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#### Abstract

In this note, we prove that for homogeneous polynomial optimization on the sphere, if the objective f is generic in the input space, all feasible points satisfying the first order and second order necessary optimality conditions are local minimizers, which addresses an issue raised in the recent work by Lasserre (Optimization Letters, 2021). As a corollary, this implies that Lasserre's hierarchy has finite convergence when f is generic.

**Keywords:** Homogeneous polynomials  $\cdot$  Optimization on the unit sphere  $\cdot$  Optimality conditions

#### 1 Introduction

Consider the optimization problem

$$\begin{cases}
\min f(x) \\
s.t. \ x \in \mathbb{S}^{n-1},
\end{cases}$$
(1.1)

where f(x) is a homogeneous polynomial of degree d and  $\mathbb{S}^{n-1}$  denotes n-dimensional unit sphere, i.e.,

$$\mathbb{S}^{n-1} := \{ x \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 = 1 \}.$$

This problem has broad applications in quantum entanglement, tensor decompositions and so on, referring to [1–3, 6] for details.

As a special case of general polynomial optimization problems, the classical Lasserre type Moment-SOS hiearchy of semidefinite relaxations [5] is efficient for solving (1.1) globally, i.e., in general the optimal value and global minimizers can be computed efficiently. Asymptotic convergence is always guaranteed since the quadratic module generated by the constraining polynomial is archimedean. Convergence rate of Lasserre's hierarchy has been studied in [3, 8]. For general polynomial optimization problems, it was shown in [7] that the Lasserre's hierarchy converges in finite steps generically under the archimedeanness. To be more specific, Nie [7] proved that the Lasserre's hierarchy has finite convergence if the linear independence constraint qualification, strict complementarity and second order sufficient conditions hold at every global minimizer and these optimality conditions hold at every local minimizer generically (we say a property holds generically if it holds except a zero measure set in the input space). Recently, Lasserre [6] has characterized all points that satisfy first and second order necessary optimality conditions, only in terms of f, its gradient and the two smallest eigenvalues of its Hessian, and he also conjectured that generically all feasible points of (1.1) satisfying first and second order necessary optimality conditions are local minimizers, for fixed degree d.

In this paper, we show that for fixed degree d, all feasible points of (1.1) satisfying the first and second order necessary optimality conditions also satisfy the second order sufficient condition except on a zero measure set in the input space. This result gives a positive answer to the issue raised by Lasserre, since every feasible point of (1.1) satisfying the first and second order sufficient optimality conditions is a local minimizer. As a direct corollary, Lasserre's hierarchy has finite convergence for optimizing homogeneous polynomials on the unit sphere generically. We would like to remark that the result of Nie [7] can not be applied directly. For fixed degree d, suppose d and d suppose d suppose d and d suppose d suppose d and d suppose d supp

$$\min f(x) \quad \text{s.t. } g(x) = 0, \tag{1.2}$$

it is true that the second order sufficient condition holds at every feasible point of (1.2) satisfying the first and second order necessary optimality conditions when f is generic in the space of polynomials with degree  $\leq d$  and g is generic in the space of polynomials with degree  $\leq 2$ . When it specializes to the case where f is required to be homogeneous of degree d and g is the fixed polynomial  $||x||^2 - 1$ , Nie's result can not be applied. This is because the set of homogeneous polynomials of degree d is already a zero measure set in the space of polynomials with degree  $\leq d$  and the same for  $||x||^2 - 1$ . The similar observation was also found by Lasserre in [6]. Throughout the paper, we assume  $d \geq 1$  because the case d = 0 is trivial.

In Section 2, we address required preliminaries and the main results are presented in Section 3.

