A Survey on Hypergraph Products

Marc Hellmuth, Lydia Ostermeier and Peter F. Stadler

Erratum: *In the accepted version of this survey* [36] *it is mistakenly stated that the direct products* $\widehat{\times}$ *and* $\widetilde{\times}$ *and the strong product* $\widehat{\boxtimes}$ *are associative. In* [32]*, we gave counterexamples for these cases and proved associativity of the hypergraph products* $\breve{\times}$, $\breve{\boxtimes}$.

Abstract. A surprising diversity of different products of hypergraphs have been discussed in the literature. Most of the hypergraph products can be viewed as generalizations of one of the four standard graph products. The most widely studied variant, the so-called square product, does not have this property, however. Here we survey the literature on hypergraph products with an emphasis on comparing the alternative generalizations of graph products and the relationships among them. In this context the so-called 2-sections and L2-sections are considered. These constructions are closely linked to related colored graph structures that seem to be a useful tool for the prime factor decompositions w.r.t. specific hypergraph products. We summarize the current knowledge on the propagation of hypergraph invariants under the different hypergraph multiplications. While the overwhelming majority of the material concerns finite (undirected) hypergraphs, the survey also covers a summary of the few results on products of infinite and directed hypergraphs.

Mathematics Subject Classification (2000). Primary 99Z99; Secondary 00A00.

Keywords. Hypergraph invariants, products, set systems.

Part 1. Introduction

There are only four "standard graph products" that preserve the salient structure of their factors and behave in an algebraically reasonable way. Their structural

This work was supported in part by the *Deutsche Forschungsgemeinschaft* (DFG) Project STA850/11-1 within the EUROCORES Programme EuroGIGA (project GReGAS) of the European Science Foundation.

features have been studied extensively over the last decades. It is well known how many of the important graph invariants propagate under product formation, and efficient algorithms have been devised to decompose graph products into their prime factors. Several monographs cover the topic in substantial detail and serve as standard references [40, 41, 31].

In contrast, very little is known about product structures of hypergraphs, even though hypergraphs have become increasingly important models of network structures. Here we survey the existing literature, focusing on the basic properties of the various hypergraph products and their mutual relationships. In this introductory part we will first investigate in which sense the standard graph products have distinguished properties. After introducing the necessary notation, and defining the most interesting hypergraph invariants we proceed to discuss a set of desirable properties of hypergraph products that generalize the situation in graphs. Much of the published literature is concerned with the so-called square product, which does *not* arise as a natural generalization of a graph product. Most of the other constructions, albeit less well investigated so far, can be described as generalizations of a corresponding graph product. The link between hypergraph products is also stressed by constructions such as 2-sections and L2-sections [6, 10]. We therefore choose to emphasize the generalizations of graph products in our survey. The following sections are then concerned with a review of the literature on the individual notions of hypergraph products. The literature is complemented by several new results that bridge some of the obvious gaps in particular for the rarely studied products. Our survey also includes a complementary recursive exposition of several new constructions and their basic properties [36].

1. Graph Products

Graph products are natural structures in discrete mathematics [30, 50] that arise in a variety of different contexts, from computer science [3, 33, 34] and computational engineering [45, 46] to theoretical biology [21, 22, 12, 60, 62]. In this section we briefly outline the commonly investigated graph products and their most salient properties to provide a frame of reference for our subsequent discussion of hypergraph products.

We consider only finite and undirected graphs $G = (V, E)$ with non-empty vertex set V and edge set E. A graph is *non-trivial* if it has at least two vertices. Graph products can be constructed in many different ways. For example, different constructions arise depending on whether loops are considered or not. There are, however, three basic properties that are required for any meaningful definition of a graph product:

- (P1) The vertex set of a product is the Cartesian product of the vertex sets of the factors.
- (P2) Adjacency in the product depends on the adjacency properties of the projections of pairs of vertices into the factors.

(P3) The product of a simple graph is a simple graph.

As shown in [39], there are 256 different possibilities to define a graph product satisfying (P1), (P2), and (P3). Only six of them are commutative, associative and have a unit. Only four products satisfy the following additional condition:

(P4) At least one of the projections of a product onto its factors is a so-called weak homomorphism (edges are mapped to edges or to vertices).

These four products are known as the *standard graph products* [40, 31]: the *Cartesian* product $\Box$, the *direct* product $\times$, the *strong* product $\boxtimes$, and the *lexicographic* product $\circ$.

In all products the vertex set $V(G_1 \circledast G_2)$ is defined as the Cartesian product $V(G_1) \times V(G_2)$, $\circledast \in \{\Box,\ \times,\ \boxtimes,\ \circ\}$. Two vertices (x_1, x_2), (y_1, y_2) are adjacent in $G_1 \boxtimes G_2$ if one of the following conditions is satisfied:

(i) $(x_1, y_1) \in E(G_1)$ and $x_2 = y_2$,
(ii) $(x_2, y_2) \in E(G_2)$ and $x_1 = y_1$,
(iii) $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2)$.

In the *Cartesian product* vertices are adjacent if and only if they satisfy (i) or (ii). Consequently, the edges of a strong product that satisfy (i) or (ii) are called *Cartesian* edge, the others are the *non-Cartesian* edges. In the direct product vertices are only adjacent if they satisfy (iii). Thus, the edge set of the strong product is the union of edges in the Cartesian and the direct product. In the lexicographic product vertices are adjacent if and only if $(x_1, y_1) \in E(G_1)$ or they satisfy (ii).

Three of these products, the Cartesian, the direct and the strong product are commutative, associative, and distributive with respect to the disjoint union. The lexicographic product is associative, not commutative, and only left-distributive with respect to the disjoint union. All products have a unit element, that is the single vertex graph K_1 for the Cartesian, the strong and the lexicographic product, and the single vertex graph with a loop $\mathcal{L}K_1$ for the direct product.

Connectedness of the products depends on the connectedness of the factors. The Cartesian and the strong product is connected if and only if all of its factors are connected. The direct product of non-trivial connected factors is connected if and only if at most one factor is bipartite. The lexicographic product $\circ_{i=1}^{n} G_i$ is connected if and only if G_1 is connected. The *costrong product* $G_1 * G_2$, with edge set $E(G_1 \circ G_2) \cup E(G_2 \circ G_1)$, can be seen as a symmetrized version of the lexicographic product. It is also closely related to the strong produce by virtue of the identity $G_1 * G_2 = \overline{\overline{G_1} \boxtimes \overline{G_2}}$ [25].

Connected graphs have a unique *Prime Factor Decomposition* (PFD) w.r.t. the strong and the Cartesian product and connected non-bipartite graphs have a unique PFD w.r.t. the direct product. The PFD w.r.t. the lexicographic product is unique only under strict conditions w.r.t. connectivity properties based on the prime factors [31].

2. Hypergraphs

2.1. Basic Definitions

Hypergraphs are a natural generalization of undirected graphs in which "edges" may consist of more than 2 vertices. More precisely, a *(finite) hypergraph* $H = (V, E)$ consists of a (finite) set V and a collection E of non-empty subsets of V.

The elements of V are called *vertices* and the elements of E are called *hyperedges*, or simply *edges* of the hypergraph. Throughout this survey, we only consider hypergraphs without multiple edges and thus, being E a usual set. If there is a risk of confusion we will denote the vertex set and the edge set of a hypergraph H explicitly by $V(H)$ and $E(H)$, respectively.

A hypergraph $H = (V, E)$ is *simple* if no edge is contained in any other edge and $|e| \geq 2$ for all $e \in E$. The dual H^* of a hypergraph $H = (V, E)$ is the hypergraph whose vertices and edges are interchanged, so that $V(H^*) = \{e_i^* \mid e_i \in E\}$ and edge set $E(H^*) = \{v_i^* \mid v_i \in V\}$ with $v_i^* = \{e_j^* \mid v_i \in e_j\}$.

For a (simple) hypergraph $H = (V, E)$ let $\mathcal{L}H := (V, E \cup \{\{x\} \mid x \in V\})$ denote the hypergraph which is formed from H by adding a loop to each vertex of H. Conversely, for a hypergraph $H' = (V', E')$ let $\mathcal{N}H' := (V', E' \setminus \{\{x\} \mid x \in V'\})$ denote the hypergraph which emerges from H' by deleting all loops.

Two vertices u and v are *adjacent* in $H = (V, E)$ if there is an edge $e \in E$ such that $u, v \in e$. If for two edges $e, f \in E$ holds $e \cap f \neq \emptyset$, we say that e and f are *adjacent*. A vertex v and an edge e of H are *incident* if $v \in e$. The *degree* $\deg(v)$ of a vertex $v \in V$ is the number of edges incident to v. The *maximum degree* $\max_{v \in V} \deg(v)$ is denoted by $\Delta(H)$.

The *rank* of a hypergraph $H = (V, E)$ is $r(H) = \max_{e \in E} |e|$, the *anti-rank* is $s(H) = \min_{e \in E} |e|$. A *uniform hypergraph* H is a hypergraph such that $r(H) = s(H)$. A simple uniform hypergraph of rank r will be called r*-uniform*. A hypergraph with $r(H) \leq 2$ is a *graph*. A 2-uniform hypergraph is usually known as a *simple graph*.

A *partial hypergraph* $H' = (V', E')$ of a hypergraph $H = (V, E)$, denoted by $H' \subseteq H$, is a hypergraph such that $V' \subseteq V$ and $E' \subseteq E$. In the class of graphs partial hypergraphs are called *subgraphs*. The partial hypergraph $H' = (V', E')$ is *induced* if $E' = \{e \in E \mid e \subseteq V'\}$. Induced hypergraphs will be denoted by $\langle V' \rangle$. A partial hypergraph of a simple hypergraph is always simple.

A *walk* in a hypergraph $H = (V, E)$ is a sequence $P_{v_0,v_k} = (v_0, e_1, v_1, e_2, \dots, e_k, v_k)$, where $e_1, \dots, e_k \in E$ and $v_0, \dots, v_k \in V$, such that each $v_{i-1} \neq v_i$ and $v_{i-1}, v_i \in e_i$ for all $i = 1, \dots, k$. The walk P_{v_0,v_k} is said to *join* the vertices v_0 and v_k. A p*-path* is a walk where the vertices $v_0, \dots, v_k$ are all distinct and for all $r, s \in \{1, \dots, k\}, r \leq s$ with $e_r = e_s$ follows that $s - r \leq p - 1$ and $e_r = e_{r+1} = \dots = e_s$. A path between two edges e_i and e_j is any path $P_{v_i v_j}$ joining vertices $v_i \in e_i$ and $v_j \in e_j$. A 1-path is just called a *path*, i.e., all vertices and all edges are different. The minimum number of pairwise vertex and edge disjoint paths of a hypergraph H whose union contains all vertices of H is called *vertex* *path partition number* and will be denoted by $\wp(H)$. Note, the path partion number satisfies $\wp(H) \leq |V(H)|$, since there is always a partition of a hypergraph into paths of length 0. A *cycle* is a sequence $(v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_0)$, such that $P_{v_0, v_{k-1}}$ is a path. A p-path or a cycle is *Hamiltonian* in H if it contains all vertices of H. The *length of a path* or *a cycle* is the number of edges contained in the path or cycle, resp.

The *distance* $d_H(v, v')$ between two vertices v_0, v_k of H is the length of a shortest path joining them. We set $d_H(v, v') = \infty$ if there is no such path. A hypergraph $H = (V, E)$ is called *connected*, if any two vertices are joined by a path. A partial hypergraph $H' \subseteq H$ is called *convex*, if all shortest paths in H between two vertices in H' are also contained in H'.

2.2. Homomorphisms and Covering Constructions

For two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ a *homomorphism* from H_1 into H_2 is a mapping $\varphi : V_1 \to V_2$ such that $\varphi(e) = \{\varphi(v_1), \ldots, \varphi(v_r)\}$ is an edge in H_2, if $e = \{v_1, \ldots, v_r\}$ is an edge in H_1. Note, a homomorphism from H_1 into H_2 implies also a mapping $\varphi_{\mathcal{E}} : E_1 \to E_2$. A mapping $\varphi : V_1 \to V_2$ is a *weak homomorphism* if edges are mapped either on edges or on vertices.

A homomorphism φ that is bijective is called an *isomorphism* if holds $\varphi(e) \in E_2$ if and only if $e \in E_1$. We say, H_1 and H_2 are *isomorphic*, in symbols $H_1 \cong H_2$ if there exists an isomorphism between them. An isomorphism from a hypergraph H onto itself is an *automorphism*.

The hypergraph $H' = (V', E')$ is a *k-fold covering* of a hypergraph $H = (V, E)$ if there is a surjective homomorphism $\pi : H' \to H$ for which

1. $|\pi^{-1}(v)| = |\pi_{\mathcal{E}}^{-1}(e)| = k$ for all $v \in V$, $e \in E$ and
2. $e' \cap f' = \emptyset$ for all distinct e', f' in $\pi_{\mathcal{E}}^{-1}(e)$, $e \in E$.

H is then called the *quotient hypergraph* of H' and π is called the *covering projection* [17]. If $k = 2$, H' is called *double cover* [16].

2.3. $L2$-sections

The notion of so-called $L2$-sections has proved to be an extremely useful tool for hypergraph product recognition algorithms. In the following we therefore consider the 2-section and $L2$-section of hypergraphs [6, 10] in some detail. In the context of 2-sections and $L2$-sections we will consider only hypergraphs without loops throughout this survey.

The 2-section $[H]_2$ of a hypergraph $H = (V, E)$ is the graph (V, E') with $E' = \{\{x, y\} \subseteq V \mid x \neq y, \exists\, e \in E : \{x, y\} \subseteq e\}$, that is, two vertices are adjacent in $[H]_2$ if they belong to the same hyperedge in H. Thus, every hyperedge of H is a clique in $[H]_2$. Note, the 2-section $[H]_2$ of a hypergraph $H = (V, E)$ is only uniquely determined if H is *conformal*, that is, for every subset $W \subseteq V$ holds that if $\langle W \rangle$ is a clique in $[H]_2$ then $W \in E$.

Let $\mathbb{P}(X)$ denotes the power set of the set X. The $L2$-section $[H]_{L2}$ of a hypergraph $H = (V, E)$ is its 2-section together with a mapping $\mathcal{L} : E' \to \mathbb{P}(E)$ with $\mathcal{L}(\{x, y\}) = \{e \in E \mid \{x, y\} \subseteq e\}$. Usually, the $L2$-section $[H]_{L2}$ is written as the triple $\Gamma = (V, E([H]_2), \mathcal{L})$. In addition to 2-sections, the $L2$-section also provides the possibility to trace back the information which of the edges of $[H]_2$ is associated to which of the hyperedges in H. Thus, the original hypergraph can be reconstructed from its $L2$-section. The inverse $[\Gamma]^{-1}_{L2} = (V, E)$ of an $L2$-section $\Gamma = (V, E', \mathcal{L})$ is the hypergraph with $E = \bigcup_{e \in E'} \mathcal{L}(e)$. Hence, the inverse $[\Gamma]^{-1}_{L2}$ of an $L2$-section is the hypergraph $H = (V, E)$ that has $L2$-section Γ.

Two $L2$-sections $\Gamma_1 = (V_1, E_1, \mathcal{L}_1)$ and $\Gamma_2 = (V_2, E_2, \mathcal{L}_2)$ are isomorphic, in symbols $\Gamma_1 \cong \Gamma_2$, if there is an isomorphism φ between the graphs (V_1, E_1) and (V_2, E_2) such that $e \in \mathcal{L}_1(\{x, y\})$ if and only if $\{\varphi(z) \mid z \in e\} \in L_2(\{\varphi(x), \varphi(y)\})$ for all $x, y \in V_1$ and $e \subseteq V_1$. Every hypergraph is uniquely (up to isomorphism) determined by its $L2$-section and *vice versa* [11, 10], i.e., $H \cong H'$ if and only if $[H]_{L2} \cong [H']_{L2}$.

A very useful property of the 2-section is the following:

Lemma 2.1 (Distance Formula). *Let $H = (V, E)$ be a hypergraph and $x, y \in V$. Then the distances between x and y in H and in $[H]_2$ are the same.*

Proof. Note, x and y are in different connected components of H if and only if x and y are in different connected components of $[H]_2$ and hence, $d_H(x, y) = d_{[H]_2}(x, y) = \infty$. Thus, w.l.o.g. assume H (and hence $[H]_2$) to be connected. Let $P = (x, e_1, v_1, \ldots, v_{k-1}, e_k, y)$ denote a shortest path between x and y in H. By construction of $[H]_2$ there is a walk $P' = (x, e'_1, v_1, \ldots, v_{k-1}, e'_k, y)$ in $[H]_2$. Thus, $k = d_H(x, y) \geq d_{[H]_2}(x, y) = l$. Assume, $k > l$. Then there is a path $Q' = (x, f'_1, v_1, \ldots, v_{k-1}, f'_l, y)$ in $[H]_2$. Thus, for all f'_i there is an edge $f_i \in E(H)$ such that $f'_i \subseteq f_i$ and hence, a walk of length l in H, a contradiction. □

2.4. Invariants

In the following paragraphs we briefly introduce the hypergraph invariants that are most commonly studied in the context of hypergraph products. We will assume throughout that $H = (V, E)$ is a given hypergraph.

