# A spectrahedral representation of the first derivative relaxation of the positive semidefinite cone 

James Saunderson*

September 5, 2018


#### Abstract

If $X$ is an $n \times n$ symmetric matrix, then the directional derivative of $X \mapsto \operatorname{det}(X)$ in the direction $I$ is the elementary symmetric polynomial of degree $n-1$ in the eigenvalues of $X$. This is a polynomial in the entries of $X$ with the property that it is hyperbolic with respect to the direction $I$. The corresponding hyperbolicity cone is a relaxation of the positive semidefinite (PSD) cone known as the first derivative relaxation (or Renegar derivative) of the PSD cone. A spectrahedal cone is a convex cone that has a representation as the intersection of a subspace with the cone of PSD matrices in some dimension. We show that the first derivative relaxation of the PSD cone is a spectrahedral cone, and give an explicit spectrahedral description of size $\binom{n+1}{2}-1$. The construction provides a new explicit example of a hyperbolicity cone that is also a spectrahedron. This is consistent with the generalized Lax conjecture, which conjectures that every hyperbolicity cone is a spectrahedron.


## 1 Introduction

### 1.1 Preliminaries

Hyperbolic polynomials, hyperbolicity cones, and spectrahedra A multivariate polynomial $p$, homogeneous of degree $d$ in $n$ variables, is hyperbolic with respect to $e \in \mathbb{R}^{n}$ if $p(e) \neq 0$ and for all $x$, the univariate polynomial $t \mapsto p(x-t e)$ has only real roots. Associated with such a polynomial is a cone

$$
\Lambda_{+}(p, e)=\left\{x \in \mathbb{R}^{n}: \text { all roots of } t \mapsto p(x-t e) \text { are non-negative }\right\} .
$$

A foundational result of Gårding [Går59] is that $\Lambda_{+}(p, e)$ is actually a convex cone, called the closed hyperbolicity cone associated with $p$ and $e$.

For example $p(x)=\prod_{i=1}^{n} x_{i}$ is hyperbolic with respect to $1_{n}$, the vector of all ones, and the corresponding closed hyperbolicity cone is the non-negative orthant, $\mathbb{R}_{+}^{n}$. Similarly $p(X)=\operatorname{det}(X)$ (where $X$ is a symmetric $n \times n$ matrix), is hyperbolic with respect to the identity matrix $I$, and the corresponding closed hyperbolicity cone is the positive semidefinite cone $\mathcal{S}_{+}^{n}$.

If a polynomial $p$ has a representation of the form

$$
\begin{equation*}
p(x)=\operatorname{det}\left(\sum_{i=1}^{n} A_{i} x_{i}\right) \tag{1}
\end{equation*}
$$

[^0]for symmetric matrices $A_{1}, \ldots, A_{n}$, and there exists $e \in \mathbb{R}^{n}$ such that $\sum_{i=1}^{n} A_{i} e_{i}$ is positive definite, we say that $p$ has a definite determinantal representation. In this case $p$ is hyperbolic with respect to $e$. The associated closed hyperbolicity cone is
\[

$$
\begin{equation*}
K=\left\{x \in \mathbb{R}^{n}: \sum_{i=1}^{n} A_{i} x_{i} \succeq 0\right\} \tag{2}
\end{equation*}
$$

\]

where we write $X \succeq 0$ to indicate that $X$ is positive semidefinite (and $X \succ 0$ to indicate that $X$ is positive definite). Such convex cones are called spectrahedral cones. If the matrices $A_{1}, A_{2}, \ldots, A_{n}$ are $d \times d$ we call (2) a spectrahedral representation of size $d$.

Derivative relaxations One way to produce new hyperbolic polynomials is to take directional derivatives of hyperbolic polynomials in directions of hyperbolicity [ABG70, Section 3.10], a construction emphasized in the context of optimization by Renegar [Ren06]. If $p$ has degree $d$ and is hyperbolic with respect to $e$, then for $k=0,1, \ldots, d$, the $k$ th directional derivative in the direction $e$, i.e.,

$$
D_{e}^{(k)} p(x)=\left.\frac{d^{k}}{d t^{k}} p(x+t e)\right|_{t=0}
$$

is also hyperbolic with respect to $e$. Moreover

$$
\Lambda_{+}\left(D_{e}^{(k)} p, e\right) \supseteq \Lambda_{+}\left(D_{e}^{(k-1)} p, e\right) \supseteq \cdots \supseteq \Lambda_{+}(p, e)
$$

so the hyperbolicity cones of the directional derivatives form a sequence of relaxations of the original hyperbolcity cone.

- Suppose $p(x)=\prod_{i=1}^{n} x_{i}$ and $e=1_{n}$. Then, for $k=0,1, \ldots, n$,

$$
D_{1_{n}}^{(k)} p(x)=k!e_{n-k}(x)
$$

where $e_{n-k}$ is the elementary symmetric polynomial of degree $n-k$ in $n$ variables. We use the notation $\mathbb{R}_{+}^{n,(k)}$ for $\Lambda_{+}\left(e_{n-k}, 1_{n}\right)$, the closed hyperbolicity cone corresponding to $e_{n-k}$.

