# A Sequence Construction of Cyclic Codes over Finite Fields 

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#### Abstract

Due to their efficient encoding and decoding algorithms, cyclic codes, a subclass of linear codes, have applications in communication systems, consumer electronics, and data storage systems. There are several approaches to constructing all cyclic codes over finite fields, including the generator matrix approach, the generator polynomial approach, and the generating idempotent approach. Another one is a sequence approach, which has been intensively investigated in the past decade. The objective of this paper is to survey the progress in this direction in the past decade. Many open problems are also presented in this paper.


Keywords Dickson polynomial • cyclic code $\cdot$ linear code $\cdot$ planar function $\cdot$ sequence

## 1 Introduction

Let $q$ be a power of a prime $p$. An $[n, k, d]$ code over $\operatorname{GF}(q)$ is a $k$-dimensional subspace of $\mathrm{GF}(q)^{n}$ with minimum (Hamming) nonzero weight $d$. Let $A_{i}$ denote the number of codewords with Hamming weight $i$ in a linear code $\mathcal{C}$ of length $n$. The weight enumerator of $\mathcal{C}$ is defined by

$$
1+A_{1} z+A_{2} z^{2}+\cdots+A_{n} z^{n}
$$

The weight distribution of $\mathcal{C}$ is the sequence $\left(1, A_{1}, \ldots, A_{n}\right)$.
An $[n, k, d]$ code over $\mathrm{GF}(q)$ is called optimal if there is no $[n, k, d+1]$ or $[n, k+1, d]$ code over $\operatorname{GF}(q)$. The optimality of a cyclic code may be proved by a bound on linear codes or by an exhaustive computer search on all linear codes over $\mathrm{GF}(q)$ with fixed length $n$ and fixed dimension $k$ or fixed length $n$ and fixed minimum distance $d$. An $[n, k, d]$ code is said to be almost optimal if a linear code with parameters $[n, k+1, d]$ or $[n, k, d+1]$ is optimal.

A vector $\left(c_{0}, c_{1}, \cdots, c_{n-1}\right) \in \mathrm{GF}(q)^{n}$ is said to be even-like if $\sum_{i=0}^{n-1} c_{i}=0$, and is odd-like otherwise. The even-like subcode of a linear code consists of all the even-like codewords of this linear code.
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An $[n, k]$ code is called cyclic if $\left(c_{0}, c_{1}, \cdots, c_{n-1}\right) \in \mathcal{C}$ implies $\left(c_{n-1}, c_{0}, c_{1}, \cdots, c_{n-2}\right) \in \mathcal{C}$. Let $\operatorname{gcd}(n, q)=1$. By identifying a vector $\left(c_{0}, c_{1}, \cdots, c_{n-1}\right) \in \operatorname{GF}(q)^{n}$ with

$$
c_{0}+c_{1} x+c_{2} x^{2}+\cdots+c_{n-1} x^{n-1} \in \mathrm{GF}(q)[x] /\left(x^{n}-1\right)
$$

a code $\mathcal{C}$ of length $n$ over $\operatorname{GF}(q)$ corresponds to a subset of $\operatorname{GF}(q)[x] /\left(x^{n}-1\right)$. The linear code $C$ is cyclic if and only if the corresponding subset in $\operatorname{GF}(q)[x] /\left(x^{n}-1\right)$ is an ideal of the ring $\operatorname{GF}(q)[x] /\left(x^{n}-1\right)$. It is well known that every ideal of $\operatorname{GF}(q)[x] /\left(x^{n}-1\right)$ is principal. Let $\mathcal{C}=(g(x))$ be a cyclic code, where $g$ is monic and has the least degree. Then $g(x)$ is called the generator polynomial and $h(x)=\left(x^{n}-1\right) / g(x)$ is referred to as the check polynomial of $\mathcal{C}$. The dual code, denoted by $\mathcal{C}^{\perp}$, of $\mathcal{C}$ has generator polynomial $\bar{h}(x)$, which is the reciprocal of $h(x)$. The complement code, denoted by $\mathcal{C}^{c}$, is generated by $h(x)$. It is known that $\mathcal{C}^{\perp}$ and $\mathcal{C}^{c}$ have the same weight distribution.

The error correcting capability of cyclic codes may not be as good as some other linear codes in general. However, cyclic codes have wide applications in storage and communication systems because they have efficient encoding and decoding algorithms [5],17,|22].

Cyclic codes have been studied for decades and a lot of progress has been made (see, for example, [4] 19] for information). The total number of cyclic codes over $\operatorname{GF}(q)$ and their constructions are closely related to cyclotomic cosets modulo $n$, and thus many areas of number theory. One way of constructing cyclic codes over $\mathrm{GF}(q)$ with length $n$ is to use the generator polynomial

$$
\begin{equation*}
\frac{x^{n}-1}{\operatorname{gcd}\left(S(x), x^{n}-1\right)} \tag{1}
\end{equation*}
$$

where

$$
S(x)=\sum_{i=0}^{n-1} s_{i} x^{i} \in \mathrm{GF}(q)[x]
$$

and $s^{\infty}=\left(s_{i}\right)_{i=0}^{\infty}$ is a sequence of period $n$ over $\mathrm{GF}(q)$. Throughout this paper, we call the cyclic code $\mathcal{C}_{s}$ with the generator polynomial of (1) the code defined by the sequence $s^{\infty}$, and the sequence $s^{\infty}$ the defining sequence of the cyclic code $\mathcal{C}_{s}$.

It can be seen that every cyclic code of length $n$ over $\mathrm{GF}(q)$ can be expressed as $\mathcal{C}_{s}$ for some sequence $s^{\infty}$ of period $n$ over $\operatorname{GF}(q)$. Because of this, this construction of cyclic codes is said to be fundamental. An impressive progress in the construction of cyclic codes with this approach has been made in the past decade (see, for example, [8, 9, 10, 11, 16, 23||24]).

The objective of this paper is to give a survey of recent development in this sequence construction of cyclic codes over finite fields. In view that this topic is huge, we have to do a selective survey. Our idea is that this survey paper complements the monograph [11], so that the two references together could give a well rounded treatment of the sequence construction of cyclic codes over finite fields. It is hoped that this survey could stimulate further investigation into this sequence approach.

## 2 Preliminaries

In this section, we present basic notation and results of $q$-cyclotomic cosets modulo $n$, planar and almost perfect nonlinear functions, and sequences that will be employed in subsequent sections.
2.1 Some notation and symbols fixed throughout this paper

Throughout this paper, we adopt the following notation unless otherwise stated:

- $p$ is a prime, $q$ is a positive power of $p$, and $m$ is a positive integer.
- $\mathbb{Z}_{n}=\{0,1, \cdots, n-1\}$, the ring of integers modulo $n$.
- $\mathbb{N}_{q}(x)$ is a function defined by $\mathbb{N}_{q}(x)=0$ if $x \equiv 0(\bmod q)$ and $\mathbb{N}_{q}(x)=1$ otherwise.
- $\alpha$ is a generator of $\operatorname{GF}\left(q^{m}\right)^{*}$, the multiplicative group of $\operatorname{GF}(q)$.
- $m_{a}(x)$ is the minimal polynomial of $a \in \mathrm{GF}\left(q^{m}\right)$ over $\mathrm{GF}(q)$.
- $\operatorname{Tr}(x)$ is the trace function from $\operatorname{GF}\left(q^{m}\right)$ to $\operatorname{GF}(q)$.
- $\delta(x)$ is a function on $\operatorname{GF}\left(q^{m}\right)$ defined by $\delta(x)=0$ if $\operatorname{Tr}(x)=0$ and $\delta(x)=1$ otherwise.
- $C_{i}$ denotes the $q$-cyclotomic coset modulo $n$ containing $i$.
- $\Gamma$ is the set of all coset leaders of the $q$-cyclotomic cosets modulo $n$.
- For any polynomial $g(x) \in \operatorname{GF}(q)[x]$ with $g(0) \neq 0, \bar{g}(x)$ denotes the reciprocal of $g(x)$.
- For a cyclic code $\mathcal{C}$ of length $n$ over $\operatorname{GF}(q)$ with generator polynomial $g(x), \mathcal{C}^{c}$ denotes its complement code that is generated by $h(x):=\left(x^{n}-1\right) / g(x)$, and $\mathcal{C}^{\perp}$ denotes its dual code with generator polynomial $\bar{h}(x)$,


### 2.2 Planar and APN polynomials

A function $f: \mathrm{GF}\left(q^{m}\right) \rightarrow \mathrm{GF}\left(q^{m}\right)$ is called almost perfect nonlinear (APN) if

$$
\max _{a \in \mathrm{GF}\left(q^{m}\right)^{*} b \in \operatorname{GF}\left(q^{m}\right)} \max \left|\left\{x \in \mathrm{GF}\left(q^{m}\right): f(x+a)-f(x)=b\right\}\right|=2,
$$

and is referred to as perfect nonlinear or planar if

$$
\max _{a \in \operatorname{GF}\left(q^{m}\right)^{*}} \max _{b \in \operatorname{GF}\left(q^{m}\right)}\left|\left\{x \in \mathrm{GF}\left(q^{m}\right): f(x+a)-f(x)=b\right\}\right|=1 .
$$

There is no perfect nonlinear (planar) function on $\operatorname{GF}\left(q^{m}\right)$ for even $q$. However, there are APN functions on $\operatorname{GF}\left(2^{m}\right)$. Both planar and APN functions over $\mathrm{GF}\left(q^{m}\right)$ for odd $q$ exist. Some planar and APN monomials will be employed to construct cyclic codes in subsequent sections.

