Singly even self-dual codes of length 24k + 10and minimum weight 4k + 2

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October 16, 2018

Abstract

Currently, the existence of an extremal singly even self-dual code of length 24k + 10 is unknown for all nonnegative integers k. In this note, we study singly even self-dual [24k + 10, 12k + 5, 4k + 2] codes. We give some restrictions on the possible weight enumerators of singly even self-dual [24k + 10, 12k + 5, 4k + 2] codes with shadows of minimum weight at least 5 for k = 2, 3, 4, 5. We discuss a method for constructing singly even self-dual codes with minimal shadow. As an example, a singly even self-dual [82, 41, 14] code with minimal shadow is constructed for the first time. In addition, as neighbors of the code, we construct singly even self-dual [82, 41, 14] codes with weight enumerator for which no singly even self-dual code was previously known to exist.

1 Introduction

Extremal self-dual codes are an important class of linear codes for both theoretical and practical reasons. It is a fundamental problem to determine the largest minimum weight among self-dual codes of that length, and much work has been done concerning this problem.

A (binary) code C of length n is a vector subspace of \mathbb{F}_2^n , where \mathbb{F}_2 denotes the finite field of order 2. All codes in this note are binary. The *dual* code

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 C^{\perp} of C is defined as $C^{\perp} = \{x \in \mathbb{F}_2^n \mid x \cdot y = 0 \text{ for all } y \in C\}$, where $x \cdot y$ is the standard inner product. A code C is called *self-dual* if $C = C^{\perp}$. Self-dual codes are divided into two classes. A self-dual code C is *doubly even* if all codewords x of C have weight $\operatorname{wt}(x) \equiv 0 \pmod{4}$, and *singly even* if there is at least one codeword of weight $\equiv 2 \pmod{4}$. A doubly even self-dual code of length n exists if and only if $n \equiv 0 \pmod{8}$, while a singly even self-dual code of length n exists if and only if n is even.

Let C be a singly even self-dual code. Let C_0 denote the subcode of C consisting of codewords x having weight $\operatorname{wt}(x) \equiv 0 \pmod{4}$. The shadow S of C is defined to be $C_0^{\perp} \setminus C$. A singly even self-dual code of length n is called a code with *minimal shadow* if the minimum weight of the shadow is 4, 1, 2 and 3 if $n \equiv 0, 2, 4$ and 6 (mod 8), respectively. The concept of singly even self-dual codes with minimal shadow was introduced in [4].

Rains [14] showed that the minimum weight d of a self-dual code of length n is bounded by $d \leq 4\lfloor n/24 \rfloor + 4$ unless $n \equiv 22 \pmod{24}$ when $d \leq 4\lfloor n/24 \rfloor + 6$. A self-dual code meeting the upper bound is called *extremal*. We say that a self-dual code is *optimal* if it has the largest minimum weight among all self-dual codes of that length. For length $24k + 10 \ (k = 0, 1, \ldots, 5)$, we give the current information on the largest minimum weight d(24k + 10):

$$d(10) = 2, d(34) = 6, d(58) = 10, d(82) = 14$$
 or 16,
 $d(106) = 16$ or $18, d(130) = 20, 22$ or $24,$

(see [6, Table I], [7, Table VI], [11, Table I]). Currently, the existence of an extremal singly even self-dual code of length 24k + 10 is unknown for all nonnegative integers k. In addition, Han and Lee [9] conjecture that there is no extremal singly even self-dual code of length 24k + 10 for all nonnegative integers k. It was shown in [5] that there is no extremal singly even self-dual code with minimal shadow for length 24k + 10. These motivate our interest in singly even self-dual [24k + 10, 12k + 5, 4k + 2] codes.

This note is organized as follows. In Section 2, the possible weight enumerators of singly even self-dual [82, 41, 14] codes are determined. In addition, in Section 3, we give some restrictions on the possible weight enumerators of singly even self-dual [24k + 10, 12k + 5, 4k + 2] codes with shadows of minimum weight at least 5 for k = 2, 3, 4, 5. In Section 4, we discuss a method for constructing singly even self-dual codes with minimal shadow. As an example, a singly even self-dual [82, 41, 14] code C_{82} with minimal shadow is constructed for the first time. Finally, in Section 5, as neighbors of C_{82} , we construct singly even self-dual [82, 41, 14] codes with weight enumerator for which no singly even self-dual code was previously known to exist. It is a fundamental problem to find which weight enumerators actually occur for the possible weight enumerators. We emphasize that singly even self-dual [82, 41, 14] codes with shadows of minimum weight 1, 5, 9 are constructed for the first time.

All computer calculations in this note were done with the help of the algebra software MAGMA [2] and the mathematical software MATHEMATICA.