- Suppose $p(X)=\operatorname{det}(X)$ is the determinant restricted to $n \times n$ symmetric matrices, and $e=I_{n}$ is the $n \times n$ identity matrix. Then, for $k=0,1, \ldots, n$,

$$
D_{I_{n}}^{(k)} p(X)=k!E_{n-k}(X)=k!e_{n-k}(\lambda(X))
$$

where $E_{n-k}(X)$ is the elementary symmetric polynomial of degree $n-k$ in the eigenvalues of $X$ or, equivalently, the coefficient of $t^{k}$ in $\operatorname{det}\left(X+t I_{n}\right)$. We use the notation $\mathcal{S}_{+}^{n,(k)}$ for $\Lambda_{+}\left(E_{n-k}, I_{n}\right)$, the closed hyperbolicity cone corresponding to $E_{n-k}$. We use the notation $\lambda(X)$ for the eigenvalues of a symmetric matrix $X$ ordered so that $\left|\lambda_{1}(X)\right| \geq\left|\lambda_{2}(X)\right| \geq \cdots \geq$ $\left|\lambda_{n}(X)\right|$. We use this order so that $\lambda_{i}\left(X^{2}\right)=\lambda_{i}(X)^{2}$ for all $i$.

The focus of this paper is the cone $\mathcal{S}_{+}^{n,(1)}$, the hyperbolicity cone associated with $E_{n-1}$. In particular, we consider whether $\mathcal{S}_{+}^{n,(1)}$ can be expressed as a 'slice' of some higher dimensional positive semidefinite cone. Such a description allows one to reformulate hyperbolic programs with respect to $\mathcal{S}_{+}^{n,(1)}$ (linear optimization over affine 'slices' of $\mathcal{S}_{+}^{n,(1)}$ ) as semidefinite programs.

Generalized Lax conjecture We have seen that every spectrahedral cone is a closed hyperbolicity cone. The generalized Lax conjecture asks whether the converse holds, i.e., whether every closed hyperbolicity cone is also a spectrahedral cone. The original Lax conjecture, now a theorem due to Helton and Vinnikov [HV07] (see also [LPR05]), states that if $p$ is a trivariate polynomial, homogeneous of degree $d$, and hyperbolic with respect to $e \in \mathbb{R}^{3}$, then $p$ has a definite determinantal representation. While a direct generalization of this algebraic result does not hold in higher dimensions [Brä11], the following geometric conjecture remains open.

Conjecture 1 (Generalized Lax Conjecture (geometric version)). Every closed hyperbolicity cone is spectrahedral.

An equivalent algebraic formulation of this conjecture is as follows.
Conjecture 2 (Generalized Lax Conjecture (algebraic version)). If $p$ is hyperbolic with respect to $e \in \mathbb{R}^{n}$, then there exists a polynomial $q$, hyperbolic with respect to $e \in \mathbb{R}^{n}$, such that qp has a definite determinantal representation and $\Lambda_{+}(q, e) \supseteq \Lambda_{+}(p, e)$.

The algebraic version of the conjecture implies the geometric version because it implies the existence of a multiplier $q$ such that the hyperbolicity cone associated with $q p$ is spectrahedral and $\Lambda_{+}(q p, e)=\Lambda_{+}(p, e) \cap \Lambda_{+}(q, e)=\Lambda_{+}(p, e)$. To see that the geometric version implies the algebraic version requires more algebraic machinery, and is discussed, for instance, in [Vin12, Section 2].

### 1.2 Main result: a spectrahedral representation of $\mathcal{S}_{+}^{n,(1)}$

In this paper, we show that $\mathcal{S}_{+}^{n,(1)}$, the first derivative relaxation of the positive semidefinite cone, is spectrahedral. We give an explicit spectrahedral representation of $\mathcal{S}_{+}^{n,(1)}$ (see Theorem 1 to follow). Moreover, in Theorem 3 in Section 2 we find an explicit hyperbolic polynomial $q$ such that $q(X) E_{n-1}(X)$ has a definite determinantal representation and $\Lambda_{+}(q, I) \supseteq \mathcal{S}_{+}^{n,(1)}$.
Theorem 1. Let $d=\binom{n+1}{2}-1$ and let $B_{1}, \ldots, B_{d}$ be any basis for the $d$-dimensional space of real symmetric $n \times n$ matrices with trace zero. If $\mathcal{B}(X)$ is the $d \times d$ symmetric matrix with $i, j$ entry equal to $\operatorname{tr}\left(B_{i} X B_{j}\right)$ then

$$
\begin{equation*}
\mathcal{S}_{+}^{n,(1)}=\left\{X \in \mathcal{S}^{n}: \mathcal{B}(X) \succeq 0\right\} . \tag{3}
\end{equation*}
$$

Section 2 is devoted to the proof of this result. At this stage we make a few remarks about the statement and some of its consequences.

