

Goethals–Seidel difference families with symmetric or skew base blocks

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Abstract

We single out a class of difference families which is widely used in some constructions of Hadamard matrices and which we call Goethals–Seidel (GS) difference families. They consist of four subsets (base blocks) of a finite abelian group of order v , which can be used to construct Hadamard matrices via the well-known Goethals–Seidel array. We consider the special class of these families in cyclic groups, where each base block is either symmetric or skew.

We omit the well-known case where all four blocks are symmetric. By extending previous computations by several authors, we complete the classification of GS-difference families of this type for odd $v < 50$. In particular, we have constructed the first examples of so called good matrices, G-matrices and best matrices of order 43, and good matrices and G-matrices of order 45.

We also point out some errors in one of the cited references.

Keywords: Goethals–Seidel array; difference families; good matrices; G-matrices; best matrices

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1 Introduction

The well-known Goethals–Seidel array (GS-array)

$$\begin{bmatrix} Z_0 & Z_1R & Z_2R & Z_3R \\ -Z_1R & Z_0 & -Z_3^T R & Z_2^T R \\ -Z_2R & Z_3^T R & Z_0 & -Z_1^T R \\ -Z_3R & -Z_2^T R & Z_1^T R & Z_0 \end{bmatrix}, \quad (1)$$

is a powerful tool for the construction of Hadamard matrices and in particular those of skew type. The four blocks Z_i of size v , needed to plug into the GS-array to obtain a Hadamard matrix of order $4v$, are usually constructed by using a suitable difference family in a finite abelian group G of order v . (The matrix R will be defined later in the case when this abelian group is cyclic.)

We recall that the subsets $X_1, X_2, \dots, X_t \subseteq G$ form a *difference family* if for every $c \in G \setminus \{0\}$ there are exactly λ ordered triples $(a, b, i) \in G \times G \times \{1, 2, \dots, t\}$ such that $\{a, b\} \subseteq X_i$ and

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$a - b = c$. We say that $(v; k_1, k_2, \dots, k_t; \lambda)$ is the *set of parameters* of that family, where $k_i = |X_i|$ is the cardinality of X_i . In the Hadamard matrix community these difference families are better known as *supplementary difference sets*, see e.g. [16, Definition 1.5, p. 281]. Note that we must have

$$\sum_{i=1}^t k_i(k_i - 1) = \lambda(v - 1). \quad (2)$$

We remark that if in a difference family (X_i) we replace one of the blocks X_i by its complement $G \setminus X_i$, then we still have a difference family; its parameter set is obtained from the original one by replacing k_i with $v - k_i$ and λ by $\lambda + (v - 2k_i)$.

The difference families that we need have $t = 4$, i.e., they are ordered quadruples of four base blocks X_1, X_2, X_3, X_4 , such that their parameter sets satisfy the condition

$$\sum_{i=1}^4 k_i = \lambda + v. \quad (3)$$

We say that the parameter sets, with $t = 4$, satisfying the conditions (2) and (3) are *GS-parameter sets* and that the corresponding difference families are *GS-difference families*. While the condition (2) is necessary for the existence of the difference family with the above mentioned parameter set, the two additional conditions that $t = 4$ and (3) are required for the construction of Hadamard matrices by using the GS-array. Indeed, assuming that $t = 4$, the condition (3) is equivalent to the condition (sum-A) below, see [16, Lemma 1.20].

In this paper the group G will be cyclic, $G = \mathbf{Z}_v = \{0, 1, \dots, v - 1\}$, $v > 1$, and the difference families will be of GS-type. Since we can permute the X_i and replace any X_i with its complement, we shall assume that

$$v/2 \geq k_1 \geq k_2 \geq k_3 \geq k_4 \geq 0. \quad (4)$$

The blocks Z_i will be circulant matrices while the matrix R in (1) will be the back-circulant identity matrix of order v ,

$$R = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & & 1 & 0 \\ \vdots & & & & \\ 0 & 1 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \end{bmatrix}.$$

We say that a subset $X \subseteq \mathbf{Z}_v$ is *symmetric* if $-X = X$, and that it is *skew* if v is odd, $|X| = (v - 1)/2$ and $(-X) \cap X = \emptyset$.

We consider only the GS-difference families (X_i) , $i = 1, 2, 3, 4$, in \mathbf{Z}_v such that each X_i is either symmetric or skew. We have four *symmetry types* for such families: (ssss), (ksss), (kkss), (kkks). (The type (kkkk) will be dismissed below.) The letter “s” stands for “symmetric” and “k” for “skew”. For instance, a difference family is of (kkss) type if two of the base blocks are symmetric and the other two are skew.

To any subset $X \subseteq \mathbf{Z}_v$ we associate the binary sequence (i.e., a sequence with entries $+1$ and -1) of length v , say $(x_0, x_1, \dots, x_{v-1})$, where $x_i = -1$ if and only if $i \in X$. By abuse of language, we shall use the symbol X to denote also the binary sequence associated to the subset X . Further, let A_i be the circulant matrix having the sequence X_i as its first row.

For any GS-difference family (X_i) the four circulants A_i satisfy the equation

$$\sum_{i=1}^4 A_i^T A_i = 4vI_v, \quad (5)$$

where I_v is the identity matrix of order v . This equation is used to verify that, after setting $Z_i = A_{i+1}$, $i = 0, \dots, 3$, into the GS-array, we obtain a Hadamard matrix. Thus the equality (5) is an essential feature of the GS-families. For the sake of convenience, let us say that four circulant $\{\pm 1\}$ -matrices (A_i) of order v are *GS-matrices* if they satisfy the equation (5).

Let (A_i) be GS-matrices. If they have the symmetry type (ssss) then they are the well-known Williamson matrices (with four symmetric circulants). If they have symmetry type (ksss), with A_1 of skew type, i.e. $A_1 + A_1^T = 2I_v$, then the quadruple consisting of the circulant A_1 and the back-circulants $B_i = A_i R$, $i = 2, 3, 4$, are good matrices. Alternatively, for $i = 2, 3, 4$ one can take B_i to be the back-circulant matrix having the same first row as A_i . (Note that each back-circulant is a symmetric matrix.)

As the reader may not be familiar with the good matrices, let us define them precisely. We say that two matrices M, N of order v are *amicable* if $MN^T = NM^T$. Let (M_i) be a quadruple of $\{\pm 1\}$ -matrices of order v . We say that the M_i are *Williamson type matrices* if they are pairwise amicable and satisfy the equation (5). Finally, we say that the matrices M_i are *good matrices* if they are Williamson type matrices, one of which is of skew type and the other three symmetric.

The G-matrices over a finite abelian group have been introduced long ago by J. Seberry Walis [12, p. 154]. (The condition (iii) of that definition is redundant as any two matrices of type 1 commute.) In the case that we consider, the group G is cyclic and G-matrices are simply the GS-matrices having the symmetry type (kkss).

Similarly, the GS-matrices of symmetry type (kkks) have been given the name of *best matrices* (see [7]).

By eliminating the parameter λ from the equations (2) and (3), we obtain that

$$\sum_{i=1}^4 (v - 2k_i)^2 = 4v. \quad (6)$$

It is also true that equations (2) and (6) imply the equation (3).