2 Weight enumerators of singly even self-dual [82, 41, 14] codes

Let C be a singly even self-dual code of length n with shadow S. Let A_i and B_i be the numbers of vectors of weight i in C and S, respectively. The weight enumerators W_C and W_S of C and S are given by $\sum_{i=0}^n A_i y^i$ and $\sum_{i=d(S)}^{n-d(S)} B_i y^i$, respectively, where d(S) denotes the minimum weight of S. If we write

$$W_C = \sum_{j=0}^{\lfloor n/8 \rfloor} a_j (1+y^2)^{n/2-4j} (y^2(1-y^2)^2)^j,$$

for suitable integers a_j , then

$$W_S = \sum_{j=0}^{\lfloor n/8 \rfloor} (-1)^j a_j 2^{n/2 - 6j} y^{n/2 - 4j} (1 - y^4)^{2j},$$

[6, (10), (11)]. Suppose that C is a singly even self-dual [82, 41, 14] code. Since the minimum weight is 14, we have

$$a_0 = 1, a_1 = -41, a_2 = 615, a_3 = -4182,$$

 $a_4 = 13161, a_5 = -18040, a_6 = 9512.$

Then the weight enumerator of the shadow S is written as:

$$\frac{a_{10}}{524288}y + \left(-\frac{a_9}{8192} - \frac{5a_{10}}{131072}\right)y^5 + \left(\frac{a_8}{128} + \frac{9a_9}{4096} + \frac{95a_{10}}{262144}\right)y^9 + \left(-\frac{a_7}{2} - \frac{a_8}{8} - \frac{153a_9}{8192} - \frac{285a_{10}}{131072}\right)y^{13} + \cdots$$

• d(S) = 1: From [6, (6)], S has a unique vector of weight 1 and S has no vector of weights 5 and 9. Hence, $a_{10} = 524288$, $a_9 = -163840$ and $a_8 = 21760$. Since $A_{14} = B_{13}$ by [10],

$$3280 + a_7 = -800 - \frac{a_7}{2}$$

Thus, we have that $a_7 = -2720$. Therefore, we have the following possible weight enumerators

$$W_{82,1}^C = 1 + 560y^{14} + 60724y^{16} + 233545y^{18} + \cdots,$$

$$W_{82,1}^S = y + 560y^{13} + 294269y^{17} + 33367568y^{21} + \cdots,$$

respectively.

• d(S) = 5: From [6, (6)], we have $a_{10} = 0$ and $a_9 = -8192$. Then we have that a_8 is divisible by 128, say $a_8 = 128\alpha$ and a_7 is divisible by 2, say $a_7 = 2\beta$, where α and β are integers. Therefore, we have the following possible weight enumerators

$$W_{82,2}^{C} = 1 + (3280 + 2\beta)y^{14} + (36244 + 128\alpha - 2\beta)y^{16} + (506153 - 896\alpha - 26\beta)y^{18} + \cdots, W_{82,2}^{S} = y^{5} + (-18 + \alpha)y^{9} + (153 - 16\alpha - \beta)y^{13} + (303568 + 120\alpha + 14\beta)y^{17} + \cdots,$$

respectively. Note that $\alpha \geq 18$ and $\beta \leq 153 - 16\alpha \leq -135$. In Section 3, it is shown that β is an even integer (see Proposition 2).

• $d(S) \ge 9$: Then we have $a_{10} = a_9 = 0$. We have that a_8 is divisible by 128, say $a_8 = 128\alpha$ and a_7 is divisible by 2, say $a_7 = 2\beta$, where α and β are integers. Therefore, we have the following possible weight enumerators

$$W_{82,3}^{C} = 1 + (3280 + 2\beta)y^{14} + (36244 + 128\alpha - 2\beta)y^{16} + (514345 - 896\alpha - 26\beta)y^{18} + \cdots,$$
$$W_{82,3}^{S} = \alpha y^{9} + (-16\alpha - \beta)y^{13} + (304384 + 120\alpha + 14\beta)y^{17} + (33293312 - 560\alpha - 91\beta)y^{21} + \cdots,$$

respectively. Note that $\alpha \ge 0$ and $\beta \le -16\alpha \le 0$. In Section 3, it is shown that β is an even integer (see Proposition 2).

It is unknown whether there is a singly even self-dual [82, 41, 16] code. The first example of a singly even self-dual [82, 41, 14] code was found in [7, Section V]. The weight enumerators of the code and its shadow were given, however unfortunately the weight enumerator of the shadow was incorrectly stated and the correct weight enumerator is

$$656y^{13} + 295200y^{17} + 33353008y^{21} + \cdots$$

This code has weight enumerator $W^C_{82,3}$ with

$$(\alpha, \beta) = (0, -656). \tag{1}$$

The code was the only previously known singly even self-dual [82, 41, 14] code. In Sections 4 and 5, we construct singly even self-dual [82, 41, 14] codes with weight enumerator for which no singly even self-dual code was previously known to exist.