- The spectrahedral representation of $\mathcal{S}_{+}^{n,(1)}$ in Theorem 1 has size $d=\binom{n+1}{2}-1=\frac{1}{2}(n+$ $2)(n-1)$. This is about half the size of the smallest previously known projected spectrahedral representation of $\mathcal{S}_{+}^{n,(1)}$, i.e., representation as the image of a spectrahedral cone under a linear map [SP15].
- A straightforward extension of this result shows that if $p$ has a definite determinantal representation and $e$ is a direction of hyperbolicity for $p$, then the hyperbolicity cone associated with the directional derivative $D_{e} p$ is spectrahedral. We discuss this in Section 3.1.
- It also follows from Theorem 1 that $\mathbb{R}_{+}^{n,(2)}$, the second derivative relaxation of the orthant in the direction $1_{n}$, has a spectrahedral representation of size $\binom{n}{2}-1$. We discuss this in Section 3.1. This representation is significantly smaller than the size $O\left(n^{n-3}\right)$ representation constructed by Brändén [Brä14], and about half the size of the smallest previously known projected spectrahedral representation of $\mathbb{R}_{+}^{n,(2)}[\mathrm{SP} 15]$.


### 1.3 Related work

We briefly summarize related work on spectrahedral and projected spectrahedral representations of the hyperbolicity cones $\mathbb{R}_{+}^{n,(k)}$ and $\mathcal{S}_{+}^{n,(k)}$. Sanyal [San13] showed that $\mathbb{R}_{+}^{n,(1)}$ is spectrahedral by giving the following explicit definite determinantal representation of $e_{n-1}(x)$, which we use repeatedly in the paper.

Proposition 1. If $1_{n}^{\perp}=\left\{x \in \mathbb{R}^{n}: 1_{n}^{T} x=0\right\}$, and $V_{n}$ is a $n \times(n-1)$ matrix with columns spanning $1_{n}^{\perp}$, then there is a positive constant $c$ such that

$$
c e_{n-1}(x)=\operatorname{det}\left(V_{n}^{T} \operatorname{diag}(x) V_{n}\right) \text { and so } \mathbb{R}_{+}^{n,(1)}=\left\{x \in \mathbb{R}^{n}: V_{n}^{T} \operatorname{diag}(x) V_{n} \succeq 0\right\} .
$$

This representation is also implicit in the work of Choe, Oxley, Sokal, and Wagner [COSW04]. Zinchenko [Zin08], gave a projected spectrahedral representation of $\mathbb{R}_{+}^{n,(1)}$. Brändén [Brä14], established that each of the cones $\mathbb{R}_{+}^{n,(k)}$ are spectrahedral by constructing graphs $G$ with edges weighted by linear forms in $x$, such that the edge weighted Laplacian $L_{G}(x)$ is positive semidefinite if and only if $x \in \mathbb{R}_{+}^{n,(k)}$. Amini showed that the hyperbolicity cones associated with certain multivariate matching polynomials are spectrahedral [Ami16], and used these to find new spectrahedral representations of the cones $\mathbb{R}_{+}^{n,(k)}$ of size $\frac{(n-1)!}{(k-1)!}+1$.

Explicit projected spectrahedral representations of the cones $\mathcal{S}_{+}^{n,(k)}$ of size $O\left(n^{2} \min \{k, n-k\}\right)$ were given by Saunderson and Parrilo [SP15], leaving open (except in the cases $k=n-2, n-1$ ) the question of whether these cones are spectrahedra. The main result of this paper is that $\mathcal{S}_{+}^{n,(1)}$ is a spectrahedron.

## 2 Proof of Theorem 1

In this section we give two proofs of Theorem 1. The first proof is convex geometric in nature whereas the second is algebraic in nature. Both arguments are self-contained. We present the geometric argument first because it suggests the choice of multiplier $q$ for the algebraic argument.

Both arguments take advantage of the fact that the cone $\mathcal{S}_{+}^{n,(1)}$ satisfies $Q \mathcal{S}_{+}^{n,(1)} Q^{T}=\mathcal{S}_{+}^{n,(1)}$ for all $Q \in O(n)$. One way to see this is to observe that the hyperbolic polynomial $E_{n-1}(X)$ that determines the cone satisfies $E_{n-1}\left(Q X Q^{T}\right)=E_{n-1}(X)$ for all $Q \in O(n)$ and the direction of hyperbolicity (the identity) is also invariant under this group action.

