# Singly even self-dual codes of length $24 k+10$ and minimum weight $4 k+2$ 

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#### Abstract

Currently, the existence of an extremal singly even self-dual code of length $24 k+10$ is unknown for all nonnegative integers $k$. In this note, we study singly even self-dual $[24 k+10,12 k+5,4 k+2]$ codes. We give some restrictions on the possible weight enumerators of singly even self-dual $[24 k+10,12 k+5,4 k+2]$ codes with shadows of minimum weight at least 5 for $k=2,3,4,5$. We discuss a method for constructing singly even self-dual codes with minimal shadow. As an example, a singly even self-dual $[82,41,14]$ code with minimal shadow is constructed for the first time. In addition, as neighbors of the code, we construct singly even self-dual $[82,41,14]$ codes with weight enumerator for which no singly even self-dual code was previously known to exist.


## 1 Introduction

Extremal self-dual codes are an important class of linear codes for both theoretical and practical reasons. It is a fundamental problem to determine the largest minimum weight among self-dual codes of that length, and much work has been done concerning this problem.

A (binary) code $C$ of length $n$ is a vector subspace of $\mathbb{F}_{2}^{n}$, where $\mathbb{F}_{2}$ denotes the finite field of order 2 . All codes in this note are binary. The dual code

[^0]$C^{\perp}$ of $C$ is defined as $C^{\perp}=\left\{x \in \mathbb{F}_{2}^{n} \mid x \cdot y=0\right.$ for all $\left.y \in C\right\}$, where $x \cdot y$ is the standard inner product. A code $C$ is called self-dual if $C=C^{\perp}$. Self-dual codes are divided into two classes. A self-dual code $C$ is doubly even if all codewords $x$ of $C$ have weight $\mathrm{wt}(x) \equiv 0(\bmod 4)$, and singly even if there is at least one codeword of weight $\equiv 2(\bmod 4)$. A doubly even self-dual code of length $n$ exists if and only if $n \equiv 0(\bmod 8)$, while a singly even self-dual code of length $n$ exists if and only if $n$ is even.

Let $C$ be a singly even self-dual code. Let $C_{0}$ denote the subcode of $C$ consisting of codewords $x$ having weight $\mathrm{wt}(x) \equiv 0(\bmod 4)$. The shadow $S$ of $C$ is defined to be $C_{0}^{\perp} \backslash C$. A singly even self-dual code of length $n$ is called a code with minimal shadow if the minimum weight of the shadow is $4,1,2$ and 3 if $n \equiv 0,2,4$ and $6(\bmod 8)$, respectively. The concept of singly even self-dual codes with minimal shadow was introduced in [4].

Rains [14] showed that the minimum weight $d$ of a self-dual code of length $n$ is bounded by $d \leq 4\lfloor n / 24\rfloor+4$ unless $n \equiv 22(\bmod 24)$ when $d \leq 4\lfloor n / 24\rfloor+$ 6. A self-dual code meeting the upper bound is called extremal. We say that a self-dual code is optimal if it has the largest minimum weight among all self-dual codes of that length. For length $24 k+10(k=0,1, \ldots, 5)$, we give the current information on the largest minimum weight $d(24 k+10)$ :

$$
\begin{aligned}
d(10)=2, d(34)=6, d(58)=10, d(82) & =14 \text { or } 16, \\
d(106) & =16 \text { or } 18, d(130)=20,22 \text { or } 24,
\end{aligned}
$$

(see [6, Table I], [7, Table VI], [11, Table I]). Currently, the existence of an extremal singly even self-dual code of length $24 k+10$ is unknown for all nonnegative integers $k$. In addition, Han and Lee [9] conjecture that there is no extremal singly even self-dual code of length $24 k+10$ for all nonnegative integers $k$. It was shown in [5] that there is no extremal singly even self-dual code with minimal shadow for length $24 k+10$. These motivate our interest in singly even self-dual $[24 k+10,12 k+5,4 k+2]$ codes.

This note is organized as follows. In Section 2, the possible weight enumerators of singly even self-dual [82, 41, 14] codes are determined. In addition, in Section 3, we give some restrictions on the possible weight enumerators of singly even self-dual $[24 k+10,12 k+5,4 k+2]$ codes with shadows of minimum weight at least 5 for $k=2,3,4,5$. In Section 4 , we discuss a method for constructing singly even self-dual codes with minimal shadow. As an example, a singly even self-dual [82, 41, 14] code $C_{82}$ with minimal shadow is constructed for the first time. Finally, in Section 5, as neighbors of $C_{82}$,
we construct singly even self-dual $[82,41,14]$ codes with weight enumerator for which no singly even self-dual code was previously known to exist. It is a fundamental problem to find which weight enumerators actually occur for the possible weight enumerators. We emphasize that singly even self-dual [ $82,41,14$ ] codes with shadows of minimum weight $1,5,9$ are constructed for the first time.