Hence, if all X_i are skew then necessarily $v = 1$. For that reason we have omitted the (kkkk) type. Note also that if $k_4 = 0$ then (6) implies that $v \leq 4$. Since the Williamson matrices are treated extensively in [9], we omit the (ssss) type as well.

In order to find GS-matrices of a given order v , one has to compute first the possible GS-parameter sets. This is easily accomplished by solving the Diophantine equation (6) subject to the conditions (4).

Our main objective is to extend and complete the classification (up to equivalence) of the GS-difference families of types (ksss), (kkss) and (kkks) for odd $v < 50$. See Table 1 in section 5 for the summary of old and new results in this range.

2 GS-parameter sets

The following proposition follows immediately from the equation (6).

Proposition 1 *The GS-parameter sets $(v; k_1, k_2, k_3, k_4; \lambda)$ with $v/2 \geq k_1 \geq k_2 \geq k_3 \geq k_4 \geq 0$ are parametrized by four-odd-square decompositions $4v = \sum_{i=1}^4 s_i^2$ when v is odd, and by four-square decompositions $v = \sum_{i=1}^4 s_i^2$ when v is even. In both cases we require that $0 \leq s_1 \leq s_2 \leq s_3 \leq s_4$.*

The parameter sets needed for the construction of GS-difference families of symmetry types (ksss), (kkss) and (kkks) always exist. We treat the three cases separately.

Proposition 2 *For a given odd $v \geq 1$, there is at least one GS-parameter set with $(v-1)/2 = k_1 \geq k_2 \geq k_3 \geq k_4$.*

Proof By setting $k_1 = (v-1)/2$ the equation (6) reduces to

$$\sum_{i=2}^4 (v - 2k_i)^2 = 4v - 1.$$

Since v is odd, $4v-1 \equiv 3 \pmod{8}$. It is known that every positive integer congruent to 3 modulo 8 is a sum of three odd squares. Indeed, this claim is equivalent to the well-known theorem of Gauss that every positive integer is a sum of at most three triangular numbers [4]. Thus there exist integers k_2, k_3, k_4 satisfying the above equation and such that $(v-1)/2 \geq k_2 \geq k_3 \geq k_4 \geq 0$. Consequently, the k_i , $i = 1, \dots, 4$, satisfy the equation (6) which implies that $\lambda \geq 0$.

In the case (ksss) there are only finitely many known GS-difference families, namely for odd $v \leq 39$ and $v = 127$. The cases $v = 43, 45$ are new and appear here for the first time. Our exhaustive search did not find any GS-difference families of type (ksss) for $v = 41, 47, 49$.

In the case (kkss) there is an infinite series of GS-difference families constructed by E. Spence [14].

For type (kkss), the GS-parameter sets are given by the following proposition.

Proposition 3 *The GS-parameter sets with $(v-1)/2 = k_1 = k_2 \geq k_3 \geq k_4$ exist if and only if $2v-1$ is a sum of two squares. They have the form*

$$(v; k_1 = k_2 = (v-1)/2, (v-r+s)/2, (v-r-s)/2; v-r-1),$$

where r, s are integers, $r > s \geq 0$ and $r^2 + s^2 = 2v-1$.

Proof Assume first that $k_1 = k_2 = (v-1)/2$. The equation (3) gives $\lambda = k_3 + k_4 - 1$ and from the equation (2) we obtain that

$$k_3(k_3 - 1) + k_4(k_4 - 1) = (v-1) \left(k_3 + k_4 - \frac{v-1}{2} \right).$$

This equation can be written as $(v - 2k_3)^2 + (v - 2k_4)^2 = 2(2v - 1)$. This implies that $2v - 1$ is a sum of two squares.

Conversely, assume that $2v - 1 = r^2 + s^2$ where $r \geq s \geq 0$ are integers. In fact we must have $r > s$. If $v = 3$ we have $r = 1$ and $s = 0$. If $v = 5$ we have $r = 3$ and $s = 0$. So, the assertion holds in these two cases. We may assume that $v > 5$. We claim that $r \leq (v-1)/2$. Otherwise we would have $2r \geq v+1$ and $4(2v-1) = 4r^2 + 4s^2 \geq (v+1)^2$ which gives the contradiction $(v-1)(v-5) \leq 0$. As r and s have different parity, we can set $k_3 = (v-r+s)/2$, $k_4 = (v-r-s)/2$ and $\lambda = v-r-1 \geq 0$. It is easy to verify that we obtain a valid GS-parameter set.

For type (kkks), the GS-parameter sets are given by the following proposition.

Proposition 4 *The GS-parameter sets with $(v-1)/2 = k_1 = k_2 = k_3 \geq k_4$ have the form*

$$(v = r^2 + r + 1; k_1 = k_2 = k_3 = r(r+1)/2, k_4 = r(r-1)/2; \lambda = r^2 - 1),$$

where r is a positive integer.

Proof Set $v = 2s + 1$, and so $k_1 = k_2 = k_3 = s$. From (3) we obtain that $k_4 = \lambda - s + 1$. Since $\sum k_i(k_i - 1) = \lambda(v-1) = 2s\lambda$, we deduce that λ satisfies the quadratic equation $\lambda^2 - (4s-1)\lambda + 4s(s-1) = 0$. The discriminant $8s+1$ must be an odd square, say $(2r+1)^2$. Thus, $s = r(r+1)/2$. The two roots of the quadratic equation are $r^2 - 1$ and $2r + r^2$. We omit the second root because it gives that $k_4 > s = (v-1)/2$. This completes the proof.

We observe that the formula $v = r^2 + r + 1$ implies that $4v - 3$ is an odd square (see also [7]).

It is not known whether for each of these parameter sets there exists a GS-difference family of type (kkks). They exist for the first six values of r , $r = 1, 2, \dots, 6$.

3 Elementary transformations and equivalence relation

Let (X_i) be a difference family over a finite abelian group G of odd order v (written additively). We fix a GS-parameter set such that (4) holds. The set of GS-families with this parameter set is invariant under the following elementary transformations:

- (a) For some i replace X_i by a translate $g + X_i$, $g \in G$.
- (b) For some i replace X_i by $-X_i$.
- (c) For all i replace X_i by its image $\alpha(X_i)$ under an automorphism α of G .
- (d) Exchange X_i and X_j provided that $|X_i| = |X_j|$.

In the case $G = \mathbf{Z}_v$, α is just the multiplication modulo v by some integer k relatively prime to v .

Definition 1 *We say that two difference families with the same parameter set satisfying (4) are equivalent if one can be transformed to the other by a finite sequence of elementary transformations.*

Note that the symmetry properties of the base blocks X_i may be destroyed by operations (a), and changed by operations (d). Nevertheless we shall use the above definition of equivalence for the classification of GS-difference families with symmetry.