2.4.1. Independence, Matching and Cover. A set $S \subseteq V$ is *independent* if it contains no edge of E; the maximum cardinality of an independent set is denoted by $\beta(H)$ and is called the *independence number* of H. Some of the older literature, e.g. [7, 61] use the term *stable* and *stability number* for this concept.

A set $T \subseteq V$ is called a *cover* of H if it intersects every edge of H, i.e., $T \cap e \neq \emptyset$ for all $e \in E$. The minimum cardinality of the covers is denoted by $\tau(H)$, and called the *covering number* of H. Cover and covering number are also known as *transversal* or *transversal number* [7].

A *fractional cover* of H is a mapping $t : V \to \mathbb{R}^+_0$ such that $\sum_{v \in e} t(v) \geq 1$ for all $e \in E$. The value $\min_t \sum_{v \in V} t(v)$ over all fractional covers t is called *fractional covering number* and denoted by $\tau^*(H)$. Note that every cover induces a fractional cover by defining $t(v) = 1$ if $v \in T$ and $t(v) = 0$ else [6].

A subset $M \subseteq E$ is a *matching* if every pair of edges from M has an empty intersection. The maximum cardinality of a matching M is called the *matching number*, denoted by $\nu(H)$ [6].

The partition number $\rho(H)$ of H denotes the minimal number of pairwise disjoint edges of E which together cover V if such a partition exists, else we set $\rho(H) = \infty$ [2, 1].

2.4.2. Coloring. A *coloring* of a hypergraph H is mapping c from either V or E into a set of colors $C = \{1, \ldots, k\}$. We refer to $c : E \to C$ as an *edge-coloring* and to $c : V \to C$ as a *vertex-coloring* or simply *coloring*.

A *proper* coloring of a hypergraph H is a coloring $c : V \to C$ such that $\{v \mid c(v) = i\}$ is an independent set for all $i \in C$. The *chromatic number* $\chi(H)$ is the minimal number of colors that admit a proper coloring of H. Hence, the *chromatic number* $\chi(H)$ is the minimum number of independent sets $V_1, \cdots, V_{\chi(H)}$ into which V can be partitioned. A proper *strong* coloring of a hypergraph H is a proper coloring such that for all edges $e \in E$ holds that $c(v) \neq c(w)$ for all distinct vertices $v, w \in e$. The *strong* chromatic number $\chi_s(H)$ is the minimal number k of colors that admit a strong k-coloring of H.

The *(k-color) discrepancy* of a hypergraph measures the deviation of a coloring c from a so-called balanced coloring, that is a coloring in which each hyperedge contains same number of vertices of each color. More formally, the *discrepancy of a coloring* c and the *k-color discrepancy of* $H = (V, E)$ are defined as follows:

$$\operatorname{disc}(H, c) = \max_{e \in E} \max_{1 \leq i \leq k} \left| |c^{-1}(i) \cap e| - \frac{1}{k}|e| \right|$$

and

$$\operatorname{disc}(H, k) = \min_{c: V \to \{1,\ldots,k\}} \operatorname{disc}(H, c)\,.$$

A *proper* edge-coloring of a hypergraph H is a coloring $c : E \to C$ such that $c(e) \neq c(f)$ for all distinct incident edges $e, f \in E$. The *chromatic index* $q(H)$ of H is the minimum number of colors that admit a proper edge-coloring. Clearly, $q(H) \geq \Delta(H)$. A hypergraph has the *colored hyperedge property* if $q(H) = \Delta(H)$.

2.4.3. Helly Property. For $v \in V$, a *star* $H(v)$ of H with center v is the set of all edges $e \in E$ such that $v \in e$. For a given simple hypergraph H a subset E' of E is an *intersecting family* if every pair of hyperedges of E' have a non-empty intersection. A hypergraph has the *Helly property* if each intersecting family is a star. An interesting characterization of Helly hypergraphs can be found in [5]: A hypergraph has the Helly property if and only if its dual is conformal.

3. Basic Properties of Hypergraph Products

Definitions of hypergraph products, to our knowledge, have never been compared systematically in a way similar to graph products. Most of the hypergraph products can be viewed as a generalization of respective graph products. However, one of the most studied hypergraph product, the so-called square product, does not provide this property. Therefore, it appears useful to make explicit the desirable properties of hypergraph products. We begin with the direct generalization of the requirements for graph products:

(P1) The vertex set of a product is the Cartesian product of the vertex sets of the factors.
(P2) Adjacency in the product depends only on the adjacency properties of the projections of pairs of vertices into the factors.
(P3) The product of simple hypergraphs is again a simple hypergraph.
(P4) At least one of the projections of a product onto its factors is a weak homomorphism.

Since graphs can be interpreted as the special hypergraphs with $|e| \leq 2$ for all $e \in E$, we would like to consider hypergraph products that specialize to graph products:

(P5) The hypergraph product of two graphs is again a graph.

For hypergraphs, these requirements appear to give more freedom than for graphs. Property (P2) posits that the presence of an edge $(x_1, x_2) \sim (y_1, y_2)$ must be determined by the presence or absence of the adjacencies $x_1 \sim y_1$ and $x_2 \sim y_2$ and a rule deciding whether $x_1 = y_1$ and $x_2 = y_2$ is to be treated like an edge or its absence, leading to $2^8 = 256$ distinct operations, see [39]. For hypergraphs, however, this leads only to a restriction on edges but does not provide a complete recipe for the construction of the edge set of the product. As a consequence, it is possible to find several non-equivalent generalizations of the standard graph products as we shall see throughout this survey.

As in the case of usual graph products at least associativity is desirable. All products that are treated in this survey are associative. We omit the proofs for this, since they can be done equivalently to the proofs as in [31]. Thus, the hypergraph products of finitely many factors are well defined and it suffices to prove the results for two (not necessarily prime) factors only. Furthermore, all products, except the lexicographic product are commutative.

Before we proceed with our analysis of hypergraph products, we need to introduce some specific notations:
Let $\circledast_{i=1}^{n} H_i = (V, E) = (\times_{i=1}^{n} V(H_i), E(\circledast_{i=1}^{n} H_i))$ be an arbitrary hypergraph product. The *projection* $p_j : V \to V(H_j)$ is defined by $v = (v_1, \dots, v_n) \mapsto v_j$. We will call v_j the *j-th coordinate* of the vertex $v \in V$. For a given vertex $w \in V(H)$ the *H_j-layer through w* is the partial hypergraph of H

$$H_j^w = \langle \{v \in V(H) \mid p_k(v) = p_k(w) \text{ for } k \neq j\} \rangle .$$

If for a hypergraph product $\circledast_{i=1}^{n} H_i$ holds $H_i \cong H$ for all $i = 1, \dots, n$ we will denote this hypergraph simply by $H^{\circledast n}$.

Let U denote the unit element, if one exists, of an arbitrary product $\circledast$, i.e., $H = H \circledast U$ for all hypergraphs H. Since all hypergraph products considered here have vertex set $V_1 \times V_2$ the unit must always be a hypergraph with a single vertex. A hypergraph is said to be *prime* if the identity $H = H_1 \circledast H_2$ implies that $H_1 \cong U$

or $H_2 \cong U$. Not all hypergraph products have a unit element. Prime factors and a prime factor decomposition cannot be meaningfully defined unless there is a unit.

Part 2. Cartesian Product

4. The Cartesian Product

The Cartesian product of hypergraphs has been investigated by several authors since the 1960s [37, 38, 11, 9, 10, 17, 49, 53]. It is probably the best-studied construction.

4.1. Definition and Basic Properties

Definition 4.1 (Cartesian Product of Hypergraphs). The *Cartesian product* $H = H_1 \Box H_2$ of two hypergraphs H_1 and H_2 has vertex set $V(H) = V(H_1) \times V(H_2)$ and the edge set

$$\begin{aligned} E(H) = &\big\{\{x\} \times f : x \in V(H_1), f \in E(H_2)\big\} \\ &\cup \big\{e \times \{y\} : e \in E(H_1), y \in V(H_2)\big\}. \end{aligned}$$

The Cartesian product is associative, commutative, distributive with respect to the disjoint union and has the single vertex graph K_1 as a unit element [37]. The Cartesian product of two simple hypergraphs is a simple hypergraph. A Cartesian product hypergraph, furthermore, is connected if and only if all of its factors are connected [37]. For the rank and anti-rank, respectively, of a Cartesian product hypergraph $H_1 \Box H_2$ holds:

$$\begin{aligned} r(H_1 \Box H_2) &= \max\{r(H_1), r(H_2)\} \\ s(H_1 \Box H_2) &= \min\{s(H_1), s(H_2)\}. \end{aligned}$$

The projections onto the factors are weak homomorphisms. According to [52] the Cartesian product of hypergraphs can be described in terms of projections as follows: For $H = H_1 \Box H_2$, with $H_i = (V_i, E_i)$, $i = 1, 2$ and $e \subseteq V(H)$ we have $e \in E(H)$ if and only if there is an $i \in \{1, 2\}$, s.t.

(i) $p_i(e) \in E_i$ and
(ii) $|p_j(e)| = 1$ for $j \neq i$.

Furthermore, $|p_i(e)| = |e|$.

The H_j*-layer through* w of a Cartesian product H is induced by all vertices of H that differ from $w \in V(H)$ exactly in the j-th coordinate. Moreover, $H_j^w \cong H_j$.

4.2. Relationships with Graph Products

The restriction of the Cartesian product to graphs coincides with the usual Cartesian graph product. The 2-sections of hypergraphs are also well-behaved:

Proposition 4.2 ([11]). *The* 2*-section of* $H = H' \Box H''$ *is the Cartesian product of the* 2*-section of* H' *and the* 2*-section of* H''*, more formally:*

$$[H' \Box H'']_2 = [H']_2 \Box [H'']_2.$$

This observation suggested the definition of the Cartesian product of $L2$-sections by constructing an appropriate labeling function for the product:

Definition 4.3 (The Cartesian Product of L2-sections [11, 10]**).** Let $\Gamma_i = (V_i, E'_i, \mathcal{L}_i)$ be the $L2$-section of the hypergraphs $H_i = (V_i, E_i)$, $i = 1, 2$. The Cartesian product of the $L2$-sections $\Gamma_1 \Box \Gamma_2 = (V, E', \mathcal{L})$ consists of the graph $(V, E') = (V_1, E'_1) \Box (V_2, E'_2)$ and a labeling function

$$\mathcal{L} = \mathcal{L}_1 \Box \mathcal{L}_2 : E' \to \mathbb{P}(E(H_1 \Box H_2))$$

with

$$\mathcal{L}\left(\{(x_1, y_1), (x_2, y_2)\}\right) = \begin{cases} \{\{x_1\} \times e \mid e \in \mathcal{L}_2\left(\{y_1, y_2\}\right)\}, & \text{if } x_1 = x_2 \\ \{e \times \{y_1\} \mid e \in \mathcal{L}_1\left(\{x_1, x_2\}\right)\}, & \text{if } y_1 = y_2 \end{cases}$$

Lemma 4.4 ([11, 10]). *For all hypergraphs* H, H' *we have:*

1. $[H \Box H']_{L_2} = [H]_{L2} \Box [H']_{L2}$
2. $[[H]_{L2} \Box [H']_{L2}]^{-1}_{L2} = [[H]_{L2}]^{-1}_{L2} \Box [[H']_{L2}]^{-1}_{L2}$

Lemma 4.5 (Distance Formula). *For all hypergraphs* H, H' *we have:*

$$d_{H \Box H'}((x, a), (y, b)) = d_H(x, y) + d_{H'}(a, b)$$

Proof. Combining the results of Lemma 2.1, Lemma 4.2 and the well-known Distance Formula for the Cartesian graph product (Corollary 5.2 in [31]) yields to the result. $\Box$

4.3. Prime Factor Decomposition

Theorem 4.6 (UPFD [37]**).** *Every connected hypergraph has a unique prime factor decomposition w.r.t. the Cartesian product.*

The PFD of disconnected hypergraphs is in general not unique [40, 31]. Theorem 4.6 was also obtained in [49] using a different approach that generalizes this result to infinite and directed hypergraphs, see Part 7.

Imrich and Peterin devised an algorithm for computing the PFD of connected graphs (V, E) in $O(|E|)$ time and space [42]. Bretto and Silvestre adapted this algorithm for the recognition of Cartesian products of hypergraphs [10]. To this end, the $L2$-sections of hypergraphs are used. We give here a short outline of this algorithm. For a given a connected hypergraph H its $L2$-section $[H]_{L2}$ is computed. Using the algorithm of Imrich and Peterin one gets the PFD of $[H]_2$. This results in an edge coloring of $[H]_2$, i.e., edges are colored with respect to the copies of the corresponding prime factors. After this, on has to check if the factors of $[H]_2$ are the labeled prime factors of $[H]_{L2}$ and has to merge factors if necessary. Finally, using the inverse $L2$-sections the prime factors of H are built back. Although the PFD of the 2-section $[H]_2 = (V, E')$ can be computed in $O(r(H)^2 \cdot |E|) = O(|E'|)$

time, the check-and-merging-process together with the build back part for the PFD of H is more time-consuming and one ends in an overall time complexity of $O((\log_2|V|)^2 \cdot r(H)^3 \cdot |E| \cdot \Delta(H)^2)$ for a given hypergraph H. The PFD of connected simple hypergraphs $H = (V, E)$ with fixed maximum degree and fixed rank can then be computed in $O(|V||E|)$ time [10]. The currently fastest algorithm is due to Hellmuth and Lehner [35]. In distinction from the method of Bretto et al. this algorithm is in a sense conceptually simpler, as *(1)* it is not needed to transform the hypergraph H into its so-called L2-section and back and *(2)* the test which (collections) of the putative factors are prime factors of H follows a complete new idea based on increments of fixed vertex-coordinate positions, that allows an easy and efficient check to determine the PFD of H.

Theorem 4.7 ([35])**.** *The PFD w.r.t. the Cartesian product of a hypergraph $H = (V, E)$ with rank r can be computed in $O(r^2|V||E|)$ time. If we assume that H has bounded rank, then this time-complexity can be reduced to $O(|E|\log^2(|V|))$.*

4.4. Invariants

Much of the literature on Cartesian hypergraph products is concerned with relationships of invariants of the factors with those of the product. In this section we compile the most salient results.

Theorem 4.8 (Automorphism Group [37]**).** *The automorphism group of the Cartesian product of connected prime hypergraphs is isomorphic to the automorphism group of the disjoint union of the factors.*

Theorem 4.9 (k-fold Covering [17]**).** *Let $H'_i = (V'_i, E'_i)$ be a k_i-fold covering of the hypergraph $H_i = (V_i, E_i)$ via a covering projection π_i, $i = 1, 2$. Then $H'_1 \Box H'_2$ is a k_1k_2-fold cover of $H_1 \Box H_2$ via a covering projection π induced naturally by π_1 and π_2, i.e, define π by:*

$$\pi((x, y)) = (\pi_1(x), \pi_2(y)),\ \textit{for}\ (x, y) \in V'_1 \times V'_2,$$
$$\pi(\{x\} \times e_2) = \{\pi_1(x)\} \times \pi_2(e_2),\ \textit{for}\ x \in V'_1,\ e_2 \in E'_2,$$
$$\pi(e_1 \times \{y\}) = \pi_1(e_1) \times \{\pi_2(y)\},\ \textit{for}\ e_1 \in E'_1,\ y \in V'_2.$$

Theorem 4.10 (Conformal Hypergraphs [9]**).** *$H = H_1 \Box H_2$ is conformal if and only if H_1 and H_2 are conformal.*

Theorem 4.11 (Helly Property [9]**).** *$H = H_1 \Box H_2$ has the Helly property if and only if H_1 and H_2 have the Helly property.*

Theorem 4.12 (Colored Hyperedge Property [11]**).** *If H_1 and H_2 have the colored hyperedge property then $H = H_1 \Box H_2$ has the colored hyperedge property.*

Theorem 4.13 ((Strong) Chromatic Number [11]**).** *Let χ_i and χ (respectively γ_i and γ) be the chromatic (resp. strong chromatic number) of H_i and $H = \Box_{i=1}^n$. Then $\chi = \max_i\{\chi_i\}$ and $\gamma = \max_i\{\gamma_i\}$.*

Theorem 4.14 (Hamiltonicity I [53]**).** *Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs that contain a Hamiltonian path. Then $H_1 \Box H_2$ contains an Hamiltonian cycle if and only if $|V_1||V_2|$ is even or at least one of H_1 or H_2 is not a bipartite graph.*

Theorem 4.15 (Hamiltonicity II [53]**).** *Let H_1 be a hypergraph with $n_1 \geq 4$ vertices containing an Hamiltonian cycle and H_2) be a hypergraph with that contains a Hamiltonian p-path with $p \leq 2\lfloor n_1/4 \rfloor$. Then $H_1 \Box H_2$ contains a Hamiltonian cycle.*

Part 3. Direct Products

In contrast to the Cartesian product, there are several different possibilities to construct a direct product. We will consider four constructions in detail: The direct product $\boxed{\mathrm{r}}$, which is closed under the restriction on r-uniform hypergraphs, the direct product $\breve{\times}$, which preserves the minimal rank of the factors, the direct product $\overset{\frown}{\times}$, which preserves the maximal rank of the factors and the direct product $\widetilde{\times}$, which does not preserve any rank of its factors.

