# THE SECOND WEIGHT OF GENERALIZED REED-MULLER CODES IN MOST CASES 

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#### Abstract

The second weight of the Generalized Reed-Muller code of length $q^{n}$ and order $d$ over the finite field with $q$ elements is now known for $d<q$ and $d>(n-1)(q-1)$. In this paper, we determine the second weight for the other values of $d$ which are not multiples of $q-1$ plus 1 . For the special case $d=a(q-1)+1$ we give an estimate.


## 1. Introduction - Notations

Let $\mathbb{F}_{q}$ be the finite field with $q$ elements and $n \geq 1$ an integer. Let $d$ be an integer such that $1 \leq d<n(q-1)$. The generalized Reed-Muller code of order $d$ is the following subspace of the space $\mathbb{F}_{q}^{\left(q^{n}\right)}$ :

$$
\operatorname{RM}_{q}(d, n)=\left\{(f(x))_{x \in \mathbb{F}_{q}^{n}} \mid f \in \mathbb{F}_{q}\left[X_{1}, \ldots, X_{n}\right] \text { and } \operatorname{deg}(f) \leq d\right\}
$$

It may be remarked that the polynomials $f$ determining this code are viewed as polynomial functions. Hence each codeword is associated with a unique reduced polynomial, namely a polynomial whose partial degrees are $\leq q-1$. We will denote by $\mathcal{F}(q, d, n)$ the space of the reduced polynomials $f \in \mathbb{F}_{q}\left[X_{1}, \ldots, X_{n}\right]$ such that $\operatorname{deg} f \leq d$. From a geometric point of view a polynomial $f$ defines a hypersurface in $\mathbb{F}_{q}^{n}$ and the number of points $N(f)$ of this hypersurface (the number of zeros of $f)$ is related to the weight of the associated codeword by the following formula:

$$
W(f)=q^{n}-N(f) .
$$

The code $\mathrm{RM}_{q}(d, n)$ has the following parameters:
(1) length $m=q^{n}$,
(2) dimension $k=\sum_{t=0}^{d} \sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\binom{t-j q+n-1}{t-j q}$,
(3) minimum distance $W_{1}=(q-b) q^{n-a-1}$, where $a$ and $b$ are the quotient and the remainder in the Euclidian division of $d$ by $q-1$, namely $d=a(q-1)+b$ and $0 \leq b<q-1$.

Remark 1.1. Be carefull not to confuse symbols. With our notations, the ReedMuller code of order $d$ has length $m$, dimemsion $k$ and minimum distance $W_{1}$. Namely it is a $\left[m, k, W_{1}\right]$-code. The integer $n$ is the number of variables of the polynomials defining the words and the order $d$ is the maximum total degree of these polynomials.

[^0]The minimum distance was given by T. Kasami, S. Lin, W. Peterson in [7. The words reaching this bound were characterized by P. Delsarte, J. Goethals and F. MacWilliams in [3. Let us denote by $W_{2}$, the second weight, namely the weight just above the minimum distance. If $d=1$, we know that the code has only three weights: 0 , the minimum distance $W_{1}=q^{n}-q^{n-1}$ and the second weight $W_{2}=q^{n}$. For $d=2$ and $q=2$ the weight distribution is more or less a consequence of the investigation of quadratic forms done by L. Dickson in 4 and was also done by E. Berlekamp and N. Sloane in an unpublished paper. For $d=2$ and any $q$ (including $q=2$ ) the weight distribution was given by R. McEliece in 9 . For $q=2$, for any $n$ and any $d$, the weight distribution is known in the range [ $W_{1}, 2.5 W_{1}$ ] by a result of Kasami, Tokora, Azumi [8]. In particular, the second weight is $W_{2}=3 \times 2^{n-d-1}$. For $d \geq n(q-1)$ the code $\operatorname{RM}_{q}(d, n)$ is the whole $\mathcal{F}(q, d, n)$, hence any integer $0 \leq t \leq q^{m}$ is a weight. The second weight was first studied by J.-P. Cherdieu and R. Rolland in [1] who proved that when $q>2$ is fixed, for $d<q$ sufficiently small the second weight is

$$
W_{2}=q^{n}-d q^{n-1}-(d-1) q^{n-2}
$$

Their result was improved by A. Sboui in [11], who proved the formula for $d \leq q / 2$. The methods in [1] and 11] are of a geometric nature by means of which the codewords reaching this weight can be determined. These codewords are hyperplane arrangements. Recently, O. Geil in [5], using Gröbner basis methods, proved the formula for $d<q$. Moreover as an application of his method, he gave a new proof of the Kasami-Lin-Peterson minimum distance formula and determined, when $d>(n-1)(q-1)$, the first $d+1-(n-1)(q-1)$ weights. However the Gröbner basis method does not determine all the codewords reaching the second weight.

To summarize the state of the art, let us note the following main points
(1) for $q=2$, the second weight is known;
(2) for $n=2$, the second weight is known for all values of $d$;
(3) for $n>2$, the second weight is known for $d<q$ and for $d>(n-1)(q-1)$.

Here and subsequently, $a$ and $b$ are respectively the quotient and the remainder in the Euclidian division of $d$ by $q-1$. In this paper, we determine for $n \geq 3, q \geq 3$ and $b \neq 1$ the second weight $W_{2}$ (or the second number of points of a hypersurface $N_{2}=q^{n}-W_{2}$ ) of the generalized Reed-Muller code and for $b=1$ we give a lower bound on this second weight. This work is done for all the other values of $d$ not yet handled, namely $q \leq d \leq(n-1)(q-1)$ for $q \geq 3$. Let us remark that for such a $d$, we have $1 \leq a \leq(n-1)$. Moreover, if $a=(n-1)$ then $b=0$. If $f \in \mathcal{F}(q, d, n) \backslash\{0\}$ we will denote by $N(f)$ the number of zeros of $f$ i.e. the number of points of the hypersurface defined by $f$, and by $W(f)=q^{n}-N(f)$ the weight of the associated codeword. If $f=h_{1} h_{2} \ldots h_{d}$ where $h_{i}\left(X_{1}, \ldots, X_{n}\right)$ is a polynomial of degree 1, we consider the hyperplane arrangement $\mathcal{A}=\left\{H_{i}\right\}_{i=1, \ldots, d}$ where $H_{i}$ is the affine hyperplane defined by $h_{i}\left(X_{1}, \ldots, X_{n}\right)=0$. The hypersurface defined by $f$ is the union of the hyperplanes $H_{i}$. We will set

$$
N(\mathcal{A})=N(f)=\# \cup_{i=1}^{d} H_{i}, \quad W(\mathcal{A})=q^{n}-N(\mathcal{A}) .
$$

The paper is organized as follows. We begin in Section 2 with a result on some special hypersurfaces: those which are unions of affine hyperplanes defined by linearly independant linear forms. We determine the configurations of this class having the minimal weight among those which do not reach the minimum distance (i.e. which are not maximal). It turns out that these particular hypersurfaces reach
the second weight except possibly for the case $d=a(q-1)+1$. In Section 3 we state and prove the main theorem on the value of the second weight for general hypersurfaces. The proof which follows the method introduced by O. Geil in [5] is based on Gröbner basis techniques. It also uses a tedious combinatorial lemma whose proof is done in the appendix. We point out in Section 4 some open questions related to the case $d=a(q-1)+1$ not solved in this paper and to the determination of the codewords reaching the second weight.

## 2. BLOCKS OF HYPERPLANE ARRANGEMENTS

2.1. Basic facts. Let us suppose that $d=d_{1}+d_{2}+\ldots+d_{k}$ where

$$
\left\{\begin{array}{l}
1 \leq d_{i} \leq q-1 \\
1 \leq k \leq n
\end{array}\right.
$$

Let us denote by $f_{1}, f_{2}, \ldots, f_{k}, k$ independant linear forms on $E=\mathbb{F}_{q}^{n}$, and let us consider the following hyperplane arrangement: for each $f_{i}$ we have $d_{i}$ distinct parallel hyperplanes defined by

$$
f_{i}(x)=u_{i, j} \quad 1 \leq j \leq d_{i}
$$

This arrangement of $d$ hyperplanes is consists of $k$ blocks of parallel hyperplanes, the $k$ directions of the blocks being linearly independant. The set of such hyperplane arrangements will be called $\mathcal{L}$.