### 2.3 The $q$-cyclotomic cosets modulo $n$

Let $\operatorname{gcd}(n, q)=1$. The $q$-cyclotomic coset containing $j$ modulo $n$ is defined by

$$
C_{j}=\left\{j, q j, q^{2} j, \cdots, q^{\ell_{j}-1} j\right\} \bmod n \subset \mathbb{Z}_{n}
$$

where $\ell_{j}$ is the smallest positive integer such that $q^{\ell_{j}} j \equiv j(\bmod n)$, and is called the size of $C_{j}$. It is known that $\ell_{j}$ divides $m$. The smallest integer in $C_{j}$ is called the coset leader of $C_{j}$. Let $\Gamma$ denote the set of all coset leaders. By definition, we have

$$
\bigcup_{j \in \Gamma} C_{j}=\mathbb{Z}_{n}
$$

It is well known that $\prod_{j \in C_{i}}\left(x-\alpha^{j}\right)$ is an irreducible polynomial of degree $\ell_{i}$ over $\operatorname{GF}(q)$ and is the minimal polynomial of $\alpha^{i}$ over $\operatorname{GF}(q)$. Furthermore, the canonical factorization of $x^{n}-1$ over $\operatorname{GF}(q)$ is given by

$$
x^{n}-1=\prod_{i \in \Gamma} \prod_{j \in C_{i}}\left(x-\alpha^{j}\right) .
$$

2.4 The linear span and minimal polynomial of sequences

Let $s^{L}=s_{0} s_{1} \cdots s_{L-1}$ be a sequence over $\operatorname{GF}(q)$. The linear span (also called linear complexity) of $s^{L}$ is defined to be the smallest positive integer $\ell$ such that there are constants $c_{0}=1, c_{1}, \cdots, c_{\ell} \in \mathrm{GF}(q)$ satisfying

$$
-c_{0} s_{i}=c_{1} s_{i-1}+c_{2} s_{i-2}+\cdots+c_{l} s_{i-\ell} \text { for all } \ell \leq i<L .
$$

In engineering terms, such a polynomial $c(x)=c_{0}+c_{1} x+\cdots+c_{l} x^{l}$ is called the feedback polynomial of a shortest linear feedback shift register (LFSR) that generates $s^{L}$. Such an integer always exists for finite sequences $s^{L}$. When $L$ is $\infty$, a sequence $s^{\infty}$ is called a semiinfinite sequence. If there is no such an integer for a semi-infinite sequence $s^{\infty}$, its linear span is defined to be $\infty$. The linear span of the zero sequence is defined to be zero. For ultimately periodic semi-infinite sequences such an $\ell$ always exists.

Let $s^{\infty}$ be a sequence of period $L$ over $\operatorname{GF}(q)$. Any feedback polynomial of $s^{\infty}$ is called a characteristic polynomial. The characteristic polynomial with the smallest degree is called the minimal polynomial of the periodic sequence $s^{\infty}$. Since we require that the constant term of any characteristic polynomial be 1 , the minimal polynomial of any periodic sequence $s^{\infty}$ must be unique. In addition, any characteristic polynomial must be a multiple of the minimal polynomial.

For periodic sequences, there are two ways to determine their linear span and minimal polynomials. One of them is given in (11) (see [14) Theorem 5.3] for a proof). The other one is described in [1]

## 3 Cyclic codes from combinatorial sequences

### 3.1 The classical cyclic code $\mathcal{C}_{\mathrm{GF}(q)}(D)$ of a subset $D \in \mathbb{Z}_{n}$

Let $n$ be a positive integer, and let $D$ be a subset of $\mathbb{Z}_{n}$. Define $B_{i}=D+i$ for all $i \in \mathbb{Z}_{n}$. Then the pair $\left(\mathbb{Z}_{n}, \mathcal{B}\right)$ is called an incidence structure, where $\mathcal{B}=\left\{B_{0}, B_{1}, \cdots, B_{n-1}\right\}$. The incidence matrix $M_{D}=(i, j)$ of this incidence structure is an $n \times n$ matrix, where $m_{i j}=1$ if $j \in B_{i}$ and $m_{i j}=0$ otherwise. By definition, $M_{D}=(i, j)$ is a binary matrix. When $M_{D}=(i, j)$ is viewed as a matrix over $\mathrm{GF}(q)$, its row vectors span a cyclic code of length $n$ over $\mathrm{GF}(q)$, which is denoted by $\mathcal{C}_{\mathrm{GF}(q)}(D)$ and called the classical code of $D$. It is easily seen that the generator polynomial of $\mathcal{C}_{\mathrm{GF}(q)}(D)$ is given by

$$
\begin{equation*}
\operatorname{gcd}\left(x^{n}-1, \sum_{i \in D} x^{i}\right) \tag{2}
\end{equation*}
$$

where the greatest common divisor is computed over $\operatorname{GF}(q)$.
When $D$ has certain combinatorial structures, the cyclic code $\mathcal{C}_{\mathrm{GF}(q)}(D)$ has been well studied in the literature ([2], [11]). This code is closely related to the code dealt with in the next section.
3.2 The cyclic code of the characteristic sequence of a subset $D \in \mathbb{Z}_{n}$

Let $D$ be a subset of $\mathbb{Z}_{n}$. The characteristic sequence $s(D)^{\infty}$ of $D$ is given by

$$
s(D)_{i}= \begin{cases}1 & \text { if } i \bmod n \in D \\ 0 & \text { otherwise }\end{cases}
$$

The binary sequence $s(D)^{\infty}$ can be viewed as a sequence of period $n$ over any field $\mathrm{GF}(q)$, and can be employed to construct the code $\mathcal{C}_{s(D)}$ over $\mathrm{GF}(q)$. For any given pair of $n$ and $q$ with $\operatorname{gcd}(q, n)=1$, the subset $D$ must be chosen properly, in order to construct a cyclic code $\mathcal{C}_{s(D)}$ with desirable parameters. Intuitively, a good choice may be to select a subset $D$ of $\mathbb{Z}_{n}$ with certain combinatorial structures. It follows from the discussions in Section 3.1 that

$$
\begin{equation*}
\mathcal{C}_{s(D)}=\mathcal{C}_{\mathrm{GF}(q)}(D)^{c} . \tag{3}
\end{equation*}
$$

Hence, in the case that the sequence $s^{\infty}$ over $\mathrm{GF}(q)$ has only entries 0 and 1 , the sequence code $\mathcal{C}_{s(D)}$ is the complement code of the classical code of its support set. This is a collection between the classical construction of cyclic codes with incidence structures and the sequence construction of this paper in the special case. However, the two approaches do not include each other.

Let $D$ be a $\kappa$-subset of $\mathbb{Z}_{n}$. The set $D$ is an $(n, \kappa, \lambda)$ difference set in $\left(\mathbb{Z}_{n},+\right)$ if the multiset

$$
\{x-y \mid x, y \in D\}
$$

contains every nonzero element of $\mathbb{Z}_{n}$ exactly $\lambda$ times.
Let $D$ be a $\kappa$-subset of $\mathbb{Z}_{n}$. The set $D$ is an ( $n, \kappa, \lambda, t$ ) almost difference set (ADS) in $\left(\mathbb{Z}_{n},+\right)$ if the multiset

$$
\{x-y \mid x, y \in D\}
$$

contains $t$ nonzero elements of $\mathbb{Z}_{n}$ exactly $\lambda$ times each and the remaining $n-1-t$ nonzero elements $\lambda+1$ times each.

Example 1 The Singer difference set in $\left(\mathbb{Z}_{n},+\right)$ is given by $D=\log _{\alpha}\left\{x \in \operatorname{GF}\left(2^{m}\right): \operatorname{Tr}(x)=\right.$ $1\} \subset \mathbb{Z}_{n}$, and has parameters $\left(2^{m}-1,2^{m-1}, 2^{m-2}\right)$, where $\alpha$ is generator of $\operatorname{GF}\left(2^{m}\right)^{*}$ and $n=2^{m}-1$.

Its characteristic sequence $\left(s_{t}\right)_{t=0}^{\infty}$, where $s_{t}=\operatorname{Tr}\left(\alpha^{t}\right)$ for any $t \geq 0$, a maximum-length sequence of period $2^{m}-1$. The minimal polynomial of the Singer sequence is equal to the minimal polynomial $m_{\alpha^{-1}}(x)$ of $\alpha^{-1}$ over $\mathrm{GF}(2)$, and its linear span is $m$. The cyclic code $\mathcal{C}_{s(D)}$ defined by the characteristic sequence of the Singer sequence is equivalent to the Hamming code with parameters $\left[2^{m}-1,2^{m}-1-m, 3\right]$ and has generator polynomial $m_{\alpha^{-1}}(x)$. The code is optimal (perfect).

A proof of the following results can be found in [11][p. 193].
Example 2 Let $q=p^{s}$ be a prime power, where $p$ is a prime, and $s$ is a positive integer, and let $m \geq 3$ be a positive integer. Let $\alpha$ be a generator of $\mathrm{GF}\left(q^{m}\right)^{*}$. Put $n=\left(q^{m}-1\right) /(q-1)$. Recall that

$$
D=\left\{0 \leq i<n: \operatorname{Tr}_{q^{m} / q}\left(\alpha^{i}\right)=0\right\} \subset \mathbb{Z}_{n}
$$

is the Singer difference set in $\left(\mathbb{Z}_{n},+\right)$ with parameters

$$
\left(\frac{q^{m}-1}{q-1}, \frac{q^{m-1}-1}{q-1}, \frac{q^{m-2}-1}{q-1}\right) .
$$

Let $s(D)^{\infty}$ be the characteristic sequence of $D$. Then the cyclic code $\mathcal{C}_{s(D)}^{c}$ has parameters

$$
\begin{equation*}
\left[\frac{q^{m}-1}{q-1},\binom{p+m-2}{m-1}^{s}+1, \frac{q^{m-1}-1}{q-1}\right] . \tag{4}
\end{equation*}
$$

Although the parameters of $\mathcal{C}_{s(D)}^{c}$ are known, the following problem is still open.
Open Problem 1 Determine the minimum distance of the code $\mathcal{C}_{s(D)}$ from the characteristic sequence of the Singer difference set $D$.

There are many families of difference sets and almost difference sets $D$ in $\left(\mathbb{Z}_{n},+\right)$. In many cases, the dimension and generator polynomial of the classical code $\mathcal{C}_{\mathrm{GF}(q)}(D)$ or its complement $\mathcal{C}_{\mathrm{GF}(q)}(D)^{c}$ (hence, $\left.\mathcal{C}_{\mathrm{GF}(q)}(D)^{\perp}\right)$ are known. However, their minimum distances are open in general. Because of the relation in (3), the dimension and generator polynomial of the sequence code $\mathcal{C}_{s(D)}$ are known in many cases, but its minimum distance is known only in some special cases. As this is a huge topic with a lot of results, it would be infeasible to survey the developments here. Thus, we refer the reader to the monograph [11] for detailed information.