An alternative product, which we prefer to call the *square product* following the work of Nešetřil and Rödl [44], is also often called "direct product" in the literature. It will be discussed in detail in section 14.

5. The Direct Product for r-uniform Hypergraphs

An early construction of a direct hypergraph product [20] was motivated by the investigation of a category of hypergraphs. The following product is categorical in the category of r-uniform hypergraphs and is only defined for r-uniform hypergraphs.

5.1. Definition and Basic Properties

Definition 5.1 (r-uniform direct product). For two r-uniform hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ their direct product $H_1 \boxed{\mathrm{r}} H_2$ has vertex set $V_1 \times V_2$ and the edge set

$$E(H_1 \boxed{\mathrm{r}} H_2) := \left\{ e \in \binom{e_1 \times e_2}{r} \mid e_i \in E_i \text{ and } p_i(e) \in E_i,\ i = 1, 2 \right\}. \tag{5.1}$$

This product is the restriction of minimal and maximal rank preserving products to r-uniform hypergraphs, defined in the following two sections. Most of the properties of the two products can indeed be inferred from the corresponding results for the $\boxed{\mathrm{r}}$-product.

5.2. Relationships with Graph Products

For $r = 2$, $\boxed{\mathrm{r}}$ is the direct graph product in the simple graph. However, since it is only defined on r-uniform hypergraphs, it cannot be generalized on the class of graphs with loops. Since $\boxed{\mathrm{r}}$ coincides with the minimal rank preserving product $\breve{\times}$ on r-uniform hypergraphs all results concerning 2-sections and $L2$-sections can be inferred from the respective results of $\breve{\times}$. In general there is no unit element for r-uniform hypergraphs, hence the term *prime* cannot be defined for this product. As far as we know, nothing is known about the behavior of hypergraph invariants under this product.

6. The Minimal Rank Preserving Direct Product

If one considers the direct product $H_1 \boxed{\mathrm{r}} H_2$ of two r-uniform hypergraphs H_1 and H_2, one observes that an edge in $E(H_1 \boxed{\mathrm{r}} H_2)$ satisfies the following two properties:

(E1) All vertices of an edge differ in each coordinate.
(E2) The projection of an edge is an edge in the respective factor.

If one tries to generalize the product $\boxed{\mathrm{r}}$ to arbitrary non-uniform hypergraphs, one always encounters edges in the corresponding hypergraph product that cannot satisfy both (E1) and (E2). Hence, a natural question is how to extend the direct product $\boxed{\mathrm{r}}$ to a product of two arbitrary, non-uniform hypergraphs in such a way that it satisfies at least one the properties.

If we insist on Property (E1) we enforce an additional constraint, that is, the projections of an edge of the product hypergraph into the factors is an edge in at least one factor and subsets of edges in the other factors. ¿From that point, we observe that the rank of the hypergraph product equals the minimal rank of the factors.

6.1. Definition and Basic Properties

This product was first defined by Sonntag in [57] using the term "Cartesian Product".

Definition 6.1 (Minimal Rank Preserving Direct Product [57]**).** For two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ their direct product $H_1 \breve{\times} H_2$ has vertex set $V_1 \times V_2$. For two edges $e_1 \in E_1$ and $e_2 \in E_2$ let $r^-_{e_1,e_2} = \min\{|e_1|, |e_2|\}$. The edge set is the defined as:

$$E(H_1 \breve{\times} H_2) := \left\{ e \in \binom{e_1 \times e_2}{r^-_{e_1,e_2}} \mid e_i \in E_i \text{ and } |p_i(e)| = r^-_{e_1,e_2},\ i = 1, 2 \right\}. \quad (6.1)$$

In other words, a subset $e = \{(x_1, y_1), \dots, (x_r, y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \breve{\times} H_2$ if and only if

(i) $\{x_1, \dots, x_r\}$ is an edge in H_1 and $\{y_1, \dots, y_r\}$ is the subset of an edge in H_2, or
(ii) $\{x_1, \dots, x_r\}$ is the subset of an edge in H_1 and $\{y_1, \dots, y_r\}$ is an edge in H_2

We have $p_1(e) = e_1$ and $p_2(e) = e_2$ provided $|e_1| = |e_2|$. If $|e_i| < |e_j|$, then $p_i(e) = e_i$ and $p_j(e) \subset e_j$, $i, j \in \{1, 2\}$. Thus, the projections need not to be (weak) homomorphisms in general, but they preserve adjacency, i.e., two vertices in a direct product $\breve{\times}$ hypergraph are adjacent, whenever they are adjacent in both of the factors.

The direct product $\breve{\times}$ is associative, it is commutative as an immediate consequence of the symmetry of the definition. Simple set-theoretic considerations show that direct product $\breve{\times}$ is left and right distributive with respect to the disjoint union. The direct product $\breve{\times}$ of two connected hypergraphs is not necessarily connected, as one can observe for the simple case $K_2 \breve{\times} K_2$. For the rank and anti-rank, respectively, of the hypergraph $H = H_1 \breve{\times} H_2$ holds:

$$r(H_1 \breve{\times} H_2) = \min\{r(H_1), r(H_2)\}$$
$$s(H_1 \breve{\times} H_2) = \min\{s(H_1), s(H_2)\}.$$

Lemma 6.2. *The direct product $H_1 \breve{\times} H_2$ of simple hypergraphs is simple.*

Proof. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two simple hypergraphs. Hence $s(H_i) \geq 2$ for $i = 1, 2$, and therefore $s(H_1 \breve{\times} H_2) = \min\{s(H_1), s(H_2)\} \geq 2$, which implies that $E(H_1 \breve{\times} H_2)$ contains no loops. Assume that there is an edge e contained in an edge e', where $e \in \binom{e_1 \times e_2}{r^-_{e_1,e_2}}$ and $e' \in \binom{e'_1 \times e'_2}{r^-_{e'_1,e'_2}}$ $e_i, e'_i \in E_i$. Notice, that $p_i(e) \subseteq p_i(e')$, $i = 1, 2$ must hold. W.l.o.g. suppose $|e_1| \leq |e_2|$, which implies $p_1(e) = e_1$. It follows $e_1 = p_1(e) \subseteq p_1(e') \subseteq e'_1$, and since H_1 is simple, $e_1 = e'_1$ must hold and hence $p_1(e') = e'_1$. From this we can conclude $|e| = |e_1| = |e'_1| = |p_1(e')| = |e'|$ and therefore $e = e'$. $\square$

The direct product $\breve{\times}$ does not have a unit, both in the class of simple and non-simple hypergraphs, i.e., neither the one vertex hypergraph K_1, nor the vertex with a loop, $\mathcal{L}K_1$, is a unit for the direct product $\breve{\times}$.

6.2. Relationships with Graph Products

The restriction of the direct product $\breve{\times}$ to r-uniform hypergraphs is the product $\boxed{\mathrm{r}}$ defined in Equation (5.1), hence the restriction of $\breve{\times}$ simple graphs coincides with the direct graph product. But since the direct product $\breve{\times}$ has no unit, we can conclude that it does not coincide with the direct graph product in the class of graphs with loops.

Lemma 6.3. *The 2-section of the direct product $H = H' \breve{\times} H''$ is the direct product of the 2-section of H' and the 2-section of H'', more formally:*

$$[H' \breve{\times} H'']_2 = [H']_2 \times [H'']_2.$$

Proof. Let p_1 and p_2 denote $p_{H'}$ and $p_{H''}$, respectively. By definition of the 2-section and the direct product, $[H]_2$ and $[H']_2 \times [H'']_2$ have the same vertex set. Thus we need to show that the identity mapping $V([H]_2) \to V([H']_2 \times [H'']_2)$ is an isomorphism. We have: $\{x,y\} \in E([H]_2) \Leftrightarrow \exists e \in E(H) : \{x,y\} \subseteq e,\ x \neq y \Leftrightarrow \exists e' \in E(H'), e'' \in E(H'') : \{p_1(x), p_1(y)\} \subseteq p_1(e) \subseteq e',\ p_1(x) \neq p_1(y)$ and $\{p_2(x), p_2(y)\} \subseteq p_2(e) \subseteq e'',\ p_2(x) \neq p_2(y) \Leftrightarrow \{p_1(x), p_1(y)\} \in E([H']_2), \{p_2(x), p_2(y)\} \in E([H'']_2) \Leftrightarrow \{x,y\} \in E([H']_2 \times [H'']_2)$. □

Definition 6.4 (The Direct Product of L2-sections). Let $\Gamma_i = (V_i, E_i', \mathcal{L}_i)$ be the $L2$-section of the hypergraphs $H_i = (V_i, E_i)$, $i = 1, 2$. The direct product of the $L2$-sections $\Gamma_1 \breve{\times} \Gamma_2 = (V, E', \mathcal{L})$ consists of the graph $(V, E') = (V_1, E_1') \times (V_2, E_2')$ and a labeling function

$$\mathcal{L} = \mathcal{L}_1 \breve{\times} \mathcal{L}_2 : E' \to \mathbb{P}(E(H_1 \breve{\times} H_2))$$

assigning to each edge $e' = \{(x_1, y_1), (x_2, y_2)\} \in E'$ a label

$$\mathcal{L}\,(e') = \{e' \cup A \mid A \in \mathcal{A}(e', e_1, e_2), e_i \in \mathcal{L}_i(e_i'), |p_i(A)| = r^-_{e_1,e_2} - 2, i = 1, 2\}$$

with $e_1' = \{x_1, x_2\}$, $e_2' = \{y_1, y_2\}$ and

$$\mathcal{A}(e', e_1, e_2) = \binom{(e_1 \setminus e_1') \times (e_2 \setminus e_2')}{r^-_{e_1,e_2} - 2}.$$

A short direct computation shows that

$$[H \breve{\times} H']_{L_2} = [H]_{L2} \breve{\times} [H']_{L2}$$

holds for all simple hypergraphs H, H'.

Lemma 6.5 (Distance Formula). *Let (x_1, x_2) and (y_1, y_2) be two vertices of the direct product $H = H_1 \breve{\times} H_2$. Then*

$d_H((x_1, x_2), (y_1, y_2)) = \min\{n \in \mathbb{N} \mid$ *each factor H_i has a walk P_{x_i,y_i} of length* $n\}$,

where it is understood that $d_H((x_1, x_2), (y_1, y_2)) = \infty$ if no such n exists.

Proof. Combining the results of Lemma 2.1, Lemma 6.3 and the Distance Formula for the direct graph product (Proposition 5.8 in [31]) yields to the result. □

Corollary 6.6. *Let (x_1, x_2) and (y_1, y_2) be two vertices of the direct product $H = H_1 \boxed{r} H_2$. Then*

$d_H((x_1, x_2), (y_1, y_2)) = \min\{n \in \mathbb{N} \mid$ *each factor H_i has a walk P_{x_i,y_i} of length* $n\}$,

where it is understood that $d_H((x_1, x_2), (y_1, y_2)) = \infty$ if no such n exists.

The absence of a unit implies that there is no meaningful PFD. To the authors' knowledge, hypergraphs invariants have not been studied for the product $\breve{\times}$.

7. The Maximal Rank Preserving Direct Product

As discussed in Section 6, if one tries to generalize the product $\boxed{\mathrm{r}}$ to arbitrary non-uniform hypergraphs, one always encounters edges in the corresponding hypergraph product that cannot satisfy both (E1) and (E2):

(E1) All vertices of an edge differ in each coordinate.
(E2) The projection of an edge is an edge in the respective factor.

In order to extend the direct product $\boxed{\mathrm{r}}$ to a product of two arbitrary, non-uniform hypergraphs in such a way that it satisfies at least one of the properties, we insist now on condition (E2) and claim that the size of an edge in the product hypergraph coincides with the size of at least one of its projections and thus, the rank of the product hypergraph is exactly the maximal rank of one of its factors.

7.1. Definition and Basic Properties

Definition 7.1 (Maximal Rank Preserving Direct Product). For two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ their direct product $H_1 \mathbin{\widehat{\times}} H_2$ has vertex set $V_1 \times V_2$. For two edges $e_1 \in E_1$ and $e_2 \in E_2$ let $r^+_{e_1,e_2} = \max\{|e_1|, |e_2|\}$. The edge set is the defined as:

$$E(H_1 \mathbin{\widehat{\times}} H_2) := \left\{ e \in \binom{e_1 \times e_2}{r^+_{e_1,e_2}} \mid e_i \in E_i \text{ and } p_i(e) = e_i,\ i = 1, 2 \right\}. \tag{7.1}$$

In other words, a subset $e = \{(x_1, y_1), \dots, (x_r, y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \mathbin{\widehat{\times}} H_2$ if and only if

(i) $\{x_1, \dots, x_r\}$ is an edge in H_1 and there is an edge $f \in E_2$ of H_2 such that $\{y_1, \dots y_r\}$ is a multiset of elements of f, and $f \subseteq \{y_1, \dots, y_r\}$, or
(ii) $\{y_1, \dots, y_r\}$ is an edge in H_2 and there is an edge $e \in E_1$ of H_1 such that $\{x_1, \dots x_r\}$ is a multiset of elements of e, and $e \subseteq \{x_1, \dots, x_r\}$.

For $e \in E(H_1 \mathbin{\widehat{\times}} H_2)$ holds: if $|e| = |p_i(e)|$, $i \in \{1, 2\}$, then $p_i(x) \neq p_i(y)$ for all $x, y \in e$ with $x \neq y$.

The direct product $\widehat{\times}$ is associative, commutative, and distributive with respect to the disjoint union. Contrary to the direct product $\breve{\times}$, projections of a product hypergraph into the factors are, by definition, homomorphisms, i.e., projections of hyperedges are hyperedges in the respective factors.

The one vertex hypergraph with a loop $\mathcal{L}K_1$ is a unit for the direct product $\widehat{\times}$ in the class of hypergraphs with loops. In the class of simple hypergraphs, this product has no unit. The direct product $\widehat{\times}$ of two connected hypergraphs is not necessarily connected, since it need not to be connected in the class of graphs. For the rank and anti-rank, respectively, of the hypergraph $H_1 \mathbin{\widehat{\times}} H_2$ holds:

$$r(H_1 \mathbin{\widehat{\times}} H_2) = \max\{r(H_1), r(H_2)\}$$
$$s(H_1 \mathbin{\widehat{\times}} H_2) = \max\{s(H_1), s(H_2)\}.$$

Lemma 7.2. *The direct product $H_1 \mathbin{\widehat{\times}} H_2$ of simple hypergraphs is simple.*

Proof. Therefore, let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two simple hypergraphs. Hence $s(H_i) \geq 2$ for $i = 1, 2$, and therefore it holds $s(H_1 \mathbin{\widehat{\times}} H_2) = \max\{s(H_1), s(H_2)\} \geq 2$, which implies that $E(H_1 \mathbin{\widehat{\times}} H_2)$ contains no loops. Assume that there is an edge e contained in an edge e', where $e \in \binom{e_1 \times e_2}{r^+_{e_1,e_2}}$ and $e' \in \binom{e'_1 \times e'_2}{r^+_{e'_1,e'_2}}$, $e_i, e'_i \in E_i$. Notice, that $p_i(e) \subseteq p_i(e')$, $i = 1, 2$ must hold. Hence, $e_i = p_i(e) \subseteq p_i(e') = e'_i$, which implies $e_i = e'_i$, since H_1 and H_2 are simple. It follows $|e| = |e'|$ and therefore $e = e'$. Thus, $H_1 \mathbin{\widehat{\times}} H_2$ is simple. $\square$

7.2. Relationships with Graph Products

If we restrict the direct product $\widehat{\times}$ to r-uniform hypergraphs, we recover the product $\boxed{\mathrm{r}}$ defined by Equation (5.1). In particular, this product coincides with the direct graph product in the class of simple graphs. Moreover, the restriction of this product to graphs coincides with the direct graph product in general, also in the class of not necessarily simple graphs with loops.