Theorem 2.1. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathcal{L}$ and let us set

$$
A=\bigcup_{H \in \mathcal{A}} H
$$

Then, the number of points of $A$ is

$$
N(\mathcal{A})=\# A=q^{n}-q^{n-k} \prod_{i=1}^{k}\left(q-d_{i}\right)
$$

Proof. We can suppose that $f_{i}(x)=x_{i}$. The points which are not in $A$ satisfy the following conditions:

```
    x
and
    x 
and
    \vdots
```

and

$$
x_{k} \neq u_{k, 1}, u_{k, 2}, \ldots, u_{k, d_{k}} .
$$

Moreover for $u>k$, the $x_{u}$ are arbitrary. Hence the number of points which are not in $A$ is

$$
q^{n-k} \prod_{i=1}^{k}\left(q-d_{i}\right)
$$

Example 2.2. Let $k=a+1, d_{i}=q-1$ for $i=1,2, \ldots, a$ and $d_{a+1}=b$. We know that these configurations are the maximal configurations, namely the configurations $\mathcal{A}$ such that $N(\mathcal{A})=q^{n}-W_{1}=N_{1}$.

Remark 2.3. The number $N(\mathcal{A})$ depends only on $k$ and $d_{1}, d_{2}, \ldots, d_{k}$. These values define a type $T$ (i.e. the set of all arrangements in $\mathcal{L}$ with the same values $k$ and $d_{1}, d_{2}, \ldots, d_{k}$ ). We will denote by $N(T)$ the common number of points of all the type $T$ arrangements.
2.2. Modification of a maximal configuration when $q \geq 3$. Let us start from a maximal configuration $\mathcal{A}$, then

$$
N(\mathcal{A})=\# \bigcup_{H \in \mathcal{A}} H=N_{1}
$$

We know (cf. [3]) that a maximal configuration is given by $a+1$ linearly independant linear forms $f_{1} f_{2}, \ldots f_{a+1}$ such that the $d=a(q-1)+b$ hyperplanes are constituted by the following blocks:
(1) $a$ blocks of $q-1$ parallel hyperplanes:
for each $i \in\{1, \ldots, a\}$ let $A_{i}=\left\{u_{i, j}\right\}_{1 \leq j \leq q-1}$ be a subset of $\mathbb{F}_{q}$ such that $\# A_{i}=q-1$. We denote by $\mathcal{A}_{i}$ the block of the $q-1$ distinct parallel hyperplanes $H_{i, j}$ defined by

$$
H_{i, j}=\left\{x \in E \mid f_{i}(x)=u_{i, j}\right\}
$$

(2) one block of $b$ parallel hyperplanes:
let $B=\left\{v_{j}\right\}_{1 \leq j \leq b}$ be a subset of $\mathbb{F}_{q}$ such that $\# B=b$. We denote by $\mathcal{B}$ the bloc of $b$ distinct parallel hyperplanes $P_{j}$ defined by

$$
P_{j}=\left\{x \in E \mid f_{a+1}(x)=v_{j}\right\}
$$

Let us remark that if $b=0$, then $B=\emptyset$ and the block $\mathcal{B}$ is void.
A maximal configuration is in $\mathcal{L}$.
2.2.1. Type 1 exchange. The type 1 exchange replaces one hyperplane of a complete block by a hyperplane in the last block. The so obtained configuration is in $\mathcal{L}$ and is not maximal by the characterization of P. Delsarte, J. Goethals and F. MacWilliams.

More precisely, we suppose that $1 \leq a \leq n-1$ and $0 \leq b<q-2$. (For $b=q-2$ this exchange gives another maximal arrangement.) Let us define the following transform of the configuration $\mathcal{A}$. Choose $i \in\{1, \ldots, a\}, j \in\{1, \ldots, q-1\}$ and $v_{b+1} \in \mathbb{F}_{q} \backslash B$. Replace the hyperplane $H_{i, j}$ by the hyperplane $P_{b+1}=\{x \in$ $\left.E \mid f_{a+1}(x)=v_{b+1}\right\}$. We call $T_{1}$ the type of the obtained configuration.

Proposition 2.4. For $1 \leq a \leq n-1$ and $0 \leq b<q-2$, the following formulas hold:

$$
\begin{gathered}
N\left(T_{1}\right)=q^{n}-2 q^{n-a-1}(q-b-1) \\
N_{1}-N\left(T_{1}\right)=q^{(n-a-1)}(q-b-2)>0
\end{gathered}
$$

Proof. The first formula is a direct consequence of Theorem 2.1. A direct computation gives us the second formula.
2.2.2. Type 2 exchange. The type 2 exchange replaces one hyperplane of a complete block by a hyperplane defined by a new linear form, linearly independant from the $a+1$ original ones. The obtained configuration is in $\mathcal{L}$ and is not maximal.

More precisely, we suppose that $1 \leq a<n-1$ and $1 \leq b<q-1$. (for $a=n-1$ the type 2 exchange cannot be done, and for $b=0$ it is the type 1 exchange). Choose a linear form $f_{a+2}$ that together with the linear forms

$$
f_{1}, \ldots, f_{a+1}, f_{a+2}
$$

forms a linearly independent system. Choose $i \in\{1, \ldots, a\}, j \in\{1, \ldots, q-1\}, w \in$ $\mathbb{F}_{q}$ and replace the hyperplane $H_{i, j}$ by the hyperplane $Q=\left\{x \in E \mid f_{a+2}(x)=w\right\}$. We call $T_{2}$ the type of the new obtained arrangement.
Proposition 2.5. For $1 \leq a<n-1$ and $1 \leq b<q-1$ the following formulas hold:

$$
\begin{gathered}
N\left(T_{2}\right)=q^{n}-2 q^{n-a-2}(q-1)(q-b) \\
N_{1}-N\left(T_{2}\right)=q^{(n-a-2)}(q-b)(q-2)>0
\end{gathered}
$$

Proof. The first formula is a direct consequence of Theorem 2.1. A direct computation gives the second formula.

Now let us compare $N\left(T_{1}\right)$ and $N\left(T_{2}\right)$ for $d$ such that

$$
\begin{aligned}
& 1 \leq a \leq n-2, \\
& 1 \leq b<q-2 .
\end{aligned}
$$

A simple computation gives the following:
Proposition 2.6. For $1 \leq a \leq n-2$ and $1 \leq b<q-2$, we get

$$
N\left(T_{1}\right)-N\left(T_{2}\right)=2 q^{(n-a-2)} b>0
$$

2.2.3. Type 3 exchange. The type 3 exchange replaces one hyperplane of the last block by a hyperplane defined by a new linear form, linearly independant from the $a+1$ original ones. The obtained configuration is in $\mathcal{L}$ and is not maximal.

We suppose that $1 \leq a<n-1$ and $2 \leq b<q-1$. (For $b=1$, the exchange does not change the type of the configuration). Choose a linear form $f_{a+2}$ which constitutes with the linear forms $f_{1}, \ldots, f_{a+1}$ a linearly independant system. Choose $j \in\{1, \ldots, b\}$ and $w \in \mathbb{F}_{q}$. Replace the hyperplane $P_{j}$ by the hyperplane $Q=\left\{x \in E \mid f_{a+2}(x)=w\right\}$. We call $T_{3}$ the type of the new obtained arrangement.
Proposition 2.7. For $1 \leq a<n-1$ and $2 \leq b<q-1$ the following formulas hold:

$$
\begin{gathered}
N\left(T_{3}\right)=q^{n}-q^{n-a-2}(q-1)(q-b+1), \\
N_{1}-N\left(T_{3}\right)=q^{(n-a-2)}(b-1)>0 .
\end{gathered}
$$

Proof. The first formula is a direct consequence of Theorem 2.1. A direct computation gives the second formula.