#### 2 Preliminaries

We review some basic results on optimality conditions for homogeneous polynomial optimization on the sphere. For every  $x \in \mathbb{S}^{n-1}$ , let

$$x^{\perp} := \{ u \in \mathbb{S}^{n-1} : u^T x = 0 \}.$$

**Proposition 2.1** ([6], Proposition 2.1) If  $x^* \in \mathbb{S}^{n-1}$  is a local minimizer of (1.1), then there exists  $\lambda^* \in \mathbb{R}$  such that:

(i) The first order necessary condition (FONC) holds:

$$\nabla f(x^*) - \lambda^* x^* = 0. \tag{2.1}$$

(ii) The second order necessary condition (SONC) holds:

$$u^T \nabla^2 f(x^*) u - \lambda^* \ge 0, \quad \forall u \in (x^*)^{\perp}. \tag{2.2}$$

Conversely, if  $x^* \in \mathbb{S}^{n-1}$  satisfies (2.1) and the second order sufficient condition (SOSC)

$$u^{T} \nabla^{2} f(x^{*}) u - \lambda^{*} > 0, \quad \forall u \in (x^{*})^{\perp},$$
 (2.3)

then  $x^*$  is a local minimizer of (1.1).

A point  $x^* \in \mathbb{S}^{n-1}$  is called an SONC (resp., SOSC) point of (1.1) if  $x^*$  satisfies the FONC and SONC (resp., SOSC). We need the elimination theorem for general homogeneous polynomial systems to prove our main result.

**Theorem 2.2** ([4], Theorem 5.7A, Chapter 1) Let  $f_1, \ldots, f_r$  be homogeneous polynomials in  $x_0, \ldots, x_n$ , having indeterminate coefficients  $a_{ij}$ . Then there is a set  $g_1, \ldots, g_t$  of polynomials in the  $a_{ij}$ , with integer coefficients, which are homogeneous in the coefficients of each  $f_i$  separately, with the following property: for any field k, and for any set of special values of the  $a_{ij} \in k$ , a necessary and sufficient condition for the  $f_i$  to have a common zero different from  $(0, \ldots, 0)$  is that the  $a_{ij}$  are a common zero of the polynomials  $g_i$ .

#### 3 Main result

In this section, we prove that for a fixed degree d, every SONC point of (1.1) satisfies the SOSC except on a zero measure set in the input space. The following is a useful lemma.

**Lemma 3.1** Suppose  $x^* \in \mathbb{S}^{n-1}$ . If  $x^*$  is an SONC point of (1.1) and the SOSC fails at  $x^*$ , then there exists a nonzero  $y^* \in \mathbb{R}^n$  such that

$$\operatorname{rank} \begin{bmatrix} \nabla f(x^*) & x^* & 0 \\ \nabla^2 f(x^*) y^* & y^* & x^* \end{bmatrix} \le 2, \quad (y^*)^T x^* = 0. \tag{3.1}$$

Conversely, if (3.1) holds for a nonzero  $y^* \in \mathbb{R}^n$ , then the FONC holds at  $x^*$  while the SOSC fails.

*Proof* Since  $x^*$  is an SONC point, we have  $\nabla f(x^*) = \lambda^* x^*$  for some  $\lambda^* \in \mathbb{R}$ , by Proposition 2.1. If the SOSC fails at  $x^*$ , then there exists  $0 \neq y^* \in \mathbb{S}^{n-1}$  satisfying

$$(y^*)^T \nabla^2 f(x^*) y^* - \lambda^* = 0, \quad (y^*)^T x^* = 0.$$

It implies that  $y^*$  is a minimizer of the problem  $\min_{z \in (x^*)^{\perp}} z^T \nabla^2 f(x^*) z$ . By the first order optimality condition, we have  $\nabla^2 f(x^*) y^* = \lambda^* y^* + \beta x^*$ , for some  $\beta \in \mathbb{R}$ . Thus  $(x^*, v)$  satisfies (3.1).

For the converse, suppose (3.1) holds for a nonzero  $y^* \in \mathbb{R}^n$ . Then there exists a nonzero  $\beta := (\beta_1, \beta_2, \beta_3)$  such that

$$\beta_1 \nabla f(x^*) + \beta_2 x^* = 0, \ \beta_1 \nabla^2 f(x^*) y^* + \beta_2 y^* + \beta_3 x^* = 0.$$

If  $\beta_1 = 0$ , then  $\beta_2 = \beta_3 = 0$  since  $x^* \in \mathbb{S}^{n-1}$ . Thus  $\beta_1 \neq 0$ , and we have

$$\nabla f(x^*) + \frac{\beta_2}{\beta_1} x^* = 0, \ (y^*)^T \nabla^2 f(x^*) y^* + \frac{\beta_2}{\beta_1} ||y^*||^2 = 0.$$

It implies that the FONC holds at  $x^*$ , while the SOSC fails.