### 2.1 Geometric argument

We begin by stating a slight reformulation of Sanyal's spectrahedral representation (Proposition 1).
Proposition 2. Let $1_{n}^{\perp}=\left\{y \in \mathbb{R}^{n}: 1_{n}^{T} y=0\right\}$ be the subspace of $\mathbb{R}^{n}$ orthogonal to $1_{n}$. Then

$$
\mathbb{R}_{+}^{n,(1)}=\left\{x \in \mathbb{R}^{n}: y^{T} \operatorname{diag}(x) y \geq 0 \text { for all } y \in 1_{n}^{\perp}\right\}
$$

Proof. This follows from Proposition 1 since $V_{n}^{T} \operatorname{diag}(x) V_{n} \succeq 0$ holds if and only if $u^{T} V_{n}^{T} \operatorname{diag}(x) V_{n} u \geq$ 0 for all $u \in \mathbb{R}^{n-1}$ which holds if and only if $y^{T} \operatorname{diag}(x) y \geq 0$ for all $y \in 1_{n}^{\perp}$.

In this section we establish a 'matrix' analogue of Proposition 2.
Theorem 2. Let $I_{n}^{\perp}=\left\{Y \in \mathcal{S}^{n}: \operatorname{tr}(Y)=0\right\}$ be the subspace of $n \times n$ symmetric matrices with trace zero. Then

$$
\begin{equation*}
\mathcal{S}_{+}^{n,(1)}=\left\{X \in \mathcal{S}^{n}: \operatorname{tr}(Y X Y) \geq 0, \text { for all } Y \in I_{n}^{\perp}\right\} . \tag{4}
\end{equation*}
$$

The concrete spectrahedral description given in Theorem 1 follows immediately from Theorem 2. Indeed if $B_{1}, B_{2}, \ldots, B_{d}$ are a basis for $I_{n}^{\perp}$ then an arbitrary $Y \in I_{n}^{\perp}$ can be written as $Y=$ $\sum_{i=1}^{d} y_{i} B_{i}$. The condition $\operatorname{tr}(Y X Y) \geq 0$ for all $Y \in I_{n}^{\perp}$ is equivalent to

$$
\sum_{i, j=1}^{d} y_{i} y_{j} \operatorname{tr}\left(B_{i} X B_{j}\right) \geq 0 \text { for all } y \in \mathbb{R}^{d} \text { which holds if and only if } \mathcal{B}(X) \succeq 0 .
$$

of Theorem 2. The convex cone $\mathcal{S}_{+}^{n,(1)}$ is invariant under the action of the orthogonal group on $n \times n$ symmetric matrices by congruence transformations. Similarly, the convex cone

$$
\left\{X \in \mathcal{S}^{n}: \operatorname{tr}(Y X Y) \geq 0 \quad \text { for all } Y \in I_{n}^{\perp}\right\}
$$

is invariant under the same action of the orthogonal group. This is because $X \in I_{n}^{\perp}$ if and only if $Q X Q^{T} \in I^{\perp}$ for any orthogonal matrix $Q$.

Because of these invariance properties, the following (straightforward) result tells us that we can establish Theorem 2 by showing that the diagonal 'slices' of these two convex cones agree.

Lemma 1. Let $K_{1}, K_{2} \subset \mathcal{S}^{n}$ be such that $Q K_{1} Q^{T}=K_{1}$ for all $Q \in O(n)$ and $Q K_{2} Q^{T}=K_{2}$ for all $Q \in O(n)$. If $\left\{x \in \mathbb{R}^{n}: \operatorname{diag}(x) \in K_{1}\right\}=\left\{x \in \mathbb{R}^{n}: \operatorname{diag}(x) \in K_{2}\right\}$ then $K_{1}=K_{2}$.

Proof. Assume that $X \in K_{1}$. Then there exists $Q$ such that $Q X Q^{T}=\operatorname{diag}(\lambda(X))$. Since $K_{1}$ is invariant under orthogonal congruence, $\operatorname{diag}(\lambda(X)) \in K_{1}$. By assumption, it follows that $\operatorname{diag}(\lambda(X)) \in K_{2}$. Since $K_{2}$ is invariant under orthogonal congruence, $X=Q^{T} \operatorname{diag}(\lambda(X)) Q \in K_{2}$. This establishes that $K_{1} \subseteq K_{2}$. Reversing the roles of $K_{1}$ and $K_{2}$ completes the argument.

Relating the diagonal slices To complete the proof of Theorem 2, it suffices (by Lemma 1) to show that the diagonal slices of the left- and right-hand sides of (4) are equal. Since the diagonal slice of $\mathcal{S}_{+}^{n,(1)}$ is $\mathbb{R}_{+}^{n,(1)}$, it is enough (by Proposition 2) to establish the following result.

## Lemma 2.

$$
\begin{aligned}
& \left\{x \in \mathbb{R}^{n}: \operatorname{tr}(Y \operatorname{diag}(x) Y) \geq 0 \text { for all } Y \in I_{n}^{\perp}\right\}= \\
& \qquad\left\{x \in \mathbb{R}^{n}: y^{T} \operatorname{diag}(x) y \geq 0 \text { for all } y \in 1_{n}^{\perp}\right\} .
\end{aligned}
$$

Proof. Suppose that $\operatorname{tr}(Y \operatorname{diag}(x) Y) \geq 0$ for all $Y \in I_{n}^{\perp}$. Let $y \in 1_{n}^{\perp}$. Then $\operatorname{diag}(y) \in I_{n}^{\perp}$ and so it follows that $\operatorname{tr}(\operatorname{diag}(y) \operatorname{diag}(x) \operatorname{diag}(y))=y^{T} \operatorname{diag}(x) y \geq 0$. This shows that the left hand side is a subset of the right hand side.