In contrast to the direct product $\breve{\times}$, however, the 2-section of the direct product of two arbitrary hypergraphs is not the direct graph product of the 2-sections of the hypergraphs, except in the special case of r-uniform hypergraphs. To see this, consider as an example the product $K_2 \mathbin{\widehat{\times}} (V, \{V\})$ with $|V| = 3$.

Despite its appealing properties, the product $\widehat{\times}$ has not been studied in the literature. It is unknown, in particular, under which conditions it admits a unique PFD. The prime factor theorems for the direct product need non-trivial preconditions even in the class of graphs. We do not expect that it will be a particularly simple problem to establish a general UPFD theorem.

8. A Direct Product that does not preserve Rank

For the sake of completeness, and as an example for the degrees of freedom inherent in the definition of hypergraph products that generalize graph products, we consider the product $\widetilde{\times}$. Its restriction to 2-uniform hypergraphs coincides with the direct graph product. However, it does not preserve r-uniformity in general. For brevity we omit proofs, which can be found in [27], throughout this section.

8.1. Definition and Basic Properties

Definition 8.1 (non-rank-preserving Direct Product). For two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, we define their *direct product* $\widetilde{\times}$ by the edge set

$$E(H_1 \mathbin{\widetilde{\times}} H_2) := \Big\{ \{(x, y)\} \cup \big((e \setminus \{x\}) \times (f \setminus \{y\}) \big) \mid x \in e \in E_1;\ y \in f \in E_2 \Big\}$$

The projections p_1 and p_2 of a product hypergraph $H = H_1 \mathbin{\widetilde{\times}} H_2$ into its factors H_1 and H_2, respectively, are homomorphisms.

It is not hard to verify that the direct product $\widetilde{\times}$ is associative, commutative, and both left and right distributive together with the disjoint union as addition. The direct product $H_1 \widetilde{\times} H_2$ of simple hypergraphs H_1, H_2 is simple. The direct product $\widetilde{\times}$ does not have a unit, neither in the class of simple hypergraphs, nor in the class of non-simple hypergraphs. The direct product $\widetilde{\times}$ of two connected hypergraphs is not necessarily connected, as one can observe for the simple case $K_2 \widetilde{\times} K_2$. For the rank and anti-rank, respectively, of the hypergraph product $H_1 \widetilde{\times} H_2$ holds:

$$r(H_1 \widetilde{\times} H_2) = (r(H_1) - 1)(r(H_2) - 1) + 1$$
$$s(H_1 \widetilde{\times} H_2) = (s(H_1) - 1)(s(H_2) - 1) + 1$$

In general, therefore, the (anti-)rank of a product will not be the (anti-)rank of one of its factors.

8.2. Relationships with Graph Products

If we restrict the definition of this product to 2-uniform hypergraphs, i.e., simple graphs, we have: $e \subseteq V(G_1 \widetilde{\times} G_2)$ is an edge in $G_1 \widetilde{\times} G_2$ iff $E = \{(x, y)(x', y')\}$ and $\{x, x'\}$ is an edge in G_1 and $\{y, y'\}$ is an edge in G_2. This is exactly the definition of the direct graph product.

Similar to the direct product $\widehat{\times}$, the 2-section of the direct product of two arbitrary hypergraphs is not the direct graph product of the 2-sections of the hypergraphs. This can be easily verified on the product $K_2 \widetilde{\times} (V, \{V\})$ with $|V| = 3$.

Since the direct product $\widetilde{\times}$ has no unit, we can conclude that it does not coincide with the direct graph product in the class of graphs with loops. The absence of a unit implies that there is no meaningful PFD. To our knowledge, no further results are available on this product.

Part 4. Strong Products

The strong product of graphs can be interpreted as a superposition of the edges of the Cartesian and the direct graph products. Here we explore the corresponding constructions for hypergraphs: Let the edge set of a strong product $H = H_1 \overset{*}{\boxtimes} H_2$, $* = \frown, \smile$ of two hypergraphs H_1 and H_2 be

$$E(H_1 \overset{*}{\boxtimes} H_2) = E(H_1 \Box H_2) \cup E(H_1 \overset{*}{\times} H_2),$$

where $E(H_1 \overset{*}{\times} H_2)$ is the edge set of one of the respective direct products discussed in the previous section.

9. The Normal Product

This particular strong product was first introduced by Sonntag [54, 55, 58]. Following the terminology of Sonntag we call this product *normal product* $\breve{\boxtimes}$.

9.1. Definition and Basic Properties

A subset $e = \{(x_1, y_1), \ldots, (x_r, y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \breve{\boxtimes} H_2$ if and only if

(i) $\{x_1, \ldots, x_r\}$ is an edge in H_1 and $y_1 = \ldots = y_r \in V(H_2)$, or
(ii) $\{y_1, \ldots, y_r\}$ is an edge in H_2 and $x_1 = \ldots = x_r \in V(H_1)$, or
(iii) $\{x_1, \ldots, x_r\}$ is an edge in H_1 and $\{y_1, \ldots, y_r\}$ is the subset of an edge in H_2, or
(iv) $\{y_1, \ldots, y_r\}$ is an edge in H_2 and $\{x_1, \ldots, x_r\}$ is the subset of an edge in H_1.

An edge e that is of type *(i)* or *(ii)* is called *Cartesian edge* and it holds $e \in E(H_1 \Box H_2)$, an edge e of type *(iii)* or *(iv)* is called *non-Cartesian edge* and it holds $e \in E(H_1 \breve{\times} H_2)$. Notice that $|e \cap e'| \leq 1$ holds for all $e \in E(H_1 \Box H_2)$ and $e' \in E(H_1 \breve{\times} H_2)$, hence $E(H_1 \Box H_2) \cap E(H_1 \breve{\times} H_2) = \emptyset$ if H_1, H_2 contain no loops.

For the same reasons as for the direct product $\breve{\times}$, the projections need not to be (weak) homomorphisms in general, but they preserve adjacency or adjacent vertices are mapped into the same vertex. The normal product $\breve{\boxtimes}$ is associative, commutative, and distributive w.r.t the disjoint union and has K_1 as unit element. Since $E(H_1 \Box H_2)$ is a spanning partial hypergraph of $E(H_1 \breve{\boxtimes} H_2)$, we can conclude that the normal product $H_1 \breve{\boxtimes} H_2$ is connected if and only if H_1 and H_2 are connected hypergraphs. For the rank and anti-rank, respectively, of a normal product hypergraph $H_1 \breve{\boxtimes} H_2$ holds:

$$r(H_1 \breve{\boxtimes} H_2) = \max\{r(H_1), r(H_2)\}$$

$$s(H_1 \breve{\boxtimes} H_2) = \min\{s(H_1), s(H_2)\}.$$

Lemma 9.1. *The normal product $H_1 \breve{\boxtimes} H_2$ of simple hypergraphs H_1, H_2 is simple.*

Proof. This follows immediately from Lemma 6.2, the fact that the Cartesian product of simple hypergraphs is simple and that the intersection of a Cartesian and a non-Cartesian edge contains at most one vertex. □

9.2. Relationships with Graph Products

The restriction of the normal product $\breve{\boxtimes}$ to 2-uniform hypergraphs coincides with the strong graph product. But it does not coincide with the strong graph product in the class of graphs with loops since the direct product $\breve{\times}$ does not coincide with the direct graph product within this class.

In the class of graphs there is a well known relation between the direct and the strong graph product. The strong product can be considered as a special case of the direct product [40]: for a graph $G \in \Gamma$ let $\mathcal{L}G$ denote the graph in Γ_0, which is formed from G by adding a loop to each vertex of G. On the other hand, for a graph $G' \in \Gamma_0$ let $\mathcal{N}G'$ denote the graph in Γ which emerges from G' by deleting all loops. Then we have for $G_1, G_2 \in \Gamma$:

$$G_1 \boxtimes G_2 = \mathcal{N}(\mathcal{L}G_1 \times \mathcal{L}G_2) \tag{9.1}$$

This relationship, however, does not exist between the direct product $\breve{\times}$ and the normal product $\breve{\boxtimes}$.

The next statement follows immediately from Proposition 4.2, Lemma 6.3 and the definition of the normal product $\breve{\boxtimes}$:

Lemma 9.2. *The* 2*-section of the normal product* $H = H' \breve{\boxtimes} H''$ *is the strong product of the* 2*-section of* H' *and the* 2*-section of* H''*, more formally:*

$$[H' \breve{\boxtimes} H'']_2 = [H']_2 \boxtimes [H'']_2.$$

One can therefore define a meaningful normal product of L2-sections:

Definition 9.3 (The Normal Product of L2-sections). Let $\Gamma_i = (V_i, E'_i, \mathcal{L}_i)$ be the $L2$-section of the hypergraphs $H_i = (V_i, E_i)$, $i = 1, 2$. The normal product of the $L2$-sections $\Gamma_1 \breve{\boxtimes} \Gamma_2 = (V, E', \mathcal{L})$ consists of the graph $(V, E') = (V_1, E'_1) \boxtimes (V_2, E'_2)$ and a labeling function assigning to each edge $e' = \{(x_1, y_1), (x_2, y_2)\} \in E'$ a label

$$\mathcal{L} = \mathcal{L}_1 \breve{\boxtimes} \mathcal{L}_2 : E' \to \mathbb{P}(E(H_1 \breve{\boxtimes} H_2))$$

with

$$\mathcal{L}\left(\{(x_1, y_1), (x_2, y_2)\}\right) = \begin{cases} \{\{x_1\} \times e \mid e \in \mathcal{L}_2\left(\{y_1, y_2\}\right)\}, & \text{if } x_1 = x_2 \\ \{e \times \{y_1\} \mid e \in \mathcal{L}_1\left(\{x_1, x_2\}\right)\}, & \text{if } y_1 = y_2 \\ \mathcal{L}_1 \breve{\times} \mathcal{L}_2\left(\{(x_1, y_1), (x_2, y_2)\}\right), & \text{otherwise.} \end{cases}$$

A straightforward but tedious computation shows $[H \breve{\boxtimes} H']_{L_2} = [H]_{L2} \breve{\boxtimes} [H']_{L2}$ for all simple hypergraphs H, H'.

Lemma 9.4 (Distance Formula). *For all hypergraphs* H, H' *we have:*

$$d_{H \breve{\boxtimes} H'}((x, a), (y, b)) = \max\{d_H(x, y), d_{H'}(a, b)\}$$

Proof. Combining the results of Lemma 2.1, Lemma 9.2 and the well-known Distance Formula for the strong graph product (Corollary 5.5 in [31]) yields to the result. $\square$

9.3. Invariants

The normal product has not received much attention since its introduction by Sonntag [54]. In particular, it has not yet been investigated regarding PFDs.

Theorem 9.5 (Hamiltonicity I [54]**).** *Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two non-trivial hypergraphs s.t. H_1 contains a Hamiltonian p-path, $p \in \mathbb{N}^+$, H_2 contains a Hamiltonian 2-path. Suppose that one of the following conditions is satisfied*

(1) $|V_2|$ *is even or* $|V_1| = 2$.
(2) $|V_1| = 3$ *and* H_i *is not isomorphic to* P_3 *or* H_j *contains no Hamiltonian 2-path or* $|E_j| \neq \left\lfloor \frac{|V_j|}{2} \right\rfloor$, $i, j \in \{1, 2\}, i \neq j$.
(3) $|V_1| \geq 4$ *and there exists a Hamiltonian 2-path* $P = (v_0, e_1, v_1, e_s, \ldots, e_{|V_2|-1}, v_{|V_2|-1})$ *in* H_2, *s.t. there is an edge* $e \in E_2$, *and even indices* i, i', $0 \leq i < i' \leq |V_2| - 1$ *with* $\{v_i, v_{i'}\} \subseteq e$.
(4) $|V_1| \geq 4$ *and* $|V_2| \geq 2 \left\lceil r / \left(\left\lfloor \frac{|V_1|-2}{p} + 1 \right\rfloor \right) \right\rceil - 1$.

Then $H_1 \breve{\boxtimes} H_2$ contains a Hamiltonian cycle.

Theorem 9.6 (Hamiltonicity II [54]**).** *Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two non-trivial hypergraphs s.t. H_1 contains a Hamiltonian p-path, $p \in \mathbb{N}^+$, H_2 contains a Hamiltonian 2-path and $|E_1| = \left\lfloor \frac{|V_1|-2}{p} \right\rfloor + 1$. Then $H_1 \breve{\boxtimes} H_2$ contains a Hamiltonian cycle if and only if at least one of the conditions $(1) - (4)$ of Theorem 9.5 is satisfied.*

10. The Strong Product

Given the "nice" properties of the direct product $\widehat{\times}$, the best candidate for a standard strong product of hypergraphs is $\widehat{\boxtimes}$ with edge set $E(H_1 \Box H_2) \cup E(H_1 \widehat{\times} H_2)$.

10.1. Definition and Basic Properties

A subset $e = \{(x_1, y_1), \ldots, (x_r, y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \widehat{\boxtimes} H_2$ if and only if

(i) $\{x_1, \ldots, x_r\} \in E(H_1)$ and $y_1 = \ldots = y_r \in V(H_2)$, or
(ii) $\{y_1, \ldots, y_r\} \in E(H_2)$ and $x_1 = \ldots = x_r \in V(H_1)$, or
(iii) $\{x_1, \ldots, x_r\} \in E(H_1)$ and there is an edge $f \in E(H_2)$ such that $\{y_1, \ldots, y_r\}$ is a multiset of elements of f, and $f \subseteq \{y_1, \ldots, y_r\}$, or
(iv) $\{y_1, \ldots, y_r\} \in E(H_2)$ and there is an edge $f \in E(H_1)$ such that $\{x_1, \ldots, x_r\}$ is a multiset of elements of f, and $f \subseteq \{x_1, \ldots, x_r\}$.

An edge e that is of type (i) or (ii) is called *Cartesian edge* and it holds $e \in E(H_1 \Box H_2)$, an edge e of type (iii) or (iv) is called *non-Cartesian edge* and it holds $e \in E(H_1 \widehat{\times} H_2)$.

The strong product $\widehat{\boxtimes}$ is associative, commutative, and distributive w.r.t. the disjoint union and has K_1 as unit element. The projections into the factors are weak homomorphisms. Since $E(H_1 \Box H_2)$ is a spanning partial hypergraph of $E(H_1 \mathbin{\widehat{\boxtimes}} H_2)$, we can conclude that strong product $H_1 \mathbin{\widehat{\boxtimes}} H_2$ is connected if and only if H_1 and H_2 are connected hypergraphs. For the rank and anti-rank, respectively, of a strong product hypergraph $H_1 \mathbin{\widehat{\boxtimes}} H_2$ holds:

$$r(H_1 \mathbin{\widehat{\boxtimes}} H_2) = \max\{r(H_1), r(H_2)\}$$

$$s(H_1 \mathbin{\widehat{\boxtimes}} H_2) = \min\{s(H_1), s(H_2)\}.$$

Lemma 10.1. *The strong product $H_1 \mathbin{\widehat{\boxtimes}} H_2$ of simple hypergraphs H_1 and H_2 is simple.*

Proof. Due to the fact that the Cartesian product and the direct product $\widehat{\times}$ of simple hypergraphs is simple, it remains to show, that no Cartesian edge is contained in any non-Cartesian edge or vice versa. Therefore, let e be a Cartesian edge and e' a non-Cartesian edge with $p_i(e') = e'_i$ for some $e'_i \in E_i$, $i = 1, 2$. Suppose first, $e' \subseteq e$. Thus $|p_i(e)| = 1$ for an $i \in \{1, 2\}$ and therefore $|p_i(e')| = 1$, but $p_i(e')$ must be an edge in H_i. Hence, one of the factors would not be simple. Now let $e \subseteq e'$. W.l.o.g. suppose $|p_1(e)| = 1$, hence $p_2(e) = e_2$ for some $e_2 \in E_2$. Since $p_2(e) \subseteq p_2(e') = e'_2$ and H_2 is simple, we can conclude $e_2 = e'_2$. If $|e'| = |e'_1|$, then $p_1(x) \neq p_1(y)$ must hold for all $x, y \in e'$ with $x \neq y'$, and therefore $|e| = 1$ and hence $|e_2| = 1$, which contradicts the fact that H_2 is simple. If conversely $|e'| = |e'_2|$, we can conclude that $|e'| = |e|$, hence $e = e'$ and $|e'_1| = |p_1(e')| = 1$ which implies that H_1 is not simple, a contradiction. $\Box$

¿From the arguments in the proof we can conclude that $E(H_1 \Box H_2) \cap E(H_1 \mathbin{\widehat{\times}} H_2) = \emptyset$ holds if H_1 and H_2 are simple. Moreover, one can show that $E(H_1 \Box H_2) \cap E(H_1 \mathbin{\widehat{\times}} H_2) = \emptyset$ provided that H_1 and H_2 are loopless.