Hence, if the SOSC fails at an SONC point of the problem (1.1), the following system

$$\operatorname{rank} \begin{bmatrix} \nabla f(x) & x & 0 \\ \nabla^2 f(x) y & y & x \end{bmatrix} \le 2, \quad y^T x = 0. \tag{3.2}$$

has a solution  $(x,y) \in \mathbb{C}^{2n}$  with  $x \neq 0$ ,  $y \neq 0$ . This is because that if  $x^*$  is such an SONC point of (1.1) (i.e., SOSC fails at  $x^*$ ), it follows from Lemma 3.1 that there exists a nonzero vector  $y^*$  such that (3.1) holds. Clearly,  $(x^*, y^*)$  is a solution of (3.2) with  $x^* \neq 0$ ,  $y^* \neq 0$ .

Next we investigate when the system (3.2) has a pair of solution  $(x,y) \in \mathbb{C}^n \times \mathbb{C}^n$  with  $x \neq 0$ ,  $y \neq 0$ . When n = 1, the rank condition in (3.2) always holds and can be dropped. When n > 1, we can replace the rank condition by the vanishing of all maximal minors. Thus, (3.2) is equivalent to

$$Q_1(x,y) = \dots = Q_t(x,y) = y^T x = 0$$

for some polynomials  $Q_1, \ldots, Q_t$ , which are homogeneous in both x and y, and their coefficients are also homogeneous in the coefficients of f. By applying Theorem 2.2 in x first, and then in y, there exist polynomials  $\phi_i(f)$  (i = 1)

 $1, \ldots, s$ ) with integer coefficients, homogeneous in the coefficients of f, such that there exist  $0 \neq x \in \mathbb{C}^n$ ,  $0 \neq y \in \mathbb{C}^n$  satisfying (3.2) if and only if

$$\phi_1(f) = \dots = \phi_s(f) = 0.$$

We would like to remark that the property of polynomials  $Q_1, \ldots, Q_t$  being homogeneous in both x and y is important. This is because it allows us to apply the elimination theorem twice, separately in x, y.

Denote by  $\mathbb{R}[x]_{=d}$  the set of all homogeneous polynomials of degree d. Let

$$\phi(f) = \phi_1^2(f) + \dots + \phi_s^2(f)$$
.

Note that  $\phi(f)$  is also a polynomial in the coefficients of f. Proposition 3.2 is directly implied by the analysis above.

**Proposition 3.2** Suppose the polynomial  $f \in \mathbb{R}[x]_{=d}$ . Then for this fixed f, there exist  $0 \neq x \in \mathbb{C}^n$ ,  $0 \neq y \in \mathbb{C}^n$  satisfying (3.2) if and only if  $\phi(f) = 0$ .

If  $\phi(f) = 0$ , then there exist  $0 \neq x^* \in \mathbb{C}^n$ ,  $0 \neq y^* \in \mathbb{C}^n$  satisfying (3.2). It implies that

$$\nabla f(x^*) - \lambda^* x^* = 0, \ \nabla^2 f(x^*) y^* - \lambda^* y^* - \mu^* x^* = 0, \ (y^*)^T x^* = 0,$$

for  $\lambda^* \in \mathbb{C}$ ,  $\mu^* \in \mathbb{C}$ . Note that the vector  $(y^*, \mu^*)$  is nonzero and we have  $H(x^*, \lambda^*)(y^*, \mu^*) = 0$ , where

$$H(x,\lambda) = \begin{bmatrix} \nabla^2 f(x) - \lambda I_n & x \\ x^T & 0 \end{bmatrix}. \tag{3.3}$$

It implies that  $\det(H(x^*, \lambda^*)) = 0$ . Hence, if  $\phi(f) = 0$ , there exist  $0 \neq x^* \in \mathbb{C}^n$ ,  $\lambda^* \in \mathbb{C}$  such that

$$\nabla f(x^*) - \lambda^* x^* = 0, \ \det(H(x^*, \lambda^*)) = 0.$$
 (3.4)

For a complex number z, |z| denotes its modulus. In the following, we prove that  $\phi(f)$  does not identically vanish.