For the reverse inclusion suppose that $y^{T} \operatorname{diag}(x) y \geq 0$ for all $y \in 1_{n}^{\perp}$. Let $Y \in I_{n}^{\perp}$. Suppose the symmetric group on $n$ symbols, $S_{n}$, acts on $\mathbb{R}^{n}$ by permutations. Then for every $\sigma \in S_{n}$, we have that $\sigma \cdot \lambda(Y) \in 1_{n}^{\perp}$ and thus

$$
\operatorname{tr}\left(\operatorname{diag}\left(\sigma \cdot \lambda\left(Y^{2}\right)\right) \operatorname{diag}(x)\right)=(\sigma \cdot \lambda(Y))^{T} \operatorname{diag}(x)(\sigma \cdot \lambda(Y)) \geq 0
$$

(Here we have used $\lambda_{i}\left(Y^{2}\right)=\lambda_{i}(Y)^{2}$, by our definition of $\lambda(\cdot)$.)
The diagonal of a symmetric matrix is a convex combination of permutations of its eigenvalues, a result due to Schur [Sch23] (see also, e.g., [MOA79]). Hence $\operatorname{diag}\left(Y^{2}\right)$ is a convex combination of permutations of $\lambda\left(Y^{2}\right)$, i.e.,

$$
\operatorname{diag}\left(Y^{2}\right)=\sum_{\sigma \in S_{n}} \eta_{\sigma}\left(\sigma \cdot \lambda\left(Y^{2}\right)\right)
$$

where the $\eta_{\sigma}$ satisfy $\eta_{\sigma} \geq 0$ and $\sum_{\sigma \in S_{n}} \eta_{\sigma}=1$. It then follows that

$$
\operatorname{tr}(Y \operatorname{diag}(x) Y)=\operatorname{tr}\left(\operatorname{diag}\left(Y^{2}\right) \operatorname{diag}(x)\right)=\sum_{\sigma \in S_{n}} \eta_{\sigma} \operatorname{tr}\left(\operatorname{diag}\left(\sigma \cdot \lambda\left(Y^{2}\right)\right) \operatorname{diag}(x)\right) \geq 0
$$

This shows that the right hand side is a subset of the left hand side.
This completes the proof of Theorem 2.

### 2.2 Algebraic argument

In this section, we establish the following algebraic version of Theorem 1.
Theorem 3. Let $n \geq 2$ and $B_{1}, \ldots, B_{d}$ be a basis for $I_{n}^{\perp}$, the subspace of $n \times n$ symmetric matrices with trace zero. Then there is a positive constant c (depending on the choice of basis) such that

1. $q(X)=\prod_{1 \leq i<j \leq n}\left(\lambda_{i}(X)+\lambda_{j}(X)\right)$ is hyperbolic with respect to $I_{n}$;
2. the hyperbolicity cone associated with $q$ satisfies

$$
\Lambda_{+}\left(q, I_{n}\right)=\left\{X \in \mathcal{S}^{n}: \lambda_{i}(X)+\lambda_{j}(X) \geq 0 \text { for all } 1 \leq i<j \leq n\right\} \supseteq \mathcal{S}_{+}^{n,(1)}
$$

3. $q(X) E_{n-1}(X)$ has a definite determinantal representation as

$$
c q(X) E_{n-1}(X)=\operatorname{det}(\mathcal{B}(X)) .
$$

We remark that $q(X)$ is defined as a symmetric polynomial in the eigenvalues of $X$, and so can be expressed as a polynomial in the entries of $X$. Although our argument does not use this fact, it can be shown that $q(X)=\operatorname{det}\left(\mathcal{L}_{2}(X)\right)$ where $\mathcal{L}_{2}(X)$ is the second additive compound matrix of $X$ [Fie74]. This means that $q$ is not only hyperbolic with respect to $I_{n}$, but also has a definite determinantal representation.
of Theorem 3. The three items in the statement of Theorem 3 are established in the following three Lemmas (Lemmas 3, 4, and 5).
Lemma 3. If $q(X)=\prod_{1 \leq i<j \leq n}\left(\lambda_{i}(X)+\lambda_{j}(X)\right)$ then $q$ is hyperbolic with respect to $I_{n}$.
Proof. First observe that $q\left(I_{n}\right)=2^{\binom{n}{2}} \neq 0$. Moreover, for any real $t$,

$$
q\left(X-t I_{n}\right)=\prod_{1 \leq i<j \leq n}\left(\lambda_{i}\left(X-t I_{n}\right)+\lambda_{j}\left(X-t I_{n}\right)\right)=\prod_{1 \leq i<j \leq n}\left(\lambda_{i}(X)+\lambda_{j}(X)-2 t\right)
$$

which has $\binom{n}{2}$ real roots given by $\frac{1}{2}\left(\lambda_{i}(X)+\lambda_{j}(X)\right)$ for $1 \leq i<j \leq n$. Hence $q$ is hyperbolic with respect to $I_{n}$.