3 Restrictions on weight enumerators of singly even self-dual [24k + 10, 12k + 5, 4k + 2] codes

It was shown in [3] that the weight enumerator of a singly even self-dual [24k + 10, 12k + 5, 4k + 2] code with minimal shadow is uniquely determined. In this section, we give some restrictions on the possible weight enumerators of singly even self-dual [24k + 10, 12k + 5, 4k + 2] codes with shadows of minimum weight at least 5 for k = 2, 3, 4, 5. It is a key idea to consider the possible weight enumerator of C_1 .

3.1 Possible weight enumerators of C_1

Let C be a singly even self-dual [24k + 10, 12k + 5, 4k + 2] code. Let $W^{(1)}$ and $W^{(3)}$ denote the weight enumerators of C_1 and C_3 , respectively. By [6, Theorem 5, 5)], the possible weight enumerators $W^{(1)} - W^{(3)}$ are written as:

$$W^{(1)} - W^{(3)} = \sum_{i=0}^{k-1} b_i (1 + 14y^4 + y^8)^{3k-1-3i} (y^4(1 - y^4)^4)^i f(y)$$
(2)

where $f(y) = y - 34y^5 + 34y^{13} - y^{17}$ and $b_0, b_1, \ldots, b_{k-1}$ are integers. Combined with the possible weight enumerators of the shadow, using (2), the possible weight enumerators of C_1 are determined.

3.2 Optimal singly even self-dual [58, 29, 10] codes with shadows of minimum weight at least 5

The possible weight enumerators of optimal singly even self-dual [58, 29, 10] codes with shadow of minimum weight at least 5 and the shadows are known as follows:

$$W_{58}^C = 1 + (319 - 24\beta - 2\gamma)y^{10} + (3132 + 152\beta + 2\gamma)y^{12} + \cdots,$$

$$W_{58}^S = \beta y^5 + \gamma y^9 + (24128 - 54\beta - 10\gamma)y^{13} + (1469952 + 320\beta + 45\gamma)y^{17} + \cdots,$$

respectively, where β, γ are integers [6]. If there is an optimal singly even self-dual [58, 29, 10] code with weight enumerator W_{58}^C , then $\beta \in \{0, 1, 2\}$ [12]. An optimal singly even self-dual code with weight enumerator W_{58}^C is known for

$$\begin{aligned} \beta &= 0 \text{ and } \gamma \in \{2m \mid m = 0, 1, \dots, 65, 68, 71, 79\}, \\ \beta &= 1 \text{ and } \gamma \in \{2m \mid m = 8, 9, \dots, 58, 63\}, \\ \beta &= 2 \text{ and } \gamma \in \{2m \mid m = 0, 4, 6, \dots, 55\} \end{aligned}$$

(see [10]).

Theorem 1. If there is an optimal singly even self-dual [58, 29, 10] code with weight enumerator W_{58}^C , then γ must be an even integer.

Proof. Let C be an optimal singly even self-dual [58, 29, 10] code with weight enumerator W_{58}^C . From (2), we obtain

$$W^{(1)} - W^{(3)} = b_0 y + (36b_0 + b_1)y^5 + (-415b_0 - 10b_1)y^9 + (-39056b_0 - 724b_1)y^{13} + (-742131b_0 - 3694b_1)y^{17} + \cdots$$

Since the shadow contains no vector of weight 1, we have that $b_0 = 0$. Hence, we have

$$W^{(1)} = \frac{1}{2}(b_1 + \beta)y^5 + \left(-5b_1 + \frac{\gamma}{2}\right)y^9 + (12064 - 362b_1 - 27\beta - 5\gamma)y^{13} + \left(734976 - 1847b_1 + 160\beta + \frac{45\gamma}{2}\right)y^{17} + \cdots$$

The result follows.

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3.3 Singly even self-dual [82, 41, 14] codes with shadows of minimum weight at least 5

Proposition 2. If there is a singly even self-dual [82, 41, 14] code with weight enumerator $W_{82,i}^C$, then β must be an even integer for i = 2, 3.