Cyclotomic classes were employed to define binary sequences, which can be viewed as sequences over $\mathrm{GF}(q)$ for any prime power $q$. Such sequences give cyclic codes $\mathcal{C}_{s(D)}$ over $\mathrm{GF}(q)$. The reader is referred to [8 10, 11] for detailed information.

## 4 Cyclic codes from a construction of sequences from polynomials over $\mathrm{GF}\left(q^{m}\right)$

Given a polynomial $f(x)$ on $\mathrm{GF}\left(q^{m}\right)$, we define its associated sequence $s^{\infty}$ by

$$
\begin{equation*}
s_{i}=\operatorname{Tr}\left(f\left(\alpha^{i}+1\right)\right) \tag{5}
\end{equation*}
$$

for all $i \geq 0$, where $\alpha$ is a generator of $\operatorname{GF}\left(q^{m}\right)^{*}$ and $\operatorname{Tr}(x)$ denotes the trace function from $\mathrm{GF}\left(q^{m}\right)$ to $\mathrm{GF}(q)$. The code $C_{s}$ defined by the sequence $s^{\infty}$ in (5) is called the code from the polynomial $f(x)$ for simplicity.

It was demonstrated in [10, 16 23] that the code $\mathcal{C}_{s}$ may have interesting parameters if the polynomial $f$ is properly chosen. The objective of this section is to survey cyclic codes $\mathcal{C}_{s}$ defined by special polynomials $f$ over $\operatorname{GF}\left(q^{m}\right)$.

### 4.1 Cyclic codes from special monomials

The following is a list of monomials over $\operatorname{GF}\left(q^{m}\right)$ with good nonlinearity (see [11] Section 1.7] for definition and details).

- $f(x)=x^{q^{m}-2}$ over GF( $q^{m}$ (APN).
- $f(x)=x^{q^{k}+1}$ over $\mathrm{GF}\left(q^{m}\right)$, where $m / \operatorname{gcd}(m, k)$ and $q$ are odd (planar).
- $f(x)=x^{\left(q^{h}-1\right) /(q-1)}$ over $\mathrm{GF}\left(q^{m}\right)$.
- $f(x)=x^{\left(3^{h}+1\right) / 2}$ over $\operatorname{GF}\left(3^{m}\right)$ (planar when $\left.\operatorname{gcd}(h, m)=1\right)$.
- $f(x)=x^{2^{t}+3}$ over GF $\left(2^{m}\right)$ (APN).
- $f(x)=x^{e}$ over $\mathrm{GF}\left(2^{m}\right), e=2^{(m-1) / 2}+2^{(m-1) / 4}-1$ and $m \equiv 1(\bmod 4)(\mathrm{APN})$.

When they are plugged into (5], sequences over $\operatorname{GF}(q)$ with certain properties are obtained. The corresponding sequence codes have interesting parameters. The objective of this section is to introduce the parameters of these cyclic codes.
4.1.1 Binary cyclic codes from $f(x)=x^{2^{t}+3}$

The monomial $f(x)=x^{2^{t}+3}$ is APN over $\mathrm{GF}\left(2^{2 t+1}\right)$. Both the sequence in (5) defined by this monomial and the code $\mathcal{C}_{s}$ are interesting.

Theorem 1 [16] Let $m=2 t+1 \geq 7$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=x^{2^{t}+3}$. Then the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is equal to $5 m+1$ and the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\begin{equation*}
\mathbb{M}_{s}(x)=(x-1) m_{\alpha^{-1}}(x) m_{\alpha^{-3}}(x) m_{\alpha^{-\left(2^{t}+1\right)}}(x) m_{\alpha^{-\left(2^{t}+2\right)}}(x) m_{\alpha^{-\left(2^{t}+3\right)}}(x) . \tag{6}
\end{equation*}
$$

The binary code $\mathcal{C}_{s}$ has parameters $\left[2^{m}-1,2^{m}-2-5 m, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$ of (6), where $d \geq 8$.

The code $\mathcal{C}_{s}$ in Theorem 1 could be optimal in some cases [16]. It would be interesting to settle the following problem.

Open Problem 2 Determine the minimum distance of the code $\mathcal{C}_{s}$ in Theorem $\square$
4.1.2 Binary cyclic codes from $f(x)=x^{2^{h}-1}$

Consider the monomial $f(x)=x^{2^{h}-1}$ over $\mathrm{GF}\left(2^{m}\right)$, where $h$ is a positive integer with $1 \leq$ $h \leq\left\lceil\frac{m}{2}\right\rceil$. As will be demonstrated below, it gives a binary sequence and binary code with special parameters.

Theorem 2 [16] Let $s^{\infty}$ be the sequence of (5), where $f(x)=x^{2^{h}-1}, 2 \leq h \leq\left\lceil\frac{m}{2}\right\rceil$. Then the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}=\left\{\begin{array}{l}
\frac{m\left(2^{h}+(-1)^{h-1}\right)}{3}, \text { if } m \text { is even }  \tag{7}\\
\frac{m\left(2^{h}+(-1)^{h-1}\right)+3}{3}, \text { if } m \text { is odd. } .
\end{array}\right.
$$

The minimal polynomial

$$
\begin{equation*}
\mathbb{M}_{s}(x)=(x-1)^{\mathbb{N}_{2}(m)} \prod_{\substack{1 \leq 2 j+1 \leq l^{h-1} \\ \text { and } \\ k_{2 j+1}=1}} m_{\alpha^{-(2 j+1)}}(x), \tag{8}
\end{equation*}
$$

$\mathbb{N}_{2}(i)=0$ if $i \equiv 0(\bmod 2)$ and $\mathbb{N}_{2}(i)=1$ otherwise.
Let $h \geq 2$. Then the binary code $\mathcal{C}_{s}$ has parameters $\left[2^{m}-1,2^{m}-1-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where

$$
d \geq\left\{\begin{array}{l}
2^{h-2}+2 \text { if } m \text { is odd and } h>2 \\
2^{h-2}+1
\end{array}\right.
$$

The code $\mathcal{C}_{s}$ in Theorem 2 could be optimal in some cases [16]. The lower bounds on $d$ given in Theorem 2 are quite tight. Nevertheless, it would be nice if the following problem could be solved.

Open Problem 3 Determine the minimum distance of the code $C_{s}$ in Theorem 2
4.1.3 Binary cyclic codes from $f(x)=x^{e}, e=2^{(m-1) / 2}+2^{(m-1) / 4}-1$ and $m \equiv 1(\bmod 4)$

Let $f(x)=x^{e}$, where $e=2^{(m-1) / 2}+2^{(m-1) / 4}-1$ and $m \equiv 1(\bmod 4)$. It is known that $f(x)$ is a permutation of $\operatorname{GF}\left(2^{m}\right)$ and is APN.

Let $t$ be a positive integer. We define $T=2^{t}-1$. For any odd $a \in\{1,2,3, \cdots, T\}$, define

$$
\varepsilon_{a}^{(t)}= \begin{cases}1, & \text { if } a=2^{h}-1, \\ \left\lceil\log _{2} \frac{T}{a}\right\rceil \bmod 2, & \text { if } 1 \leq a<2^{h}-1\end{cases}
$$

and

$$
\begin{equation*}
\kappa_{a}^{(t)}=\varepsilon_{a}^{(t)} \bmod 2 \tag{9}
\end{equation*}
$$

Properties of the binary sequence and binary code defined by $f(x)=x^{e}$ are documented in the following theorem.

Theorem 3 [16] Let $m \geq 9$ be odd. Let $s^{\infty}$ be the sequence of (5). Then the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}=\left\{\begin{array}{l}
\frac{m\left(2^{(m+7) / 4}+(-1)^{(m-5) / 4}\right)+3}{3}, \text { if } m \equiv 1 \quad(\bmod 8),  \tag{10}\\
\frac{m\left(2^{(m+7) / 4}+(-1)^{(m-5) / 4}-6\right)+3}{3}, \text { if } m \equiv 5 \quad(\bmod 8) .
\end{array}\right.
$$

The minimal polynomial

$$
\mathbb{M}_{s}(x)=(x-1) \prod_{i=0}^{2 \frac{m-1}{4}-1} m_{\alpha^{-i-2} \frac{m-1}{2}}(x) \prod_{\substack{1 \leq 2 j 1 \leq 2 \\ \text { an-1 } \\ k_{2 j+1}^{(m-1 / 4)}=1}} m_{\alpha^{-2 j-1}}(x)
$$

if $m \equiv 1(\bmod 8)$; and

$$
\mathbb{M}_{s}(x)=(x-1) \prod_{i=1}^{2 \frac{m-1}{4}-1} m_{\alpha^{-i-2} \frac{m-1}{2}}(x) \prod_{\substack{3 \leq 2 j 1 \leq 2 \\ \text { an-1 } \\ x_{2 j+1}^{(m-1 / 4)}=1}} m_{\alpha^{-2 j-1}}(x)
$$

if $m \equiv 5(\bmod 8)$, where $\kappa_{2 j+1}^{(h)}$ was defined in (9).
The binary code $\mathcal{C}_{s}$ has parameters $\left[2^{m}-1,2^{m}-1-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, and the minimum weight $d$ has the following bounds:

$$
d \geq \begin{cases}2^{(m-1) / 4}+2 & \text { if } m \equiv 1  \tag{11}\\ 2^{(m-1) / 4} & (\bmod 8), \\ \text { if }^{2} \equiv 5 & (\bmod 8) .\end{cases}
$$

The code $\mathcal{C}_{s}$ in Theorem 3] could be optimal in some cases [16]. The lower bounds on $d$ given in Theorem 2 are reasonably good. It would be interesting to work on the following problem.