Lemma 4. If $n \geq 2$ then

$$
\Lambda_{+}\left(q, I_{n}\right)=\left\{X \in \mathcal{S}^{n}: \lambda_{i}(X)+\lambda_{j}(X) \geq 0 \text { for all } 1 \leq i<j \leq n\right\} \supseteq \mathcal{S}_{+}^{n,(1)} .
$$

Proof. Since the roots of $t \mapsto q\left(X-t I_{n}\right)$ are $\frac{1}{2}\left(\lambda_{i}(X)+\lambda_{j}(X)\right)$, the description of $\Lambda_{+}\left(q, I_{n}\right)$ is immediate. Both sides of the inclusion are invariant under congruence by orthogonal matrices. By Lemma 1 it is enough to show that the inclusion holds for the diagonal slices of both sides. Note that

$$
\left\{x \in \mathbb{R}^{n}: \operatorname{diag}(x) \in \Lambda_{+}\left(q, I_{n}\right)\right\}=\left\{x \in \mathbb{R}^{n}: x_{i}+x_{j} \geq 0 \text { for all } 1 \leq i<j \leq n\right\} .
$$

Hence it is enough to establish that

$$
\begin{equation*}
\left\{x \in \mathbb{R}^{n}: x_{i}+x_{j} \geq 0 \text { for all } 1 \leq i<j \leq n\right\} \supseteq \mathbb{R}_{+}^{n,(1)} . \tag{5}
\end{equation*}
$$

To do so, we use the characterization of $\mathbb{R}_{+}^{n,(1)}$ from Proposition 2. This tells us that if $x \in \mathbb{R}_{+}^{n,(1)}$ then $v^{T} \operatorname{diag}(x) v=\sum_{\ell=1}^{n} x_{\ell} v_{\ell}^{2} \geq 0$ for all $v \in 1_{n}^{\perp}$. In particular, let $v$ be the element of $1_{n}^{\perp}$ with $v_{i}=1$ and $v_{j}=-1$ and $v_{k}=0$ for $k \notin\{i, j\}$. Then, if $x \in \mathbb{R}_{+}^{n,(1)}$ it follows that $\sum_{\ell=1}^{n} x_{\ell} v_{\ell}^{2}=x_{i}+x_{j} \geq 0$. This completes the proof.

Lemma 5. If $B_{1}, \ldots, B_{d}$ is a basis for $I_{n}^{\perp}$, then there is a positive constant $c$ (depending on the choice of basis) such that

$$
c q(X) E_{n-1}(X)=\operatorname{det}(\mathcal{B}(X))
$$

Proof. Since both sides are invariant under orthogonal congruence, it is enough to show that the identity holds for diagonal matrices. In other words, it is enough to show that

$$
c \prod_{1 \leq i<j \leq n}\left(x_{i}+x_{j}\right) e_{n-1}(x)=\operatorname{det}(\mathcal{B}(\operatorname{diag}(x))) .
$$

Since a change of basis for the subspace of symmetric matrices with trace zero only changes $\operatorname{det}(\mathcal{B}(X))$ by a positive constant (which is one if the change of basis is orthogonal with respect to the trace inner product), it is enough to choose a particular basis for the subspace of symmetric matrices with trace zero, and show that the identity holds for a particular constant.

Let $v_{1}, v_{2}, \ldots, v_{n-1}$ be a basis for $1_{n}^{\perp}=\left\{x \in \mathbb{R}^{n}: \sum_{i=1}^{n} x_{i}=0\right\}$. Let $M_{i j}$ be the $n \times n$ matrix with a one in the $(i, j)$ and the $(j, i)$ entry, and zeros elsewhere. Clearly the $M_{i j}$ for $1 \leq i<j \leq n$ form a basis for the subspace of symmetric matrices with zero diagonal. Together $\operatorname{diag}\left(v_{1}\right), \operatorname{diag}\left(v_{2}\right), \ldots, \operatorname{diag}\left(v_{n-1}\right)$ and $M_{i j}$ for $1 \leq i<j \leq n$ form a basis for the subspace of symmetric matrices with trace zero.