Proof. Let C be a singly even self-dual [82, 41, 14] code with shadow S of minimum weight at least 5. From $W_{82,i}^C$ and $W_{82,i}^S$ (i = 2, 3), the possible weight enumerators of C and S are written using integers a, b, c

$$\begin{split} W^C_{82} = &1 + (3280 + 2c)y^{14} + (36244 + 128b - 2c)y^{16} \\ &+ (514345 - 8192a - 896b - 26c)y^{18} + \cdots, \\ W^S_{82} = &ay^5 + (-18a + b)y^9 + (153a - 16b - c)y^{13} \\ &+ (304384 - 816a + 120b + 14c)y^{17} + \cdots, \end{split}$$

respectively. From (2), we obtain

$$W^{(1)} - W^{(3)} = b_0 y + (78b_0 + b_1)y^5 + (1688b_0 + 32b_1 + b_2)y^9 + (-32382b_0 - 553b_1 - 14b_2)y^{13} + (-2525349b_0 - 37184b_1 - 678b_2)y^{17} + \cdots$$

Since the shadow contains no vector of weight 1, we have that $b_0 = 0$. Hence, we have

$$W^{(1)} = \frac{a+b_1}{2}y^5 + \left(-9a+16b_1 + \frac{b_2+b}{2}\right)y^9 + \left(\frac{153a-553b_1-c}{2} - 7b_2 - 8b\right)y^{13} + (152192 - 408a - 18592b_1 - 339b_2 + 60b + 7c)y^{17} + \cdots$$

Since $a + b_1$ is even, c must be an even integer. The result follows.

3.4 Singly even self-dual [106, 53, 18] codes with shadows of minimum weight at least 5

By a method similar to that given in Section 2, the possible weight enumerators of singly even self-dual [106, 53, 18] codes with shadows of minimum weight at least 5 and the shadows are determined as follows:

$$\begin{split} W_{106}^C = &1 + (35245 + 2d)y^{18} + (416262 + 128c - 2d)y^{20} \\ &+ (6586310 + 8192b - 896c - 34d)y^{22} \\ &+ (86626645 + 524288a - 106496b + 1024c + 34d)y^{24} + \cdots, \\ W_{106}^S = &ay^5 + (-24a - b)y^9 + (276a + 22b + c)y^{13} \\ &+ (-2024a - 231b - 20c - d)y^{17} + \cdots, \end{split}$$

respectively, where a, b, c, d are integers. If a = 0, then $-b \in \{0, 1, 2\}$ by [12, Lemma 2].

Proposition 3. If there is a singly even self-dual [106, 53, 18] code with weight enumerator W_{106}^C , then d must be an even integer.

Proof. Let C be a singly even self-dual [106, 53, 18] code with weight enumerator W_{106}^C . From (2), we obtain

$$W^{(1)} - W^{(3)} = b_0 y + (120b_0 + b_1)y^5 + (5555b_0 + 74b_1 + b_2)y^9 + (87440b_0 + 1382b_1 + 28b_2 + b_3)y^{13} + (-2666610b_0 - 38670b_1 - 675b_2 - 18b_3)y^{17} + \cdots$$

Since the shadow has minimum weight at least 5, we have $b_0 = 0$. Hence, we have

$$W^{(1)} = \frac{a+b_1}{2}y^5 + \left(-12a+37b_1+\frac{b_2-b}{2}\right)y^9 + \left(138a+691b_1+14b_2+11b+\frac{b_3+c}{2}\right)y^{13} + \left(-1012a-19335b_1-9b_3-10c-\frac{675b_2+231b+d}{2}\right)y^{17} + \cdots$$

Since $b_2 - b$ is even, d must be even. The result follows.

It is unknown whether there is a singly even self-dual [106, 53, 18] code or not (see [11, Table I]).

3.5 Singly even self-dual [130, 65, 22] codes with shadows of minimum weight at least 5

By a method similar to that given in Section 2, the possible weight enumerators of singly even self-dual [130, 65, 22] codes with shadows of minimum weight at least 5 and the shadows are determined as follows:

$$\begin{split} W^C_{130} =& 1 + (388700 + 2e)y^{22} + (4791150 + 128d - 2e)y^{24} \\&+ (81082890 + 8192c - 896d - 42e)y^{26} \\&+ (1200197180 + 524288b - 106496c + 512d + 42e)y^{28} \\&+ (14196225992 - 33554432a - 9961472b + 532480c \\&+ 10752d + 420e)y^{30} + \cdots, \end{split}$$

$$W^S_{130} =& ay^5 + (-30a + b)y^9 + (435a - 28b - c)y^{13} \\&+ (-4060a + 378b + 26c + d)y^{17} \\&+ (27405a - 3276b - 325c - 24d - e)y^{21} + \cdots, \end{split}$$

respectively, where a, b, c, d, e are integers.

Proposition 4. If there is a singly even self-dual [130, 65, 22] code with weight enumerator W_{130}^C , then e must be an even integer.