Now let us compare $N\left(T_{1}\right)$ and $N\left(T_{3}\right)$ for $d$ such that $1 \leq a \leq n-2,2 \leq b<q-2$. A simple computation gives the following:
Proposition 2.8. For $1 \leq a \leq n-2$ and $2 \leq b<q-2$, we get

$$
N\left(T_{3}\right)-N\left(T_{1}\right)=q^{(n-a-2)}\left(q^{2}-(b+2) q-b+1\right)>0 .
$$

For $b=q-2$, the type 1 transform is not valuable (it gives $N\left(T_{1}\right)=N_{1}$ ) so we must compare $N\left(T_{3}\right)$ and $N\left(T_{2}\right)$. A direct computation gives the following:

Proposition 2.9. For $1 \leq a \leq n-2$ and $b=q-2$,

$$
N\left(T_{3}\right)-N\left(T_{2}\right)=q^{(n-a-2)}(q-1)>0
$$

holds.
2.2.4. Type 4 exchange. The type 4 exchange, used when $b=1$, deletes the unique hyperplane of the last block. Let us denote by $T_{4}$ the type of the new obtained arrangement. Let us remark that this configuration is the maximal configuration related to the degree $d-1$, namely gives the minimal distance for the Reed-Muller code of order $d-1$. Then by a direct computation the following proposition holds:

Proposition 2.10. For $1 \leq a<n-1$ and $b=1$ the following formulas hold:

$$
\begin{gathered}
N\left(T_{4}\right)=q^{n}-q^{n-a}, \\
N_{1}-N\left(T_{4}\right)=q^{(n-a-1)}>0 .
\end{gathered}
$$

Now let us compare, for $b=1$ and $q=3, N\left(T_{2}\right)$ and $N\left(T_{4}\right)$. A simple computation gives the following:

Proposition 2.11. For $q=3,1 \leq a \leq n-2$ and $b=1$, we get

$$
N\left(T_{2}\right)-N\left(T_{4}\right)=3^{n-a-2} .
$$

Let us also compare, for $b=1$ and $q \geq 4, N\left(T_{1}\right)$ and $N\left(T_{4}\right)$. A simple computation gives the following:

Proposition 2.12. For $q \geq 4,1 \leq a \leq n-2$ and $b=1$, we get

$$
N\left(T_{4}\right)-N\left(T_{1}\right)=q^{n-a-1}(q-4) \geq 0
$$

2.2.5. The best case for a type $T_{1}$ or $T_{2}$ or $T_{3}$ or $T_{4}$ arrangement. Let us set $N_{2}^{\prime}=$ $\max \left(N\left(T_{1}\right), N\left(T_{2}\right), N\left(T_{3}\right), N\left(T_{4}\right)\right.$ ) (if $N\left(T_{i}\right)$ is not defined we don't consider it in the max). $\quad N_{2}^{\prime}$ is the largest number of zeros for a type $T_{1}$ or $T_{2}$ or $T_{3}$ or $T_{4}$ arrangement. We summarize the results of this subsection in the following theorem. We will denote by $W_{2}^{\prime}$ the second weight for the arrangements of the previous type, namely $W_{2}^{\prime}=q^{n}-N_{2}^{\prime}$.

Theorem 2.13. The values of $N_{2}^{\prime}$ and $W_{2}^{\prime}$ are:
(1) Let us suppose that $q \geq 4$.
(a) For $1 \leq a<n-1$ and $2 \leq b<q-1$, the maximal number of points $N_{2}^{\prime}$ is reached by the type $T_{3}$, hence

$$
\begin{gathered}
N_{2}^{\prime}=N\left(T_{3}\right)=q^{n}-q^{n-a-2}(q-1)(q-b+1), \\
W_{2}^{\prime}=q^{n-a-2}(q-1)(q-b+1)
\end{gathered}
$$

(b) For $1 \leq a<n-1$ and $b=1$, the maximal number of points $N_{2}^{\prime}$ is reached by the type $T_{4}$, hence

$$
\begin{gathered}
N_{2}^{\prime}=N\left(T_{4}\right)=q^{n}-q^{n-a}, \\
W_{2}^{\prime}=q^{n-a} .
\end{gathered}
$$

(c) For $1 \leq a \leq n-1$ and $b=0$, the maximal number of points $N_{2}^{\prime}$ is reached by the type $T_{1}$, hence

$$
\begin{gathered}
N_{2}^{\prime}=N\left(T_{1}\right)=q^{n}-2 q^{n-a-1}(q-b-1), \\
W_{2}^{\prime}=2 q^{n-a-1}(q-b-1)
\end{gathered}
$$

(2) Let us now suppose that $q=3$.
(a) For $1 \leq a \leq n-1$ and $b=0$, the maximal number of points $N_{2}^{\prime}$ is reached by the type $T_{1}$, hence

$$
\begin{aligned}
N_{2}^{\prime}=N\left(T_{1}\right) & =q^{n}-2 q^{n-a-1}(q-1) \\
W_{2}^{\prime} & =4 \times 3^{n-a-1}
\end{aligned}
$$

(b) For $1 \leq a<n-1$ and $b=1$, the maximal number of points $N_{2}^{\prime}$ is reached by the type $T_{2}$, hence

$$
\begin{gathered}
N_{2}^{\prime}=N\left(T_{2}\right)=q^{n}-2 q^{n-a-2}(q-1)^{2}, \\
W_{2}^{\prime}=8 \times 3^{n-a-2}
\end{gathered}
$$

### 2.3. The best case for a $\mathcal{L}$ arrangement.

Theorem 2.14. Let $\mathcal{B}$ a hyperplane arrangement in $\mathcal{L}$. Suppose that $\mathcal{B}$ is not maximal and not in $T_{1}$ nor in $T_{2}$ nor in $T_{3}$ nor in $T_{4}$. Then $N(\mathcal{B})<N_{2}^{\prime}$.

Proof. Let us denote by $k, d_{1}, \ldots, d_{k}$ the values defining the type of this arrangement. Then

$$
N(\mathcal{B})=q^{n}-q^{n-k} \prod_{i=1}^{k}\left(q-d_{i}\right)
$$

Let us set $d^{\prime}=\sum_{i=1}^{k} d_{i}=a^{\prime}(q-1)+b^{\prime}$.
(1) If we can find two distinct indices $i_{1}$ and $i_{2}$ such that

$$
1 \leq d_{i_{1}} \leq d_{i_{2}} \leq q-2
$$

let us replace one hyperplane of the block $i_{1}$ by a new hyperplane (not in $\mathcal{B})$ added to the block $i_{2}$. We obtain the arrangement $\mathcal{B}^{\prime}$. As $\mathcal{B}$ is not in $T_{1}$ nor in $T_{2}$ nor in $T_{3}, \mathcal{B}^{\prime}$ is not a maximal arrangement. Moreover

$$
\begin{gathered}
N\left(\mathcal{B}^{\prime}\right)-N(\mathcal{B})=K\left(\left(q-d_{i_{1}}\right)\left(q-d_{i_{2}}\right)-\left(q-d_{i_{1}}+1\right)\left(q-d_{i_{2}}-1\right)\right) \\
=K\left(d_{i_{2}}-d_{i_{1}}+1\right)>0
\end{gathered}
$$

where $K=q^{n-k} \prod_{i \neq i_{1}, i_{2}}\left(q-d_{i}\right)$. Then $\mathcal{B}$ is not maximal among the $\mathcal{L}$ arrangements not reaching $N_{1}$.
(2) If all the $d_{i}$ but $d_{i_{1}}$ are 0 or $q-1$, namely $\mathcal{B}$ consists of $a^{\prime}$ complete blocks containing $q-1$ hyperplanes and one block of $b^{\prime}$ hyperplanes. As $\mathcal{B}$ is not a maximal configuration then either $a^{\prime}<a$ holds or $a^{\prime}=a$ and $b^{\prime}<b$ holds. In both cases $d^{\prime}=a^{\prime}(q-1)+b^{\prime}<d$ and we can add a new direction, linearly independant from the previous $a^{\prime}$ directions and one hyperplane in this new direction. The obtained configuration $\mathcal{B}^{\prime}$ is not maximal and $N\left(\mathcal{B}^{\prime}\right)>N(\mathcal{B})$, then $\mathcal{B}$ is not maximal among the $\mathcal{L}$ arrangements not reaching $N_{1}$.
(3) If all the $d_{i}$ are 0 or $q-1$, namely $\mathcal{B}$ is contituted by $a^{\prime}$ complete blocks containing $q-1$ hyperplanes. As $\mathcal{B}$ is not maximal, $d^{\prime}<d$ holds. Let us add a new hyperplane in a new direction linearly independant from the $a^{\prime}$ previous directions. As $\mathcal{B}$ is not a $T_{4}$ configuration, the obtained configuration $\mathcal{B}^{\prime}$ is not maximal. Moreover $N\left(\mathcal{B}^{\prime}\right)>N(\mathcal{B})$, then $\mathcal{B}$ is not maximal among the $\mathcal{L}$ arrangements not reaching $N_{1}$.