Open Problem 4 Determine the minimum distance of the code $\mathcal{C}_{s}$ in Theorem 3 or improve the lower bounds in (11).
4.1.4 Binary cyclic codes from $f(x)=x^{2^{2 h}-2^{h}+1}$, where $\operatorname{gcd}(m, h)=1$

Define $f(x)=x^{e}$, where $e=2^{2 h}-2^{h}+1$ and $\operatorname{gcd}(m, h)=1$. It is known that $f$ is APN under these conditions. In this section, we restrict $h$ to the following range:

$$
1 \leq h \leq \begin{cases}\frac{m-1}{4} \text { if } m \equiv 1 & (\bmod 4)  \tag{12}\\ \frac{m-3}{4} \text { if } m \equiv 3 & (\bmod 4) \\ \frac{m-4}{4} \text { if } m \equiv 0 & (\bmod 4), \\ \frac{m-2}{4} \text { if } m \equiv 2 & (\bmod 4)\end{cases}
$$

Some parameters of the binary sequence and the code defined by $f(x)=x^{e}$ are given in the following theorem.

Theorem 4 [16] Let $h$ satisfy the conditions of (12]. Let $s^{\infty}$ be the sequence of (5). Then the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}=\left\{\begin{array}{l}
\frac{m\left(2^{(h+2}+(-1)^{h-1}\right)+3}{3} \text { if } h \text { is even }  \tag{13}\\
\frac{m\left(2^{h+2}+(-1)^{h-1}-6\right)+3}{3} \text { if } h \text { is odd. }
\end{array}\right.
$$

The minimal polynomial

$$
\mathbb{M}_{s}(x)=(x-1) \prod_{i=0}^{2^{h}-1} m_{\alpha^{-i-2^{m-h}}}(x) \prod_{\substack{1 \leq 2 j+1 \leq \leq^{h-1} \\ x_{2}^{h}+1^{-1}}} m_{\alpha^{-2 j-1}}(x)
$$

if $h$ is even; and

$$
\mathbb{M}_{s}(x)=(x-1) \prod_{i=1}^{2^{h}-1} m_{\alpha^{-i-22^{m-h}}}(x) \prod_{\substack{3 \leq 2 j+1 \leq l^{h}-1 \\ k_{2 j+1}=1}} m_{\alpha^{-2 j-1}}(x)
$$

if $h$ is odd, where $\kappa_{2 j+1}^{(h)}$ was defined in (9).
The code $\mathcal{C}_{s}$ has parameters $\left[2^{m}-1,2^{m}-1-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, and the minimum weight $d$ has the following bounds:

$$
d \geq\left\{\begin{array}{l}
2^{h}+2 \text { if } h \text { is even }  \tag{14}\\
2^{h} \quad \text { if } h \text { is odd } .
\end{array}\right.
$$

The code $\mathcal{C}_{s}$ in Theorem 4 could be optimal in some cases [16]. It would be interesting to attack the following two problems.

Open Problem 5 Determine the minimum distance of the code $C_{s}$ in Theorem 4
Open Problem 6 Determine the dimension and the minimum weight of the code $\mathcal{C}_{s}$ of this section when $h$ satisfies

$$
\begin{cases}\frac{m-1}{2} \geq h>\frac{m-1}{4} \text { if } m \equiv 1 \quad(\bmod 4),  \tag{15}\\ \frac{m-3}{2} \geq h>\frac{m-3}{4} \text { if } m \equiv 3 \quad(\bmod 4), \\ \frac{m-4}{2} \geq h>\frac{m-4}{4} \text { if } m \equiv 0 \quad(\bmod 4), \\ \frac{m^{2}-2}{2} \geq h>\frac{m-2}{4} \text { if } m \equiv 2 \quad(\bmod 4) .\end{cases}
$$

4.1.5 Binary cyclic codes from $f(x)=x^{2^{m}-2}$ over $\operatorname{GF}\left(2^{m}\right)$

Let $\rho_{i}$ denote the total number of even integers in the 2 -cyclotomic coset $C_{i}$ modulo $2^{m}-1$. We then define

$$
\begin{equation*}
v_{i}=\frac{m \rho_{i}}{\ell_{i}} \bmod 2 \tag{16}
\end{equation*}
$$

for each $i \in \Gamma$, where $\ell_{i}=\left|C_{i}\right|$ and $\Gamma$ denotes the set of coset leaders modulo $n=2^{m}-1$.
It is known that $f(x)=x^{2^{m}-2}$ over $\mathrm{GF}\left(2^{m}\right)$ is APN. For the binary sequence and code defined by this monomial, we have the following.

Theorem 5 [9] Let $s^{\infty}$ be the sequence of (5), where $f(x)=x^{2^{m}-2}$. Then the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is equal to $(n+1) / 2$ and the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\begin{equation*}
\mathbb{M}_{s}(x)=\prod_{j \in \Gamma, v_{j}=1} m_{\alpha^{-j}}(x) . \tag{17}
\end{equation*}
$$

The binary code $C_{s}$ has parameters $\left[2^{m}-1,2^{m-1}-1, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$. If $m$ is odd, the minimum distance $d$ of $\mathcal{C}_{s}$ is even and satisfies $d^{2}-d+1 \geq n$, and the dual code $\mathcal{C}_{s}^{\perp}$ has parameters $\left[2^{m}-1,2^{m-1}, d^{\perp}\right]$, where $d^{\perp}$ satisfies that $\left(d^{\perp}\right)^{2}-d^{\perp}+1 \geq n$.

When $f(x)=x^{q^{m}-2}$ and $q>2$, the dimension of the code $\mathcal{C}_{s}$ over $\mathrm{GF}(q)$ was settled in [23]. But no lower bound on the minimum distance of $C_{s}$ is developed.
4.1.6 Cyclic codes from $f(x)=x^{q^{\alpha}+1}$, where $m / \operatorname{gcd}(m, \kappa)$ and $q$ are odd

Let $f(x)=x^{q^{\kappa}+1}$, where $m / \operatorname{gcd}(m, \kappa)$ and $q$ are odd. It is known that $f$ is planar. Properties of the sequence and code defined by this monomial are described below.

Theorem 6 [9] Let $m$ be odd. Let $s^{\infty}$ be the sequence of (5), where $f(x)=x^{q^{k}+1}$. Then the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is equal to $2 m+\mathbb{N}_{p}(m)$ and the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\begin{equation*}
\mathbb{M}_{s}(x)=(x-1)^{\mathbb{N}_{p}(m)} m_{\alpha^{-1}}(x) m_{\alpha^{-\left(p^{\alpha+1}\right)}}(x), \tag{18}
\end{equation*}
$$

where $\mathbb{N}_{p}(i)=0$ if $i \equiv 0(\bmod p)$ and $\mathbb{N}_{p}(i)=1$ otherwise.
The code $\mathcal{C}_{s}$ has parameters $\left[n, n-2 m-\mathbb{N}_{p}(m), d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where

$$
\begin{cases}d=4 & \text { if } q=3 \text { and } m \equiv 0 \\ 4 \leq d \leq 5 & (\bmod p), \\ d=3 & \text { if } q>3 \text { and } m \neq 0 \\ \hline(\bmod p), \\ 3 \leq d \leq 4 & \text { if } q>3 \text { and } m \neq 0 \\ (\bmod p), \\ (\bmod p) .\end{cases}
$$

Extending the work of [25], one can determine the weight distribution of $\mathcal{C}_{s}^{\perp}$. With the MacWilliams identity, one can settle the minimum distance of the code $\mathcal{C}_{s}$ in Theorem6
4.1.7 Cyclic codes from $f(x)=x^{\left(q^{h}-1\right) /(q-1)}$

Let $h$ be a positive integer satistying the following condition:

$$
1 \leq h \leq\left\{\begin{array}{l}
(m-1) / 2 \text { if } m \text { is odd and }  \tag{19}\\
m / 2 \text { if } m \text { is even. }
\end{array}\right.
$$

Let $J \geq t \geq 2$, and let $\mathbb{N}(J, t)$ denote the total number of vectors $\left(i_{1}, i_{2}, \cdots, i_{t-1}\right)$ with $1 \leq i_{1}<i_{2}<\cdots<i_{t-1}<J$. By definition, we have the following recursive formula:

$$
\begin{equation*}
\mathbb{N}(J, t)=\sum_{j=t-1}^{J-1} \mathbb{N}(j, t-1) \tag{20}
\end{equation*}
$$

It is easily seen that

$$
\begin{equation*}
\mathbb{N}(J, 2)=J-1 \text { for all } J \geq 2 \tag{21}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbb{N}(J, 3)=\frac{(J-1)(J-2)}{2} \text { for all } J \geq 3 \tag{22}
\end{equation*}
$$

It then follows from (20), (21) and (22) that

$$
\begin{equation*}
\mathbb{N}(J, 4)=\sum_{j=3}^{J-1} \mathbb{N}(j, 3)=\sum_{j=3}^{J-1} \frac{(J-1)(J-2)}{2}=\frac{J^{3}-6 J^{2}+11 J-6}{6} \tag{23}
\end{equation*}
$$

By definition, we have

$$
\begin{equation*}
\mathbb{N}(t, t)=1 \text { for all } t \geq 2 \tag{24}
\end{equation*}
$$

For convenience, we define $\mathbb{N}(J, 1)=1$ for all $J \geq 1$.
Theorem 7 Let h satisfy the condition of (19). Let $s^{\infty}$ be the sequence of (5), where $f(x)=$ $x^{\left(q^{h}-1\right) /(q-1)}$. Then the linear span $\mathbb{L}_{s}$ and minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ are given by

$$
\mathbb{L}_{s}=\left(\mathbb{N}_{p}(h)+\sum_{t=1}^{h-1} \sum_{u=1}^{h-1} \mathbb{N}_{p}(h-u) \mathbb{N}(u, t)\right) m+\mathbb{N}_{p}(m)
$$

and

$$
\begin{aligned}
\mathbb{M}_{s}(x)= & (x-1)^{\mathbb{N}_{p}(m)} m_{\alpha^{-1}}(x)^{\mathbb{N}_{p}(h)} \prod_{\substack{1 \leq u \leq h-1 \\
\mathbb{N}_{p}(h-u)=1}} m_{\alpha^{-\left(q^{0}+q^{u}\right)}}(x) \times \\
& \prod_{t=2}^{h-1} \prod_{\substack{t \leq u \leq h-1 \\
\mathbb{N}_{p}(h-u)=1}} \prod_{1 \leq i_{1}<\cdots<i_{t-1}<u} m_{\alpha^{-\left(q^{0}+\sum_{j=1}^{t-1} q^{j}+q^{j}+q^{u}\right)}}(x) .
\end{aligned}
$$

The code $\mathcal{C}_{s}$ has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$.
Open Problem 7 Determine the minimum distance of the code $\mathcal{C}_{s}$ in Theorem 7 or develop a tight lower bound on it.