All computer calculations in this note were done with the help of the algebra software Magma [2] and the mathematical software Mathematica.

## 2 Weight enumerators of singly even self-dual [82, 41, 14] codes

Let $C$ be a singly even self-dual code of length $n$ with shadow $S$. Let $A_{i}$ and $B_{i}$ be the numbers of vectors of weight $i$ in $C$ and $S$, respectively. The weight enumerators $W_{C}$ and $W_{S}$ of $C$ and $S$ are given by $\sum_{i=0}^{n} A_{i} y^{i}$ and $\sum_{i=d(S)}^{n-d(S)} B_{i} y^{i}$, respectively, where $d(S)$ denotes the minimum weight of $S$. If we write

$$
W_{C}=\sum_{j=0}^{\lfloor n / 8\rfloor} a_{j}\left(1+y^{2}\right)^{n / 2-4 j}\left(y^{2}\left(1-y^{2}\right)^{2}\right)^{j},
$$

for suitable integers $a_{j}$, then

$$
W_{S}=\sum_{j=0}^{\lfloor n / 8\rfloor}(-1)^{j} a_{j} 2^{n / 2-6 j} y^{n / 2-4 j}\left(1-y^{4}\right)^{2 j}
$$

[6, (10), (11)]. Suppose that $C$ is a singly even self-dual [82, 41, 14] code. Since the minimum weight is 14 , we have

$$
\begin{aligned}
a_{0}=1, a_{1}=-41, a_{2}=615, a_{3}=-4182 & \\
& a_{4}=13161, a_{5}=-18040, a_{6}=9512 .
\end{aligned}
$$

Then the weight enumerator of the shadow $S$ is written as:

$$
\begin{aligned}
& \frac{a_{10}}{524288} y+\left(-\frac{a_{9}}{8192}-\frac{5 a_{10}}{131072}\right) y^{5}+\left(\frac{a_{8}}{128}+\frac{9 a_{9}}{4096}+\frac{95 a_{10}}{262144}\right) y^{9} \\
& +\left(-\frac{a_{7}}{2}-\frac{a_{8}}{8}-\frac{153 a_{9}}{8192}-\frac{285 a_{10}}{131072}\right) y^{13}+\cdots .
\end{aligned}
$$

- $d(S)=1$ : From [6, (6)], $S$ has a unique vector of weight 1 and $S$ has no vector of weights 5 and 9 . Hence, $a_{10}=524288, a_{9}=-163840$ and $a_{8}=21760$. Since $A_{14}=B_{13}$ by [10],

$$
3280+a_{7}=-800-\frac{a_{7}}{2}
$$

Thus, we have that $a_{7}=-2720$. Therefore, we have the following possible weight enumerators

$$
\begin{aligned}
& W_{82,1}^{C}=1+560 y^{14}+60724 y^{16}+233545 y^{18}+\cdots \\
& W_{82,1}^{S}=y+560 y^{13}+294269 y^{17}+33367568 y^{21}+\cdots
\end{aligned}
$$

respectively.

- $d(S)=5:$ From [6, (6)], we have $a_{10}=0$ and $a_{9}=-8192$. Then we have that $a_{8}$ is divisible by 128, say $a_{8}=128 \alpha$ and $a_{7}$ is divisible by 2 , say $a_{7}=2 \beta$, where $\alpha$ and $\beta$ are integers. Therefore, we have the following possible weight enumerators

$$
\begin{aligned}
W_{82,2}^{C}= & 1+(3280+2 \beta) y^{14}+(36244+128 \alpha-2 \beta) y^{16} \\
& +(506153-896 \alpha-26 \beta) y^{18}+\cdots, \\
W_{82,2}^{S}= & y^{5}+(-18+\alpha) y^{9}+(153-16 \alpha-\beta) y^{13} \\
& +(303568+120 \alpha+14 \beta) y^{17}+\cdots,
\end{aligned}
$$

respectively. Note that $\alpha \geq 18$ and $\beta \leq 153-16 \alpha \leq-135$. In Section 3, it is shown that $\beta$ is an even integer (see Proposition (2).

- $d(S) \geq 9$ : Then we have $a_{10}=a_{9}=0$. We have that $a_{8}$ is divisible by 128 , say $a_{8}=128 \alpha$ and $a_{7}$ is divisible by 2 , say $a_{7}=2 \beta$, where $\alpha$ and $\beta$ are integers. Therefore, we have the following possible weight enumerators

$$
\begin{aligned}
W_{82,3}^{C}= & 1+(3280+2 \beta) y^{14}+(36244+128 \alpha-2 \beta) y^{16} \\
& +(514345-896 \alpha-26 \beta) y^{18}+\cdots, \\
W_{82,3}^{S}= & \alpha y^{9}+(-16 \alpha-\beta) y^{13}+(304384+120 \alpha+14 \beta) y^{17} \\
& +(33293312-560 \alpha-91 \beta) y^{21}+\cdots,
\end{aligned}
$$

respectively. Note that $\alpha \geq 0$ and $\beta \leq-16 \alpha \leq 0$. In Section 3, it is shown that $\beta$ is an even integer (see Proposition (2).