Proof. Let C be a singly even self-dual [130, 65, 22] code with weight enumerator W_{130}^C . From (2), we obtain

$$W^{(1)} - W^{(3)} = b_0 y + (162b_0 + b_1)y^5 + (11186b_0 + 116b_1 + b_2)y^9 + (394498b_0 + 5081b_1 + 70b_2 + b_3)y^{13} + (4628826b_0 + 65936b_1 + 1092b_2 + 24b_3 + b_4)y^{17} + (-226397710b_0 - 2983519b_1 - 43758b_2 - 781b_3 - 22b_4)y^{21} + \cdots$$

Since the shadow has minimum weight at least 5, we have $b_0 = 0$. Hence, we

have

$$W^{(1)} = \frac{a+b_1}{2}y^5 + \left(-15a+58b_1+\frac{b_2}{2}+\frac{b}{2}\right)y^9$$

+ $\left(\frac{435a}{2}+\frac{5081b_1}{2}+35b_2+\frac{b_3}{2}-14b-\frac{c}{2}\right)y^{13}$
+ $\left(-2030a+32968b_1+546b_2+12b_3+\frac{b_4}{2}+189b+13c+\frac{d}{2}\right)y^{17}$
+ $\left(\frac{27405a}{2}-\frac{2983519b_1}{2}-21879b_2-\frac{781b_3}{2}-11b_4-1638b\right)$
 $-\frac{325c}{2}-12d-\frac{e}{2}y^{21}+\cdots$

From the coefficients of y^{13} and y^{21} , e must be even. The result follows. \Box

It is unknown whether there is a singly even self-dual [130, 65, 22] code or not (see [11, Table I]).

4 Construction of singly even self-dual codes with minimal shadow

The following method for constructing singly even self-dual codes was given in [15]. Let C be a doubly even self-dual code of length 8t. Let x be a vector of odd weight. Let C^0 denote the subcode of C consisting of all codewords which are orthogonal to x. Then there are cosets C^1, C^2, C^3 of C^0 such that $C^{0\perp} = C^0 \cup C^1 \cup C^2 \cup C^3$, where $C = C^0 \cup C^2$ and $x + C = C^1 \cup C^3$. Then

$$C(x) = (0, 0, C^{0}) \cup (1, 1, C^{2}) \cup (1, 0, C^{1}) \cup (0, 1, C^{3})$$
(3)

is a singly even self-dual code of length 8t + 2. Using this method, a singly even self-dual code with minimal shadow was constructed in [15] for the parameters [42, 21, 8] and [58, 29, 10]. This may be generalized as follows.

Theorem 5. Let C be an extremal doubly even self-dual code of length 8t with covering radius R. Then there is a vector x of weight $2\lfloor \frac{R+1}{2} \rfloor - 1$ such that C(x) in (3) is a singly even self-dual $[8t+2, 4t+1, \min\{4\lfloor \frac{t}{3} \rfloor + 4, 2\lfloor \frac{R+1}{2} \rfloor\}]$ code with minimal shadow.

Proof. Since there is a coset of minimum weight R, there is a coset of minimum weight $2\lfloor \frac{R+1}{2} \rfloor - 1$ (see [1, Fact 4]). We denote the coset by x + C, where x has weight $2\lfloor \frac{R+1}{2} \rfloor - 1$. Then the code C(x) in (3) is a self-dual code of length 8t + 2 [15]. The minimum weight of $C^0 \cup C^2$ is $4\lfloor \frac{t}{3} \rfloor + 4$. The minimum weight of $C^1 \cup C^3$ is $2\lfloor \frac{R+1}{2} \rfloor - 1$. Hence, C(x) has minimum weight $\min\{4\lfloor \frac{t}{3} \rfloor + 4, 2\lfloor \frac{R+1}{2} \rfloor\}$.

It remains to show that C(x) has shadow of minimum weight 1. Without loss of generality, we may assume that $x \in C^1$. Let v be a vector of C^1 . Then v is written as x + c, where $c \in C^0$. Since $c \cdot x = 0$, we obtain

$$\operatorname{wt}(x+c) \equiv \operatorname{wt}(x) \pmod{4}.$$
 (4)

Let w be a vector of C^3 . Then w is written as x + c + c', where $c \in C^0$ and $c' \in C^2$. From (4) and $(x + c) \cdot c' = 1$, we obtain

$$\operatorname{wt}(x+c+c') \equiv \operatorname{wt}(x) + 2 \pmod{4}.$$

Suppose that $\operatorname{wt}(x) \equiv 1 \pmod{4}$ (resp. $\operatorname{wt}(x) \equiv 3 \pmod{4}$). Then $(0, 0, C^0) \cup (0, 1, C^3)$ (resp. $(0, 0, C^0) \cup (1, 0, C^1)$) is the doubly even subcode of C(x). In addition, the vector $(1, 0, \ldots, 0)$ (resp. $(0, 1, 0, \ldots, 0)$) is orthogonal to any vector of the doubly even subcode. This shows that the shadow has minimum weight 1.