## 3. Main Result for general hypersurfaces

3.1. Gröbner basis techniques. We will use a Gröbner basis theoretical method similar to the one used by O. Geil in [5] to compute the second weight of the generalized Reed-Muller code $R M_{q}(d, n)(q \geq 3$ and $q \leq d \leq(n-1)(q-1))$. For the convenience of the reader we recall some general definitions and results on Gröbner basis which can be found in [2]. We repeat the relevant material from [5] and [6], where the details can be found.

Let $\mathcal{M}$ the set of monomials of $\mathbb{F}_{q}\left[X_{1}, \ldots, X_{n}\right]$

$$
M\left(X_{1}, X_{2}, \ldots, X_{n}\right)=\prod_{i=1}^{n} X_{i}^{\alpha_{i}}
$$

where $\alpha_{i} \in \mathbb{N}$. Let $\prec$ be a monomial ordering on $\mathcal{M}$. If $f \in \mathbb{F}_{q}\left[X_{1}, \ldots, X_{n}\right]$, we will denote by $\operatorname{lm}(f)$ its leading monomial and by $\operatorname{lt}(f)$ its leading term. We will denote by $\operatorname{lcm}(\mathrm{f}, \mathrm{g})$ the low common multiple of $f$ and $g$. If $\operatorname{lm}(f)=\prod_{i=1}^{n} X_{i}^{\alpha_{i}}$, the multidegree of $f$, denoted by multideg $(f)$, is $\left(\alpha_{1}, \cdots, \alpha_{n}\right)$.

The first main tool is the division algorithm of a polynomial $f \in \mathbb{F}_{q}\left[X_{1}, \ldots, X_{n}\right]$ by an ordered set $\left(f_{1}, \cdots, f_{s}\right)$ of polynomials. Using this algorithm, $f$ can be written

$$
f=a_{1} f_{1}+\ldots+a_{s} f_{s}+r
$$

where $r \in \mathbb{F}_{q}\left[X_{1}, \ldots, X_{n}\right]$ and either $r=0$ or $r$ is a linear combination, with coefficients in $\mathbb{F}_{q}$, of monomials, none of which is divisible by any of $l t\left(f_{1}\right), \ldots, l t\left(f_{s}\right)$. Moreover if $a_{i} f_{i} \neq 0$, then we have multideg $(f) \preceq \operatorname{multideg}\left(a_{i} f_{i}\right)$. Note that the result depends on the monomial ordering and on the ordering of the $s$-tuple of polynomials $\left(f_{1}, \cdots, f_{s}\right)$.
Definition 3.1. Let $\prec$ be a monomial ordering. A finite subset $\left\{g_{1}, \ldots, g_{s}\right\}$ of an ideal I is said to be a Gröbner basis if

$$
\left\langle\operatorname{lt}\left(g_{1}\right), \ldots, \operatorname{lt}\left(g_{s}\right)\right\rangle=\langle\operatorname{lt}(I)\rangle
$$

The Buchberger's algorithm provides a way to decide if a basis $\left\{g_{1}, \ldots, g_{s}\right\}$ is a Gröbner basis or not. It uses the following notion of $S$-polynomial.
Definition 3.2. Let $f, g \in \mathbb{F}_{q}\left[X_{1}, \ldots, X_{n}\right]$ be two nonzero polynomials. The $S$ polynomial of $f$ and $g$ is

$$
S(f, g)=\frac{\operatorname{lcm}(\operatorname{lm}(\mathrm{f}), \operatorname{lm}(\mathrm{g}))}{\operatorname{lt}(\mathrm{f})} f-\frac{\operatorname{lcm}(\operatorname{lm}(\mathrm{f}), \operatorname{lm}(\mathrm{g}))}{\operatorname{lt}(\mathrm{g})} g
$$

Theorem 3.3 (Bruchberber's algorithm). A set $\left\{g_{1}, \ldots, g_{s}\right\}$ is a Gröbner basis for the ideal $\left\langle g_{1}, \ldots, g_{s}\right\rangle$ if and only if for all pair $i \neq j$ the remainder on division of $S\left(g_{i}, g_{j}\right)$ by $\left\{g_{1}, \ldots, g_{s}\right\}$ listed in some order is zero.

Remark 3.4. The previous algorithm can be simplified by the following remark: if $\operatorname{lm}\left(g_{i}\right)$ and $\operatorname{lm}\left(g_{j}\right)$ are relatively prime, then the remainder on division of $S\left(g_{i}, g_{j}\right)$ by $\left\{g_{1}, \ldots, g_{s}\right\}$ listed in some order is zero.
Definition 3.5. Let $I$ be an ideal of $\mathbb{F}_{q}\left[X_{1}, \ldots, X_{n}\right]$. The footprint of $I$ is

$$
\Delta(I)=\{M \in \mathcal{M} \mid
$$

$M$ is not the leading monomial of any polynomial in $I\}$.
We will use the following result which can be found in [6]:

Theorem 3.6. Let us consider the following ideal I of $\mathbb{F}_{q}\left[X_{1}, \ldots, X_{n}\right]$ :

$$
I=\left\langle F_{1}, \ldots, F_{k}, X_{1}^{q}-X_{1}, \ldots, X_{n}^{q}-X_{n}\right\rangle
$$

Then the footprint $\Delta(I)$ is finite and

$$
\# \Delta(I)=\# V_{q}(I)
$$

where $V_{q}(I)$ is the set of the $\mathbb{F}_{q}$-rational points of the variety defined by the ideal $I$.
If we know a Gröbner basis of the ideal $I$, the footprint is easy to determine.
Theorem 3.7. Let $I$ be an ideal and $\left\{g_{1}, \ldots, g_{s}\right\}$ a Gröbner basis of $I$. Let $J$ be the ideal $\left\langle\operatorname{lm}\left(g_{1}\right), \ldots, \operatorname{lm}\left(g_{s}\right)\right\rangle$. Then

$$
\Delta(I)=\Delta(J)
$$

In the following, we will restrict $\prec$ to be the graded lexicographic ordering on $\mathcal{M}$ defined by

$$
\prod_{i=1}^{n} X_{i}^{\alpha_{i}} \prec \prod_{i=1}^{n} X_{i}^{\beta_{i}}
$$

if $\left(\alpha_{1}, \ldots, \alpha_{n}\right) \neq\left(\beta_{1}, \ldots, \beta_{n}\right)$ and either $\sum_{i=1}^{n} \alpha_{i}<\sum_{i=1}^{n} \beta_{i}$ holds or $\sum_{i=1}^{n} \alpha_{i}=$ $\sum_{i=1}^{n} \beta_{i}$ with the first non-zero entry of $\left(\beta_{1}-\alpha_{1}, \ldots, \beta_{n}-\alpha_{n}\right)$ being positive holds.

### 3.2. The second weight.

Theorem 3.8. For $n \geq 3, q \geq 3$ and $q-1<d \leq(n-1)(q-1)$ the second weight $W_{2}$ of the generalized Reed-Muller code $\operatorname{RM}_{q}(d, n)$ satisfies
(1) if $1 \leq a \leq n-1$ and $b=0$ then

$$
W_{2}=W_{2}^{\prime}=2 q^{n-a-1}(q-1)
$$

(2) if $1 \leq a<n-1$ and $b=1$ then
(a) if $a<n-2$ then

$$
q^{n-a}-q^{n-a-1}+q^{n-a-2}-q^{n-a-3} \leq W_{2} \leq q^{n-a}=W_{2}^{\prime}
$$

(b) if $a=n-2$ then

$$
q^{2}-2 \leq W_{2} \leq q^{2}=W_{2}^{\prime}
$$

(3) if $1 \leq a<n-1$ and $2 \leq b<q-1$ then

$$
W_{2}=W_{2}^{\prime}=q^{n-a-2}(q-1)(q-b+1)
$$

Proof. Let $F\left(X_{1}, X_{2}, \ldots, X_{n}\right)$ be a reduced polynomial of degree $d, \operatorname{lm}(F)=$ $X_{1}^{u_{1}} X_{2}^{u_{2}} \ldots X_{n}^{u_{n}}$ its leading monomial. We suppose that the variables $X_{i}$ are numbered in such a way that $u_{1} \geq u_{2} \ldots \geq u_{n}$. Let us consider the ideals

$$
I=\left\langle F, X_{1}^{q}-X_{1}, \ldots, X_{n}^{q}-X_{n}\right\rangle
$$

and

$$
J=\left\langle X_{1}^{u_{1}} X_{2}^{u_{2}} \ldots X_{n}^{u_{n}}, X_{1}^{q}, \ldots, X_{n}^{q}\right\rangle
$$