The definition of equivalence for the special case of cyclic GS-difference families of type (ksss), (kkss), and (kkks) given in [6, 7, 8] uses only elementary transformations of type (c). Apparently their GS-difference families (also known as supplementary difference sets) are non-ordered quadruples, in which case they do not need the elementary operations of type (d). For convenience, we shall refer to their equivalence classes as *small classes*. If X_i is symmetric then $-X_i = X_i$ but this fails in the case when X_i is skew. Hence, for GS-difference families of type (ksss) the elementary operation (b) can be viewed as a special case of (c). However, this is not the case for GS-difference families of type (kkss) or (kkks). For that reason, our equivalence class may contain more than one small class. In the (kkss) case an equivalence class may contain two small classes, and in the (kkks) case it may contain up to four small classes.

Table 1 of [6] contains the list of representatives of the small equivalence classes of supplementary difference sets (i.e., GS-difference families) of type (kkss) for all odd $v \leq 33$. It should be used with care as it contains several errors. For example, let us consider the very first item in that table, namely the parameter set (3;1,1,2,0;1). It is claimed there that there is only one small equivalence class, namely the one with the base blocks $S_0 = \{1\}$, $S_1 = \{1\}$, $S_2 = \{1, 2\}$, $S_3 = \emptyset$. However, if we replace S_0 with $-S_0 = \{2\}$ then, according to the definition given in [6, p. 207], the families $-S_0, S_1, S_2, S_3$ and S_0, S_1, S_2, S_3 are not equivalent. Note that they are equivalent according to our Definition 1. For more examples see our comment at the end of subsection 7.1 of the appendix.

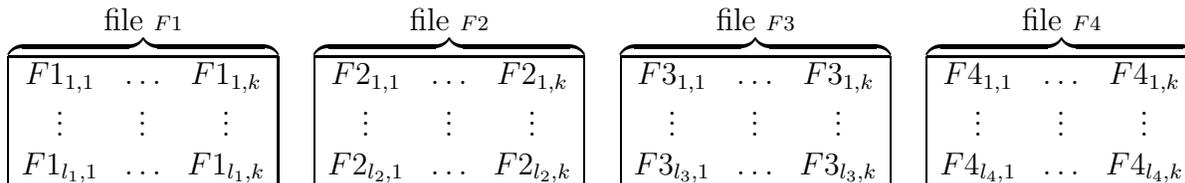
4 Algorithmic issues

All computations for GS-difference families (X_1, X_2, X_3, X_4) with parameters $(v; k_1, k_2, k_3, k_4; \lambda)$, v odd, in this paper have been performed via a two-phase algorithmic scheme. The first phase is the data collection phase. The second phase is the matching phase. We provide a more detailed description of these two phases below.

4.1 Data collection

During the data collection phase, we collect four data files $F1, F2, F3, F4$, corresponding to the four base blocks X_1, X_2, X_3, X_4 of the GS-difference family. Each data file contains positive integers arranged in k columns where $k = (v - 1)/2$. Each data file is typically made up of several millions of rows, and corresponds to an exhaustive search for sequences arising from the corresponding base block. The usual power spectral density filtering (see [5]) is applied, during the data collection phase. The positive integers in the four data files represent the multiplicities of the various differences arising from the corresponding base blocks. More specifically, a

combinatorial generation algorithm is used to generate exhaustively all possible base blocks that are not eliminated from the power spectral density filtering. If we denote by $X_{1,i}$ the i th candidate for the first base block X_1 , the i th row of file $F1$ contains a list of how often each group element is represented as a difference $g_1 - g_2$ where $g_1, g_2 \in X_{1,i}$. The same principle applies to files $F2, F3, F4$ as well. Here is a schematic representation of the four data files:



where l_1, l_2, l_3, l_4 are the total numbers of lines of the files $F1, F2, F3, F4$. Due to the fact that the differences d and $v - d$ have the same multiplicity, we record these multiplicities only for $d = 1, 2, \dots, k$ where $k = (v - 1)/2$.

4.2 Matching

During the matching phase, we process the four data files $F1, F2, F3, F4$, with the aim to find quadruples of lines (one in each file), so that the k element-wise sums are all equal to the constant λ , defined in the beginning of the Introduction. More specifically, we are looking for a line ℓ_1 in file $F1$, a line ℓ_2 in file $F2$, a line ℓ_3 in file $F3$, a line ℓ_4 in file $F4$, such that the following k conditions are satisfied:

$$\begin{cases} F1_{\ell_1,1} + F2_{\ell_2,1} + F3_{\ell_3,1} + F4_{\ell_4,1} = \lambda \\ \vdots \\ F1_{\ell_1,k} + F2_{\ell_2,k} + F3_{\ell_3,k} + F4_{\ell_4,k} = \lambda \end{cases}$$

It may be the case that such quadruples do not exist, which proves the non-existence of the corresponding GS-difference families. If such quadruples do exist, then we compute all of them by using a bins-based technique and a parallel backtracking algorithm [11]. We use two different algorithms and two different implementations, to validate the results.

The bins-based technique works by performing a pre-processing step which consists in calculating the different values that appear on the first column of each of the four data files, and subsequently splitting the four data files in smaller parts, so that each part contains only those rows that start with a specific number. Since we know that numbers in the first column must sum to λ , we are able to compute all possible cases for which this can happen, given all the possible values appearing in the first columns of $F1, F2, F3, F4$. This process typically establishes a few hundreds of possible cases C_i , each of which can then be processed in parallel. A typical case C_i is identified by four numbers n_1, n_2, n_3, n_4 such that $n_1 + n_2 + n_3 + n_4 = \lambda$ and four files $F1_{C_i}, F2_{C_i}, F3_{C_i}, F4_{C_i}$, such that $F1_{C_i}$ contains all lines in $F1$ that start with n_1 and similarly for the other three files. It is also possible that the number of cases is zero, and in that case it is clear the problem does not have a solution. If the number of cases is not zero,

then for each case C_i we examine the total numbers of lines in the corresponding four files, $F1_{C_i}, F2_{C_i}, F3_{C_i}, F4_{C_i}$ and if the following condition is satisfied

$$(\# \text{lines in } F1_{C_i}) \times (\# \text{lines in } F4_{C_i}) < 10^7 \ \& \ (\# \text{lines in } F2_{C_i}) \times (\# \text{lines in } F3_{C_i}) < 10^7$$

then case C_i is dealt with by a serial C program. Note that this condition is checked, once we have rearranged the files $F1_{C_i}, F2_{C_i}, F3_{C_i}, F4_{C_i}$ in increasing order, according to their total numbers of lines. If the above condition is not satisfied for case C_i , then the four files $F1_{C_i}, F2_{C_i}, F3_{C_i}, F4_{C_i}$ are split in smaller parts, by using the bins based on the second column. The threshold value 10^7 has been determined experimentally.

5 Summary of computational results and conclusions

There are several known infinite classes of difference families with four base blocks which can be used to construct Hadamard matrices via the Goethals–Seidel array. For that reason we refer to these families as GS-difference families. As an example let us mention one such family constructed in [17] which is especially interesting as it gives skew Hadamard matrices. It was observed recently (see [1]) that by using a different array (different from the GS-array) this infinite series gives also symmetric Hadamard matrices of the same orders. We are not aware of any GS-parameter set for which there is no cyclic difference family having the parameters specified by this set.