10.2. Relationships with Graph Products

The restriction of the strong product $\widehat{\boxtimes}$ to (not necessarily simple) graphs (with or without loops) coincides with the strong graph product. For simple hypergraphs H_1 and H_2 without loops, furthermore, we have

$$H_1 \mathbin{\widehat{\boxtimes}} H_2 = \mathcal{N}(\mathcal{L}H_1 \mathbin{\widehat{\times}} \mathcal{L}H_2) \tag{10.1}$$

Thus, the strong product can be considered as a special case of the direct product, generalizing the well-known results about the direct and strong graph product, see Equ.(9.1).

In contrast to the direct product $\widehat{\times}$, the 2-section of the strong product $\widehat{\boxtimes}$ coincides with the strong graph product of the 2-sections of its factors.

Lemma 10.2. *The 2-section of the strong product $H = H' \mathbin{\widehat{\boxtimes}} H''$ is the strong product of the 2-section of H' and the 2-section of H'', more formally:*

$$[H' \mathbin{\widehat{\boxtimes}} H'']_2 = [H']_2 \boxtimes [H'']_2.$$

Proof. Let p_1 and p_2 denote $p_{H'}$ and $p_{H''}$, respectively. By definition of the 2-section and the strong product, $[H]_2$ and $[H']_2 \boxtimes [H'']_2$ have the same vertex set. Thus we need to show that the identity mapping $V([H]_2) \to V([H']_2 \times [H'']_2)$ is an isomorphism. We have: $\{x, y\} \in E([H]_2) \Leftrightarrow \exists\, e \in E(H) : \{x, y\} \subseteq e,\ x \neq y \Leftrightarrow$

1. $p_1(x) = p_1(y)$ and $(p_2(x), p_2(y)) \subseteq p_2(e) \in E(H'')$, $p_2(x) \neq p_2(y)$ or
2. $p_2(x) = p_2(y)$ and $(p_1(x), p_1(y)) \subseteq p_1(e) \in E(H')$, $p_1(x) \neq p_1(y)$ or
3. $(p_1(x), p_1(y)) \subseteq p_1(e) \in E(H')$ and $(p_2(x), p_2(y)) \subseteq p_2(e) \in E(H'')$, $p_i(x) \neq p_i(y)$, $i = 1, 2$.

$\Leftrightarrow$

1. $p_1(x) = p_1(y)$ and $(p_2(x), p_2(y)) \in E([H'']_2)$ or
2. $p_2(x) = p_2(y)$ and $(p_1(x), p_1(y)) \in E([H']_2)$ or
3. $(p_1(x), p_1(y)) \in E([H']_2)$ and $(p_2(x), p_2(y)) \in E([H'']_2)$.

$\Leftrightarrow \{x, y\} \in E([H']_2 \boxtimes [H'']_2)$. □

Definition 10.3 (The Strong Product of L2-sections). Let $\Gamma_i = (V_i, E'_i, \mathcal{L}_i)$ be the $L2$-section of the hypergraphs $H_i = (V_i, E_i)$, $i = 1, 2$. The strong product of the $L2$-sections $\Gamma_1 \widehat{\boxtimes} \Gamma_2 = (V, E', \mathcal{L})$ consists of the graph $(V, E') = (V_1, E'_1) \boxtimes (V_2, E'_2)$ and a labeling function

$$\mathcal{L} = \mathcal{L}_1 \widehat{\boxtimes} \mathcal{L}_2 : E' \to \mathbb{P}(E(H_1 \widehat{\boxtimes} H_2))$$

assigning to each edge $e' = \{(x_1, y_1), (x_2, y_2)\} \in E'$ a label

$$\mathcal{L}(e') = (\mathcal{L}_1 \Box \mathcal{L}_2)(e') \cup \widehat{\mathcal{L}}(e')$$

where $(\mathcal{L}_1 \Box \mathcal{L}_2)(e') = \emptyset$ if $x_1 \neq x_2$ and $y_1 \neq y_2$ and

$$\widehat{\mathcal{L}}(e') = \begin{cases} \{e' \cup B \mid B \in \mathcal{B}(x_1, e'_2, e_2), e_2 \in \mathcal{L}_2(e'_2)\}, & \text{if } x_1 = x_2 \\ \{e' \cup C \mid C \in \mathcal{C}(y_1, e'_1, e_1), e_1 \in \mathcal{L}_1(e'_1)\}, & \text{if } y_1 = y_2 \\ \{e' \cup D \mid D \in \mathcal{D}(e', e_1, e_2), e_1 \in \mathcal{L}_1(e'_1), e_2 \in \mathcal{L}_2(e'_2)\}, & \text{else} \end{cases}$$

with $e'_1 = \{x_1, x_2\}$, $e'_2 = \{y_1, y_2\}$ and

$$\mathcal{B}(x_1, e'_2, e_2) = \left\{ B \in \binom{e_1 \times (e_2 \setminus e'_2)}{|e_2| - 2} \mid x_1 \in e_1 \in E_1, |e_1| < |e_2|, p_i(B \cup e') = e_i, i = 1, 2 \right\}$$

and

$$\mathcal{C}(y_1, e'_1, e_1) = \left\{ C \in \binom{(e_1 \setminus e'_1) \times e_2}{|e_1| - 2} \mid y_1 \in e_2 \in E_2, |e_2| < |e_1|, p_i(C \cup e') = e_i, i = 1, 2 \right\}$$

and

$$\mathcal{D}(e', e_1, e_2) = \left\{ D \in \binom{(e_1 \times e_2) \setminus (e'_1 \times e'_2)}{r^+_{e_1, e_2} - 2} \mid p_i(D \cup e') = e_i, i = 1, 2 \right\}$$

Again, one can show that $[H \widehat{\boxtimes} H']_{L_2} = [H]_{L2} \widehat{\boxtimes} [H']_{L2}$ holds for all simple hypergraphs H, H'.

Lemma 10.4 (Distance Formula). *For all hypergraphs H, H' without loops we have:*

$$d_{H \widehat{\boxtimes} H'}((x,a),(y,b)) = \max\{d_H(x,y), d_{H'}(a,b)\}$$

Proof. Combining the results of Lemma 2.1, Lemma 10.2 and the well-known Distance Formula for the strong graph product (Corollary 5.5 in [31]) yields to the result. □

Although $\widehat{\boxtimes}$ appears to be the most promising strong product of hypergraphs, it has not been investigated in any detail so far.

11. Alternative Constructions Generalizing the Strong Graph Product

In order to generalize the strong graph product one can draw on an abundance of ways to define strong hypergraph products. To complete this part, we suggest a few of possibilities that have not been considered in the literature so far but might warrant further attention.

1. $E(H_1 \widetilde{\boxtimes} H_2) = E(H_1 \Box H_2) \cup E(H_1 \widetilde{\times} H_2)$
2. $E(H_1 \boxtimes H_2) = \left\{ \binom{e\times e'}{r^-_{e,e'}} \mid e \in E_1, e' \in E_2 \right\}$
3. $E(H_1 \boxtimes H_2) = \left\{ \binom{e\times e'}{r^+_{e,e'}} \mid e \in E_1, e' \in E_2 \right\}$
4. $E(H_1 \boxtimes H_2) = \left\{ \binom{e\times e'}{|e|} \mid e \in E_1, e' \in E_2 \right\} \cup \left\{ \binom{e\times e'}{|e'|} \mid e \in E_1, e' \in E_2 \right\}$
5. $E(H_1 \boxtimes H_2) = \left\{ \binom{e\times e'}{k} \mid e \in E_1, e' \in E_2, r^-_{e,e'} \leq k \leq r^+_{e,e'} \right\}$

Part 5. Lexicographic and Costrong Products

12. The Lexicographic Product

The lexicographic product is the only non-commutative product treated in this survey. The lexicographic product of hypergraphs has received considerable attention in the literature [56, 58, 26, 19, 29, 28, 8, 16].

12.1. Definition and Basic Properties

Definition 12.1 (The Lexicographic Product [19]**).** Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two hypergraphs. The *lexicographic product* $H = H_1 \circ H_2$ has vertex set $V(H) = V_1 \times V_2$ and edge set

$$E(H) = \{e \subseteq V(H) : p_1(e) \in E_1, |p_1(e)| = |e|\} \cup \{\{x\} \times e_2 \mid x \in V_1, e_2 \in E_2\}.$$

Since $|p_1(e)| = |e|$ there are $|e|$ vertices of e that have pairwise different first coordinates. A related construction was also explored in [19]:

Definition 12.2 (X-join of hypergraphs). Let $X = (V(X), E(X))$ be a hypergraph and let $\{H(x) \mid x \in V(X)\}$ be a set of arbitrary pairwise disjoint hypergraphs, each of them associated with a vertex $x \in V(X)$. The X*-join* of $\{H(x) \mid x \in V(X)\}$ is the hypergraph $Z = (V(Z), E(Z))$ with

$$\begin{aligned} V(Z) = &\bigcup_{x \in V(X)} V(H(x)) \quad \text{and} \\ e \in E(Z) \Leftrightarrow &e \in E(H(x)), \quad \text{or} \\ &|e \cap V(H(x))| \leq 1 \text{ and } \{x \mid e \cap V(H(x)) \neq \emptyset\} \in E(X). \end{aligned}$$

If $X \cong K_2$, then Z is also called *join* of H_1 and H_2, $Z = H_1 \oplus H_2$. The X-join is a generalization of the lexicographic product in the following sense: If Z is an X-join of hypergraphs $\{H(x) \mid x \in V(X)\}$ and if $H(x) \cong H$ for all $x \in V(X)$ then Z is equivalent to the lexicographic product $X \circ H$.

The lexicographic product of two simple hypergraphs is simple. It is associative, has the single vertex graph K_1 as a unit element, and is right-distributive with respect to the join and the disjoint union of hypergraphs. Additionally, we have the following left-distributive properties w.r.t. join and disjoint union:

$$\begin{aligned} \overline{K}_n \circ (H_1 + H_2) &= \overline{K}_n \circ H_1 + \overline{K}_n \circ H_2 \\ K_n \circ (H_1 \oplus H_2) &= K_n \circ H_1 \oplus K_n \circ H_2 \end{aligned}$$

for all hypergraphs H_1, H_2 [19, 26]. The lexicographic product is not commutative in general.

Theorem 12.3 ([19]). *Let H_1 and H_2 be two non-trivial connected finite hypergraphs. Then $H_1 \circ H_2 \cong H_2 \circ H_1$ only if H_1 and H_2 are complete graphs or H_1 and H_2 are both powers of some hypergraph H.*

Connectedness of the lexicographic product depends only on the first factor. More precisely, $H_1 \circ H_2$ is a connected hypergraph if and only if H_1 is connected. For the rank and anti-rank, respectively, of a lexicographic product hypergraph $H_1 \circ H_2$ holds:

$$\begin{aligned} r(H_1 \circ H_2) &= \max\{r(H_1), r(H_2)\} \\ s(H_1 \circ H_2) &= \min\{s(H_1), s(H_2)\}. \end{aligned}$$

The projection p_1 of a lexicographic product $H = H_1 \circ H_2$ of two hypergraphs H_1, H_2 into the first factor H_1 is a weak homomorphism. The H_2-layer through w, H_2^w, is the partial hypergraph of H induced by all vertices of H which differ from a given vertex $w \in V(H)$ exactly in the second coordinate, and it holds:

$$H_2^w = \langle \{(p_1(w), t) \mid t \in V(H_2)\} \rangle \cong H_2.$$

12.2. Relationships with Graph Products

The restriction of the lexicographic product on graphs coincides with the usual lexicographic graph product.

Lemma 12.4. *The 2-section of $H = H' \circ H''$ is the lexicographic product of the 2-section of H' and the 2-section of H'', more formal:*

$$[H' \circ H'']_2 = [H']_2 \circ [H'']_2.$$

Proof. Let p_1 and p_2 denote $p_{H'}$ and $p_{H''}$, respectively. By definition of the 2-section and the lexicographic product, $[H]_2$ and $[H']_2 \boxtimes [H'']_2$ have the same vertex set. Thus, we need to show that the identity mapping $V([H]_2) \to V([H']_2 \times [H'']_2)$ is an isomorphism. We have: $\{x, y\} \in E([H]_2) \Leftrightarrow \exists\, e \in E(H) : \{x, y\} \subseteq e,\ x \neq y$ $\Leftrightarrow$

1. $(p_1(x), p_1(y)) \subseteq p_1(e) \in E(H')$ such that $p_1(x) \neq p_1(y)$ or
2. $p_1(x) = p_1(y)$ and $(p_2(x), p_2(y)) \subseteq p_2(e) \in E(H'')$ such that $p_2(x) \neq p_2(y)$.

$\Leftrightarrow$

1. $(p_1(x), p_1(y)) \in E([H']_2)$ or
2. $p_1(x) = p_1(y)$ and $(p_2(x), p_2(y)) \in E([H'']_2)$.

$\Leftrightarrow \{x, y\} \in E([H']_2 \circ [H'']_2)$. □

Definition 12.5 (The Lexicographic Product of $L2$-sections). Let $\Gamma_i = (V_i, E_i', \mathcal{L}_i)$ be the $L2$-section of the hypergraphs $H_i = (V_i, E_i)$, $i = 1, 2$. The lexicographic product of the $L2$-sections $\Gamma_1 \circ \Gamma_2 = (V, E', \mathcal{L})$ consists of the graph $(V, E') = (V_1, E_1') \circ (V_2, E_2')$ and a labeling function

$$\mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_2 : E' \to \mathbb{P}(E(H_1 \circ H_2))$$

assigning to each edge $e' = \{(x_1, y_1), (x_2, y_2)\} \in E'$ a label

$$\mathcal{L}(\{(x_1, y_1), (x_2, y_2)\}) = \begin{cases} \{(e, f(e)) \mid e \in \mathcal{L}_1(\{x_1, x_2\}), f \in F(e, e')\}, & \text{if } x_1 \neq x_2 \\ \{\{x_1\} \times e \mid e \in \mathcal{L}_2(\{y_1, y_2\})\}, & \text{otherwise} \end{cases}$$

where $F(e, e') = \{f : e \to V_2 \mid \mathrm{f}(x_1) = y_1, f(x_2) = y_2\}$ and $(e, f(e))$ denotes the set $\{(x_1, f(x_1)), \ldots, (x_k, f(x_k))\}$ for $e = \{x_1, \ldots, x_k\}$.

A straightforward computation shows that $[H \circ H']_{L_2} = [H]_{L2} \circ [H']_{L2}$ holds for all simple hypergraphs H, H'.

Lemma 12.6 (Distance Formula). *Let (x_1, x_2) and (y_1, y_2) be two vertices of the lexicographic product $H = H_1 \circ H_2$. Then*

$$d_H((x_1, x_2), (y_1, y_2)) = \begin{cases} d_{H_1}(x_1, y_1), & \textit{if } x_1 \neq y_1 \\ d_{H_2}(x_2, y_2), & \textit{if } x_1 = y_1 \textit{ and } \deg_{H_1}(x_1) = 0 \\ \min\{d_{H_2}(x_2, y_2), 2\}, & \textit{if } x_1 = y_1 \textit{ and } \deg_{H_1}(x_1) \neq 0 \end{cases}$$

Proof. Combining the results of Lemma 2.1, Lemma 12.4 and the Distance Formula for the lexicographic graph product (Proposition 5.12 in [31]) yields to the result. □

12.3. Prime Factor Decomposition

Similar conditions as for graphs are known for the uniqueness of the PFD of a hypergraph w.r.t. the lexicographic product:

Lemma 12.7 ([26]). *Let H be a hypergraph without isolated vertices and m, n natural numbers. Then $\overline{K}_n \circ H + \overline{K}_m$ is prime with respect to the lexicographic product if and only if $H \circ \overline{K}_n + \overline{K}_m$ is prime.*

If H has no trivial join-components, then $K_n \circ H \oplus K_m$ is prime with respect to the lexicographic product if and only if $H \circ K_n \oplus K_m$ is prime.