Using the footprint of $I$ and $J$ we get

$$
\# \Delta(I) \leq \# \Delta(J)=q^{n}-\prod_{i=1}^{n}\left(q-u_{i}\right)
$$

We remark that this last value is the number of points of a hyperplane arrangement $\mathcal{A}$ which is in $\mathcal{L}$. Then, if $\left(u_{1}, u_{2}, \ldots, u_{n}\right) \neq(q-1, q-1, \ldots, q-1, b, 0 \ldots, 0)$, the arrangement $\mathcal{A}$ is not maximal and consequently

$$
\# \Delta(I) \leq \# \Delta(J) \leq N_{2}^{\prime}
$$

If $\left(u_{1}, u_{2}, \ldots, u_{n}\right)=(q-1, q-1, \ldots, q-1, b, 0 \ldots, 0)$, let us compute for each $1 \leq i \leq a+1$ (or $1 \leq i \leq a$ if $b=0)$

$$
\begin{gathered}
H_{i}\left(X_{1}, X_{2}, \ldots, X_{n}\right)=\frac{\operatorname{lcm}\left(\operatorname{lm}(\mathrm{F}), \mathrm{X}_{\mathrm{i}}^{\mathrm{q}}\right)}{X_{i}^{q}}\left(X_{i}^{q}-X_{i}\right)-\frac{\operatorname{lcm}\left(\operatorname{lm}(\mathrm{F}), \mathrm{X}_{\mathrm{i}}^{\mathrm{q}}\right)}{\operatorname{lt}(F)} F \\
H_{i}\left(X_{1}, X_{2}, \ldots, X_{n}\right)=-X_{1}^{u_{1}} \ldots X_{i-1}^{u_{i-1}} X_{i} X_{i+1}^{u_{i+1}} \ldots X_{n}^{u_{n}}-X_{i}^{q-u_{i}} G
\end{gathered}
$$

where $G=F-\operatorname{lt}(F)$. Then, let us set $R_{i}$ the remainder of the division of $H_{i}$ by $\left(F, X_{1}^{q}-X_{1}, \ldots, X_{n}^{q}-X_{n}\right)$. By Bruchberger's algorithm3.3 and Remark3.4 ( $X_{i}^{q}$ and $X_{j}^{q}$ are relatively prime if $i \neq j$ ) if all the $R_{i}$ are null, then $\left\{F, X_{1}^{q}-X_{1}, \ldots, X_{n}^{q}-X_{n}\right\}$ is a Gröbner basis. Hence by Theorem 3.7

$$
\# \Delta(I)=\# \Delta(J)=q^{n}-\prod_{i=1}^{n}\left(q-u_{i}\right)=q^{n}-(q-b) q^{n-a-1}
$$

We conclude that in this case the hypersurface defined by $F$ is maximal.
If one of the $R_{i}$ is not zero, let us consider

$$
M=\operatorname{lm}\left(R_{i}\right)=X_{1}^{\alpha_{1}} \ldots X_{n}^{\alpha_{n}}
$$

If the index $i$ is such that $1 \leq i \leq a$ we can suppose that $i=1$. In this case we have $X_{1}^{q-u_{i}}=X_{1}$. Then we have the following constraints on the exponents $\left(\alpha_{1}, \ldots, \alpha_{n}\right):$
(1) $\sum_{i=1}^{n} \alpha_{i} \leq d+1$,
(2) $0 \leq \alpha_{i} \leq q-1$,
(3) $X_{1}^{q-1} \ldots X_{a}^{q-1} X_{a+1}^{b}$ does not divide $M$.

If $i=a+1$ then $X_{i}^{q-u_{i}}=X_{a+1}^{q-b}$. In this case we have the following constraints on the exponents $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ :
(1) $\sum_{i=1}^{n} \alpha_{i} \leq d+q-b$,
(2) $0 \leq \alpha_{i} \leq q-1$,
(3) $X_{1}^{q-1} \ldots X_{a}^{q-1} X_{a+1}^{b}$ does not divide $M$.

Remark 3.9. Let us remark that if $b=\alpha_{a+1}=0$ the first constraint on the $\alpha_{i}$ is always $\sum_{i=1}^{n} \alpha_{i} \leq d+1$.

Now we have

$$
I=\left\langle F, R_{i}, X_{1}^{q}-X_{1}, \ldots, X_{n}^{q}-X_{n}\right\rangle
$$

so, if we set

$$
J_{1}=\left\langle X_{1}^{q-1} \ldots X_{a}^{q-1} X_{a+1}^{b}, M, X_{1}^{q}, \ldots, X_{n}^{q}\right\rangle
$$

we get

$$
\# \Delta(I) \leq \# \Delta\left(J_{1}\right)
$$

Let us consider

$$
\begin{gathered}
A_{1}=\left\{\beta=\left(\beta_{1}, \ldots, \beta_{n}\right) \mid \beta_{1}=q-1, \ldots=\beta_{a}=q-1, \beta_{a+1} \geq b\right. \\
\left.0 \leq \beta_{a+2} \leq q-1, \ldots, 0 \leq \beta_{n} \leq q-1\right\} \\
A_{2}=\left\{\beta=\left(\beta_{1}, \ldots, \beta_{n}\right) \mid \alpha_{1} \leq \beta_{1} \leq q-1, \ldots, \alpha_{n} \leq \beta_{n} \leq q-1\right\}
\end{gathered}
$$

and

$$
\begin{gathered}
A_{1} \cap A_{2}=\left\{\beta=\left(\beta_{1}, \ldots, \beta_{n}\right) \mid \beta_{1}=\ldots=\beta_{a}=q-1, \beta_{a+1} \geq \gamma\right. \\
\left.\alpha_{a+2} \leq \beta_{a+2} \leq q-1, \ldots, \alpha_{n} \leq \beta_{n} \leq q-1\right\}
\end{gathered}
$$

where $\gamma=\max \left(b, \alpha_{a+1}\right)$. Then

$$
\begin{gathered}
N(F) \leq \# \Delta\left(J_{1}\right)=q^{n}-\# A_{1}-\# A_{2}+\# A_{1} \cap A_{2}, \\
W(F) \geq \# A_{1}+\# A_{2}-\# A_{1} \cap A_{2} \\
W(F) \geq(q-b) q^{n-a-1}+\prod_{i=1}^{n}\left(q-\alpha_{i}\right)-(q-\gamma) \prod_{i=a+2}^{n}\left(q-\alpha_{i}\right) .
\end{gathered}
$$

The following lemma 3.10 is exactly what we need to compute the minimum $\mu$ of $\# A_{2}-\# A_{1} \cap A_{2}$. Then, a lower bound of $W_{2}$ is $\mu+(q-b) q^{n-a-1}$. In most cases, namely when $b \neq 1$, this lower bound is effectively reached by a hyperplane arrangement and we have $W_{2}=W_{2}^{\prime}$.
Lemma 3.10. Let $q$, $n$, $d$ be integers such that $q \geq 3, n \geq 3, q \leq d \leq(n-1)(q-1)$. We denote by $a$ and $b$ the quotient and the remainder on division of $d$ by $q-1$, namely $d=a(q-1)+b$ where $0 \leq b<q-1$.

We denote by $V$ the set of the finite sequences of integers $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, of length $n$, such that
(1) for $i=1, \ldots$, n we have $0 \leq \alpha_{i} \leq q-1$;
(2) $\sum_{i=1}^{n} \alpha_{i} \leq K$ where $K=d+1$ if $b=0$ and $K=d+q-b$ if $b>0$;
(3) if $\alpha_{1}=\alpha_{2}=\ldots=\alpha_{a}=q-1$, then $\alpha_{a+1}<b$.