The cyclic GS-difference families with parameter set $(v; k_1, k_2, k_3, k_4; \lambda)$ of symmetry type (ksss), (kkss) and (kkks) provide quadruples of matrices of order v known as good matrices, G-matrices and best matrices, respectively. Some sporadic examples of such matrices are given, and systematic exhaustive searches for some small values of v have been carried out in [2, 3, 6, 7, 8, 14, 15].

There are only finitely many good matrices that have been constructed so far. Their orders are odd integers up to 39 and the integers 43, 45 and 127. The ones of orders 43 and 45 appear first in this paper. The order 41 is the smallest odd integer for which there are no GS-families of (ksss) symmetry type. Our exhaustive searches show that such difference families do not exist also in cyclic groups of orders 47 and 49.

There is an infinite series of circulant G-matrices due to E. Spence [14]. This fact is not well-known. For instance it is not mentioned in the paper [6] devoted to G-matrices. In that paper the authors have classified the GS-difference families of symmetry type (kkss) for odd $v \leq 33$ and given a few examples for $v = 37, 41$. In the appendix (see subsection 7.1) we point out that this classification has some errors. We have performed exhaustive searches for $v = 37, 41, 43, 45$ and 49. We found that there are 7,3,4,1 equivalence classes for the first four of these orders, and none for $v = 49$. There are no GS-parameter sets for G-matrices when $v = 35, 39, 47$.

The GS-parameter sets for best matrices (symmetry type (kkks)) form an infinite series, see Proposition 4. Apparently there is no known infinite series of best matrices so far. There is a simple necessary arithmetic condition for the existence of best matrices of order v , namely $4v - 3$ must be an odd square. For $v = 3, 7, 13, 21, 31$ the small equivalence classes are listed

in [7]. However, we warn the reader that these lists may contain errors just like in the case of G-matrices. For instance when $v = 3$ there are two small equivalence classes while only one is recorded in [7]. Our exhaustive search for $v = 43$ found that there are five equivalence classes of best matrices.

A short summary of the known facts about the existence of good matrices, G-matrices and best matrices for odd orders $v < 50$ is given in Table 1 below. We write “yes” if the matrices exist and “no” if the exhaustive search did not find any such matrices. The entry “ \times ” means that the parameter set is not suitable for the symmetry type specified for the column. The entries that we obtained in this work are shown in boldface in Table 1.

v	k_1	k_2	k_3	k_4	λ	ksss	kkss	kkks	v	k_1	k_2	k_3	k_4	λ	ksss	kkss	kkks
3	1	1	1	0	0	yes	yes	yes	31	15	15	15	10	24	yes	yes	yes
5	2	2	1	1	1	yes	yes	\times		15	13	12	12	21	yes	\times	\times
7	3	3	3	1	3	yes	yes	yes	33	16	16	15	11	25	yes	yes	\times
	3	2	2	2	2	yes	\times	\times		16	16	13	12	24	yes	yes	\times
9	4	4	3	2	4	yes	yes	\times		16	14	14	12	23	yes	\times	\times
11	5	4	4	3	5	yes	\times	\times	35	17	16	16	12	26	yes	\times	\times
13	6	6	6	3	8	yes	no	yes		17	16	14	13	25	yes	\times	\times
	6	6	4	4	7	yes	yes	\times	37	18	18	16	13	28	yes	yes	\times
15	7	7	6	4	9	yes	yes	\times		18	15	15	15	26	yes	\times	\times
	7	6	5	5	8	yes	\times	\times	39	19	18	17	14	29	yes	\times	\times
17	8	7	7	5	10	yes	\times	\times		19	17	16	15	28	yes	\times	\times
19	9	9	7	6	12	yes	yes	\times	41	20	20	16	16	31	no	yes	\times
	9	7	7	7	11	yes	\times	\times	43	21	21	21	15	35	no	yes	yes
21	10	10	10	6	15	yes	yes	yes		21	21	18	16	33	no	yes	\times
	10	9	8	7	13	yes	\times	\times		21	19	19	16	32	yes	\times	\times
23	11	11	10	7	16	yes	yes	\times		21	20	17	17	32	no	\times	\times
25	12	12	9	9	17	no	yes	\times	45	22	22	21	16	36	yes	yes	\times
	12	11	11	8	17	yes	\times	\times		22	21	19	17	34	yes	\times	\times
	12	10	10	9	16	yes	\times	\times		22	19	19	18	33	yes	\times	\times
27	13	13	11	9	19	yes	yes	\times	47	23	22	22	17	37	no	\times	\times
	13	12	10	10	18	yes	\times	\times		23	21	19	19	35	no	\times	\times
29	14	13	12	10	20	yes	\times	\times	49	24	24	22	18	39	no	no	\times
										24	22	21	19	37	no	\times	\times

Table 1. Existence of GS-difference families with symmetry

More details, including the listing of representatives of equivalence classes of GS-difference families for $33 \leq v \leq 49$, are provided in the appendix.

For applications of the good, best and G-matrices to construction of the various combinatorial structures see [2, 6, 7, 8]. Let us just mention two of them. First, good matrices of order v give the complex Hadamard matrices of order $2v$. Second, by Theorems 1 and 2 of [7], if there exist best matrices of order v , then there exist orthogonal designs $OD(8v; 1, 1, 2, 2, 4v-3, 4v-3)$

and $OD(4v; 1, 1, 1, 4v - 3)$. In particular, this holds for $v = 43$ as we have constructed best matrices of order 43. For the definition of OD's see the same paper or [13].

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References

- [1] O. Di Mateo, D. Ž. Đoković, I. S. Kotsireas, Symmetric Hadamard matrices of order 116 and 172 exist, *Special Matrices* 3 (2015), 227–234.
- [2] D. Ž. Đoković, Good matrices of orders 33, 35 and 127, *JCMCC* 14 (1993), 145–152.
- [3] D. Ž. Đoković, Supplementary difference sets with symmetry for Hadamard matrices. *Operators and Matrices*, 3 (2009), 557–569.
- [4] W. Duke, Some old problems and new results about quadratic forms. *Not. Amer. Math. Soc.* 44, (1997), 190–196.
- [5] R. J. Fletcher, M. Gysin, J. Seberry, Application of the discrete Fourier transform to the search for generalised Legendre pairs and Hadamard matrices. *Australas. J. Combin.* 23 (2001), 75–86.
- [6] S. Georgiou and C. Koukouvinos, On circulant G-matrices, *JCMCC* 40 (2002), 205–225.
- [7] S. Georgiou, C. Koukouvinos and J. Seberry, On circulant best matrices and their applications, *Linear and Multilinear Algebra*, 48 (2001), 263–274.
- [8] S. Georgiou, C. Koukouvinos and S. Stylianou, On good matrices, skew Hadamard matrices and optimal designs, *Computational Statistics & Data Analysis* 41 (2002), 171–184.
- [9] W. H. Holzmann, H. Kharaghani and B. Tayfeh-Rezaie, Williamson matrices up to order 59, *Designs, Codes and Cryptography*, 46 (2008), 343–352.
- [10] C. Koukouvinos and J. Seberry, On G-matrices, *Bull. Inst. Combin. Appl.* 9 (1993), 40–44.
- [11] J. Pearl, *Heuristics: Intelligent Search Strategies for Computer Problem Solving* (Addison-Wesley, 1984).