It is unknown whether there is a singly even self-dual [82, 41, 16] code. The first example of a singly even self-dual $[82,41,14]$ code was found in [7, Section V]. The weight enumerators of the code and its shadow were given, however unfortunately the weight enumerator of the shadow was incorrectly stated and the correct weight enumerator is

$$
656 y^{13}+295200 y^{17}+33353008 y^{21}+\cdots
$$

This code has weight enumerator $W_{82,3}^{C}$ with

$$
\begin{equation*}
(\alpha, \beta)=(0,-656) \tag{1}
\end{equation*}
$$

The code was the only previously known singly even self-dual [82, 41, 14] code. In Sections 4 and 5, we construct singly even self-dual [82, 41, 14] codes with weight enumerator for which no singly even self-dual code was previously known to exist.

## 3 Restrictions on weight enumerators of singly even self-dual $[24 k+10,12 k+5,4 k+2]$ codes

It was shown in [3] that the weight enumerator of a singly even self-dual $[24 k+10,12 k+5,4 k+2]$ code with minimal shadow is uniquely determined. In this section, we give some restrictions on the possible weight enumerators of singly even self-dual $[24 k+10,12 k+5,4 k+2]$ codes with shadows of minimum weight at least 5 for $k=2,3,4,5$. It is a key idea to consider the possible weight enumerator of $C_{1}$.

### 3.1 Possible weight enumerators of $C_{1}$

Let $C$ be a singly even self-dual $[24 k+10,12 k+5,4 k+2]$ code. Let $W^{(1)}$ and $W^{(3)}$ denote the weight enumerators of $C_{1}$ and $C_{3}$, respectively. By [6, Theorem 5,5$)]$, the possible weight enumerators $W^{(1)}-W^{(3)}$ are written as:

$$
\begin{equation*}
W^{(1)}-W^{(3)}=\sum_{i=0}^{k-1} b_{i}\left(1+14 y^{4}+y^{8}\right)^{3 k-1-3 i}\left(y^{4}\left(1-y^{4}\right)^{4}\right)^{i} f(y) \tag{2}
\end{equation*}
$$

where $f(y)=y-34 y^{5}+34 y^{13}-y^{17}$ and $b_{0}, b_{1}, \ldots, b_{k-1}$ are integers. Combined with the possible weight enumerators of the shadow, using (2), the possible weight enumerators of $C_{1}$ are determined.

### 3.2 Optimal singly even self-dual [58, 29, 10] codes with shadows of minimum weight at least 5

The possible weight enumerators of optimal singly even self-dual [58, 29, 10] codes with shadow of minimum weight at least 5 and the shadows are known as follows:

$$
\begin{aligned}
W_{58}^{C}= & 1+(319-24 \beta-2 \gamma) y^{10}+(3132+152 \beta+2 \gamma) y^{12}+\cdots \\
W_{58}^{S}= & \beta y^{5}+\gamma y^{9}+(24128-54 \beta-10 \gamma) y^{13} \\
& +(1469952+320 \beta+45 \gamma) y^{17}+\cdots
\end{aligned}
$$

respectively, where $\beta, \gamma$ are integers [6]. If there is an optimal singly even self-dual $[58,29,10]$ code with weight enumerator $W_{58}^{C}$, then $\beta \in\{0,1,2\}$ [12]. An optimal singly even self-dual code with weight enumerator $W_{58}^{C}$ is known for

$$
\begin{aligned}
& \beta=0 \text { and } \gamma \in\{2 m \mid m=0,1, \ldots, 65,68,71,79\}, \\
& \beta=1 \text { and } \gamma \in\{2 m \mid m=8,9, \ldots, 58,63\}, \\
& \beta=2 \text { and } \gamma \in\{2 m \mid m=0,4,6, \ldots, 55\}
\end{aligned}
$$

(see [10]).
Theorem 1. If there is an optimal singly even self-dual [58, 29, 10] code with weight enumerator $W_{58}^{C}$, then $\gamma$ must be an even integer.