Let $H = P \circ \overline{K}_q$ be a PFD of H with $P = \overline{K}_q \circ A + \overline{K}_m$ such that A has no non-trivial components. Then $H = \overline{K}_q \circ R$ with $R = A \circ \overline{K}_q + \overline{K}_m$ is also a PFD of H that arises by *transposition* of $\overline{K}_q$ from $H = P \circ \overline{K}_q$. The transposition of K_q is defined analoguously.

Theorem 12.8 ([26]). *Any prime factor decomposition of a graph can be transformed into any other one by transpositions of totally disconnected or complete factors.*

Corollary 12.9 ([26]). *All prime factor decompositions of a hypergraph H with respect to the lexicographic product have the same number of factors.*

If there is a prime factorization of H without complete or totally disconnected graphs as factors, then H has a unique prime factor decomposition.

If there is a prime factorization of H in which only complete graphs as factors have trivial join-components and only totally disconnected factors have trivial components, then H has a unique prime factor decomposition.

12.4. Invariants

Definition 12.10 (Wreath Product of Automorphism Groups [19, 26]). Let $H = (V, E)$ and $H'(V', E')$ be hypergraphs. The *wreath product* of their automorphism groups is defined as

$$\begin{aligned} Aut(H) \circ Aut(H') := &\{\varphi \in Aut(H \circ H') \mid \exists\, \psi \in Aut(H),\ \forall\, v \in V\ \exists\, \psi'_v \in Aut(H'),\\ &\text{s.t. } \varphi((v, v')) = (\psi(v), \psi'_v(v'))\ \forall\, (v, v') \in V \times V'\} \end{aligned}$$

Hence $Aut(H) \circ Aut(H')$ forms a subgroup of $Aut(H \circ H')$. The elements of $Aut(H) \circ Aut(H')$ map H'-layer onto H'-layer and are therefore often called *natural* automorphisms. In [19] and [29] it is shown, under which conditions holds $Aut(H) \circ Aut(H') = Aut(H \circ H')$.

Theorem 12.11 (Double Covers [16]). *If the hypergraphs H_1, H_2 are double cover hypergraphs then so is their lexicographic product $H_1 \circ H_2$.*

Theorem 12.12 (Hamiltonicity I [56]). *Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$, $|V_2| \geq 2$ be two hypergraphs. Then their lexicographic product $H_1 \circ H_2$ contains a Hamiltonian path if and only if there exists a walk $W = (v_0, e_1, v_1, \ldots, e_{k-1}, v_k)$ in H_1 with $V(W) = V_1$ such that $\wp(H_2) \leq \min_{v \in V_1} \{|j| \mid v_j = v, j \in \{0, \ldots, k\}\}$ and $\max_{v \in V_1} \{|j| \mid v_j = v, j \in \{0, \ldots, k\}\} \leq |V_2|$.*

Theorem 12.13 (Hamiltonicity II [56]**).** *Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be two non-trivial hypergraphs. Then their lexicographic product $H_1 \circ H_2$ contains a Hamiltonian cycle if and only if there exists a walk $W = (v_0, e_1, v_1, \dots, e_{k-1}, v_k)$ in H_1 with $V(W) = V_1$ such that $\wp(H_2) \leq \min_{v\in V_1} \{|j| \mid v_j = v, j \in \{0, \dots, k\}\}$, $\max_{v\in V_1} \{|j| \mid v_j = v, j \in \{0, \dots, k\}\} \leq |V_2|$ and $\max_{v\in V_1} \{|j| \mid v_j = v, j \in \{0, \dots, k\}\} = |V_2|$ implies that $v_0 \neq v_k$ and there is an edge e in E_1 containing both v_0 and v_k.*

13. Costrong Product

13.1. Definition and Basic Properties

Since the lexicographic product is not commutative, it appears natural to consider "symmetrized" variants of the lexicographic product. The *costrong product* $H_1 * H_2$ [26] has the edge set

$$E(H_1 * H_2) = E(H_1 \circ H_2) \cup E(H_2 \circ H_1).$$

The costrong product is associative, commutative and has K_1 as unit [26]. The costrong product of two simple hypergraphs is not simple, unless both factors are r-uniform. The projections into the factors are neither (weak) homomorphisms nor preserve adjacency. Rank and anti-rank of the costrong hypergraph product $H_1 * H_2$ satisfy

$$r(H_1 * H_2) = \max\{r(H_1), r(H_2)\}$$
$$s(H_1 * H_2) = \min\{s(H_1), s(H_2)\}.$$

A hypergraph $H = (V, E)$ is said to be *coconnected* if for each pair of vertices $u, v \in V$ there exists a sequence of pairwise distinct vertices $u = u_1, \dots, u_k = v$ such that consecutive vertices u_i, u_{i+1} are not both contained in any edge of H. A costrong product of two hypergraphs is coconnected if and only if both of the factors are.

Theorem 13.1 (UPFD [26]**).** *Every finite coconnected hypergraph has a unique PFD w.r.t. the costrong product.*

In [59] it is shown under which (quite complex) conditions the costrong product is Hamiltonian. Automorphism groups of costrong products have been considered by Gaszt and Imrich, [26].

13.2. Relationships with Graph Products

The restriction of the costrong product to graphs coincides with the respective costrong graph product. The costrong product of graphs can be obtained from the strong product by virtue of the identity $G_1 * G_2 = \overline{\overline{G_1} \boxtimes \overline{G_2}}$. This construction is not applicable to hypergraphs because no suitable definition of complements of hypergraphs has been proposed so far [25, 26].

The $L2$-section of costrong products can be derived in a straightforward way from the definition of the $L2$-section of the lexicographic product. We omit an explicitly description here.

Part 6. Other Hypergraph Products

The hypergraph products discussed so far reduce to graph products at least in the class of simple graphs. In this part of the survey we summarize alternative constructions that have received considerable attention but do not correspond to graph products.

14. The Square Product

The literature is by no means consistent in its use of the terms "square product" and "direct product". In particular, many authors use the term for "direct product" for the square product, see e.g. [7, 48, 51, 14, 13, 1, 61, 2, 9, 17, 23, 44, 18]. We favor the term "square product" introduced by Nešetřil and Rödl [44], since it does not reduce to the direct product on graphs. To add to the confusion, some authors also used the term "strong product", see e.g. [9]. The square product seems to be the most widely studied of the hypergraph products.

14.1. Definition and Basic Properties

Definition 14.1 (Square Product of Hypergraphs [7]). Given two hypergraphs $H_1 = (V_1, E_1)$, $H_2 = (V_2, E_2)$ the *square product* $H = H_1 \blacksquare H_2$ has vertex set $V(H) = V_1 \times V_2$ and edge set

$$E(H) = \{e_1 \times e_2 \mid e_i \in E_i\}\,.$$

The square product is associative, commutative, distributive w.r.t. the disjoint union and has the single vertex graph with loop $\mathcal{L}K_1$ as unit element. The square product of two hypergraphs is connected if and only if both of its factors are connected, and it is a uniform hypergraph if an only if both of the factors are uniform hypergraphs. The square product of two simple hypergraphs is a simple hypergraph. The projections into the factors are homomorphisms [18]. For the rank and anti-rank, respectively, of the square product holds:

$$r(H_1 \blacksquare H_2) = r(H_1)r(H_2)$$
$$s(H_1 \blacksquare H_2) = s(H_1)s(H_2)$$

Proposition 14.2 ([7]). *The dual hypergraph of a square product of two hypergraphs H_1 and H_1 is the square product of the dual hypergraphs of H_1 and H_2. More formal:*

$$(H_1 \blacksquare H_2)^* = H_1^* \blacksquare H_2^*$$

14.2. Relationships with Graph Products

The square product of two graphs is not a graph, but a 4-uniform hypergraph. Nevertheless, its 2-section has the structure of a graph product.

Lemma 14.3. *The 2-section of the square product $H = H' \blacksquare H''$ is the strong product of the 2-section of H' and the 2-section of H'', more formally:*

$$[H' \blacksquare H'']_2 = [H']_2 \boxtimes [H'']_2.$$

Proof. Let p_1 and p_2 denote $p_{H'}$ and $p_{H''}$, respectively. By definition of the 2-section and the strong graph product, $[H]_2$ and $[H']_2 \boxtimes [H'']_2$ have the same vertex set. Thus, we need to show that the identity mapping $V([H]_2) \to V([H']_2 \times [H'']_2)$ is an isomorphism. We have: $\{x, y\} \in E([H]_2) \Leftrightarrow \exists\, e \in E(H) : \{x, y\} \subseteq e = p_1(e) \times p_2(e),\ x \neq y \Leftrightarrow$

1. $(p_1(x), p_1(y)) \subseteq p_1(e) \in E(H')$ and
2. $(p_2(x), p_2(y)) \subseteq p_2(e) \in E(H'')$.

Note, in contrast to the other proofs we do not need the condition that $p_i(x) \neq p_i(y)$, $i = 1, 2$. However, since $x \neq y$ we can conclude that $p_i(x) \neq p_i(y)$ implies $p_j(x) = p_j(y)$, $i \neq j$. Thus, $\{x, y\} \in E([H]_2) \Leftrightarrow$

1. $(p_1(x), p_1(y)) \in E([H']_2)$ and $p_2(x) = p_2(y)$ or
2. $(p_2(x), p_2(y)) \in E([H'']_2)$ and $p_1(x) = p_1(y)$ or
3. $(p_1(x), p_1(y)) \in E([H']_2)$ and $(p_2(x), p_2(y)) \in E([H'']_2)$.

$\Leftrightarrow \{x, y\} \in E([H']_2 \boxtimes [H'']_2)$. □

Definition 14.4 (The Square Product of L2-sections). Let $\Gamma_i = (V_i, E'_i, \mathcal{L}_i)$ be the $L2$-section of the hypergraphs $H_i = (V_i, E_i)$, $i = 1, 2$. The square product of the $L2$-sections $\Gamma_1 \blacksquare \Gamma_2 = (V, E', \mathcal{L})$ consists of the graph $(V, E') = (V_1, E'_1) \boxtimes (V_2, E'_2)$ and a labeling function

$$\mathcal{L} = \mathcal{L}_1 \blacksquare \mathcal{L}_2 : E' \to \mathbb{P}(E(H_1 \blacksquare H_2))$$

with

$$\mathcal{L}(\{(x_1, y_1), (x_2, y_2)\}) = \{e_1 \times e_2 \mid (A1) \text{ or } (A2) \text{ or } (A3)\}$$

where

$$\begin{aligned}
&(A1) \quad x_1 = x_2 \in e_1 \in E_1, e_2 \in \mathcal{L}_2(\{y_1, y_2\}) \\
&(A2) \quad y_1 = y_2 \in e_2 \in E_2, e_1 \in \mathcal{L}_1(\{x_1, x_2\}) \\
&(A3) \quad e_1 \in \mathcal{L}_1(\{x_1, x_2\}) \text{ and } e_2 \in \mathcal{L}_2(\{y_1, y_2\})
\end{aligned}$$

One can show that $[H \blacksquare H']_{L_2} = [H]_{L2} \blacksquare [H']_{L2}$ holds for all simple hypergraphs H, H'.

Lemma 14.5 (Distance Formula). *For all hypergraphs H, H' without loops we have:*

$$d_{H \blacksquare H'}((x, a), (y, b)) = \max\{d_H(x, y), d_{H'}(a, b)\}$$

Proof. Combining the results of Lemma 2.1, Lemma 14.3 and the well-known Distance Formula for the strong graph product (Corollary 5.5 in [31]) yields to the result. □

14.3. Prime Factor Decomposition

Theorem 14.6 (UPFD [18]**).** *Every finite connected hypergraph (without multiple edges) has a unique PFD w.r.t. the square product.*

14.4. Invariants

Theorem 14.7 (Automorphism Group [18]**).** *Let $H = H_1 \blacksquare \ldots \blacksquare H_n$ be the square product of prime hypergraphs H_i fulfilling the following condition:*

> *For each pair of distinct vertices there exists an edge containing exactly one of those vertices.*

Furthermore, let $\varphi \in Aut(H)$. Then there exists a permutation π of $\{1, \ldots, n\}$ and isomorphisms $a_i : H_i \to H_{\pi(i)}$ such that the $\pi(i)$-th component of $\varphi((x_1, \ldots, x_i, \ldots, x_n))$ is $a_i(x_i)$.

Corollary 14.8. *Under the assumptions of Theorem 14.7 the automorphism group of $H = H_1 \blacksquare \ldots \blacksquare H_n$ is generated by direct products of automorphisms of the H_i and exchanges of isomorphic factors.*

Theorem 14.9 (k-fold Covering [17]**).** *Let $H'_i = (V'_i, E'_i)$ be a k_i-fold covering of hypergraphs $H_i = (V_i, E_i)$ via a covering projection p'_i, $i = 1, 2$. Then the square product $H'_1 \blacksquare H'_2$ is a k_1k_2-fold cover of $H_1 \blacksquare H_2$ via a covering projection π induced naturally by p'_1 and p'_2, i.e, define p by:*

> $p((x, y)) = (p'_1(x), p'_2(y))$, *for* $(x, y) \in V'_1 \times V'_2$,
> $p(e_1 \times e_2) = p'_1(e_1) \times p'_2(e_2)$, *for* $e_1 \in E'_1$, $e_2 \in E'_2$,

Theorem 14.10 (Conformal Hypergraphs [9]**).** *$H = H_1 \blacksquare H_2$ is conformal if and only if H_1 and H_2 are conformal.*

Theorem 14.11 (Helly Property [9]**).** *$H = H_1 \blacksquare H_2$ has the Helly property if and only if H_1 and H_2 have the Helly property.*

Theorem 14.12 (Stability Number [61, 7]**).** *For any two hypergraphs $H = (V, E)$ and $H' = (V', E')$ with stability number β and β', respectively, holds*

$$\beta\beta' + |V'|\beta + |V|\beta' \leq \beta(H \blacksquare H') \leq |V'|\beta + |E|\beta'.$$

The stability number $\beta(H)$ of a hypergraph H satisfies $\beta(H) = |V(H)| - \tau(H)$ and $\beta(H \blacksquare H') = |V(H)||V(H')| - \tau(H \blacksquare H')$. Further results for the stability number can thus be obtained from the properties of the covering number [7].

Theorem 14.13 (Matching and Covering [6]**).** *For two hypergraphs H and H' we have*

$$\begin{aligned} \nu(H)\nu(H') &\leq \nu(H \blacksquare H') \leq \tau^*(H)\nu(H') \leq \tau^*(H)\tau^*(H') \\ = \tau^*(H \blacksquare H') &\leq \tau^*(H)\tau(H') \leq \tau(H \blacksquare H') \leq \tau(H)\tau(H'). \end{aligned}$$

Theorem 14.14 (Fractional Covering Number I [7]**).** *A necessary and sufficient condition for a hypergraph H to satisfy $\tau(H \blacksquare H') = \tau(H)\tau(H')$ for all H' is that $\tau(H) = \tau^*(H)$.*

Theorem 14.15 (Fractional Covering Number II [7]**).** *Let H and H' be two hypergraphs. Then*

$$\tau(H \blacksquare H') \geq \tau(H) + \tau(H') - 1.$$

A hypergraph $H = (V, E)$ satisfies $\tau(H \blacksquare H') = \tau(H) + \tau(H') - 1$ for every hypergraph H' if and only if $\bigcap_{e\in E} \neq \emptyset$.

Theorem 14.16 (Fractional Covering Number III [47]**).** *For every hypergraph H holds:*

$$\tau^*(H) = \lim_{n\to\infty} \sqrt[n]{\tau(H^{\blacksquare n})}$$

Theorem 14.17 (Fractional Covering Number IV [23, 6]**).** *For every hypergraph H holds:*

$$\tau^*(H) = \min_{H'} \frac{\tau(H \blacksquare H')}{\tau(H')},$$

where H' runs over all hypergraphs.

Theorem 14.18 (Fractional Covering and Matching Number [6]**).** *For every hypergraph H with Helly property holds:*

$$\tau^*(H) = \min_{H'} \frac{\tau(H \blacksquare H')}{\tau(H')},$$

where H' runs over all hypergraphs.

Theorem 14.19 (Partition Number [1]**).** *Let $d \in \mathbb{N}$ and $H_i = (V_i, E_i), i = 1, \ldots, n$ be hypergraphs such that $E_i = \binom{V_i}{d} \cup \{\{v\} \mid v \in V_i\}$. Moreover, let $r_i = |V_i| \mod d$. If $d > \prod_{i:r_i\neq 0} r_i$ then*

$$\rho(\mathop{\blacksquare}_{i=1}^{n} H_i) = \prod_{i=1}^{n} \rho(H_i).$$

Theorem 14.20 (Chromatic Number I [7]**).** *For two hypergraphs H and H' we have:*

$$\chi(H \blacksquare H') \leq \min\{\chi(H), \chi(H')\}.$$

In [7] the authors asked whether the chromatic number of the square product of two hypergraphs goes to infinity if the chromatic numbers of both of the factors go to infinity. The following theorem, which is due to D. Mubayi and V. Rödl, refutes this conjecture.