- [12] J. Seberry Wallis, On Hadamard matrices, *Journal of Combinatorial Theory*, 18 (1975), 149–164.
- [13] J. Seberry, M. Yamada, Hadamard matrices, sequences, and block designs. In *Contemporary design theory*, 431-560, Wiley-Intersci. Ser. Discrete Math. Optim., Wiley, New York, 1992.
- [14] E. Spence, Skew Hadamard matrices of order $2(q + 1)$, *Discrete Mathematics* 18 (1977), 79–85.
- [15] G. Szekeres, A note on skew type orthogonal ± 1 matrices, in: Hajnal, A., Lovasz, L., Sos, V. T., (Eds.), *Combinatorics, Colloquia Mathematica Societatis Janos Bolyai*, No. 52. North Holland (1989), Amsterdam, 489–498.
- [16] J. S. Wallis, Hadamard matrices, in LNM 292, *Combinatorics: Room Squares, Sum-Free Sets, Hadamard matrices*, Springer-Verlag 1972.
- [17] M. Xia, T. Xia, J. Seberry and J. Wu, An infinite series of Goethals–Seidel arrays, *Discrete Applied Mathematics* 145 (2005), 498–504.
- [18] X. M. Zhang, G-matrices of order 19, *Bull. Inst. Combin. Appl.* 4 (1992), 95–98.

7 Appendix

For the previous work on the classification of GS-families of type (ksss) see [8], for the type (kkss) see [6], and for (kkks) see [7]. The paper [3] gives additional GS-families of all these types, as well as some multi-circulant GS-families of type (ssss).

For each odd integer v in the range $33 \leq v \leq 49$ and for each GS-parameter set $(v; k_1, k_2, k_3, k_4; \lambda)$ with $k_1 = (v - 1)/2 \geq k_2 \geq k_3 \geq k_4 \geq 0$ we list below (or provide references for) the representatives of the equivalence classes for GS-difference families having symmetry type (ksss), (kkss) or (kkks).

7.1 $v = 33$

For the symmetry type (ksss), we have verified the claim made in [8] that there are 6, 4 and 5 equivalence classes of GS-difference families with parameter sets $(33; 16, 16, 15, 11; 25)$, $(33; 16, 16, 13, 12; 24)$ and $(33; 16, 14, 14, 12; 23)$, respectively.

On the other hand, our computations for type (kkss) do not agree with those in [6]. We found that there are 20 equivalence classes. Their representatives are listed below.

(kkss) type:

(a) $(33; 16, 16, 15, 11; 25)$

$[2, 4, 6, 10, 13, 14, 15, 16, 21, 22, 24, 25, 26, 28, 30, 32]$,

$[3, 4, 5, 6, 7, 8, 9, 12, 15, 16, 19, 20, 22, 23, 31, 32]$,

[0, 2, 7, 8, 12, 13, 14, 16, 17, 19, 20, 21, 25, 26, 31] ,
[0, 2, 5, 8, 10, 11, 22, 23, 25, 28, 31] .

(b) (33; 16, 16, 15, 11; 25)
[1, 3, 5, 6, 11, 12, 14, 16, 18, 20, 23, 24, 25, 26, 29, 31] ,
[2, 4, 5, 9, 11, 12, 13, 15, 16, 19, 23, 25, 26, 27, 30, 32] ,
[0, 1, 2, 3, 5, 11, 12, 16, 17, 21, 22, 28, 30, 31, 32] ,
[0, 8, 9, 12, 15, 16, 17, 18, 21, 24, 25] .

(c) (33; 16, 16, 15, 11; 25)
[4, 5, 7, 8, 9, 10, 12, 14, 16, 18, 20, 22, 27, 30, 31, 32] ,
[3, 7, 8, 9, 10, 12, 13, 15, 16, 19, 22, 27, 28, 29, 31, 32] ,
[0, 1, 6, 7, 8, 10, 11, 14, 19, 22, 23, 25, 26, 27, 32] ,
[0, 7, 10, 12, 15, 16, 17, 18, 21, 23, 26] .

(d) (33; 16, 16, 15, 11; 25)
[3, 4, 5, 6, 7, 9, 10, 12, 14, 16, 18, 20, 22, 25, 31, 32] ,
[1, 2, 4, 5, 6, 7, 8, 13, 14, 16, 18, 21, 22, 23, 24, 30] ,
[0, 1, 3, 4, 9, 10, 11, 14, 19, 22, 23, 24, 29, 30, 32] ,
[0, 3, 4, 7, 9, 12, 21, 24, 26, 29, 30] .

(e) (33; 16, 16, 15, 11; 25)
[2, 4, 6, 8, 11, 12, 13, 16, 18, 19, 23, 24, 26, 28, 30, 32] ,
[2, 3, 4, 5, 8, 11, 16, 18, 19, 20, 21, 23, 24, 26, 27, 32] ,
[0, 1, 2, 3, 7, 8, 10, 11, 22, 23, 25, 26, 30, 31, 32] ,
[0, 4, 6, 9, 10, 16, 17, 23, 24, 27, 29] .

(f) (33; 16, 16, 15, 11; 25)
[2, 4, 5, 10, 12, 16, 18, 19, 20, 22, 24, 25, 26, 27, 30, 32] ,
[2, 5, 9, 11, 12, 13, 14, 16, 18, 23, 25, 26, 27, 29, 30, 32] ,
[0, 1, 3, 6, 7, 10, 11, 12, 21, 22, 23, 26, 27, 30, 32] ,
[0, 7, 8, 12, 15, 16, 17, 18, 21, 25, 26] .

(g) (33; 16, 16, 13, 12; 24)
[1, 3, 5, 7, 8, 10, 11, 13, 14, 15, 16, 21, 24, 27, 29, 31] ,
[1, 3, 7, 9, 10, 11, 15, 16, 19, 20, 21, 25, 27, 28, 29, 31] ,
[0, 3, 4, 5, 7, 10, 11, 22, 23, 26, 28, 29, 30] ,
[3, 8, 9, 11, 12, 13, 20, 21, 22, 24, 25, 30] .

(h) (33; 16, 16, 13, 12; 24)
[2, 3, 7, 9, 10, 12, 14, 15, 16, 20, 22, 25, 27, 28, 29, 32] ,
[1, 7, 8, 10, 12, 16, 18, 19, 20, 22, 24, 27, 28, 29, 30, 31] ,
[0, 2, 3, 4, 5, 11, 14, 19, 22, 28, 29, 30, 31] ,

[5, 7, 8, 11, 12, 16, 17, 21, 22, 25, 26, 28] .

(i) (33; 16, 16, 13, 12; 24)

[1, 3, 5, 6, 8, 9, 12, 14, 15, 16, 20, 22, 23, 26, 29, 31] ,
[1, 2, 10, 11, 14, 16, 18, 20, 21, 24, 25, 26, 27, 28, 29, 30] ,
[0, 3, 4, 8, 9, 10, 12, 21, 23, 24, 25, 29, 30] ,
[1, 2, 3, 9, 12, 14, 19, 21, 24, 30, 31, 32] .