Using this basis we evaluate the matrix $\mathcal{B}(\operatorname{diag}(x))$. We note that

$$
\begin{aligned}
\operatorname{tr}\left(\operatorname{diag}\left(v_{i}\right) \operatorname{diag}(x) \operatorname{diag}\left(v_{j}\right)\right) & =v_{i}^{T} \operatorname{diag}(x) v_{j} \quad \text { for } 1 \leq i, j \leq n \\
\operatorname{tr}\left(\operatorname{diag}\left(v_{i}\right) \operatorname{diag}(x) M_{j k}\right) & =0 \quad \text { for all } 1 \leq i \leq n \text { and } 1 \leq j<k \leq n
\end{aligned}
$$

since $M_{j k}$ has zero diagonal, and that

$$
\operatorname{tr}\left(M_{i j} \operatorname{diag}(x) M_{k \ell}\right)= \begin{cases}x_{i}+x_{j} & \text { if } i=k \text { and } j=\ell \\ 0 & \text { otherwise }\end{cases}
$$

for all $1 \leq i<j \leq n$ and $1 \leq k<\ell \leq n$. This means that $\mathcal{B}(\operatorname{diag}(x))$ is block diagonal, and so

$$
\begin{equation*}
\operatorname{det}(\mathcal{B}(\operatorname{diag}(x)))=\prod_{1 \leq i<j \leq n}\left(x_{i}+x_{j}\right) \operatorname{det}\left(V_{n}^{T} \operatorname{diag}(x) V_{n}\right) \tag{6}
\end{equation*}
$$

where $V_{n}$ is the $n \times(n-1)$ matrix with columns $v_{1}, v_{2}, \ldots, v_{n}$. By Proposition 1 , there is a positive constant $c$ such that

$$
\begin{equation*}
\operatorname{det}\left(V_{n}^{T} \operatorname{diag}(x) V_{n}\right)=c e_{n-1}(x), \tag{7}
\end{equation*}
$$

Combining (6) and (7) gives the stated result.
This completes the proof of Theorem 3.

## 3 Discussion

### 3.1 Consequences of Theorem 1

A straightforward consequence of Theorem 1 is that if $p$ has a definite determinantal representation, and $e$ is a direction of hyperbolicity for $p$, then the hyperbolicity cone associated with the directional derivative $D_{e} p$ is spectrahedral.

Corollary 1. If $p(x)=\operatorname{det}\left(\sum_{i=1}^{n} A_{i} x_{i}\right)$ for symmetric $\ell \times \ell$ matrices $A_{1}, \ldots, A_{n}$, and $A_{0}=$ $\sum_{i=1}^{m} A_{i} e_{i}$ is positive definite, then $\Lambda_{+}\left(D_{e} p, e\right)$ has a spectrahedral representation of size $\binom{\ell+1}{2}-1$.
Proof. The hyperbolicity cone $\Lambda_{+}\left(D_{e} p, e\right)$ can be expressed as

$$
\Lambda_{+}\left(D_{e} p, e\right)=\left\{x \in \mathbb{R}^{n}: \sum_{i=1}^{n} A_{0}^{-1 / 2} A_{i} A_{0}^{-1 / 2} x_{i} \in \mathcal{S}_{+}^{n,(1)}\right\}
$$

(see, e.g., [SP15, Proposition 4]). Applying Theorem 1 then gives

$$
\Lambda_{+}\left(D_{e} p, e\right)=\left\{x \in \mathbb{R}^{n}: \mathcal{B}\left(\sum_{i=1}^{n} A_{0}^{-1 / 2} A_{i} A_{0}^{-1 / 2} x_{i}\right) \succeq 0\right\} .
$$

Our main result also yields a spectrahedral representation of $\mathbb{R}_{+}^{n,(2)}$, the second derivative relaxation of the non-negative orthant, of size $\binom{n}{2}-1$. This is, in fact, a special case of Corollary 1. In the statement below, $V_{n}$ is any $n \times(n-1)$ matrix with columns that span $1_{n}^{\perp}$.

Corollary 2. The hyperbolicity cone $\mathbb{R}_{+}^{n,(2)}$ has a spectrahedral representation of size $\binom{n}{2}-1$ given by

$$
\mathbb{R}_{+}^{n,(2)}=\left\{x \in \mathbb{R}^{n}: \mathcal{B}\left(V_{n}^{T} \operatorname{diag}(x) V_{n}\right) \succeq 0\right\} .
$$

Proof. First, we use the fact that $\mathbb{R}_{+}^{n,(2)}=\Lambda_{+}\left(D_{1_{n}} e_{n-1}, 1_{n}\right)$. Then, by Sanyal's result (Proposition 1), we know that $e_{n-1}(x)$ has a definite determinantal representation. The stated result then follows directly from Corollary 1 with polynomial $p=e_{n-1}$ and direction $e=1_{n}$.