Theorem 14.21 (Chromatic Number II [48]**).** *For every integer $n \geq 2$ and $k = 2^n$ there exists a hypergraph H_k satisfying $\chi(H_k) > k$ and $\chi(H_k \blacksquare H_k) = 2$.*

Moreover, in [48] the authors conjectured that for every $r \geq 2$ there is a $c \geq 2$ such that for every positive integer k there exists r-uniform hypergraphs G_k and H_k for which $\chi(G_k) > k$, $\chi(H_k) > k$ and $\chi(G_k \blacksquare H_k) \leq c$, and that this also true for the special case $c = r = 2$.

Other results concerning the chromatic number of a special case of square products, i.e., square products of complete graphs, are due to Sterboul and can be found in [61].

Theorem 14.22 (Discrepancy [15]**).** *For any $k \in \mathbb{N}$ and any two hypergraphs H, H' it holds:*

$$disc\,(H \blacksquare H', k) \leq k \cdot disc\,(H, k) \cdot disc\,(H', k).$$

A special partial hypergraph of the square product $H^{\blacksquare d}$ of a hypergraph $H = (V, E)$ has found particular attention and is also called the *d-fold symmetric product*, defined as the subgraph $\{V^d, \{e^d, e \in E\}\}$, where M^d denotes the usual d-fold Cartesian set product of the set M. In [14, 13] the authors gave several upper and lower bounds for the discrepancy w.r.t. this product.

15. The Categorial Product

The following hypergraph product was motivated by the investigation of a category of hypergraphs [20]. It is categorical in the category of hypergraphs. It has rarely been studied since its introduction, however; to our knowledge, the only systematic account is a contribution by X. Zhu [64].

15.1. Definition and Basic Properties

Definition 15.1 (Hypergraph Product [20]**).** Let $H_i = (V_i, E_i)$, $i = 1, 2$ be two hypergraphs. Then their *product* $H = H_1 \circledcirc H_2$ has edge set $V(H) = V_1 \times V_2$ and vertex set

$$E(H) = \{e \subseteq V(H) \mid p_i(e) \in E_i,\ i = 1, 2\}$$

The categorial hypergraph product is associative, commutative, distributive w.r.t. the disjoint union and has the single vertex graph with loop $\mathcal{L}K_1$ as unit element. The projections into the factors are, by definition, homomorphisms. However, the product of two simple hypergraphs is not a simple hypergraph and the product of two non-trivial uniform hypergraphs does not result in a uniform hypergraph. For the rank and anti-rank, respectively, of a hypergraph product holds:

$$r(H_1 \circledcirc H_2) = r(H_1) r(H_2)$$
$$s(H_1 \circledcirc H_2) = \max\{s(H_1), s(H_2)\}$$

We have the following relations with other hypergraph products:

- $E(H_1 \widehat{\times} H_2) = E' \subseteq E(H_1 \circledcirc H_2)$ with $E' := \{e \in E(H_1 \circledcirc H_2) \mid \nexists e' \in E(H_1 \circledcirc H_2) \text{ s.t. } e' \subset e\}$ for simple hypergraphs H_1 and H_2 [64].
- $E(H_1 \blacksquare H_2) \subseteq E(H_1 \circledcirc H_2)$

15.2. Relationships with Graph Products

The restriction of this product to graphs does not coincide with any known graph product. Moreover, the product $G_1 \circledcirc G_2$ of two graphs G_1 and G_2 is no graph anymore, but a hypergraph of rank 4.

Lemma 15.2. *The 2-section of the product $H = H' \circledcirc H''$ is the strong product of the 2-section of H' and the 2-section of H'', more formally:*

$$[H' \circledcirc H'']_2 = [H']_2 \boxtimes [H'']_2.$$

This result can be proved analogously to the proof of Lemma 14.3.

Definition 15.3 (The Categorial Product of L2-sections). Let $\Gamma_i = (V_i, E_i', \mathcal{L}_i)$ be the $L2$-section of the hypergraphs $H_i = (V_i, E_i)$, $i = 1, 2$. The categorial product of the $L2$-sections $\Gamma_1 \circledcirc \Gamma_2 = (V, E', \mathcal{L})$ consists of the graph $(V, E') = (V_1, E_1') \boxtimes (V_2, E_2')$ and a labeling function

$$\mathcal{L} = \mathcal{L}_1 \circledcirc \mathcal{L}_2 : E' \to \mathbb{P}(E(H_1 \circledcirc H_2))$$

assigning to each edge $e' = \{(x_1, y_1), (x_2, y_2)\} \in E'$ a label with

$$\mathcal{L}\,(e') = \begin{cases} \bigcup_{e_1 \in \mathcal{L}(e_1'), e_2 \in E_2 : y_1 \in e_2} \mathcal{E}(e', e_1, e_2), & \text{if } y_1 = y_2 \\ \bigcup_{e_2 \in \mathcal{L}(e_2'), e_1 \in E_1 : x_1 \in e_1} \mathcal{E}(e', e_1, e_2), & \text{if } x_1 = x_2 \\ \bigcup_{e_1 \in \mathcal{L}(e_1'), e_2 \in \mathcal{L}(e_2')} \mathcal{E}(e', e_1, e_2), & \text{else, i.e., } x_1 \neq x_2, y_1 \neq y_2 \end{cases}$$

where

$$\mathcal{E}(e', e_1, e_2) = \{f \subseteq e_1 \times e_2 \mid e' \subseteq f, p_i(f) = e_i, i = 1, 2\}.$$

The identity $[H \circledcirc H']_{L_2} = [H]_{L2} \circledcirc [H']_{L2}$ holds for all simple hypergraphs H, H'.

Lemma 15.4 (Distance Formula). *For all hypergraphs H, H' without loops we have:*

$$d_{H \circledcirc H'}((x, a), (y, b)) = \max\{d_H(x, y), d_{H'}(a, b)\}$$

Proof. Combining the results of Lemma 2.1, Lemma 15.2 and the well-known Distance Formula for the strong graph product (Corollary 5.5 in [31]) yields to the result. □

15.3. Invariants

Only the chromatic number of the categorial product $H_1 \circledcirc H_2$ has been investigated in some detail [64].

Theorem 15.5 (Chromatic Number I [64]). *Let H_1 and H_2 be two hypergraphs such that $\chi(H_i) = n + 1$. Moreover, let H_i, $i = 1, 2$ contain a partial hypergraph $H_i' = \{V_i', \{e \subseteq V_i' \mid |e| \geq 2\}\}$ with $|V_i'| = n$ and a vertex-critical $n + 1$ chromatic partial hypergraph $H_i'' = (V_i'', E_i'')$, i.e., for any vertex $x \in V_i''$ the hypergraph induced by $V_i'' - \{x\}$ is n-colorable. Furthermore, let $V_i' \cap V_i'' \neq \emptyset$, $i = 1, 2$. Then $\chi(H_1 \circledcirc H_2) = n + 1$.*

Theorem 15.6 (Chromatic Number II [64]**).** *Let $H = (V, E)$ be a hypergraph with $\chi(H) = n + 1$ such that any $v \in V$ is contained in a partial hypergraph $H' = (V', E')$ of H with $|V'| = n$ and $E' = \{e \subseteq V' \mid |e| \geq 2\}$. Then for any hypergraph G with $\chi(G) = n + 1$ holds $\chi(H \circledcirc G) = n + 1$.*

Part 7. Beyond Finite and Undirected Hypergraphs

16. Infinite Hypergraphs

Only finite hypergraphs and products of finitely many factors have been treated so far. It is possible to extend the definitions of the products to infinitely many finite and to infinite hypergraphs. For this purpose we need the following definition. For an arbitrary family of (vertex) sets V_i, $i \in I$, their Cartesian set product $V = \times_{i\in I} V_i$ consists of the set of all functions $x : i \mapsto x_i$, $x_i \in V_i$ of I into $\bigcup_{i\in I} V_i$. Notice, that the Cartesian set product of an arbitrary family of sets is usually denoted by $\prod_{i\in I} V_i$, but to emphasize the relation to the finite case, we use the term $\times_{i\in I} V_i$ instead. In this case, the *projection* $p_j : V \to V_j$ is defined by $v \mapsto v_j$ whenever $v : j \mapsto v_j$. As before, we will call v_j the *j-th coordinate* of the vertex $v \in V$.

Several of the hypergraph products discussed in the previous section are connected provided each of the the finitely many finite factors are connected. A corresponding result can be established for finitely many connected factors of infinite size using the Distance Formula for the respective product. In contrast, connectedness results do not necessarily carry over to products of infinitely many hypergraphs. As an example consider the Cartesian product of infinitely many factors. There are vertices that differ in infinitely many coordinates and thus, by the Distance Formula cannot be connected by a path of finite length, [49, 38]. This in turn leads to difficulties concerning the prime factor decomposition. Again, consider the Cartesian product. An infinite connected hypergraph can have infinitely many prime factors. In this case it cannot be the Cartesian product of these factors, since the product is not connected, but a connected component of this product. For this purpose, the *weak Cartesian product* is presented which was first introduced by Sabidussi [52] for graphs and later generalized by Imrich [38].

Let $\{H_i \mid i \in I\}$ be a family of hypergraphs and let $a_i \in V(H_i)$ for $i \in I$. The weak Cartesian product $H = \square_{i\in I}(H_i, a_i)$ of the rooted hypergraphs (H_i, a_i) is defined by

$$V(H) = \{v \in \underset{i\in I}{\times} V(H_i) \mid p_i(v) \neq a_i \text{ for at most finitely many } i \in I\}$$

$$E(H) = \{e \subseteq V(H) \mid p_j(e) \in E(H_j) \text{ for exactly one } j \in I, |p_i(e)| = 1 \text{ for } i \neq j\}.$$

Note the weak Cartesian product does not depend on the "reference coordinates" a_i; furthermore, it reduced to the ordinary Cartesian product if I is finite. The weak Cartesian product is associative, commutative, distributive w.r.t. the disjoint union and has trivially K_1 as unit. Furthermore, the weak Cartesian product of connected hypergraphs is connected [38].

Theorem 16.1 (UPFD [49]). *Every simple connected finite or infinite hypergraph with finitely or infinitely many factors has a unique PFD w.r.t. the weak Cartesian product.*

We suspect that similar constructions can be used to define infinite versions of the other hypergraph products treated in this contribution.

Other results for infinite hypergraphs are known e.g. about the chromatic number of square products:

Theorem 16.2 ([48]). *Let H and H' be two hypergraph whose edges have finite size. Suppose that $\chi(H) = \chi(H') = \infty$. Then $\chi(H \blacksquare H') = \infty$.*

Surprisingly, Theorem 16.2 does not hold if both hypergraphs have edges that are all of infinite size. Let $V(H) = \{1, 2, \dots\}$ and $E(H)$ comprising all infinite subsets of $V(H)$. Clearly, $\chi(H) = \infty$, since any coloring with finitely many colors results in an edge colored with only one color. As mentioned in [48], in $H \blacksquare H$ there exists a proper 2-coloring assigning each vertex $(i, j) \in V(H \blacksquare H)$ one color if $i \leq j$ and the other color else.

Theorem 16.3 ([48]). *Let H and H' be two hypergraph whose edges are all of infinite size. Suppose that $\chi(H) = \chi(H') = \infty$. Then $\chi(H \blacksquare H') = 2$.*

17. Directed Hypergraphs

Directed hypergraphs play a role e.g. as models of chemical reaction networks and in transit and satisfiability problems, see [24, 63, 4] for reviews. Directed hypergraphs can be defined in various ways. Here, we refer to the most general definition. A *directed hypergraph* $\overrightarrow{H} = (V, \overrightarrow{E})$ consists of a vertex set V and a set of *hyperarcs* $\overrightarrow{E}$, where each hyperarc $\overrightarrow{e} \in \overrightarrow{E}$ is an ordered pair of nonempty, not necessarily disjoint subsets of V, $\overrightarrow{e} = (t(e), h(e))$, the *tail* and *head* of $\overrightarrow{e}$, respectively. We call a directed hypergraph *simple*, if $t(e) \subseteq t(e')$ and $h(e) \subseteq h(e')$ implies $\overrightarrow{e} = \overrightarrow{e}'$. The 2-section is then a directed graph with arc (x, y) if there is an edge $(t(e), h(e))$ with $x \in t(e)$ and $y \in h(e)$.

Product structures have not been studied extensively in a directed setting, even though there are some exceptions. The lexicographic product of directed graphs, for instance, appears in a general technique to amplify lower bounds for index coding problems [8].

The directed version of the Cartesian product was first introduced in [49]. The Cartesian product $\overrightarrow{H} = \overrightarrow{H}_1 \Box \overrightarrow{H}_2$ of two directed hypergraphs $\overrightarrow{H}_1 =$

$(V_1, \overrightarrow{E}_1), \overrightarrow{H}_2 = (V_2, \overrightarrow{E}_2)$ has edge set

$$\overrightarrow{E}(\overrightarrow{H}) = \Big\{(\{x\} \times t(f), \{x\} \times h(f)) \mid x \in V_1, \overrightarrow{f} \in \overrightarrow{E}_2\Big\} \cup \Big\{(t(e) \times \{y\}, h(e) \times \{y\}) \mid \overrightarrow{e} \in \overrightarrow{E}_1, y \in V_2\Big\}.$$

Basic properties of the Cartesian product of undirected hypergraphs can immediately be transferred to the directed case. Moreover, uniqueness of the PFD was shown in [49].

Theorem 17.1 (UPFD [49]). *Every connected (finite or infinite) directed hypergraph has a unique PFD w.r.t. the (weak) Cartesian product.*

The definition of the square product might be transferred to hypergraphs as follows: The square product $\overrightarrow{H} = \overrightarrow{H}_1 \blacksquare \overrightarrow{H}_2$ of two directed hypergraphs $\overrightarrow{H}_1 = (V_1, \overrightarrow{E}_1), \overrightarrow{H}_2 = (V_2, \overrightarrow{E}_2)$ has edge set

$$\overrightarrow{E}(\overrightarrow{H}) = \Big\{(t(e) \times t(f), h(e) \times h(f)) \mid \overrightarrow{e} \in \overrightarrow{E}_1, \overrightarrow{f} \in \overrightarrow{E}_2\Big\}$$

In [43] the authors introduce the square product of so-called $\mathcal{N}$-systems, that is a special class of directed hypergraphs. More precisely, $\overrightarrow{N} = (V, \overrightarrow{E})$ is an $\mathcal{N}$-system if $|t(e)| = 1$ and $t(e) \subseteq h(e)$ holds for all $\overrightarrow{e} \in \overrightarrow{E}$ and $\bigcup_{\overrightarrow{e} \in \overrightarrow{E}} t(e) = V$. It is shown that the square product is closed in the class of $\mathcal{N}$-systems, i.e., the square product of two $\mathcal{N}$-systems is again an $\mathcal{N}$-system.

Theorem 17.2 ([43]). *Let $\overrightarrow{N}$ be an $\mathcal{N}$-system. If $[\overrightarrow{N}]_2$ is thin, i.e., there are no two vertices $u, v \in V(\overrightarrow{N})$ with $(u, x) \in \overrightarrow{E}([\overrightarrow{N}]_2) \Leftrightarrow (v, x) \in \overrightarrow{E}([\overrightarrow{N}]_2)$ and $(x, u) \in \overrightarrow{E}([\overrightarrow{N}]_2) \Leftrightarrow (x, v) \in \overrightarrow{E}([\overrightarrow{N}]_2)$, and connected, then $\overrightarrow{N}$ has a unique PFD with unique coordinatization.*

The authors conjectured, furthermore, that the condition of thinness can be omitted as long as one is satisfied with a unique PFD without insisting on a unique coordinatization.

18. Summary

Table 1 provides an overview of the properties of the hypergraph products discussed in this survey. Table 2 shows which hypergraph invariants can be transferred from factors to products at least under some additional conditions.

We considered the following properties:

(P1) Associativity.
(P2) Commutativity.
(P3) Distributivity with respect to the disjoint union.
(P4) Existence of a unit.
(P5) $H_1 \circledast H_2$ is (co-)connected $\Leftrightarrow H_1$ and H_2 are (co-)connected.
(P6) If H_1 and H_2 are simple then $H_1 \circledast H_2$ is simple.