We concentrate on singly even self-dual [24k+10, 12k+5, 4k+2] codes with minimal shadow. There is no extremal singly even self-dual code of length 24k + 10 with minimal shadow for any nonnegative integer k [5]. Hence, we have the following proposition.

Proposition 6. If there is an extremal doubly even self-dual code of length 24k+8 with covering radius $R \ge 4k+1$, then there is a singly even self-dual [24k+10, 12k+5, 4k+2] codes with minimal shadow.

The bordered double circulant extremal doubly even self-dual [80, 40, 16] code $B_{80,4}$ in [8] has generator matrix

$$\left(\begin{array}{ccccc} & 0 & 1 & \cdots & 1 \\ & & 1 & & & \\ & I_{40} & \vdots & R & \\ & & 1 & & & \end{array}\right),$$

where I_{40} is the identity matrix of order 40 and R is the 39 × 39 circulant matrix with first row

(111100000100101111101011101001101100011).

It was shown in [13] that $B_{80,4}$ has covering radius 13, where a coset of minimum weight 13 is given by $x_{80} + B_{80,4}$ and x_{80} has the following support:

 $\{2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38\}.$

We denote the code $B_{80,4}(x_{80})$ by C_{82} .

Proposition 7. The code C_{82} is a singly even self-dual [82, 41, 14] code with minimal shadow.

For $k \ge 4$, only the extended quadratic residue code QR_{104} of length 104 is the known extremal doubly even self-dual code of length 24k + 8. It is not known whether QR_{104} has covering radius $R \ge 17$. Our computer search failed to find a coset of weight ≥ 17 in QR_{104} .

5 New singly even self-dual [82, 41, 14] codes

In this section, we continue a search to find singly even self-dual [82, 41, 14] codes with weight enumerator for which no singly even self-dual code was previously known to exist.

Two self-dual codes C and C' of length n are said to be *neighbors* if $\dim(C \cap C') = n/2 - 1$. Any self-dual code of length n can be reached from any other by taking successive neighbors (see [6]). By considering self-dual neighbors of C_{82} , we found 50 singly even self-dual [82, 41, 14] codes $N_{82,i}$ (i = 1, 2, ..., 50) with weight enumerator for which no singly even self-dual code was previously known to exist. These codes are constructed as

$$\langle (C_{82} \cap \langle x \rangle^{\perp}), x \rangle,$$

where the supports $\operatorname{supp}(x)$ of x are listed in Table 1. The weight enumerators W and the values (α, β) are also listed in the table.

Combined with the known result in [7], the results in the previous section and this section show the following:

Proposition 8. There is a singly even self-dual [82, 41, 14] code with shadow of minimum weight s for $s \in \{1, 5, 9, 13\}$.