### 3.2 Questions

Constructing spectrahedral representations It is natural to ask for which values of $k$ the cones $\mathcal{S}_{+}^{n,(k)}$ are spectrahedral. Our main result shows that $\mathcal{S}_{+}^{n,(1)}$ has a spectrahedral representation of size $d=\binom{n+1}{2}-1$. The only other cases for which spectrahedral representations are known are the straightforward cases $k=n-1$ and $k=n-2$. If $k=n-1$ then

$$
\mathcal{S}_{+}^{n,(n-1)}=\left\{X \in \mathcal{S}^{n}: \operatorname{tr}(X) \geq 0\right\}
$$

is a spectrahedron (with a representation of size 1). Since $\mathcal{S}_{+}^{n,(n-2)}$ is a quadratic cone, it is a spectrahedron. To give an explicit representation, let $d=\binom{n+1}{2}-1$ and $B_{1}, B_{2}, \ldots, B_{d}$ be an orthonormal basis (with respect to the trace inner product) for the subspace $I_{n}^{\perp}$. Now $X \in \mathcal{S}_{+}^{n,(n-2)}$ if and only if (see, e.g., [SP15, Section 5.1])

$$
\begin{equation*}
\operatorname{tr}(X) \geq 0 \text { and } \operatorname{tr}(X)^{2}-\operatorname{tr}\left(X^{2}\right)=\left[\sqrt{\frac{n-1}{n}} \operatorname{tr}(X)\right]^{2}-\sum_{i=1}^{d} \operatorname{tr}\left(B_{i} X\right)^{2} \geq 0 . \tag{8}
\end{equation*}
$$

By a well-known spectrahedral representation of the second-order cone, (8) holds if and only if

$$
\sqrt{\frac{n-1}{n}} \operatorname{tr}(X) I_{d}+\left[\begin{array}{ccccc}
\operatorname{tr}\left(B_{1} X\right) & \operatorname{tr}\left(B_{2} X\right) & \operatorname{tr}\left(B_{3} X\right) & \cdots & \operatorname{tr}\left(B_{d} X\right)  \tag{9}\\
\operatorname{tr}\left(B_{2} X\right) & -\operatorname{tr}\left(B_{1} X\right) & 0 & \cdots & 0 \\
\operatorname{tr}\left(B_{3} X\right) & 0 & -\operatorname{tr}\left(B_{1} X\right) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\operatorname{tr}\left(B_{d} X\right) & 0 & 0 & \cdots & -\operatorname{tr}\left(B_{1} X\right)
\end{array}\right] \succeq 0
$$

So we see that $\mathcal{S}_{+}^{n,(n-2)}$ has a spectrahedral representation of size $d=\binom{n+1}{2}-1$. At this stage, it is unclear how to extend the approach in this paper to the remaining cases.

Question 1. Are the cones $\mathcal{S}_{+}^{n,(k)}$ spectrahedral for $k=2,3, \ldots, n-3$ ?
At first glance, it may seem that Corollary 1 allows us to construct a spectrahedral representation for $\mathcal{S}_{+}^{n,(2)}$ from a spectrahedral representation for $\mathcal{S}_{+}^{n,(1)}$. However, this is not the case. To apply Corollary 1 to this situation, we would need a definite determinantal representation of $E_{n-1}(X)$, which our main result (Theorem 1) does not provide.

Lower bounds on size Another natural question concerns the size of spectrahedral representations of hyperbolicity cones. Given a hyperbolicity cone $K$, there is a unique (up to scaling) hyperbolic polynomial $p$ of smallest degree $d$ that vanishes on the boundary of $K$ (see, e.g., [Kum16]). Clearly any spectrahedral representation must have size at least $d$, but it seems that in some cases the smallest spectrahedral representation (if it exists at all) must have larger size.

Question 2. Is there a spectrahedral representation of $\mathcal{S}_{+}^{n,(1)}$ with size smaller than $\binom{n+1}{2}-1$ ?
Recently, there has been considerable interest in developing methods for producing lower bounds on the size of projected spectrahedral descriptions of convex sets (see, e.g., $\left[\mathrm{FGP}^{+} 15\right]$ ) . There has been much less development in the case of lower bounds on the size of spectrahedral descriptions. The main work in this direction is due to Kummer [Kum16]. For instance it follows from [Kum16, Theorem 1] that any spectrahedral representation of the quadratic cone $\mathcal{S}_{+}^{n,(n-2)}$ must have size at least $\frac{1}{2}\left[\binom{n+1}{2}-1\right]$. Furthermore, in the special case that $\binom{n+1}{2}-1=2^{k}+1$ for some $k$ (which occurs if $n=3$ and $k=2$ or $n=4$ and $k=3$ ) then Kummer's work shows that any spectrahedral representation of $\mathcal{S}_{+}^{n,(n-2)}$ must have size at least $\binom{n+1}{2}-1$. This establishes that the construction in (9) is optimal when $n=3$ and $n=4$. Furthermore, in the case $n=3$ we have that $\mathcal{S}_{+}^{n,(1)}=\mathcal{S}_{+}^{n,(n-2)}$. Hence our spectrahedral representation for $\mathcal{S}_{+}^{n,(1)}$ is also optimal if $n=3$.

## Acknowledgments

I would like to thank Hamza Fawzi for providing very helpful feedback on a draft of this paper.

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[^0]:    *Department of Electrical and Computer Systems Engineering, Monash University, VIC 3800, Australia. Email: james.saunderson@monash.edu

