_\varepsilon$. In view of (6) and because A is of type (FP), there exists a net $(a_{\alpha,\varepsilon}, a_{\alpha,\varepsilon}^*)$ in $\operatorname{gra} A$ such that $a_{\alpha,\varepsilon}^* \in U_\varepsilon$ and

(8)
$$\langle a_{\alpha,\varepsilon}, x_0^* \rangle + \langle x_\alpha, a_{\alpha,\varepsilon}^* \rangle - \langle a_{\alpha,\varepsilon}, a_{\alpha,\varepsilon}^* \rangle > \langle x_\alpha, x_0^* \rangle, \quad \forall \alpha \in I.$$

Now fix $\alpha \in I$. By (8),

$$\langle a_{\alpha,\varepsilon}, x_0^* \rangle + \langle x_0^{**}, a_{\alpha,\varepsilon}^* \rangle - \langle a_{\alpha,\varepsilon}, a_{\alpha,\varepsilon}^* \rangle > \langle x_0^{**} - x_\alpha, a_{\alpha,\varepsilon}^* \rangle + \langle x_\alpha, x_0^* \rangle.$$

Hence,

(9)
$$F_A(x_0^{**}, x_0^*) > \langle x_0^{**} - x_\alpha, a_{\alpha,\varepsilon}^* \rangle + \langle x_\alpha, x_0^* \rangle.$$

Since $a_{\alpha,\varepsilon}^* \in U_\varepsilon$, there exist

(10)
$$t_{\alpha,\varepsilon} \in [0,1] \text{ and } b_{\alpha,\varepsilon}^* \in U_{X^*}$$

such that

(11)
$$a_{\alpha,\varepsilon}^* = t_{\alpha,\varepsilon} x_0^* + \varepsilon b_{\alpha,\varepsilon}^*.$$

Using (9), (11), and (4), we deduce that

$$
\begin{aligned}
F_A(x_0^{**}, x_0^*) &> \langle x_0^{**} - x_\alpha, t_{\alpha,\varepsilon} x_0^* + \varepsilon b_{\alpha,\varepsilon}^* \rangle + \langle x_\alpha, x_0^* \rangle \\
&= t_{\alpha,\varepsilon} \langle x_0^{**} - x_\alpha, x_0^* \rangle + \varepsilon \langle x_0^{**} - x_\alpha, b_{\alpha,\varepsilon}^* \rangle + \langle x_\alpha, x_0^* \rangle \\
&\geq t_{\alpha,\varepsilon} \langle x_0^{**} - x_\alpha, x_0^* \rangle - \varepsilon \| x_0^{**} - x_\alpha \| + \langle x_\alpha, x_0^* \rangle \\
&\geq t_{\alpha,\varepsilon} \langle x_0^{**} - x_\alpha, x_0^* \rangle - \varepsilon (\| x_0^{**} \| + M) + \langle x_\alpha, x_0^* \rangle.
\end{aligned}
$$
(12)

In view of (10) and since $\alpha \in I$ was chosen arbitrarily, we take the limit in (12) and obtain with the help of (5) that

(13)
$$F_A(x_0^{**}, x_0^*) \geq -\varepsilon (\| x_0^{**} \| + M) + \langle x_0^{**}, x_0^* \rangle.$$

Next, letting $\varepsilon \to 0$ in (13), we have

(14)
$$F_A(x_0^{**}, x_0^*) \geq \langle x_0^{**}, x_0^* \rangle.$$

Therefore, (3) holds in all cases. ∎

We now obtain the promised corollary:

Corollary 3.2 *Let $A\colon X \rightrightarrows X^*$ be maximally monotone. Then the following are equivalent.*

(i) *A is of type (D).*

(ii) *A is of type (NI).*

(iii) *A is of type (FP).*

Proof. First *(i) implies (iii)* is Fact 2.2; next Theorem 3.1 shows *(iii) implies (ii)*; while Fact 2.3 implies concludes the circle with *(ii) implies (i)*. ∎

We note that while the result is now quite easy, it remained inaccessible until [12, Theorem 4.4] was available.

Remark 3.3 *Let $A\colon X \rightrightarrows X^*$ be maximally monotone.* Corollary 3.2 establishes the equivalences of the key types (D), (NI), and (FP), which as noted all hold when X is reflexive or $A = \partial f$, where $f\colon X \to \,]{-\infty}, +\infty]$ is convex, lower semicontinuous, and proper (see [6, 17, 19]).

Furthermore, these notions are also equivalent to type (ED), see [20]. For a nonlinear operator they also coincide with *uniqueness* of maximal extensions to X^{**} (see [12]). In [6, p. 454] there is discussion of this result and of the linear case.

Finally, when A is a linear relation, it has recently been established that all these notions coincide with monotonicity of the *adjoint* multifunction A^* (see [1]).

Acknowledgments

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