Table 1:	Singly	even	self-dual	[82, 4]	1, 14]	neighbors	$N_{82,i}$

Code	$\operatorname{supp}(x)$	W	(lpha,eta)
$N_{82,1}$	$\{2, 7, 10, 14, 47, 51, 54, 56, 58, 59, 62, 64, 72, 79\}$	$W^{C}_{82,2}$	(18, -750)
$N_{82,2}$	$\{2, 7, 12, 13, 14, 42, 47, 56, 57, 59, 61, 71, 73, 79\}$	$W^{C}_{82,3}$	(1, -650)
$N_{82,3}$	$\{6,9,11,42,44,47,51,56,59,61,75,77,78,79\}$	$W^{C}_{82,3}$	(1, -668)
$N_{82,4}$	$\{5, 9, 45, 49, 55, 59, 61, 63, 66, 70, 71, 72, 75, 81\}$	$W^{C}_{82,3}$	(1, -680)
$N_{82,5}$	$\{2, 3, 9, 13, 14, 39, 40, 47, 49, 56, 57, 64, 77, 82\}$	$W^{C}_{82,3}$	(1, -682)
$N_{82,6}$	$\{2, 3, 9, 11, 12, 45, 46, 49, 53, 64, 72, 75, 77, 80\}$	$W^{C}_{82,3}$	(1, -686)
$N_{82,7}$	$\{5, 43, 46, 49, 50, 51, 63, 65, 66, 71, 72, 73, 77, 81\}$	$W^{C}_{82,3}$	(1, -688)
$N_{82,8}$	$\{3, 4, 5, 7, 9, 45, 48, 55, 56, 58, 61, 66, 73, 77\}$	$W^{C'}_{82,3}$	(1, -692)
$N_{82,9}$	$\{3, 7, 40, 46, 49, 52, 54, 57, 58, 59, 72, 74, 75, 79\}$	$W^{C}_{82,3}$	(1, -694)
$N_{82,10}$	$\{3, 11, 14, 44, 45, 46, 49, 51, 59, 71, 72, 76, 77, 81\}$	$W^{C}_{82,3}$	(1, -696)
$N_{82,11}$	$\{6, 7, 10, 12, 46, 51, 53, 55, 58, 70, 71, 73, 78, 82\}$	$W^{C}_{82,3}$	(1, -698)
$N_{82,12}$	$\{5, 8, 47, 51, 52, 57, 61, 66, 67, 71, 72, 74, 79, 80\}$	$W^{C}_{82,3}$	(1, -700)
$N_{82,13}$	$\{2, 3, 7, 8, 9, 11, 40, 44, 49, 52, 55, 63, 77, 82\}$	$W^{C}_{82,3}$	(1, -702)
$N_{82,14}$	$\{11, 12, 45, 46, 49, 50, 52, 55, 60, 62, 66, 70, 71, 81\}$	$W^{C}_{82,3}$	(1, -704)
$N_{82,15}$	$\{3, 44, 45, 46, 58, 60, 62, 64, 65, 67, 68, 73, 74, 77\}$	$W^{C}_{82,3}$	(1, -712)
$N_{82,16}$	$\{2, 4, 10, 43, 45, 46, 49, 54, 64, 66, 76, 78, 80, 81\}$	$W^{C}_{82,3}$	(1, -722)
$N_{82,17}$	$\{2, 4, 9, 10, 45, 56, 57, 59, 63, 64, 67, 68, 70, 76\}$	$W^{C}_{82,3}$	(1, -738)
$N_{82,18}$	$\{3, 6, 9, 10, 40, 47, 53, 54, 55, 68, 73, 76, 80, 81\}$	$W^{C}_{82,3}$	(1, -748)
$N_{82,19}$	$\{2, 11, 13, 37, 47, 51, 52, 55, 70, 77, 78, 79, 80, 82\}$	$W^{C}_{82,3}$	(2, -672)
$N_{82,20}$	$\{3, 9, 11, 47, 49, 59, 60, 62, 67, 68, 74, 76, 81, 82\}$	$W^{C}_{82,3}$	(2, -720)
$N_{82,21}$	$\{4, 8, 9, 40, 48, 49, 52, 54, 55, 66, 67, 68, 73, 81\}$	$W^{C}_{82,3}$	(2, -732)
$N_{82,22}$	$\{5, 6, 8, 11, 44, 45, 53, 56, 57, 61, 62, 64, 65, 66\}$	$W_{82,3}^{C}$	(2, -734)
$N_{82,23}$	$\{4, 7, 8, 9, 46, 57, 58, 61, 63, 68, 71, 73, 78, 81\}$	$W_{82,3}^{0}$	(0, -640)
$N_{82,24}$	$\{2, 3, 5, 10, 40, 44, 57, 58, 60, 63, 65, 71, 76, 79\}$	$W_{82,3}^{0}$	(0, -650)
$N_{82,25}$	$\{2, 5, 6, 8, 50, 51, 58, 63, 64, 66, 67, 71, 73, 81\}$	$W_{82,3}^{\cup}$	(0, -660)
N82,26	$\{2, 3, 9, 40, 54, 50, 59, 60, 61, 62, 67, 70, 78, 82\}$	$W_{82,3}^{UVC}$	(0, -662)
N _{82,27}	$\{4, 5, 38, 40, 48, 53, 50, 57, 62, 64, 60, 69, 71, 70\}$	$W_{82,3}$	(0, -664)
N 82,28	$\{3, 7, 8, 10, 39, 50, 51, 62, 00, 07, 70, 73, 77, 82\}$	$\frac{VV_{82,3}}{WC}$	(0, -008) (0, -672)
N 82,29	$\{2, 45, 45, 40, 50, 51, 52, 55, 01, 09, 72, 74, 74, 77, 81\}$	$\frac{VV_{82,3}}{WC}$	(0, -072) (0, -676)
No2.01	$\{0, 7, 9, 40, 50, 01, 03, 70, 75, 77, 79, 60, 61, 62\}$	$W_{82,3} = W^C$	(0, -678)
Noo oo	$\{6, 11, 50, 53, 54, 56, 50, 61, 64, 68, 69, 72, 74, 76\}$	$W^{82,3}_{W^C}$	(0, -680)
Neo 22	$\{8, 11, 12, 35, 49, 50, 53, 56, 57, 58, 62, 72, 77, 82\}$	$W^{R2,3}_{W^{C2}}$	(0, -684)
No2,33	$\{5, 11, 46, 56, 57, 58, 60, 62, 63, 64, 65, 70, 71, 79\}$	$W^{82,3}_{V^{C}}$	(0, -686)
$N_{82,34}$	$\{10, 11, 13, 14, 52, 54, 60, 64, 70, 71, 72, 76, 77, 80\}$	$W^{82,3}_{82,2}$	(0, -688)
N82 36	$\{5, 9, 45, 49, 56, 57, 61, 62, 63, 64, 67, 70, 75, 81\}$	$W^{C}_{22,2}$	(0, -690)
$N_{82,37}$	$\{2, 6, 8, 9, 44, 45, 48, 56, 66, 68, 75, 77, 80, 81\}$	W_{82}^{C}	(0, -692)
$N_{82.38}$	$\{4, 8, 10, 42, 44, 54, 58, 60, 63, 65, 68, 77, 79, 80\}$	W_{823}^{C}	(0, -694)
$N_{82.39}$	$\{3, 9, 43, 44, 49, 50, 51, 52, 55, 61, 65, 71, 75, 81\}$	$W_{82,3}^{\tilde{C},3}$	(0, -696)
$N_{82,40}$	$\{6, 7, 13, 42, 44, 49, 50, 52, 54, 55, 57, 63, 72, 74\}$	$W^{\bar{C},\bar{c}}_{82,3}$	(0, -698)