As a corollary of Theorem 7 we have the following.

Corollary 1 Let $h=3$. The code $\mathcal{C}_{s}$ of Theorem $\square$ has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$ given by

$$
\mathbb{M}_{s}(x)=(x-1)^{\mathbb{N}_{p}(m)} m_{\alpha^{-1}}(x) m_{\alpha^{-1-q}}(x) m_{\alpha^{-1-q^{2}}}(x) m_{\alpha^{-1-q-q^{2}}}(x)
$$

if $p \neq 3$, and

$$
\mathbb{M}_{s}(x)=(x-1)^{\mathbb{N}_{p}(m)} m_{\alpha^{-1-q}}(x) m_{\alpha^{-1-q^{2}}}(x) m_{\alpha^{-1-q-q^{2}}}(x)
$$

if $p=3$, where

$$
\mathbb{L}_{s}=\left\{\begin{array}{l}
4 m+\mathbb{N}_{p}(m) \text { if } p \neq 3,  \tag{25}\\
3 m+\mathbb{N}_{p}(m) \text { if } p=3 .
\end{array}\right.
$$

In addition,

$$
\left\{\begin{array}{l}
3 \leq d \leq 8 \text { if } p=3 \text { and } \mathbb{N}_{p}(m)=1, \\
3 \leq d \leq 6 \text { if } p=3 \text { and } \mathbb{N}_{p}(m)=0, \\
3 \leq d \leq 8 \text { if } p>3 .
\end{array}\right.
$$

Open Problem 8 For the code $\mathcal{C}_{s}$ of Corollary $\mathbb{1}$ do the following lower bounds hold?

$$
d \geq\left\{\begin{array}{l}
5 \text { when } p=3 \text { and } \mathbb{N}_{p}(m)=1, \\
4 \text { when } p=3 \text { and } \mathbb{N}_{p}(m)=0, \\
6 \text { when } p>3 \text { and } \mathbb{N}_{p}(m)=1, \\
5 \text { when } p>3 \text { and } \mathbb{N}_{p}(m)=0 .
\end{array}\right.
$$

4.1.8 Cyclic codes from $f(x)=x^{\left(3^{h}+1\right) / 2}$

Let $h$ be a positive integer satistying the following conditions:

$$
\left\{\begin{array}{l}
h \text { is odd, }  \tag{26}\\
\operatorname{gcd}(m, h)=1, \\
3 \leq h \leq\left\{\begin{array}{l}
(m-1) / 2 \text { if } m \text { is odd and } \\
m / 2 \text { if } m \text { is even. }
\end{array}\right.
\end{array}\right.
$$

Theorem 8 Let h satisfy the third condition of (26). Let $s^{\infty}$ be the sequence of (5), where $f(x)=x^{\left(3^{h}+1\right) / 2}$. Then the linear span $\mathbb{L}_{s}$ and minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ are given by

$$
\mathbb{L}_{s}=\mathbb{N}_{3}(m)+\left(\sum_{i=0}^{h} \mathbb{N}_{3}(h-i+1)\right) m+\left(\sum_{t=2}^{h} \mathbb{N}(h, t)+\sum_{t=2}^{h-1} \sum_{i_{t}=t}^{h-1} \mathbb{N}_{3}\left(h-i_{t}+1\right) \mathbb{N}\left(i_{t}, t\right)\right) m
$$

and

$$
\begin{aligned}
& \mathbb{M}_{s}(x)=(x-1)^{\mathbb{N}_{3}(m)} m_{\alpha^{-1}}(x)^{\mathbb{N}_{3}(h+1)} m_{\alpha^{-2}}(x) \prod_{t=1}^{h-1} \prod_{1 \leq i_{1}<\cdots<i_{i} \leq h-1} m_{\alpha^{-\left(2+\sum_{j=1}^{t}\right.}}(x) \times
\end{aligned}
$$

where $\mathbb{N}_{3}(j)$ and $\mathbb{N}(j, t)$ were defined in Sections 2.1 and 4.1 .7 respectively.
Furthermore, the code $\mathcal{C}_{s}$ has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$.

As shown in Theorem [8, the linear span and the minimal polynomial of the sequence $s^{\infty}$ has a complex formula. It looks difficult to discover further properties of the code in Theorem 8

Open Problem 9 Determine the minimum distance of the code $\mathcal{C}_{s}$ in Theorem 8 or develop a tight lower bound on it.

As a corollary of Theorem 8 , we have the following.
Corollary 2 Let $h=3$. The code $\mathcal{C}_{s}$ of Theorem 8 has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and the generator polynomial $\mathbb{M}_{s}(x)$ given by

$$
\mathbb{M}_{s}(x)=(x-1)^{\mathbb{N}_{3}(m)} m_{\alpha^{-1}}(x) m_{\alpha^{-2}}(x) m_{\alpha^{-5}}(x) m_{\alpha^{-10}}(x) m_{\alpha^{-11}}(x) m_{\alpha^{-13}}(x) m_{\alpha^{-14}}(x),
$$

where $\mathbb{L}_{s}=7 m+\mathbb{N}_{3}(m)$. In addition,

$$
\left\{\begin{array}{l}
5 \leq d \leq 16 \text { if } \mathbb{N}_{3}(m)=1, \\
4 \leq d \leq 16 \text { if } \mathbb{N}_{3}(m)=0 .
\end{array}\right.
$$

Open Problem 10 For the code $\mathcal{C}_{s}$ of Corollary do the following lower bounds hold?

$$
d \geq\left\{\begin{array}{l}
9 \text { when } \mathbb{N}_{p}(m)=1, \\
8 \text { when } \mathbb{N}_{p}(m)=0 .
\end{array}\right.
$$

### 4.2 Cyclic codes from Dickson polynomials

In this section, we survey known results on cyclic codes from Dickson polynomials over finite fields. All the results presented in this subsection come from [12].

### 4.2.1 Dickson polynomials over $\mathrm{GF}\left(q^{m}\right)$

In 1896, Dickson introduced the following family of polynomials over $\mathrm{GF}\left(q^{m}\right)$ [7]:

$$
\begin{equation*}
D_{h}(x, a)=\sum_{i=0}^{\left\lfloor\frac{h}{2}\right\rfloor} \frac{h}{h-i}\binom{h-i}{i}(-a)^{i} x^{h-2 i}, \tag{27}
\end{equation*}
$$

where $a \in \mathrm{GF}\left(q^{m}\right)$ and $h \geq 0$ is called the order of the polynomial. This family is referred to as the Dickson polynomials of the first kind.

Dickson polynomials of the second kind over $\operatorname{GF}\left(q^{m}\right)$ are defined by

$$
\begin{equation*}
E_{h}(x, a)=\sum_{i=0}^{\left\lfloor\frac{h}{2}\right\rfloor}\binom{h}{h-i}(-a)^{i} x^{h-2 i}, \tag{28}
\end{equation*}
$$

where $a \in \mathrm{GF}\left(q^{m}\right)$ and $h \geq 0$ is called the order of the polynomial.
Dickson polynomials are an interesting topic of mathematics and engineering, and have many applications. For example, the Dickson polynomials $D_{5}(x, a)=x^{5}-u x-u^{2} x$ over $\mathrm{GF}\left(3^{m}\right)$ are employed to construct a family of planar functions [6, 15], and those planar functions give two families of commutative presemifields, planes, several classes of linear codes [3],25], and two families of skew Hadamard difference sets [15]. The reader is referred to [21] for detailed information about Dickson polynomials. In subsequent subsections, we survey cyclic codes derived from Dickson polynomials.
4.2.2 Cyclic codes from the Dickson polynomial $D_{p^{u}}(x, a)$

Since $q$ is a power of $p$, it is known that $D_{h p}(x, a)=D_{h}(x, a)^{p}$ [21, Lemma 2.6]. It then follows that $D_{p^{u}}(x, a)=x^{p^{u}}$ for all $a \in \mathrm{GF}\left(q^{m}\right)$.

The code $\mathcal{C}_{s}$ over $\mathrm{GF}(q)$ defined by the Dickson polynomial $f(x)=D_{p^{u}}(x, a)=x^{p^{u}}$ over $\mathrm{GF}\left(q^{m}\right)$ has the following parameters.

Theorem 9 The code $\mathcal{C}_{s}$ defined by the Dickson polynomial $D_{p^{u}}(x, a)=x^{p^{u}}$ has parameters $[n, n-m-\delta(1), d]$ and generator polynomial $\mathbb{M}_{s}(x)=(x-1)^{\delta(1)} m_{\alpha^{-p}}(x)$, where

$$
d=\left\{\begin{array}{l}
4 \text { if } q=2 \text { and } \delta(1)=1, \\
3 \text { if } q=2 \text { and } \delta(1)=0, \\
3 \text { if } q>2 \text { and } \delta(1)=1, \\
2 \text { if } q>2 \text { and } \delta(1)=0,
\end{array}\right.
$$

and the function $\delta(x)$ and the polynomial $m_{\alpha^{j}}(x)$ were defined in Section 2.1.
When $q=2$, the code of Theorem 9 is equivalent to the binary Hamming weight or its even-weight subcode, and is thus optimal. The code is either optimal or almost optimal with respect to the Sphere Packing Bound.

### 4.2.3 Cyclic codes from $D_{2}(x, a)=x^{2}-2 a$

In this section we consider the code $C_{S}$ defined by $f(x)=D_{2}(x, a)=x^{2}-2 a$ over $\operatorname{GF}\left(q^{m}\right)$. When $p=2$, this code was treated in Section 4.2.2. When $p>2$, the following theorem is a variant of Theorem 5.2 in [9], but has much stronger conclusions on the minimum distance of the code.