Let us set $\gamma=\max \left(\alpha_{a+1}, b\right)$.
Then, the following holds:

$$
\begin{equation*}
\min _{\alpha \in V}\left\{\prod_{i=1}^{n}\left(q-\alpha_{i}\right)-(q-\gamma) \prod_{i=a+2}^{n}\left(q-\alpha_{i}\right)\right\}=\mu \tag{1}
\end{equation*}
$$

where

$$
\mu=\left\{\begin{array}{lll}
(q-2) q^{n-a-1} & \text { if } & b=0 \\
(q-1) q^{n-a-3} & \text { if } & b=1, a<n-2 \\
(q-2) q^{n-a-2} & \text { if } & b=1, a=n-2 \\
(b-1) q^{n-a-2} & \text { if } & 2 \leq b<q-1
\end{array} .\right.
$$

## 4. Open questions

Now we know the second weight of a Generalized Reed-Muller code, in almost any case. It remains to determine the exact value of this second weight when $d=a(q-1)+1$. For these particular values we have just proved that

$$
q^{n-a}-q^{n-a-1}+q^{n-a-2}-q^{n-a-3} \leq W_{2} \leq q^{n-a}=W_{2}^{\prime} \quad \text { if } a<n-2
$$

and that

$$
q^{n-a}-2 q^{n-a-2} \leq W_{2} \leq q^{n-a} \quad \text { if } a=n-2
$$

It would be very surprising to find a non-maximal hypersurface of degree $d=$ $a(q-1)+1$ with strictly more than $q^{n}-q^{n-a}$ points. Then we can ask the following questions:
(1) When $d=a(q-1)+1$, what is the exact value of $W_{2}$ ?
(2) When $d=a(q-1)+1$, what is the maximal number of points of a nonmaximal hypersurface of degree $d$ given by unions of hyperplanes? (in this paper we have proved that the maximum number of points for a hyperplane configuration in $\mathcal{L}$ is $q^{n}-q^{n-a}$ ).
We have not determined in the paper which are the codewords reaching the second weight. In our opinion, these codewords are hyperplanes arrangements. But this is not proved. However, we can deduce from the results obtained in [10] on the number of points of irreducible but not absolutely irreducible hypersurfaces that such a hypersurface cannot reach the second weight. In fact a simple computation shows that the number of points of such a hypersurface is strictly less than the maximum number of points of a non-maximal hypersurface in $\mathcal{L}$ (namely the number called $N_{2}^{\prime}=q^{n}-W_{2}^{\prime}$ ) and a fortiori cannot reach the second weight.

Appendix A. Proof of lemma 3.10

## A.1. Preliminary remarks. Let us set

$$
\begin{gathered}
P_{1}=\prod_{i=1}^{n}\left(q-\alpha_{i}\right) \\
P_{2}=(q-\gamma) \prod_{i=a+2}^{n}\left(q-\alpha_{i}\right)
\end{gathered}
$$

Hence we have to study the minimum value of $P_{1}-P_{2}$. Note that in the particular case $d=(n-1)(q-1)$ the value of $a$ is $n-1$ and $P_{2}=(q-\gamma)$.

Lemma A.1. If we permute the first a elements $\alpha_{i}$ we don't change the value of $P_{1}-P_{2}$. When $\alpha_{a+1}<b$, if we permute the last $n-a-1$ elements we don't change the value of $P_{1}-P_{2}$. When $\alpha_{a+1} \geq b$, namely when $\gamma=\alpha_{a+1}$, if we permute $\alpha_{a+1}$ with one of the last $n-a-1$ elements $\alpha_{i}$ such that $\alpha_{i} \geq b$ we don't change the value of $P_{1}-P_{2}$.

Proof. This can be seen directly on the formulas giving $P_{1}$ and $P_{2}$.
Then, from now on, we will suppose that the sequences $\alpha$ are such that

$$
\begin{gathered}
\alpha_{1} \geq \alpha_{2} \geq \cdots \geq \alpha_{a-1} \geq \alpha_{a} \\
\alpha_{a+2} \geq \alpha_{a+3} \geq \cdots \geq \alpha_{n}
\end{gathered}
$$

if $\alpha_{a+1}<b$, and that

$$
\begin{gathered}
\alpha_{1} \geq \alpha_{2} \geq \cdots \geq \alpha_{a-1} \geq \alpha_{a} \\
\alpha_{a+1} \geq \alpha_{a+2} \geq \cdots \geq \alpha_{n}
\end{gathered}
$$

if $\alpha_{a+1} \geq b$. In particular, when we transform a sequence, we always reorder the new obtained sequence in this way.

Lemma A.2. If we replace $\alpha_{i}$ by $\alpha_{i}+1$ and if the new sequence is in $V$, then the new $P_{1}-P_{2}$ is lower than the old one.

Proof. When $i \leq a$ the value of $P_{1}$ decreases, the value of $P_{2}$ is not modified. When $i=a+1$ and $\alpha_{a+1}<b, P_{1}$ decreases, the value of $P_{2}$ is not modified. When $i \geq a+1$ and $\alpha_{a+1} \geq b, P_{2}$ and $P_{1}$ decreases, then we must examine more precisely
the behaviour of $P_{1}-P_{2}$. The difference between the old value of $P_{1}-P_{2}$ and the new one is

$$
\begin{aligned}
& \prod_{\substack{j=1 \\
j \neq i}}^{n}\left(q-\alpha_{j}\right)-\prod_{\substack{j=a+1 \\
j \neq i}}^{n}\left(q-\alpha_{j}\right)= \\
& \left(\prod_{j=1}^{a}\left(q-\alpha_{j}\right)-1\right) \prod_{\substack{j=a+1 \\
j \neq i}}^{n}\left(q-\alpha_{j}\right)
\end{aligned}
$$

But, as $\alpha_{a+1} \geq b, \alpha_{a} \leq q-2$ and then

$$
\prod_{j=1}^{a}\left(q-\alpha_{j}\right) \geq 2
$$

We conclude that the new value is lower than the old one. It remains to study the case where $\alpha_{a+1}<b$ and $i>a+1$. The difference between the old value of $P_{1}-P_{2}$ and the new one is now

$$
\left(\prod_{j=1}^{a+1}\left(q-\alpha_{j}\right)-(q-b)\right) \prod_{\substack{j=a+2 \\ j \neq i}}^{n}\left(q-\alpha_{j}\right)
$$

But as $\left(q-\alpha_{a+1}\right)>(q-b)$ we conclude that the new value of $P_{1}-P_{2}$ is lower than the old one.

Lemma A.3. The minimum in the equation (1) is reached for sequences $\alpha$ such that $\sum_{i=1}^{n} \alpha_{i}=K$.
Proof. It is sufficient to prove that if $\sum_{i=1}^{n} \alpha_{i}<K$ it is possible to add 1 to a well choosen $\alpha_{i}$ (and then increase the sum), and obtain a new sequence in $V$ for which the new $P_{1}-P_{2}$ is lower than the old one. Suppose that $\sum_{i=1}^{n} \alpha_{i}<K$.

1) Suppose that $\alpha_{a+1} \geq b$.
a) If $\left(\alpha_{1}, \ldots, \alpha_{a}\right) \neq(q-1, \ldots, q-1, q-2)$ then there exists a $i$ such that $1 \leq i \leq a$ which does not reach its maximal value. Then we can replace $\alpha_{i}$ by $\alpha_{i}+1$. By Lemma A. 2 we conclude that the new $P_{1}-P_{2}$ is lower than the old one.
b) Now suppose that $\left(\alpha_{1}, \ldots, \alpha_{a}\right)=(q-1, \ldots, q-1, q-2)$.
$\alpha)$ If $b=0$ then $K=d+1=a(q-1)+1$. But the sum of the first $a$ elements is $a(q-1)-1$. So that $\alpha_{a+1}$ is at most 1 (this term exists because $a \leq n-1$ ). Then we can replace $\alpha_{a+1}$ by $\alpha_{a+1}+1$ because $q \geq 3$. By Lemma A. 2 we conclude that the new $P_{1}-P_{2}$ is lower than the old one.
$\beta$ ) If $b \geq 1$ then $K=d+q-b=a(q-1)+q$. In this case we know that $a \leq n-2$. We have $\alpha_{a+1}+\alpha_{a+2}+\ldots \alpha_{n} \leq q$ then if $\alpha_{a+1}<q-1$ we can add 1 to this term, if $\alpha_{a+1}=q-1$ then $\alpha_{a+2} \leq 1$ and because $q \geq 3$ it is possible to add 1 to to this term. By Lemma A. 2 we conclude that the new $P_{1}-P_{2}$ is lower than the old one.
2) Suppose that $\alpha_{a+1}<b$. Then by Lemma A.2 if we replace $\alpha_{a+1}$ by $\alpha_{a+1}+1$, we obtain a new $P_{1}-P_{2}$ lower than the old one.