**Lemma 3.3** The polynomial  $\phi(f)$  does not vanish identically in the coefficients of  $f \in \mathbb{R}[x]_{=d}$ .

*Proof* We prove the result by considering the cases d=2 and  $d\neq 2$ . To show that  $\phi(f)$  does not vanish identically, we only need to prove that  $\phi(p)=0$  for a special p.

- 6 Optimality conditions for homogeneous polynomial optimization on the unit sphere
- (1) Suppose d=2. Let  $p(x):=\frac{1}{2}(x_1^2+2x_2^2+\cdots+nx_n^2)$ . If  $\phi(p)=0$ , the equation (3.4) holds for some  $x^*\neq 0$ ,  $\lambda^*\in\mathbb{C}$ . Thus, we have  $kx_k^*=\lambda^*x_k^*$   $(k=1,\ldots,n)$ . Note that there exists  $\ell\in\{1,\ldots,n\}$  such that

$$x_{\ell}^* \neq 0, \quad x_1^* = \dots = x_{\ell-1}^* = x_{\ell+1}^* = \dots = x_n^* = 0.$$

Otherwise, if  $x_i^* \neq 0$ ,  $x_j^* \neq 0$  for  $i \neq j$ , we have  $i = \lambda^* = j$ , which is a contradiction. Hence  $\lambda^* = \ell$ , and the following holds

$$H(x^*, \lambda^*) = \begin{bmatrix} 1 - \ell & & & & \\ & \dots & & & \\ & & 0 & & x_\ell^* \\ & & & \dots & \\ & & & n - \ell & \\ & & x_\ell^* & & 0 \end{bmatrix}.$$

Clearly, we have  $\det(H(x^*, \lambda^*)) \neq 0$ , which contradicts the second equation in (3.4). Hence,  $\phi(p) \neq 0$ .

(2) Suppose  $d \neq 2$ . Let  $\alpha := 2^{d-2}$ ,  $p := \alpha x_1^d + \alpha^2 x_2^d + \dots + \alpha^n x_n^d$ . If  $\phi(p) = 0$ , then there exist  $x^* \neq 0$ ,  $\lambda^* \in \mathbb{C}$  satisfying (3.4). Note that the multiplier  $\lambda^* \neq 0$ , otherwise  $x^*$  would vanish. Without loss of generality, assume that  $x_1^* = \dots = x_\ell^* = 0, x_{\ell+1}^* \neq 0, \dots, x_n^* \neq 0$  for  $\ell \in \{0, 1, \dots, n-1\}$ . From the equation (3.4), the following holds

$$\lambda^* = d\alpha^{\ell+1} (x_{\ell+1}^*)^{d-2} = \dots = d\alpha^n (x_n^*)^{d-2}.$$

Denote  $s^* = (x_{\ell+1}^*, ..., x_n^*)^T$ , we have

$$H(x^*, \lambda^*) = \begin{bmatrix} -\lambda^* I_{\ell} & 0 & 0\\ 0 & (d-2)\lambda^* I_{n-\ell} & s^*\\ 0 & (s^*)^T & 0 \end{bmatrix}.$$

Thus, it holds that

$$\begin{aligned} |\det(H(x^*,\lambda^*))|^{\frac{1}{2}} &= |d-2|^{n-\ell-1}|\lambda^*|^{n-1}|(x^*_{\ell+1})^2 + \dots + (x^*_n)^2| \\ &\geq |d-2|^{n-\ell-1}|\lambda^*|^{n-1}(|x^*_{\ell+1}|^2 - |x^*_{\ell+2}|^2 - \dots - |x^*_n|^2) \\ &\geq |d-2|^{n-\ell-1}|\lambda^*|^{n-1}|\frac{\lambda^*}{d}|^{\frac{2}{d-2}}(\frac{1}{4^{\ell+1}} - \frac{1}{4^{\ell+2}} - \dots - \frac{1}{4^n}) \\ &> 0. \end{aligned}$$

which contradicts the second equation in (3.4). Hence,  $\phi(p) \neq 0$ .