Theorem 10 Let $p>2$ and $m \geq 3$. The code $C_{s}$ defined by $f(x)=D_{2}(x, a)=x^{2}-2 a$ has parameters $[n, n-2 m-\delta(1-2 a), d]$ and generator polynomial

$$
\mathbb{M}_{s}(x)=(x-1)^{\delta(1-2 a)} m_{\alpha^{-1}}(x) m_{\alpha^{-2}}(x),
$$

where

$$
d= \begin{cases}4 & \text { if } q=3 \text { and } \delta(1-2 a)=0, \\ 5 & \text { if } q=3 \text { and } \delta(1-2 a)=1, \\ 3 & \text { if } q>3 \text { and } \delta(1-2 a)=0, \\ 4 & \text { if } q>3 \text { and } \delta(1-2 a)=1,\end{cases}
$$

and the function $\delta(x)$ and the polynomial $\mathbb{M}_{\alpha^{j}}(x)$ were defined in Section [2.1]
The code of Theorem 10 is either optimal or almost optimal for all $m \geq 2$.

### 4.2.4 Cyclic codes from $D_{3}(x, a)=x^{3}-3 a x$

In this section we treat the code $\mathcal{C}_{S}$ defined by the Dickson polynomial $D_{3}(x, a)=x^{3}-3 a x$. We need to distinguish among the three cases: $p=2, p=3$ and $p \geq 5$. The case that $p=3$ was covered in Section4.2.2 So we need to consider only the two remaining cases.

We first handle the case $q=p=2$ and state the following lemma.

Lemma 1 Let $q=p=2$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=D_{3}(x, a)=x^{3}-3 a x=$ $x^{3}+a x$. Then the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\mathbb{M}_{s}(x)= \begin{cases}(x-1)^{\delta(1)} m_{\alpha^{-3}}(x) & \text { if } a=0 \\ (x-1)^{\delta(1+a)} m_{\alpha^{-1}}(x) m_{\alpha^{-3}}(x) & \text { if } a \neq 0\end{cases}
$$

where $m_{\alpha^{-j}}(x)$ and the function $\delta(x)$ were defined in Section 2.1 and the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}= \begin{cases}\delta(1)+m & \text { if } a=0, \\ \delta(1+a)+2 m & \text { if } a \neq 0\end{cases}
$$

The following theorem gives information on the code $\mathcal{C}_{s}$.
Theorem 11 Let $q=p=2$ and let $m \geq 4$. Then the binary code $C_{s}$ defined by the sequence of Lemma $\square$ has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where $\mathbb{M}_{s}(x)$ and $\mathbb{L}_{s}$ are given in Lemma $\square$ and

$$
d=\left\{\begin{array}{l}
2 \text { if } a=0 \text { and } \delta(1)=0, \\
4 \text { if } a=0 \text { and } \delta(1)=1, \\
5 \text { if } a \neq 0 \text { and } \delta(1+a)=0, \\
6 \text { if } a \neq 0 \text { and } \delta(1+a)=1 .
\end{array}\right.
$$

Remark 1 When $a=0$ and $\delta(1)=1$, the code is equivalent to the even-weight subcode of the Hamming code. We are mainly interested in the case that $a \neq 0$. When $a=1$, the code $\mathcal{C}_{s}$ is a double-error correcting binary BCH code or its even-like subcode. Theorem 11 shows that well-known classes of cyclic codes can be constructed with Dickson polynomials of order 3. The code is either optimal or almost optimal.

Now we consider the case $q=p^{t}$, where $p \geq 5$ or $p=2$ and $t \geq 2$.
Lemma 2 Let $q=p^{t}$, where $p \geq 5$ or $p=2$ and $t \geq 2$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=D_{3}(x, a)=x^{3}-3 a x$. Then the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\mathbb{M}_{s}(x)= \begin{cases}(x-1)^{\delta(-2)} m_{\alpha^{-3}}(x) m_{\alpha^{-2}}(x) & \text { if } a=1, \\ (x-1)^{\delta(1-3 a)} m_{\alpha^{-3}}(x) m_{\alpha^{-2}}(x) m_{\alpha^{-1}}(x) & \text { if } a \neq 1\end{cases}
$$

where $m_{\alpha^{-j}}(x)$ and the function $\delta(x)$ were defined in Section 2.1] and the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}= \begin{cases}\delta(-2)+2 m & \text { if } a=1 \\ \delta(1+a)+3 m & \text { if } a \neq 1\end{cases}
$$

The following theorem provides information on the code $\mathcal{C}_{s}$.
Theorem 12 Let $q=p^{t}$, where $p \geq 5$ or $p=2$ and $t \geq 2$. Then the code $\mathcal{C}_{s}$ defined by the sequence of Lemma 2 has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where $\mathbb{M}_{s}(x)$ and $\mathbb{L}_{s}$ are given in Lemma 2 and

$$
\left\{\begin{array}{l}
d \geq 3 \text { if } a=1, \\
d \geq 4 \text { if } a \neq 1 \text { and } \delta(1-3 a)=0, \\
d \geq 5 \text { if } a \neq 1 \text { and } \delta(1-3 a)=1, \\
d \geq 5 \text { if } a \neq 1 \text { and } \delta(1-3 a)=0 \text { and } q=4, \\
d \geq 6 \text { if } a \neq 1 \text { and } \delta(1-3 a)=1 \text { and } q=4 .
\end{array}\right.
$$

Remark 2 The code $\mathcal{C}_{s}$ of Theorem 12 is either a BCH code or the even-like subcode of a BCH code. One can similarly show that the code is either optimal or almost optimal.

When $q=4, a \neq 1, \delta(1-3 a)=1$, and $m \geq 3$, the Sphere Packing Bound shows that $d=6$. But the minimum distance is still open in other cases.

Open Problem 11 Determine the minimum distance $d$ for the code $\mathcal{C}_{s}$ of Theorem 12
4.2.5 Cyclic codes from $D_{4}(x, a)=x^{4}-4 a x^{2}+2 a^{2}$

In this section we deal with the code $\mathcal{C}_{s}$ defined by the Dickson polynomial $D_{4}(x, a)=$ $x^{4}-4 a x^{2}+2 a^{2}$. We have to distinguish among the three cases: $p=2, p=3$ and $p \geq 5$. The case $p=2$ was covered in Section 4.2.2 So we need to consider only the two remaining cases.

We first take care of the cas $q=p=3$ and have the following lemma.

Lemma 3 Let $q=p=3$ and $m \geq 3$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=D_{4}(x, a)=$ $x^{4}-4 a x^{2}+2 a^{2}$. Then the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\mathbb{M}_{s}(x)=\left\{\begin{array}{l}
(x-1)^{\delta(1)} m_{\alpha^{-4}}(x) m_{\alpha^{-1}}(x) \text { if } a=0, \\
(x-1)^{\delta(1)} m_{\alpha^{-4}}(x) m_{\alpha^{-2}}(x) \text { if } a=1, \\
(x-1)^{\delta\left(1-a-a^{2}\right)} m_{\alpha^{-4}}(x) m_{\alpha^{-2}}(x) m_{\alpha^{-1}}(x) \text { otherwise },
\end{array}\right.
$$

where $m_{\alpha^{-j}}(x)$ and the function $\delta(x)$ were defined in Section 2.1] and the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}= \begin{cases}\delta(1)+2 m & \text { if } a=0 \\ \delta(1)+2 m & \text { if } a=1 \\ \delta\left(1-a-a^{2}\right)+3 m & \text { otherwise }\end{cases}
$$

The following theorem gives information on the code $\mathcal{C}_{s}$.

Theorem 13 Let $q=p=3$ and $m \geq 3$. Then the code $\mathcal{C}_{s}$ defined by the sequence of Lemma 3 has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where $\mathbb{M}_{s}(x)$ and $\mathbb{L}_{s}$ are given in Lemma 3 and

$$
\left\{\begin{array}{l}
d=2 \text { if } a=1, \\
d=3 \text { if } a=0 m \equiv 0(\bmod 6), \\
d \geq 4 \text { if } a=0 m \not \equiv 0(\bmod 6), \\
d \geq 5 \text { if } a^{2} \neq a \text { and } \delta\left(1-a-a^{2}\right)=0, \\
d=6 \text { if } a^{2} \neq a \text { and } \delta\left(1-a-a^{2}\right)=1 .
\end{array}\right.
$$

Remark 3 When $a=1$, the code of Theorem 13 is neither optimal nor almost optimal. The code is either optimal or almost optimal in all other cases.

Open Problem 12 Determine the minimum distance $d$ for the code $C_{s}$ of Theorem 13
Now we consider the case $q=p^{t}$, where $p \geq 5$ or $p=3$ and $t \geq 2$.

Lemma 4 Let $m \geq 2$ and $q=p^{t}$, where $p \geq 5$ or $p=3$ and $t \geq 2$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=D_{4}(x, a)=x^{4}-4 a x^{2}+2 a^{2}$. Then the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\mathbb{M}_{s}(x)=\left\{\begin{array}{l}
(x-1)^{\delta(1)} m_{\alpha^{-4}}(x) m_{\alpha^{-3}}(x) m_{\alpha^{-1}}(x) \text { if } a=\frac{3}{2} \\
(x-1)^{\delta(1)} m_{\alpha^{-4}}(x) m_{\alpha^{-3}}(x) m_{\alpha^{-2}}(x) \text { if } a=\frac{1}{2} \\
(x-1)^{\delta\left(1-4 a+2 a^{2}\right)} \prod_{i=1}^{4} m_{\alpha^{-i}}(x) \text { if } a \notin\left\{\frac{3}{2}, \frac{1}{2}\right\}
\end{array}\right.
$$

where $m_{\alpha^{-j}}(x)$ and the function $\delta(x)$ were defined in Section 2.1] and the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}= \begin{cases}\delta(1)+3 m & \text { if } a \in\left\{\frac{3}{2}, \frac{1}{2}\right\} \\ \delta\left(1-4 a+2 a^{2}\right)+4 m & \text { otherwise }\end{cases}
$$

The following theorem delivers to us information on the code $\mathcal{C}_{s}$.
Theorem 14 Let $m \geq 2$ and $q=p^{t}$, where $p \geq 5$ or $p=3$ and $t \geq 2$. Then the code $\mathcal{C}_{s}$ defined by the sequence of Lemma 4 has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where $\mathbb{M}_{s}(x)$ and $\mathbb{L}_{s}$ are given in Lemma 4 and

$$
\left\{\begin{array}{l}
d \geq 3 \text { if } a=\frac{3}{2} \\
d \geq 4 \text { if } a=\frac{1}{2} \\
d \geq 5 \text { if } a \notin\left\{\frac{3}{2}, \frac{1}{2}\right\} \text { and } \delta\left(1-4 a+a^{2}\right)=0 \\
d=6 \text { if } a \notin\left\{\frac{3}{2}, \frac{1}{2}\right\} \text { and } \delta\left(1-4 a+a^{2}\right)=1
\end{array}\right.
$$

Remark 4 Except the cases that $a \in\left\{\frac{3}{2}, \frac{1}{2}\right\}$, the code $\mathcal{C}_{s}$ of Theorem 14 is either optimal or almost optimal.