(P7) The projections $p_i : V(H_1 \circledast H_2) \to V(H_i)$ for $i \in \{1, 2\}$ are (at least weak) homomorphisms.
(P8) The projections preserve adjacency.
(P9) The adjacency properties of a product depends on those of its factors.
(P10) Unique prime factorization in special classes of hypergraphs.
(P11) Preserves uniformity
(P12) Preserves r-uniformity
(P13) The restriction of the product $\circledast$ on simple graphs coincides with the respective graph product.
(P14) The restriction of the product $\circledast$ on not necessarily simple graphs is the corresponding graph product.
(P15) The 2-section of the product coincides with the graph product of the 2-section of the factors.

	$\Box$	$\overset{\smile}{\times}$	$\overset{\frown}{\times}$	$\overset{\sim}{\times}$	$\overset{\smile}{\boxtimes}$	$\overset{\frown}{\boxtimes}$	$\circ$	$*$	$\blacksquare$	$\circledcirc$
P1	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
P2	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	–	$\surd$	$\surd$	$\surd$
P3	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	–	–	$\surd$	$\surd$
P4	K_1	–	$\mathcal{L}K_1$	–	K_1	K_1	K_1	K_1	$\mathcal{L}K_1$	$\mathcal{L}K_1$
P5	$\Leftrightarrow$	$\Rightarrow$	$\Rightarrow$	$\Rightarrow$	$\Leftrightarrow$	$\Leftrightarrow$	iff H_1	$\overset{co}{\Leftrightarrow}$	$\Leftrightarrow$	$\Leftrightarrow$
P6	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	–	$\surd$	–
P7	w	–	$\surd$	$\surd$	–	w	p_1, w	–	$\surd$	$\surd$
P8	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	p_1	–	$\surd$	$\surd$
P9	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
P10	4.6,4.7	?	?	?	?	?	12.8,12.9	13.1	14.6	?
P11	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	–
P12	$\surd$	$\surd$	$\surd$	–	$\surd$	$\surd$	$\surd$	$\surd$	–	–
P13	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	–	–
P14	$\surd$	–	$\surd$	–	–	$\surd$	$\surd$	$\surd$	–	–
P15	$\Box$	$\times$	–	–	$\boxtimes$	$\boxtimes$	$\circ$	$*$	$\boxtimes$	$\boxtimes$

TABLE 1. Properties of the hypergraph products.

We considered the following invariants:

(I1) Automorphism group
(I2) k-fold covering
(I3) Independence, matching and covers
(I4) Coloring properties
(I5) Helly property
(I6) Hamiltonicity

	$\Box$	$\breve{\times}, \frown{\times}, \tilde{\times}, \frown{\boxtimes}$	$\breve{\boxtimes}$	$\circ$	$*$	$\blacksquare$	$\circledcirc$
I1	4.8	?	?	[19, 29]	[26]	14.7, 14.8	?
I2	4.9	?	?	12.11	?	14.9	?
I3	?	?	?	?	?	14.12 to 14.19	?
I4	4.12, 4.13	?	?	?	?	14.20 to 14.22	15.5, 15.6
I5	4.10, 4.11	?	?	?	?	14.10, 14.11	?
I6	4.14, 4.15	?	9.5, 9.6	12.12, 12.13	[59]	?	?

TABLE 2. Invariants and Hypergraph Products

The two summary tables use symbol "$\surd$" to indicate that a condition is satisfied, while "$-$" means that the product does not have the property in question. The question mark "?" implies that it is unknown at present whether a particular statement is true. Numbers in brackets refer to citations, while numbers without brackets refer to theorems listed in this survey that establish the property under certain additional preconditions or provides results on particular invariants. If (P7) holds only for weak homomorphisms we indicate this with "w". If (P7) and (P8) holds only for the projection onto the first factor we indicate this by "p_1".

References

[1] R. Ahlswede and N. Cai. On partitioning and packing products with rectangles. *Comb. Probab. Comput.*, 3(4):429–434, 1994.

[2] R. Ahlswede and N. Cai. On extremal set partitions in Cartesian product spaces. Bollobás, Béla (ed.) et al., Combinatorics, geometry and probability. A tribute to Paul Erdős. Proceedings of the conference dedicated to Paul Erdős on the occasion of his 80th birthday, Cambridge, UK, 26 March 1993. Cambridge: Cambridge University Press. 23-32 (1997)., 1997.

[3] D. Archambault, T. Munzner, and D. Auber. TopoLayout: Multilevel graph layout by topological features. *IEEE Transactions on Visualization and Computer Graphics*, 13(2):305–317, 2007.

[4] G. Ausiello, P. G. Franciosa, and D. Frigioni. Directed hypergraphs: problems, algorithmic results, and a novel decremental approach. In A. Restivo, S. R. D. Rocca, and L. Roversi, editors, *ICTCS*, volume 2202 of *Lecture Notes in Computer Science*, page 312327. Springer, 2001.

[5] H.-J. Bandelt and Erich Prisner. Clique graphs and helly graphs. *J. Comb. Theory, Ser. B*, 51(1):34–45, 1991.

[6] C. Berge. *Hypergraphs: Combinatorics of finite sets*, volume 45. North-Holland, Amsterdam, 1989.

[7] C. Berge and M. Simonovitis. The coloring numbers of the direct product of two hypergraphs. In *Hypergraph Seminar*, volume 411 of *Lecture Notes in Mathematics*, pages 21–33, Berlin / Heidelberg, 1974. Springer-Verlag.

[8] A. Blasiak, R. Kleinberg, and E. Lubetzky. Lexicographic products and the power of non-linear network coding. *CoRR*, abs/1108.2489, 2011.

[9] A. Bretto. Hypergraphs and the helly property. *Ars Comb.*, 78:23–32, 2006.

[10] A. Bretto and Y. Silvestre. Factorization of Cartesian products of hypergraphs. Thai, My T. (ed.) et al., Computing and combinatorics. 16th annual international conference, COCOON 2010, Nha Trang, Vietnam, July 19–21, 2010. Proceedings. Berlin: Springer. Lecture Notes in Computer Science 6196, 173-181 (2010)., 2010.

[11] A. Bretto, Y. Silvestre, and T. Vallée. Cartesian product of hypergraphs: properties and algorithms. In *4th Athens Colloquium on Algorithms and Complexity (ACAC 2009)*, volume 4 of *EPTCS*, pages 22–28, 2009.

[12] J. Cupal, S. Kopp, and P. F. Stadler. RNA shape space topology. *Artificial Life*, 6:3–23, 2000.

[13] B. Doerr, M. Gnewuch, and N. Hebbinghaus. Discrepancy of products of hypergraphs. Felsner, Stefan (ed.), 2005 European conference on combinatorics, graph theory and applications (EuroComb '05). Extended abstracts from the conference, Technische Universität Berlin, Berlin, Germany, September 5–9, 2005. Paris: Maison de l'Informatique et des Mathématiques Discrètes (MIMD). Discrete Mathematics & Theoretical Computer Science. Proceedings. AE, 323-328, electronic only (2005)., 2005.

[14] B. Doerr, M. Gnewuch, and N. Hebbinghaus. Discrepancy of symmetric products of hypergraphs. *Discr. Math. Theor. Comp. Sci.*, AE:323–238, 2005.

[15] B. Doerr, A. Srivastav, and P. Wehr. Discrepancy of cartesian products of arithmetic progressions. *Electr. J. Comb.*, 11s:1–16, 2004.

[16] W. Dörfler. Double covers of hypergraphs and their properties. *Ars Comb.*, 6:293–313, 1978.

[17] W. Dörfler. Multiple Covers of Hypergraphs. *Annals of the New York Academy of Sciences*, 319(1):169–176, 1979.

[18] W. Dörfler. On the direct product of hypergraphs. *Ars Comb.*, 14:67–78, 1982.

[19] W. Dörfler and W. Imrich. Über die X-Summe von Mengensystemen. Combinat. Theory Appl., Colloquia Math. Soc. Janos Bolyai 4, 297-309 (1970)., 1970.

[20] W. Dörfler and D.A. Waller. A category-theoretical approach to hypergraphs. *Arch. Math.*, 34:185–192, 1980.

[21] W. Fontana and P. Schuster. Continuity in Evolution: On the Nature of Transitions. *Science*, 280:1451–1455, 1998.

[22] W. Fontana and P. Schuster. Shaping Space: The Possible and the Attainable in RNA Genotype-Phenotype Mapping. *J. Theor. Biol.*, 194:491–515, 1998.

[23] Z. Füredi. Matchings and covers in hypergraphs. *Graphs Comb.*, 4(2):115–206, 1988.

[24] G. Gallo and M. Scutellà. Directed hypergraphs as a modelling paradigm. *Decisions in Economics and Finance*, 21:97–123, 1998.

[25] G. Gaszt and W. Imrich. On the lexicographic and costrong product of set systems. *Aequationes Mathematicae*, 6:319–320, 1971.

[26] G. Gaszt and W. Imrich. Über das lexikographische und das kostarke Produkt von Mengensystemen. (On the lexicographic and the costrong product of set systems). *Aequationes Math.*, 7:82–93, 1971.

[27] L. Gringmann. *Hypergraph Products*. Diploma thesis, Fakultät für Mathematik und Informatik, Universität Leipzig, 2010.

[28] G. Hahn. *Directed hypergraphs: the group of their composition*. Ph.d. thesis, McMaster University, 1980.

[29] G. Hahn. The automorphism group of a product of hypergraphs. *J. Comb. Theory, Ser. B*, 30:276–281, 1981.

[30] R. Hammack. On direct product cancellation of graphs. *Discrete Math.*, 309(8):2538–2543, 2009.

[31] R. Hammack, W. Imrich, and S. Klavžar. *Handbook of Product Graphs*. Discrete Mathematics and its Applications. CRC Press, 2nd edition, 2011.

[32] R.H. Hammack, M. Hellmuth, L. Ostermeier, and P.F. Stadler. Associativity and non-associativity of some hypergraph products. *Math. Comp. Sci*, 10(3):403–408, 2016.

[33] C. Heine, S. Jaenicke, M. Hellmuth, P.F. Stadler, and G. Scheuermann. Visualization of graph products. *IEEE Transactions on Visualization and Computer Graphics*, 16(6):1082–1089, 2010.

[34] M. Hellmuth. A local prime factor decomposition algorithm. *Discrete Mathematics*, 311(12):944 – 965, 2011.

[35] M. Hellmuth and F. Lehner. Fast factorization of Cartesian products of (directed) hypergraphs. *J. Theor. Comp. Sci.*, 615:1–11, 2016.

[36] M. Hellmuth, L. Ostermeier, and P.F. Stadler. A survey on hypergraph products. *Math. Comput. Sci*, 6:1–32, 2012.

[37] W. Imrich. Kartesisches Produkt von Mengensystemen und Graphen. *Studia Sci. Math. Hungar.*, 2:285 – 290, 1967.

[38] W. Imrich. über das schwache Kartesische Produkt von Graphen. *Journal of Combinatorial Theory*, 11(1):1–16, 1971.

[39] W. Imrich and H. Izbicki. Associative products of graphs. *Monatshefte für Mathematik*, 80(4):277–281, 1975.

[40] W. Imrich and S. Klavžar. *Product graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[41] W. Imrich, S. Klavžar, and F. R. Douglas. *Topics in Graph Theory: Graphs and Their Cartesian Product*. AK Peters, Ltd., Wellesley, MA, 2008.

[42] W. Imrich and I. Peterin. Recognizing cartesian products in linear time. *Discrete Math.*, 307(3-5):472–483, 2007.

[43] W. Imrich and P. F. Stadler. A prime factor theorem for a generalized direct product. *Discussiones Math. Graph Th.*, 26:135–140, 2006.

[44] J. Nešetřil and V. Rödl. Products of graphs and their applications. In *Graph Theory*, volume 1018 of *Lecture Notes in Mathematics*, pages 151–160. Springer Berlin / Heidelberg, 1983.

[45] A. Kaveh and K. Koohestani. Graph products for configuration processing of space structures. *Comput. Struct.*, 86(11-12):1219–1231, 2008.

[46] A. Kaveh and H. Rahami. An efficient method for decomposition of regular structures using graph products. *Intern. J. Numer. Meth. Eng.*, 61(11):1797–1808, 2004.

[47] R. J. McEliece and E. C. Posner. Hide and seek, data storage, and entropy. *The Annals of Mathematical Statistics*, 42(5):1706–1716, 1971.

[48] D. Mubayi and V. Rödl. On the chromatic number and independence number of hypergraph products. *J. Comb. Theory, Ser. B*, 97(1):151–155, 2007.

[49] L. Ostermeier, M. Hellmuth, and P. F. Stadler. The Cartesian product of hypergraphs. *Journal of Graph Theory*, 2011.

[50] P.-J. Ostermeier, M. Hellmuth, K. Klemm, J. Leydold, and P.F. Stadler. A note on quasi-robust cycle bases. *Ars Math. Contemp.*, 2(2):231–240, 2009.

[51] R. Pemantle, J. Propp, and D. Ullman. On tensor powers of integer programs. *SIAM J. Discrete Math.*, 5(1):127–143, 1992.

[52] G. Sabidussi. Graph Multiplication. *Mathematische Zeitschrift*, 72(1):446–457, 1960.

[53] M. Sonntag. Hamiltonian properties of the Cartesian sum of hypergraphs. *J. Inf. Process. Cybern.*, 25(3):87–100, 1989.

[54] M. Sonntag. Hamiltonicity of the normal product of hypergraphs. *J. Inf. Process. Cybern.*, 26(7):415–433, 1990.

[55] M. Sonntag. Corrigendum to: “Hamiltonicity of the normal product of hypergraphs”. *J. Inf. Process. Cybern.*, 27(7):385–386, 1991.

[56] M. Sonntag. Hamiltonicity and traceability of the lexicographic product of hypergraphs. *J. Inf. Process. Cybern.*, 27(5-6):289–301, 1991.

[57] M. Sonntag. *Hamiltonsche Eigenschaften von Produkten von Hypergraphen*. Habilitation, Fakultät für Mathematik und Naturwissenschaften, Bergakademie Freiberg, 1991.

[58] M. Sonntag. Hamiltonicity of products of hypergraphs. Combinatorics, graphs and complexity, Proc. 4th Czech. Symp., Prachatice/Czech. 1990, Ann. Discrete Math. 51, 329-332 (1992)., 1992.

[59] M. Sonntag. Hamiltonicity of the disjunction of two hypergraphs. *J. Inf. Process. Cybern.*, 29(3):193–205, 1993.

[60] B. M. R. Stadler, P. F. Stadler, G. P. Wagner, and W. Fontana. The topology of the possible: Formal spaces underlying patterns of evolutionary change. *J. Theor. Biol.*, 213:241–274, 2001.

[61] F. Sterboul. On the chromatic number of the direct product of hypergraphs. Proc. 1rst Working Sem. Hypergraphs, Columbus 1972, Lect. Notes Math. 411, 165-174 (1974)., 1974.

[62] G. Wagner and P. F. Stadler. Quasi-independence, homology and the unity of type: A topological theory of characters. *J. Theor. Biol.*, 220:505–527, 2003.

[63] A. V. Zeigarnik. On hypercycles and hypercircuits in hypergraphs. In P. Hansen, P. W. Fowler, and M. Zheng, editors, *Discrete Mathematical Chemistry*, volume 51 of *DIMACS series in discrete mathematics and theoretical computer science*, pages 377–383. American Mathematical Society, Providence, RI, 2000.

[64] X. Zhu. On the chromatic number of the product of hypergraphs. *Ars Comb.*, 34:25–31, 1992.

Marc Hellmuth
Center for Bioinformatics
Saarland University
Building E 2.1, Room 413
P.O. Box 15 11 50
D - 66041 Saarbrücken
Germany
e-mail: `marc@bioinf.uni-leipzig.de`

Lydia Ostermeier
Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22,
D-04103 Leipzig,
Germany

Bioinformatics Group,
Department of Computer Science and Interdisciplinary Center for Bioinformatics
University of Leipzig,
Härtelstrasse 16-18, D-04107 Leipzig, Germany
e-mail: `glydia@bioinf.uni-leipzig.de`

Peter F. Stadler
Bioinformatics Group,
Department of Computer Science; and Interdisciplinary Center for Bioinformatics,
University of Leipzig,
Härtelstrasse 16-18, D-04107 Leipzig,Germany

Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22, D-04103 Leipzig, Germany

RNomics Group, Fraunhofer Institut für Zelltherapie und Immunologie, Deutscher Platz 5e, D-04103 Leipzig, Germany

Department of Theoretical Chemistry, University of Vienna, Währingerstraße 17, A-1090 Wien, Austria

Santa Fe Institute, 1399 Hyde Park Rd., Santa Fe, NM87501, USA
e-mail: `studla@bioinf.uni-leipzig.de`