(j) (33; 16, 16, 13, 12; 24)

[1, 3, 5, 7, 8, 11, 14, 15, 16, 20, 21, 23, 24, 27, 29, 31] ,
[1, 2, 5, 8, 10, 12, 13, 14, 15, 16, 22, 24, 26, 27, 29, 30] ,
[0, 4, 5, 6, 12, 14, 15, 18, 19, 21, 27, 28, 29] ,
[1, 2, 3, 4, 7, 14, 19, 26, 29, 30, 31, 32] .

(k) (33; 16, 16, 13, 12; 24)

[2, 3, 5, 7, 10, 14, 15, 16, 20, 21, 22, 24, 25, 27, 29, 32] ,
[2, 4, 5, 6, 11, 12, 14, 15, 16, 20, 23, 24, 25, 26, 30, 32] ,
[0, 2, 4, 7, 8, 9, 10, 23, 24, 25, 26, 29, 31] ,
[5, 8, 9, 12, 13, 15, 18, 20, 21, 24, 25, 28] .

(l) (33; 16, 16, 13, 12; 24)

[3, 4, 5, 6, 8, 10, 12, 13, 15, 16, 19, 22, 24, 26, 31, 32] ,
[2, 4, 5, 8, 9, 10, 12, 13, 14, 15, 16, 22, 26, 27, 30, 32] ,
[0, 1, 2, 3, 9, 11, 15, 18, 22, 24, 30, 31, 32] ,
[3, 6, 7, 11, 12, 14, 19, 21, 22, 26, 27, 30] .

(m) (33; 16, 16, 13, 12; 24)

[1, 2, 4, 5, 6, 8, 10, 12, 13, 15, 16, 19, 22, 24, 26, 30] ,
[1, 2, 3, 4, 6, 7, 8, 11, 13, 14, 16, 18, 21, 23, 24, 28] ,
[0, 4, 6, 9, 10, 15, 16, 17, 18, 23, 24, 27, 29] ,
[3, 4, 12, 13, 15, 16, 17, 18, 20, 21, 29, 30] .

(n) (33; 16, 16, 13, 12; 24)

[1, 5, 7, 8, 11, 13, 15, 16, 19, 21, 23, 24, 27, 29, 30, 31] ,
[2, 3, 4, 5, 7, 8, 9, 11, 12, 14, 16, 18, 20, 23, 27, 32] ,
[0, 7, 8, 9, 10, 13, 14, 19, 20, 23, 24, 25, 26] ,
[3, 4, 6, 8, 9, 16, 17, 24, 25, 27, 29, 30] .

(o) (33; 16, 16, 13, 12; 24)

[4, 5, 7, 11, 13, 15, 16, 19, 21, 23, 24, 25, 27, 30, 31, 32] ,
[1, 2, 3, 4, 5, 6, 9, 11, 12, 13, 16, 18, 19, 23, 25, 26] ,
[0, 6, 9, 11, 12, 13, 16, 17, 20, 21, 22, 24, 27] ,
[2, 6, 7, 9, 11, 12, 21, 22, 24, 26, 27, 31] .

(p) (33;16,16,13,12;24)
 [2,3,6,11,13,15,16,19,21,23,24,25,26,28,29,32] ,
 [1,3,4,5,9,16,18,19,20,21,22,23,25,26,27,31] ,
 [0,3,5,9,12,13,14,19,20,21,24,28,30] ,
 [4,6,9,10,15,16,17,18,23,24,27,29] .

(q) (33;16,16,13,12;24)
 [3,5,8,9,13,15,16,19,21,22,23,26,27,29,31,32] ,
 [1,3,4,5,7,9,13,14,15,16,21,22,23,25,27,31] ,
 [0,1,2,3,4,11,14,19,22,29,30,31,32] ,
 [3,4,8,11,13,16,17,20,22,25,29,30] .

(r) (33;16,16,13,12;24)
 [3,4,6,7,9,12,13,14,16,18,22,23,25,28,31,32] ,
 [2,3,4,5,7,9,10,11,12,15,16,19,20,25,27,32] ,
 [0,1,2,4,6,13,16,17,20,27,29,31,32] ,
 [3,4,7,9,15,16,17,18,24,26,29,30] .

(s) (33;16,16,13,12;24)
 [1,2,3,4,7,9,12,14,16,18,20,22,23,25,27,28] ,
 [2,3,4,5,6,9,10,11,12,15,16,19,20,25,26,32] ,
 [0,1,2,3,9,12,16,17,21,24,30,31,32] ,
 [2,6,10,12,13,15,18,20,21,23,27,31] .

(t) (33;16,16,13,12;24)
 [2,6,8,10,11,12,13,16,18,19,24,26,28,29,30,32] ,
 [1,3,4,5,7,8,9,10,12,15,16,19,20,22,27,31] ,
 [0,7,8,10,12,13,16,17,20,21,23,25,26] ,
 [4,5,6,8,13,14,19,20,25,27,28,29] .

The parameter set for the first 6 equivalence classes (a-f) is (33;16,16,15,11;25) and for the remaining 14 classes (g-t) is (33;16,16,13,12;24). Each of the equivalence classes (a-f) consists of two small equivalence classes. Consequently, there should be 12 small classes in [6] for the first parameter set. However, only 9 small classes have been listed in that paper. Similarly, there are 28 small classes for the second parameter set, but only 22 are listed in [6].

7.2 $v = 35$

We have verified the claim made in [8] that, for the type (ksss), there are 4 and 2 equivalence classes for the GS-parameter sets (35;17,16,16,12;26) and (35;17,16,14,13;25), respectively.

7.3 $v = 37$

For (ksss) type, we have verified the claim made in [8] that there is only one equivalence class of solutions for the parameter set $(37; 18, 18, 16, 13; 28)$ and only one for $(37; 18, 15, 15, 15; 26)$.

For (kkss) type, the family (a) below was constructed in [3] and the families (b-e) in [6]. We have completed the classification by constructing the families (f) and (g).

(kkss) type:

(a) $(37; 18, 18, 16, 13; 28)$

[1, 2, 3, 4, 6, 7, 8, 11, 14, 16, 17, 18, 22, 24, 25, 27, 28, 32] ,
[1, 3, 6, 8, 12, 15, 19, 20, 21, 23, 24, 26, 27, 28, 30, 32, 33, 35] ,
[1, 5, 8, 9, 14, 16, 17, 18, 19, 20, 21, 23, 28, 29, 32, 36] ,
[0, 4, 5, 6, 10, 12, 13, 24, 25, 27, 31, 32, 33] .

(b) $(37; 18, 18, 16, 13; 28)$

[1, 2, 3, 4, 5, 13, 18, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31] ,
[1, 2, 4, 6, 8, 9, 12, 14, 15, 18, 20, 21, 24, 26, 27, 30, 32, 34] ,
[1, 5, 6, 7, 8, 11, 15, 16, 21, 22, 26, 29, 30, 31, 32, 36] ,
[0, 1, 4, 6, 9, 13, 17, 20, 24, 28, 31, 33, 36] .