Table 1: Singly even self-dual [82, 41, 14] neighbors $N_{82,i}$ (continued)

Code	$\operatorname{supp}(x)$	W	(lpha,eta)
$N_{82,41}$	$\{2, 4, 8, 13, 45, 46, 49, 51, 58, 65, 66, 73, 74, 80\}$	$W^{C}_{82,3}$	(0, -700)
$N_{82,42}$	$\{3, 9, 12, 45, 54, 55, 59, 64, 66, 72, 74, 75, 78, 80\}$	$W^{C'}_{82,3}$	(0, -706)
$N_{82,43}$	$\{2, 4, 9, 10, 45, 55, 56, 57, 60, 64, 67, 69, 72, 74\}$	$W^{C'}_{82,3}$	(0, -708)
$N_{82,44}$	$\{4, 9, 11, 40, 45, 46, 55, 57, 63, 64, 65, 71, 72, 74\}$	$W^{C'}_{82,3}$	(0, -710)
$N_{82,45}$	$\{3, 44, 45, 46, 57, 60, 61, 62, 63, 70, 71, 74, 75, 77\}$	$W^{C'}_{82,3}$	(0, -712)
$N_{82,46}$	$\{7, 40, 44, 45, 52, 53, 55, 56, 67, 68, 71, 76, 79, 81\}$	$W^{C'}_{82,3}$	(0, -716)
$N_{82,47}$	$\{3, 5, 9, 12, 42, 45, 47, 51, 53, 55, 60, 64, 68, 75\}$	$W^{C'}_{82.3}$	(0, -718)
$N_{82,48}$	$\{6, 39, 44, 45, 54, 60, 62, 64, 65, 75, 77, 78, 79, 81\}$	$W^{C'}_{82,3}$	(0, -720)
$N_{82,49}$	$\{2, 5, 9, 43, 60, 61, 62, 64, 68, 71, 74, 76, 80, 81\}$	$W^{C'}_{82,3}$	(0, -724)
$N_{82,50}$	$\{3, 7, 9, 13, 43, 46, 48, 49, 50, 52, 58, 60, 63, 81\}$	$W^{C'}_{82,3}$	(0, -728)

It remains to determine whether there is a singly even self-dual [82, 41, 14] code with shadow of minimum weight 17.

At the end of this section, we summarize the current information on the weight enumerators which actually occur for the possible weight enumerators. A singly even self-dual [82, 41, 14] code with weight enumerator $W_{82,1}^C$ is known (see Proposition 7). A singly even self-dual [82, 41, 14] code with weight enumerator $W_{82,2}^C$ is known for $(\alpha, \beta) = (18, -750)$ (see Table 1). A singly even self-dual [82, 41, 14] code with weight enumerator $W_{82,2}^C$ is known for $(\alpha, \beta) = (18, -750)$ (see Table 1). A singly even self-dual [82, 41, 14] code with weight enumerator $W_{82,3}^C$ is known for

$$\begin{split} \alpha &= 0 \text{ and } \beta = -\ 640, -650, -656, -660, -662, -664, -668, -672, -676, \\ &-\ 678, -680, -684, -686, -688, -690, -692, -694, -696, \\ &-\ 698, \\ \alpha &= 1 \text{ and } \beta = -\ 650, -668, -680, -682, -686, -688, -692, -694, -696, \\ &-\ 698, -700, -702, -704, -712, -722, -738, -748, \\ \alpha &= 2 \text{ and } \beta = -\ 672, -720, -732, -734 \end{split}$$

(see (1) and Table 1).

Acknowledgment. This work was supported by JSPS KAKENHI Grant Number 15H03633. The author would like to thank the anonymous referee for useful comments.

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