From now on we will suppose that $\alpha$ is such that

$$
\sum_{i=1}^{n} \alpha_{i}=K
$$

Lemma A.4. Let $1 \leq i \leq a$ and $a+1 \leq j \leq n$ and suppose that $\alpha_{j}>\alpha_{i}$. If we permute these two elements, and if we obtain a sequence which is in $V$, then for the new sequence the value of $P_{1}-P_{2}$ is lower or equal to the old one.
Proof. Indeed $P_{1}$ does not change, and $P_{2}$ increases (if $j>a+1$ or if $j=a+1$ and $\alpha_{j}>b$ ) or does not change (if $j=a+1$ and $\alpha_{j} \leq b$ ).
Lemma A.5. Suppose that $1 \leq \alpha_{i} \leq \alpha_{j} \leq q-2$ and that we are in one of the following cases:
(1) $1 \leq j<i \leq a$;
(2) $a+2 \leq j<i \leq n$;
(3) $\alpha_{a+1} \geq b$ and $a+1 \leq j<i \leq n$;
(4) $1 \leq j \leq a$ and $a+2 \leq i \leq n$.

Let us replace $\alpha_{i}$ by $\alpha_{i}-1$ and $\alpha_{j}$ by $\alpha_{j}+1$. If the new sequence is in $V$, the new value of $P_{1}-P_{2}$ is lower than the old one.

Proof. 1) Case $1 \leq j<i \leq a$. The difference between the old value of $P_{1}-P_{2}$ and the new value is

$$
\left(\alpha_{j}-\alpha_{i}+1\right) \prod_{k \neq i, j}(q-k)>0
$$

2) Case $a+2 \leq j<i \leq n$. The difference between the the old value of $P_{1}-P_{2}$ and the new value is

$$
\left(\alpha_{j}-\alpha_{i}+1\right)\left(\prod_{k=1}^{a+1}\left(q-\alpha_{k}\right)-(q-\gamma)\right) \prod_{\substack{k=a+2 \\ k \neq i, j}}^{n}\left(q-\alpha_{k}\right)>0
$$

To verify that the previous expression is $>0$ note that if $\alpha_{a+1}<b$ then $\gamma=b$ and

$$
\prod_{k=1}^{a+1}\left(q-\alpha_{k}\right) \geq\left(q-\alpha_{a+1}\right)
$$

Hence

$$
\prod_{k=1}^{a+1}\left(q-\alpha_{k}\right)-(q-b) \geq\left(b-\alpha_{a+1}\right)>0
$$

If $\alpha_{a+1} \geq b$ then $\gamma=\alpha_{a+1}$ and $\alpha_{a} \leq q-2$. Then

$$
\prod_{k=1}^{a+1}\left(q-\alpha_{k}\right)-(q-\gamma) \geq 2\left(q-\alpha_{a+1}\right)-\left(q-\alpha_{a+1}\right)=\left(q-\alpha_{a+1}\right)>0
$$

3) Case $\alpha_{a+1} \geq b$ and $a+1 \leq j<i \leq n$. The formula of the difference between the the old value of $P_{1}-P_{2}$ and the new value is similar

$$
\left(\alpha_{j}-\alpha_{i}+1\right)\left(\prod_{k=1}^{a}\left(q-\alpha_{k}\right)-1\right) \prod_{\substack{k=a+1 \\ k \neq i, j}}^{n}\left(q-\alpha_{k}\right)>0
$$

To verify that the previous expression is $>0$ we have just to remark that $\alpha_{a} \leq q-2$, then $\prod_{k=1}^{a}\left(q-\alpha_{k}\right) \geq 2$.
4) Case $1 \leq j \leq a$ and $a+2 \leq i \leq n$. A simple computation shows that the difference between the old value of $P_{1}-P 2$ and the new value is

$$
\left(\alpha_{j}-\alpha_{i}+1\right) \prod_{\substack{k=1 \\ k \neq i, j}}^{n}\left(q-\alpha_{k}\right)+(q-\gamma) \prod_{\substack{k=a+2 \\ k \neq i}}^{n}\left(q-\alpha_{k}\right)>0
$$

A.2. The head of a best sequence. We give here the form of the first $a$ terms of a sequence $\alpha$ for which $P_{1}-P_{2}$ is minimum. We prove that $\alpha$ can be choosen such that one of the two following conditions holds:
(1) $\left(\alpha_{1}, \ldots \alpha_{a-1}, \alpha_{a}\right)=(q-1, \ldots, q-1, q-2)$ and $\alpha_{a+1} \geq b$;
(2) $\left(\alpha_{1}, \ldots, \alpha_{a}\right)=(q-1, \ldots, q-1)$ and $\alpha_{a+1}<b$;

1) Let us suppose first that there exists a sequence $\alpha$ such that $\alpha_{a+1}<b$ and for which $P_{1}-P_{2}$ is minimum. We will prove that for such a sequences the first $a$ terms can be set to $q-1$. Suppose that there exists a $j \leq a$ such that $\alpha_{j}<q-1$. We have

$$
\sum_{i=1}^{a+1} \alpha_{i}<a(q-1)+b=d
$$

then $\alpha_{a+2}>0$. If $\alpha_{a+2}>\alpha_{j}$ by Lemma A.4 we can permute the two terms to obtain a sequence which have a lower or equal $P_{1}-P_{2}$. If $\alpha_{a+2} \leq \alpha_{j}$, by Lemma A. 5 the sequence obtained by replacing $\alpha_{a+2}$ by $\alpha_{a+2}-1$ and $\alpha_{j}$ by $\alpha_{j}+1$ has a lower $P_{1}-P_{2}$. So we have proved that we can increase the value of $\alpha_{j}$ and obtain a lower or equal $P_{1}-P_{2}$.
2) Let us suppose now that there exists a sequence $\alpha$, such that $\alpha_{a+1} \geq b$, for which $P_{1}-P_{2}$ is minimum. We will prove that for such a sequence the first $a-1$ terms can be set to $q-1$ and $\alpha_{a}$ can be set to $q-2$. Suppose that there exists a $j \leq a$ such that $\alpha_{j}<q-1$ if $j<a$ or or that $\alpha_{j}<q-2$ if $j=a$.
a) If $b=0$ then $K=d+1=a(q-1)+1$. But $\sum_{i=1}^{a} \alpha_{i}<a(q-1)-1$. Then $\alpha_{a+1}>0$. If $\alpha_{a+1}>\alpha_{j}$ by Lemma A.4 we can permute the two terms to obtain a sequence which have a lower or equal $P_{1}-P_{2}$. If $\alpha_{a+1} \leq \alpha_{j}$, by Lemma A.5 the sequence obtained by replacing $\alpha_{a+1}$ by $\alpha_{a+1}-1$ and $\alpha_{j}$ by $\alpha_{j}+1$ has a lower $P_{1}-P_{2}$. So we have proved that we can increase the value of $\alpha_{j}$ and obtain a lower or equal $P_{1}-P_{2}$.
b) If $b>0$ then $K=d+q-b=a(q-1)+q$. But $\sum_{i=1}^{a} \alpha_{i}<a(q-1)-1+$ and then $\sum_{i=1}^{a+1} \alpha_{i}<a(q-1)+q-2+$. Hence $\alpha_{a+2}>0$. With the same method than in the previous part 1) we prove that we can increase the value of $\alpha_{j}$ and obtain a lower or equal $P_{1}-P_{2}$.
A.3. The tail of a best sequence. We give here the form of the terms $\alpha_{i}$ for $i \geq a+1$ of a sequence $\alpha$ for which $P_{1}-P_{2}$ is minimum, assuming that the head is as in the previous subsection.