The following is our main result.

**Theorem 3.4** Suppose the polynomial  $f \in \mathbb{R}[x]_{=d}$  satisfies  $\phi(f) \neq 0$ , then every SONC point of (1.1) satisfies the SOSC. Moreover, when f is generic in  $\mathbb{R}[x]_{=d}$ , every SONC point of (1.1) is an SOSC point.

Proof Suppose otherwise the SOSC fails at a SONC point of (1.1). By Lemma 3.1, the system (3.2) is feasible for some  $x^* \neq 0$ ,  $y^* \neq 0$ . It follows from Proposition 3.2 that  $\phi(f) = 0$ , which contradicts the assumption of Theorem 3.4. Since the polynomial  $\phi(f)$  does not vanish identically (cf. Lemma 3.3), the set  $\{f \in \mathbb{R}[x]_{=d} : \phi(f) = 0\}$  is a zero measure subset of  $\mathbb{R}[x]_{=d}$ . Thus, every SONC point of (1.1) is an SOSC point when f is generic in  $\mathbb{R}[x]_{=d}$ .

A direct corollary of Theorem 3.4 is that the standard Lasserre's hierarchy converges in finite steps generically.

**Corollary 3.5** Suppose f is generic in  $\mathbb{R}[x]_{=d}$ , then the Lasserre's hierarchy of (1.1) has finite convergence.

*Proof* Note that every local minimizer of (1.1) is an SONC point. By Theorem 3.4, for generic f, every local minimizer of (1.1) is an SOSC point. We can easily verify that the linear independence constraint qualification, strict complementarity conditions hold at every local minimizer since there is no inequality constraint. Thus the Lasserre's hierarchy of (1.1) has finite convergence for generic f, by Theorem 1.1, [7].

We would like to remark that Theorem 3.4 has a simple principle when d=2.

**Lemma 3.6** Suppose  $f = \frac{1}{2}x^T Ax$  for a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ , and the eigenvalues of A are ordered by  $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ . Then every SONC point of (1.1) satisfies the SOSC if and only if the least eigenvalue  $\lambda_1$  is simple.

*Proof* Note that each point satisfying the FONC is an eigenvector of A with associated eigenvalue  $2f(x^*)$ . Suppose  $x^*$  is an SONC point, then  $x^*$  must be the eigenvector associated with  $\lambda_1$ , by Corollary 2.4, [6]. If  $\lambda_1$  is not simple, there must exist a nonzero  $v \in \mathbb{S}^{n-1}$  such that

$$Av = \lambda_1 v, \quad v^T x^* = 0.$$

Hence, we have

$$v^T \nabla^2 f(x^*) v - \lambda_1 = 0,$$

which implies the SOSC fails at  $x^*$ . On the another hand, suppose  $\lambda_1$  is simple. Let  $v_2, \ldots, v_n$  be the unit orthogonal eigenvectors associated with eigenvalues  $\lambda_2, \ldots, \lambda_n$ , and we have

$$(x^*)^{\perp} = \{u \in \mathbb{S}^{n-1} : u = \mu_2 v_2 + \dots + \mu_n v_n, \ \mu_2, \dots, \mu_n \in \mathbb{R}\}.$$

For any  $u \in (x^*)^{\perp}$ , we have

$$u^{T} \nabla^{2} f(x^{*}) u - \lambda_{1} = \sum_{i=2}^{n} \lambda_{i} \mu_{i}^{2} - \lambda_{1} > \lambda_{1} \sum_{i=2}^{n} \mu_{i}^{2} - \lambda_{1} = 0.$$

Hence,  $x^*$  is an SOSC point.

Note that for generic symmetric matrix A, every eigenvalue is simple, which directly implies that for generic  $f \in \mathbb{R}[x]_{=2}$ , every SONC point of (1.1) is an SOSC point.

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