(c) $(37; 18, 18, 16, 13; 28)$

[1, 4, 5, 6, 7, 8, 13, 17, 19, 21, 22, 23, 25, 26, 27, 28, 34, 35] ,
[1, 3, 5, 7, 8, 10, 11, 14, 15, 16, 19, 20, 24, 25, 28, 31, 33, 35] ,
[2, 4, 5, 6, 8, 13, 16, 18, 19, 21, 24, 29, 31, 32, 33, 35] ,
[0, 7, 12, 13, 14, 15, 18, 19, 22, 23, 24, 25, 30] .

(d) $(37; 18, 18, 16, 13; 28)$

[1, 4, 7, 10, 11, 12, 15, 17, 19, 21, 23, 24, 28, 29, 31, 32, 34, 35] ,
[1, 3, 4, 7, 9, 11, 12, 16, 17, 19, 22, 23, 24, 27, 29, 31, 32, 35] ,
[3, 7, 8, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 29, 30, 34] ,
[0, 5, 7, 9, 10, 11, 18, 19, 26, 27, 28, 30, 32] .

(e) $(37; 18, 18, 16, 13; 28)$

[1, 2, 4, 5, 6, 7, 11, 14, 16, 17, 18, 22, 24, 25, 27, 28, 29, 34] ,
[1, 2, 3, 5, 10, 13, 15, 19, 20, 21, 23, 25, 26, 28, 29, 30, 31, 33] ,
[2, 3, 5, 6, 8, 9, 10, 14, 23, 27, 28, 29, 31, 32, 34, 35] ,
[0, 3, 4, 10, 12, 16, 18, 19, 21, 25, 27, 33, 34] .

(f) $(37; 18, 18, 16, 13; 28)$

[1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 17, 19, 21, 22, 24, 26, 30] ,
[1, 2, 3, 4, 6, 8, 10, 11, 14, 15, 18, 20, 21, 24, 25, 28, 30, 32] ,
[3, 5, 6, 11, 13, 14, 17, 18, 19, 20, 23, 24, 26, 31, 32, 34] ,
[0, 3, 10, 11, 12, 15, 16, 21, 22, 25, 26, 27, 34] .

(g) (37;18,18,16,13;28)
 [1,2,3,4,6,7,9,10,11,14,16,18,20,22,24,25,29,32] ,
 [1,2,3,4,5,8,9,10,11,13,14,16,18,20,22,25,30,31] ,
 [2,3,4,6,7,15,16,18,19,21,22,30,31,33,34,35] ,
 [0,2,5,8,12,13,18,19,24,25,29,32,35] .

7.4 $v = 39$

For (ksss) type, we have verified the claim made in [8] that there are 3 equivalence classes of solutions for the parameter set (39;19,18,17,14;29) and two classes for (39;19,17,16,15;28).

The classification of the GS-difference families for $v = 41, 43, 45, 47, 49$ and symmetry types (ksss), (kkss), (kkks) was not carried out so far. The results of our computations are given below.

7.5 $v = 41$

For (ksss) type and the parameter set (41;20,20,16,16;31) our exhaustive search did not find any solutions. For (kkss) type, there are 3 equivalence classes with representatives:

(kkss) type:

(a) (41;20,20,16,16;31)
 [2,4,7,8,10,12,13,14,16,19,20,23,24,26,30,32,35,36,38,40] ,
 [3,4,8,9,10,12,16,19,21,23,24,26,27,28,30,34,35,36,39,40] ,
 [4,5,6,7,9,13,14,20,21,27,28,32,34,35,36,37] ,
 [8,9,11,12,13,15,18,20,21,23,26,28,29,30,32,33] .

(b) (41;20,20,16,16;31)
 [2,4,6,7,8,10,11,14,15,17,20,22,23,25,28,29,32,36,38,40] ,
 [1,2,4,6,9,10,14,16,17,18,19,20,26,28,29,30,33,34,36,38] ,
 [3,4,5,6,7,10,11,16,25,30,31,34,35,36,37,38] ,
 [2,4,7,8,9,12,14,15,26,27,29,32,33,34,37,39] .

(c) (41;20,20,16,16;31)
 [2,7,9,11,13,14,16,17,20,22,23,26,29,31,33,35,36,37,38,40] ,
 [1,2,7,9,11,12,15,17,18,19,20,25,27,28,31,33,35,36,37,38] ,
 [5,6,7,9,13,14,17,18,23,24,27,28,32,34,35,36] ,
 [2,3,4,6,7,9,10,18,23,31,32,34,35,37,38,39] .

The GS-difference family (b) above is equivalent to the one found in [6]. Our exhaustive search found two more equivalence classes. Note that the second family listed in [6] for $n = 41$

is not a difference family.

7.6 $v = 43$

There are four GS-parameter sets with $v = 43$ and $k_1 = 21 \geq k_2 \geq k_3 \geq k_4 \geq 0$. Three of them have no GS-difference families of type (ksss). Only the GS-parameter set $(43; 21, 19, 19, 16; 32)$ has such families, they form just one equivalence class.

(ksss) type:

(a) $(43; 21, 19, 19, 16; 32)$

[1, 3, 4, 5, 6, 7, 10, 12, 14, 15, 16, 19, 20, 21, 25, 26, 30, 32, 34, 35, 42] ,
[0, 2, 4, 5, 7, 10, 11, 12, 13, 17, 26, 30, 31, 32, 33, 36, 38, 39, 41] ,
[0, 3, 5, 8, 9, 15, 17, 18, 19, 21, 22, 24, 25, 26, 28, 34, 35, 38, 40] ,
[4, 5, 6, 8, 12, 13, 16, 19, 24, 27, 30, 31, 35, 37, 38, 39] .

(kkss) type:

(a) $(43; 21, 21, 21, 15; 35)$

[1, 3, 5, 9, 11, 13, 14, 16, 17, 21, 23, 24, 25, 28, 31, 33, 35, 36, 37, 39, 41] ,
[4, 5, 6, 9, 14, 15, 16, 17, 18, 19, 21, 23, 30, 31, 32, 33, 35, 36, 40, 41, 42] ,
[0, 1, 3, 4, 6, 7, 8, 10, 14, 15, 20, 23, 28, 29, 33, 35, 36, 37, 39, 40, 42] ,
[0, 1, 4, 5, 6, 14, 17, 20, 23, 26, 29, 37, 38, 39, 42] .

(b) $(43; 21, 21, 18, 16; 33)$

[1, 4, 5, 7, 10, 12, 15, 17, 19, 21, 23, 25, 27, 29, 30, 32, 34, 35, 37, 40, 41] ,
[1, 2, 3, 10, 13, 14, 17, 19, 22, 23, 25, 27, 28, 31, 32, 34, 35, 36, 37, 38, 39] ,
[2, 3, 4, 7, 9, 10, 12, 13, 17, 26, 30, 31, 33, 34, 36, 39, 40, 41] ,
[3, 4, 8, 12, 13, 14, 15, 16, 27, 28, 29, 30, 31, 35, 39, 40] .