1) Let us suppose first that there exists a sequence $\alpha$ such that $\alpha_{a+1}<b$ and for which $P_{1}-P_{2}$ is minimum. We have seen in the previous subsection that we can suppose that the first $a$ terms are $q-1$. We know that $K=a(q-1)+q$. Then
$\sum_{i=a+2}^{n} \alpha_{i}=q-\alpha_{a+1}$ using Lemma A.5 we can pack the terms $\alpha_{i}$ for $i \geq a+2$ in such a way that
a) if $\alpha_{a+1}=0$ then $\alpha_{a+2}=q-1, \alpha_{a+3}=1$ and $\alpha_{i}=0$ for $i>a+3$;
b) if $b>\alpha_{a+1} \geq 1$ then $\alpha_{a+2}=q-\alpha_{a+1}$ and and $\alpha_{i}=0$ for $i>a+2$.
2) Let us suppose now that $\alpha_{a+1} \geq b$. We We have seen in the previous subsection that we can suppose that the $a-1$ first $a-1$ terms are $q-1$ and $\alpha_{a}=q-2$.
a) If $b=0$ then $K=a(q-1)+1$. Then by Lemma A.5 we can pack the terms $\alpha_{i}$ for $i \geq a+1$ in such a way that $\alpha_{a+1}=2$ and $\alpha_{i}=0$ for $i>a+1$.
b) If $b>0$ then $K=a(q-1)+q$. Then by Lemma A.5 we can pack the terms $\alpha_{i}$ for $i \geq a+1$ in such a way that $\alpha_{a+1}=q-1, \alpha_{a}+2=2$ and $\alpha_{i}=0$ for $i>a+2$. A.4. The minimum value of $P_{1}-P_{2}$.
3) Case $\mathrm{b}=0$. Then $K=a(q-1)+1$. The previous results give directly a sequence for which $P_{1}-P_{2}$ is minimum:

$$
\begin{gathered}
\alpha_{1}=\cdots=\alpha_{a-1}=q-1 \\
\alpha_{a}=q-2, \quad \alpha_{a+1}=2, \quad \alpha_{a+2}=\cdots \alpha_{n}=0
\end{gathered}
$$

For this sequence we have

$$
P_{1}=2(q-2) q^{n-a-1} \quad P_{2}=(q-2) q^{n-a-1}
$$

then the minimum value of $P_{1}-P_{2}$ is

$$
\mu=(q-2) q^{n-a-1}
$$

2) Case $\mathrm{b}=1$. Then $K=a(q-1)+q$ and $a \leq n-2$.
a) Let us test first the assumption $\alpha_{a+1}=0$. The previous results give directly a sequence reaching the minimum of $P_{1}-P_{2}$ under this assumption:

$$
\begin{gathered}
\alpha_{1}=\cdots=\alpha_{a}=q-1 \\
\alpha_{a+1}=0, \quad \alpha_{a+2}=q-1, \quad \alpha_{a+3}=1, \quad \alpha_{a+4}=\cdots \alpha_{n}=0
\end{gathered}
$$

We remark that if $a=n-2$ this case cannot occur because there is not enough room to contain all the $\alpha_{i}$. For this sequence we have

$$
P_{1}=q(q-1) q^{n-a-3}, \quad P_{2}=(q-1)(q-1) q^{n-a-3}
$$

so that the minimum of $P_{1}-P_{2}$ under this assumption is

$$
\mu_{1}=(q-1) q^{n-a-3}
$$

b) Now let us test the assumption $\alpha_{a+1} \geq b=1$. The previous results give directly a sequence reaching the minimum of $P_{1}-P_{2}$ under this assumption:

$$
\begin{gathered}
\alpha_{1}=\cdots=\alpha_{a-1}=q-1 \\
\alpha_{a}=q-2, \quad \alpha_{a+1}=q-1, \quad \alpha_{a+2}=2, \quad \alpha_{a+3}=\cdots \alpha_{n}=0
\end{gathered}
$$

For this sequence we have

$$
P_{1}=2(q-2) q^{n-a-2}, \quad P_{2}=(q-2) q^{n-a-2}
$$

so that the minimum of $P_{1}-P_{2}$ under this assumption is

$$
\mu_{2}=(q-2) q^{n-a-2}
$$

c) Conclusion on the case $b=1$. Let us compare $\mu_{1}$ and $\mu_{2}($ when $a<n-2)$ ):

$$
\begin{gathered}
\mu_{2}-\mu_{1}=q^{n-a-1}-2 q^{n-a-2}-q^{n-a-2}+q^{n-a-3} \\
\mu_{2}-\mu_{1}=q^{n-a-2}(q-3)+q^{n-a-3}
\end{gathered}
$$

But $q \geq 3$, then $\mu_{2}>\mu_{1}$. Hence the minimum value is $\mu_{1}$.
Let us summarize the obtained result in the case $b=1$ :

- if $a<n-2$ then $\mu=\mu_{1}=(q-1) q^{n-a-3}$;
- if $a=n-2$ then $\mu=\mu_{2}=(q-2) q^{n-a-2}=q-2$.

3) Case $2 \leq b<q-1$. Then $K=a(q-1)+q$ and $a \leq n-2$.
a) Test of the assumption $\alpha_{a+1}<b$.
$\alpha)$ Test of the joint assumption $\alpha_{a+1}=0$. The previous results give directly a sequence reaching the minimum of $P_{1}-P_{2}$ under this assumption:

$$
\begin{gathered}
\alpha_{1}=\cdots=\alpha_{a}=q-1 \\
\alpha_{a+1}=0, \quad \alpha_{a+2}=q-1, \quad \alpha_{a+3}=1, \quad \alpha_{a+4}=\cdots \alpha_{n}=0
\end{gathered}
$$

This case cannot occur if $a=n-2$. For this sequence we have

$$
P_{1}=q(q-1) q^{n-a-3}, \quad P_{2}=(q-b)(q-1) q^{n-a-3}
$$

Then the minimum reached by $P_{1}-P_{2}$ under this assumption is

$$
\mu_{1}=b(q-1) q^{n-a-3}
$$

$\beta$ ) Test of the joint assumption $\alpha_{a+1} \neq 0$. The previous results shows that a sequence reaching the minimum of $P_{1}-P_{2}$ under these assumptions is of the form

$$
\begin{gathered}
\alpha_{1}=\cdots=\alpha_{a}=q-1 \\
\alpha_{a+1}>0, \quad \alpha_{a+2}=q-\alpha_{a+1}, \quad \alpha_{a+3}=\cdots \alpha_{n}=0
\end{gathered}
$$

For this sequence we have

$$
P_{1}=\left(q-\alpha_{a+1}\right) \alpha_{a+1} q^{n-a-2}, \quad P_{2}=(q-b) \alpha_{a+1} q^{n-a-2}
$$

then

$$
P_{1}-P_{2}=\left(b-\alpha_{a+1}\right) \alpha_{a+1} q^{n-a-2} .
$$

The minimum of the quadratic polynomial $\left(b-\alpha_{a+1}\right) \alpha_{a+1}$ (with $1 \leq \alpha_{a+1}<b \leq$ $q-2)$ is reached for $\alpha_{a+1}=1$ which gives for minimum of $P_{1}-P_{2}$

$$
\mu_{2}=(b-1) q^{n-a-2}
$$

b) Test of the assumption $\alpha_{a+1} \geq b$. The previous results shows that a sequence reaching the minimum of $P_{1}-P_{2}$ under this assumption is

$$
\begin{gathered}
\alpha_{1}=\cdots=\alpha_{a-1}=q-1 \\
\alpha_{a}=q-2, \quad \alpha_{a+1}=q-1, \quad \alpha_{a+2}=2, \quad \alpha_{a+3}=\cdots \alpha_{n}=0
\end{gathered}
$$

For this sequence we have

$$
P_{1}=2(q-2) q^{n-a-2}, \quad P_{2}=(q-2) q^{n-a-2}
$$

so that the minimum of $P_{1}-P_{2}$ under this assumption is

$$
\mu_{3}=(q-2) q^{n-a-2}
$$

c) Conclusion of the case $2 \leq b<q-1$. The minimum of $P_{1}-P_{2}$ is

$$
\mu=\min \left(\mu_{1}, \mu_{2}, \mu_{3}\right)=\mu_{2}=(b-1) q^{n-a-2}
$$

Indeed, as $q-1>b \geq 2$, we have $q-2>b-1>0$, which prove that $\mu_{3}>\mu_{2}$. To prove that $\mu_{1}>\mu_{2}$ let us compute

$$
\mu_{1}-\mu_{2}=b(q-1) q^{n-a-3}-(b-1) q^{n-a-2}=q^{n-a-2}-b q^{n-a-3}
$$

But $b<q$, then $\mu_{1}-\mu_{2}>0$.

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