Open Problem 13 Determine the minimum distance $d$ for the code $\mathcal{C}_{s}$ of Theorem 14
4.2.6 Cyclic codes from $D_{5}(x, a)=x^{5}-5 a x^{3}+5 a^{2} x$

In this section we deal with the code $\mathcal{C}_{S}$ defined by the Dickson polynomial $D_{5}(x, a)=$ $x^{5}-5 a x^{3}+5 a^{2} x$. We have to distinguish among the three cases: $p=2, p=3$ and $p \geq 7$. The case $p=5$ was covered in Section 4.2.2 So we need to consider only the remaining cases.

We first consider the cas $q=p=2$ and have the following lemma.
Lemma 5 Let $q=p=2$ and $m \geq 5$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=D_{5}(x, a)=$ $x^{5}-5 a x^{3}+5 a^{2} x$. Then the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\mathbb{M}_{s}(x)=\left\{\begin{array}{l}
(x-1)^{\delta(1)} m_{\alpha^{-5}}(x) \text { if } a=0, \\
(x-1)^{\delta(1)} m_{\alpha-5}(x) m_{\alpha^{-3}}(x) \text { if } 1+a+a^{3}=0, \\
(x-1)^{\delta(1)} \prod_{i=0}^{2} m_{\alpha^{-(2 i+1)}}(x) \text { if } a+a^{2}+a^{4} \neq 0
\end{array}\right.
$$

where $m_{\alpha^{-j}}(x)$ and the function $\delta(x)$ were defined in Section 2.1] and the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}= \begin{cases}\delta(1)+m & \text { if } a=0 \\ \delta(1)+2 m & \text { if } 1+a+a^{3}=0 \\ \delta(1)+3 m & \text { if } a+a^{2}+a^{4} \neq 0\end{cases}
$$

The following theorem describes parameters of the code $\mathcal{C}_{s}$.
Theorem 15 Let $q=p=2$ and $m \geq 5$. Then the code $\mathcal{C}_{s}$ defined by the sequence of Lemma 5 has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where $\mathbb{M}_{s}(x)$ and $\mathbb{L}_{s}$ are given in Lemma 5 and

$$
\left\{\begin{array}{l}
d=2 \text { if } a=0 \text { and } \delta(1)=0 \text { and } \operatorname{gcd}(5, n)=5, \\
d=3 \text { if } a=0 \text { and } \delta(1)=0 \text { and } \operatorname{gcd}(5, n)=1, \\
d=4 \text { if } a=0 \text { and } \delta(1)=1, \\
d \geq 3 \text { if } 1+a+a^{3}=0 \text { and } \delta(1)=0, \\
d \geq 4 \text { if } 1+a+a^{3}=0 \text { and } \delta(1)=1, \\
d \geq 7 \text { if } a+a^{2}+a^{4} \neq 0 \text { and } \delta(1)=0, \\
d=8 \text { if } a+a^{2}+a^{4} \neq 0 \text { and } \delta(1)=1 .
\end{array}\right.
$$

Remark 5 The code of Theorem 15 is either optimal or almost optimal. The code is not a BCH code when $1+a+a^{3}=0$, and a BCH code in the remaining cases.

Open Problem 14 Determine the minimum distance d for the code $\mathcal{C}_{s}$ of Theorem 15ffor the three open cases.

We now consider the cas $(p, q)=(2,4)$ and have the following lemma.
Lemma 6 Let $(p, q)=(2,4)$ and $m \geq 3$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=$ $D_{5}(x, a)=x^{5}-5 a x^{3}+5 a^{2} x$. Then the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\mathbb{M}_{s}(x)= \begin{cases}(x-1)^{\delta(1)} m_{\alpha^{-5}}(x) & \text { if } a=0 \\ (x-1)^{\delta(1)} m_{\alpha^{-5}}(x) m_{\alpha^{-3}}(x) m_{\alpha^{-2}}(x) & \text { if } a=1, \\ (x-1)^{\delta\left(1+a+a^{2}\right)} m_{\alpha^{-5}}(x) m_{\alpha^{-3}}(x) m_{\alpha^{-2}}(x) m_{\alpha^{-1}}(x) & \text { if } a+a^{2} \neq 0\end{cases}
$$

where $m_{\alpha^{-j}}(x)$ and the function $\delta(x)$ were defined in Section 2.1] and the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}= \begin{cases}\delta(1)+m & \text { if } a=0, \\ \delta(1)+3 m & \text { if } a=1, \\ \delta(1)+4 m & \text { if } a+a^{2} \neq 0 .\end{cases}
$$

The following theorem supplies information on the code $\mathcal{C}_{s}$.
Theorem 16 Let $(p, q)=(2,4)$ and $m \geq 3$. Then the code $C_{s}$ defined by the sequence of Lemma $\square$ has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where $\mathbb{M}_{s}(x)$ and $\mathbb{L}_{s}$ are given in Lemma 6 and

$$
\left\{\begin{array}{l}
d=2 \text { if } a=0 \text { and } \delta(1)=0 \text { and } \operatorname{gcd}(5, n)=5, \\
d=3 \text { if } a=0 \text { and } \operatorname{gcd}(5, n)=1, \\
d \geq 3 \text { if } a=1, \\
d \geq 6 \text { if } a+a^{2} \neq 0 \text { and } \delta(1)=0, \\
d \geq 7 \text { if } a+a^{2} \neq 0 \text { and } \delta(1)=1 .
\end{array}\right.
$$

Examples of the code of Theorem 16 are documented in arXiv:1206.4370 and many of them are optimal.

Open Problem 15 Determine the minimum distance $d$ of the code $\mathcal{C}_{s}$ in Theorem 16
We now consider the case $(p, q)=\left(2,2^{t}\right)$, where $t \geq 3$, and state the following lemma.

Lemma 7 Let $(p, q)=\left(2,2^{t}\right)$ and $m \geq 3$, where $t \geq 3$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=D_{5}(x, a)=x^{5}-5 a x^{3}+5 a^{2} x$. Then the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\mathbb{M}_{s}(x)= \begin{cases}(x-1)^{\delta(1)} m_{\alpha^{-5}}(x) m_{\alpha^{-4}}(x) m_{\alpha^{-1}}(x) & \text { if } a=0 \\ \prod_{i=2}^{5} m_{\alpha^{-i}}(x) & \text { if } 1+a+a^{2}=0 \\ (x-1)^{\delta\left(1+a+a^{2}\right)} \prod_{i=1}^{5} m_{\alpha^{-i}}(x) & \text { if } a+a^{2}+a^{3} \neq 0\end{cases}
$$

where $m_{\alpha^{-j}}(x)$ and the function $\delta(x)$ were defined in Section 2.1 and the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}= \begin{cases}\delta(1)+3 m & \text { if } a=0 \\ \delta(1)+4 m & \text { if } 1+a+a^{2}=0 \\ \delta(1)+5 m & \text { if } a+a^{2}+a^{3} \neq 0\end{cases}
$$

The following theorem provides information on the code $\mathcal{C}_{s}$.
Theorem 17 Let $(p, q)=\left(2,2^{t}\right)$, where $t \geq 3$. Then the code $\mathcal{C}_{s}$ defined by the sequence of Lemma $Z$ has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where $\mathbb{M}_{s}(x)$ and $\mathbb{L}_{s}$ are given in Lemma $Z$ and

$$
\left\{\begin{array}{l}
d \geq 3 \text { if } a=0 \text { and } \delta(1)=0, \\
d \geq 4 \text { if } a=0 \text { and } \delta(1)=1, \\
d \geq 5 \text { if } 1+a+a^{2}=0, \\
d \geq 6 \text { if } a+a^{2}+a^{3} \neq 0 \text { and } \delta(1)=0, \\
d \geq 7 \text { if } a+a^{2}+a^{3} \neq 0 \text { and } \delta(1)=1 .
\end{array}\right.
$$

Open Problem 16 Determine the minimum distance d of the code $\mathcal{C}_{s}$ in Theorem 17
Examples of the code of Theorem 17 can be found in arXiv:1206.4370, and many of them are optimal. The code of Theorem 17is not a BCH code when $a=0$, and a BCH code otherwise.

We now consider the case $q=p=3$ and state the following lemma and theorem.
Lemma 8 Let $q=p=3$ and $m \geq 3$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=D_{5}(x, a)=$ $x^{5}-5 a x^{3}+5 a^{2} x$. Then the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\mathbb{M}_{s}(x)= \begin{cases}(x-1)^{\delta\left(1+a+2 a^{2}\right)} m_{\alpha^{-5}}(x) m_{\alpha^{-4}}(x) m_{\alpha^{-2}}(x) & \text { if } a-a^{6}=0, \\ (x-1)^{\delta\left(1+a+2 a^{2}\right)} \prod_{i=2}^{5} m_{\alpha^{-i}}(x) & \text { if } a-a^{6} \neq 0,\end{cases}
$$

where $m_{\alpha^{-j}}(x)$ and the function $\delta(x)$ were defined in Section 2.1 and the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}=\left\{\begin{array}{l}
\delta\left(1+a+2 a^{2}\right)+3 m \text { if } a-a^{6}=0 \\
\delta\left(1+a+2 a^{2}\right)+4 m \text { if } a-a^{6} \neq 0
\end{array}\right.
$$

The following theorem gives information on the code $\mathcal{C}_{s}$.
Theorem 18 Let $q=p=3$ and $m \geq 3$. Then the code $\mathcal{C}_{s}$ defined by the sequence of Lemma 8 has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where $\mathbb{M}_{s}(x)$ and $\mathbb{L}_{s}$ are given in Lemma 8 and

$$
\left\{\begin{array}{l}
d \geq 4 \text { if } a-a^{6}=0 \\
d \geq 7 \text { if } a-a^{6} \neq 0 \text { and } \delta\left(1+a+2 a^{2}\right)=0 \\
d \geq 8 \text { if } a-a^{6} \neq 0 \text { and } \delta\left(1+a+2 a^{2}\right)=1
\end{array}\right.
$$

Open Problem 17 Determine the minimum distance $d$ of the code $\mathcal{C}_{s}$ in Theorem 18 (our experimental data indicates that the lower bounds are the specific values of $d$ ).