(c) $(43; 21, 21, 18, 16; 33)$

[2, 4, 6, 9, 10, 15, 16, 18, 21, 23, 24, 26, 29, 30, 31, 32, 35, 36, 38, 40, 42] ,
[4, 7, 8, 9, 11, 12, 15, 17, 22, 23, 24, 25, 27, 29, 30, 33, 37, 38, 40, 41, 42] ,
[6, 7, 9, 13, 14, 15, 17, 18, 19, 24, 25, 26, 28, 29, 30, 34, 36, 37] ,
[7, 8, 9, 11, 13, 16, 17, 20, 23, 26, 27, 30, 32, 34, 35, 36] .

(d) $(43; 21, 21, 18, 16; 33)$

[1, 3, 8, 10, 12, 15, 16, 18, 19, 21, 23, 26, 29, 30, 32, 34, 36, 37, 38, 39, 41] ,
[1, 2, 4, 5, 6, 10, 11, 12, 13, 14, 16, 17, 19, 20, 21, 25, 28, 34, 35, 36, 40] ,
[2, 4, 6, 7, 12, 14, 15, 18, 19, 24, 25, 28, 29, 31, 36, 37, 39, 41] ,
[3, 6, 7, 13, 18, 19, 20, 21, 22, 23, 24, 25, 30, 36, 37, 40] .

(kkks) type:

(a) (43;21,21,21,15;35)

[3,5,6,8,10,13,14,15,19,21,23,25,26,27,31,32,34,36,39,41,42] ,
[3,4,7,11,13,14,17,19,20,22,25,27,28,31,33,34,35,37,38,41,42] ,
[1,2,3,13,14,16,17,19,20,21,25,28,31,32,33,34,35,36,37,38,39] ,
[0,6,10,14,15,16,17,19,24,26,27,28,29,33,37] .

(b) (43;21,21,21,15;35)

[1,3,4,6,9,10,13,14,19,21,23,25,26,27,28,31,32,35,36,38,41] ,
[1,3,7,8,10,13,14,15,17,18,19,21,23,27,31,32,34,37,38,39,41] ,
[2,4,5,6,13,14,15,16,17,20,21,24,25,31,32,33,34,35,36,40,42] ,
[0,1,2,4,5,10,12,18,25,31,33,38,39,41,42] .

(c) (43;21,21,21,15;35)

[1,2,3,5,10,13,16,17,19,21,23,25,28,29,31,32,34,35,36,37,39] ,
[1,2,4,5,6,10,11,13,14,16,19,21,23,25,26,28,31,34,35,36,40] ,
[2,4,5,6,7,10,11,12,13,17,21,23,24,25,27,28,29,34,35,40,42] ,
[0,1,2,3,6,10,16,17,26,27,33,37,40,41,42] .

(d) (43;21,21,21,15;35)

[2,4,5,6,8,9,11,12,13,17,19,21,23,25,27,28,29,33,36,40,42] ,
[2,5,6,7,8,11,13,15,16,17,18,20,21,24,29,31,33,34,39,40,42] ,
[4,5,6,9,10,11,13,14,16,17,18,19,20,21,28,31,35,36,40,41,42] ,
[0,3,10,11,13,14,15,20,23,28,29,30,32,33,40] .

(e) (43;21,21,21,15;35)

[1,3,6,8,11,12,15,17,21,23,24,25,27,29,30,33,34,36,38,39,41] ,
[6,8,9,10,11,13,16,17,20,21,24,25,28,29,31,36,38,39,40,41,42] ,
[2,3,8,9,10,11,12,13,15,16,17,18,19,20,22,29,36,37,38,39,42] ,
[0,2,4,8,9,14,15,20,23,28,29,34,35,39,41] .

7.7 $v = 45$

(ksss) type:

(a) (45;22,22,21,16;36)

[1,2,3,6,7,10,11,13,15,22,24,25,26,27,28,29,31,33,36,37,40,41] ,
[1,3,4,7,9,10,13,15,20,21,22,23,24,25,30,32,35,36,38,41,42,44] ,
[0,2,3,4,5,9,12,13,15,17,22,23,28,30,32,33,36,40,41,42,43] ,
[1,6,14,15,17,18,21,22,23,24,27,28,30,31,39,44] .

(b) (45;22,22,21,16;36)

[1, 2, 4, 8, 9, 10, 12, 14, 16, 18, 19, 21, 22, 25, 28, 30, 32, 34, 38, 39, 40, 42] ,
 [1, 2, 4, 5, 7, 10, 14, 15, 17, 21, 22, 23, 24, 28, 30, 31, 35, 38, 40, 41, 43, 44] ,
 [0, 1, 4, 5, 6, 9, 10, 11, 12, 14, 22, 23, 31, 33, 34, 35, 36, 39, 40, 41, 44] ,
 [2, 5, 9, 10, 18, 20, 21, 22, 23, 24, 25, 27, 35, 36, 40, 43] .

(c) (45; 22, 21, 19, 17; 34)

[1, 3, 6, 7, 9, 12, 16, 18, 21, 22, 25, 26, 28, 30, 31, 32, 34, 35, 37, 40, 41, 43] ,
 [0, 1, 2, 4, 8, 9, 10, 11, 12, 15, 19, 26, 30, 33, 34, 35, 36, 37, 41, 43, 44] ,
 [0, 2, 3, 5, 10, 13, 14, 18, 19, 20, 25, 26, 27, 31, 32, 35, 40, 42, 43] ,
 [0, 3, 4, 5, 7, 9, 17, 21, 22, 23, 24, 28, 36, 38, 40, 41, 42] .

(d) (45; 22, 19, 19, 18; 33)

[1, 3, 6, 8, 10, 12, 18, 21, 22, 25, 26, 28, 29, 30, 31, 32, 34, 36, 38, 40, 41, 43] ,
 [0, 1, 7, 8, 11, 13, 18, 19, 21, 22, 23, 24, 26, 27, 32, 34, 37, 38, 44] ,
 [0, 1, 2, 4, 5, 12, 13, 17, 19, 22, 23, 26, 28, 32, 33, 40, 41, 43, 44] ,
 [3, 4, 5, 6, 8, 11, 12, 17, 21, 24, 28, 33, 34, 37, 39, 40, 41, 42] .

(kkss) type:

(a) (45; 22, 22, 21, 16; 39)

[1, 2, 5, 7, 8, 12, 13, 15, 17, 19, 21, 22, 25, 27, 29, 31, 34, 35, 36, 39, 41, 42] ,
 [2, 3, 4, 5, 6, 9, 11, 12, 13, 17, 18, 20, 21, 22, 26, 29, 30, 31, 35, 37, 38, 44] ,
 [0, 1, 2, 3, 4, 6, 10, 11, 13, 19, 21, 24, 26, 32, 34, 35, 39, 41, 42, 43, 44] ,
 [3, 4, 7, 8, 17, 19, 20, 22, 23, 25, 26, 28, 37, 38, 41, 42] .

7.8 $v = 47$

In this case there are two GS-parameter sets with $k_1 = (v-1)/2$, namely (47; 23, 22, 22, 17; 37) and (47; 23, 21, 19, 19; 35). Our exhaustive search did not find any families of type (ksss).

7.9 $v = 49$

In this case there are again two GS-parameter sets with $k_1 = (v-1)/2$, namely (49; 24, 24, 22, 18; 39) and (49; 24, 22, 21, 19; 37). Our exhaustive search did not find any families of type (ksss) or (kkss).