Examples of the code of Theorem 18 are described in arXiv:1206.4370, and some of them are optimal.

We now consider the case $(p, q)=\left(3,3^{t}\right)$, where $t \geq 3$, and state the following lemma and theorem.

Lemma 9 Let $(p, q)=\left(3,3^{t}\right)$ and $m \geq 2$, where $t \geq 2$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=D_{5}(x, a)=x^{5}-5 a x^{3}+5 a^{2} x$. Then the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\mathbb{M}_{s}(x)= \begin{cases}(x-1)^{\delta(1)} m_{\alpha^{-5}}(x) m_{\alpha^{-4}}(x) m_{\alpha^{-2}}(x) m_{\alpha^{-1}}(x) & \text { if } 1+a=0, \\ (x-1)^{\delta(a-1)} m_{\alpha-5}(x) m_{\alpha^{-4}}(x) m_{\alpha^{-3}}(x) m_{\alpha^{-2}}(x) & \text { if } 1+a^{2}=0, \\ (x-1)^{\delta\left(1+a+2 a^{2}\right)} \prod_{i=1}^{5} m_{\alpha^{-i}}(x) & \text { if }(a+1)\left(a^{2}+1\right) \neq 0,\end{cases}
$$

where $m_{\alpha^{-j}}(x)$ and the function $\delta(x)$ were defined in Section 2.1 and the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}= \begin{cases}\delta(1)+4 m & \text { if } a+1=0, \\ \delta(a-1)+4 m & \text { if } a^{2}+1=0, \\ \delta\left(1+a+2 a^{2}\right)+5 m & \text { if }(a+1)\left(a^{2}+1\right) \neq 0\end{cases}
$$

The following theorem supplies information on the code $\mathcal{C}_{s}$.
Theorem 19 Let $(p, q)=\left(3,3^{t}\right)$ and $m \geq 2$, where $t \geq 2$. Then the code $\mathcal{C}_{s}$ defined by the sequence of Lemma $\backslash$ has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where $\mathbb{M}_{s}(x)$ and $\mathbb{L}_{s}$ are given in Lemma 9 and

$$
\left\{\begin{array}{l}
d \geq 3 \text { if } a=-1 \text { and } \delta(1)=0, \\
d \geq 4 \text { if } a=-1 \text { and } \delta(1)=1, \\
d \geq 5 \text { if } a^{2}=-1 \text { and } \delta(a-1)=0, \\
d \geq 6 \text { if } a^{2}=-1 \text { and } \delta(a-1)=1, \\
d \geq 6 \text { if }(a+1)\left(a^{2}+1\right) \neq 0 \text { and } \delta\left(1+a+2 a^{2}\right)=0, \\
d \geq 7 \text { if }(a+1)\left(a^{2}+1\right) \neq 0 \text { and } \delta\left(1+a+2 a^{2}\right)=1 .
\end{array}\right.
$$

Open Problem 18 Determine the minimum distance $d$ of the code $\mathcal{C}_{s}$ in Theorem 19
Examples of the code of Theorem 19 are available in arXiv:1206.4370, and some of them are optimal. The code is a BCH code, except in the case that $a=-1$.

We finally consider the case $p \geq 7$, and present the following lemma and theorem.
Lemma 10 Let $p \geq 7$ and $m \geq 2$. Let $s^{\infty}$ be the sequence of (5), where $f(x)=D_{5}(x, a)=$ $x^{5}-5 a x^{3}+5 a^{2} x$. Then the minimal polynomial $\mathbb{M}_{s}(x)$ of $s^{\infty}$ is given by

$$
\mathbb{M}_{s}(x)=\left\{\begin{array}{l}
(x-1)^{\delta\left(1-5 a+5 a^{2}\right)} m_{\alpha^{-5}}(x) m_{\alpha^{-4}}(x) m_{\alpha^{-2}}(x) m_{\alpha^{-1}}(x) \text { if } a=2, \\
(x-1)^{\delta\left(1-5 a+5 a^{2}\right)} m_{\alpha^{-5}}(x) m_{\alpha^{-4}}(x) m_{\alpha^{-3}}(x) m_{\alpha^{-1}}(x) \text { if } a=\frac{2}{3}, \\
(x-1)^{\delta\left(1-5 a+5 a^{2}\right)} m_{\alpha^{-5}}(x) m_{\alpha^{-4}}(x) m_{\alpha^{-3}}(x) m_{\alpha^{-2}}(x) \text { if } a^{2}-3 a+1=0, \\
(x-1)^{\delta\left(1-5 a+5 a^{2}\right)} \prod_{i=1}^{5} m_{\alpha^{-i}}(x) \text { if }\left(a^{2}-3 a+1\right)(a-2)(3 a-2) \neq 0,
\end{array}\right.
$$

where $m_{\alpha^{-j}}(x)$ and the function $\delta(x)$ were defined in Section 2.1] and the linear span $\mathbb{L}_{s}$ of $s^{\infty}$ is given by

$$
\mathbb{L}_{s}=\left\{\begin{array}{l}
\delta\left(1-5 a+5 a^{2}\right)+4 m, \text { if }\left(a^{2}-3 a+1\right)(a-2)(3 a-2)=0, \\
\delta\left(1-5 a+5 a^{2}\right)+5 m, \text { otherwise } .
\end{array}\right.
$$

The following theorem provides information on the code $\mathcal{C}_{s}$.
Theorem 20 Let $p \geq 7$ and $m \geq 2$. Then the code $\mathcal{C}_{s}$ defined by the sequence of Lemma 10 has parameters $\left[n, n-\mathbb{L}_{s}, d\right]$ and generator polynomial $\mathbb{M}_{s}(x)$, where $\mathbb{M}_{s}(x)$ and $\mathbb{L}_{s}$ are given in Lemma 10 and

$$
\left\{\begin{array}{l}
d \geq 3 \text { if } a=2 \text { and } \delta\left(1-5 a+5 a^{2}\right)=0, \\
d \geq 4 \text { if } a=2 \text { and } \delta\left(1-5 a+5 a^{2}\right)=1, \\
d \geq 4 \text { if } a=\frac{2}{3} \text { and } \delta\left(1-5 a+5 a^{2}\right)=0, \\
d \geq 5 \text { if } a=\frac{2}{3} \text { and } \delta\left(1-5 a+5 a^{2}\right)=1, \\
d \geq 5 \text { if } 1-3 a+a^{2}=0 \text { and } \delta\left(1-5 a+5 a^{2}\right)=0, \\
d \geq 6 \text { if } 1-3 a+a^{2}=0 \text { and } \delta\left(1-5 a+5 a^{2}\right)=1, \\
d \geq 6 \text { if }\left(a^{2}-3 a+1\right)(a-2)(3 a-2) \neq 0 \text { and } \delta\left(1-5 a+5 a^{2}\right)=0, \\
d \geq 7 \text { if }\left(a^{2}-3 a+1\right)(a-2)(3 a-2) \neq 0 \text { and } \delta\left(1-5 a+5 a^{2}\right)=1 .
\end{array}\right.
$$

Open Problem 19 Determine the minimum distance $d$ of the code $\mathcal{C}_{s}$ in Theorem 20
Examples of the code of Theorem 20 can be found in arXiv:1206.4370, and some of them are optimal. The code is a BCH code, except in the cases $a \in\{2,2 / 3\}$.

### 4.2.7 Cyclic codes from other $D_{i}(x, a)$ for $i \geq 6$

Parameters of cyclic codes from $D_{i}(x, a)$ for small $i$ could be established in a similar way. However, more cases are involved and the situation is getting more complicated when $i$ gets bigger. Examples of the code $\mathcal{C}_{5}$ from $D_{7}(x, a)$ and $D_{11}(x, a)$ can be found in arXiv:1206.4370.

### 4.2.8 Cyclic codes from Dickson polynomials of the second kind

Results on cyclic codes from Dickson polynomials of the second kind can be developed in a similar way. Experimental data indicates that the codes from the Dickson polynomials of the first kind are in general better than those from the Dickson polynomials of the second kind, though some cyclic codes from Dickson polynomials of the second kind could also be optimal or almost optimal.

### 4.2.9 Comments on the cyclic codes from Dickson polynomials

It is really amazing that in most cases the cyclic codes derived from the Dickson polynomials of small degrees within the framework of this paper are optimal or almost optimal (see arXiv:1206.4370 for examples of optimal codes).

We had to treat Dickson polynomials of small degrees case by case over finite fields with different characteristics as we do not see a way of treating them in a single strike. The generator polynomial and the dimension of the codes depend heavily on the degree of the Dickson polynomials and the characteristic of the base field.

## 5 Concluding remarks

Recall that every cyclic code over a finite field can be expressed as $\mathcal{C}_{s}$ for some sequence $s^{\infty}$. This approach can produce all cyclic codes over finite fields, including BCH codes. It is thus no surprise that some of the codes from Dickson polynomials are in fact BCH codes.

Since it is a fundamental approach, it produces both good and bad cyclic codes. It is open what sequences over a finite field give cyclic codes with optimal parameters.

Though a considerable amount of progress on this approach of constructing cyclic codes with sequences has been made, a lot of investigation should be further done, as there is a huge number of constructions of sequences in the literature. The reader is cordially invited to join the journey in this direction.

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