Proof. Let $C$ be an optimal singly even self-dual [58, 29, 10] code with weight enumerator $W_{58}^{C}$. From (22), we obtain

$$
\begin{aligned}
W^{(1)}-W^{(3)}= & b_{0} y+\left(36 b_{0}+b_{1}\right) y^{5}+\left(-415 b_{0}-10 b_{1}\right) y^{9} \\
& +\left(-39056 b_{0}-724 b_{1}\right) y^{13}+\left(-742131 b_{0}-3694 b_{1}\right) y^{17}+\cdots
\end{aligned}
$$

Since the shadow contains no vector of weight 1 , we have that $b_{0}=0$. Hence, we have

$$
\begin{aligned}
W^{(1)}= & \frac{1}{2}\left(b_{1}+\beta\right) y^{5}+\left(-5 b_{1}+\frac{\gamma}{2}\right) y^{9}+\left(12064-362 b_{1}-27 \beta-5 \gamma\right) y^{13} \\
& +\left(734976-1847 b_{1}+160 \beta+\frac{45 \gamma}{2}\right) y^{17}+\cdots
\end{aligned}
$$

The result follows.

### 3.3 Singly even self-dual [82, 41, 14] codes with shadows of minimum weight at least 5

Proposition 2. If there is a singly even self-dual $[82,41,14]$ code with weight enumerator $W_{82, i}^{C}$, then $\beta$ must be an even integer for $i=2,3$.

Proof. Let $C$ be a singly even self-dual $[82,41,14]$ code with shadow $S$ of minimum weight at least 5 . From $W_{82, i}^{C}$ and $W_{82, i}^{S}(i=2,3)$, the possible weight enumerators of $C$ and $S$ are written using integers $a, b, c$

$$
\begin{aligned}
W_{82}^{C}= & 1+(3280+2 c) y^{14}+(36244+128 b-2 c) y^{16} \\
& +(514345-8192 a-896 b-26 c) y^{18}+\cdots, \\
W_{82}^{S}= & a y^{5}+(-18 a+b) y^{9}+(153 a-16 b-c) y^{13} \\
& +(304384-816 a+120 b+14 c) y^{17}+\cdots,
\end{aligned}
$$

respectively. From (22), we obtain

$$
\begin{aligned}
W^{(1)}-W^{(3)}= & b_{0} y+\left(78 b_{0}+b_{1}\right) y^{5}+\left(1688 b_{0}+32 b_{1}+b_{2}\right) y^{9} \\
& +\left(-32382 b_{0}-553 b_{1}-14 b_{2}\right) y^{13} \\
& +\left(-2525349 b_{0}-37184 b_{1}-678 b_{2}\right) y^{17}+\cdots .
\end{aligned}
$$

Since the shadow contains no vector of weight 1 , we have that $b_{0}=0$. Hence, we have

$$
\begin{aligned}
W^{(1)}= & \frac{a+b_{1}}{2} y^{5}+\left(-9 a+16 b_{1}+\frac{b_{2}+b}{2}\right) y^{9} \\
& +\left(\frac{153 a-553 b_{1}-c}{2}-7 b_{2}-8 b\right) y^{13} \\
& +\left(152192-408 a-18592 b_{1}-339 b_{2}+60 b+7 c\right) y^{17}+\cdots .
\end{aligned}
$$

Since $a+b_{1}$ is even, $c$ must be an even integer. The result follows.

### 3.4 Singly even self-dual $[106,53,18]$ codes with shadows of minimum weight at least 5

By a method similar to that given in Section 2, the possible weight enumerators of singly even self-dual $[106,53,18]$ codes with shadows of minimum
weight at least 5 and the shadows are determined as follows:

$$
\begin{aligned}
W_{106}^{C}= & 1+(35245+2 d) y^{18}+(416262+128 c-2 d) y^{20} \\
& +(6586310+8192 b-896 c-34 d) y^{22} \\
& +(86626645+524288 a-106496 b+1024 c+34 d) y^{24}+\cdots, \\
W_{106}^{S}= & a y^{5}+(-24 a-b) y^{9}+(276 a+22 b+c) y^{13} \\
& +(-2024 a-231 b-20 c-d) y^{17}+\cdots,
\end{aligned}
$$

respectively, where $a, b, c, d$ are integers. If $a=0$, then $-b \in\{0,1,2\}$ by 12, Lemma 2].

Proposition 3. If there is a singly even self-dual $[106,53,18]$ code with weight enumerator $W_{106}^{C}$, then $d$ must be an even integer.

Proof. Let $C$ be a singly even self-dual $[106,53,18]$ code with weight enumerator $W_{106}^{C}$. From (2), we obtain

$$
\begin{aligned}
W^{(1)}-W^{(3)}= & b_{0} y+\left(120 b_{0}+b_{1}\right) y^{5}+\left(5555 b_{0}+74 b_{1}+b_{2}\right) y^{9} \\
& +\left(87440 b_{0}+1382 b_{1}+28 b_{2}+b_{3}\right) y^{13} \\
& +\left(-2666610 b_{0}-38670 b_{1}-675 b_{2}-18 b_{3}\right) y^{17}+\cdots .
\end{aligned}
$$

Since the shadow has minimum weight at least 5 , we have $b_{0}=0$. Hence, we have

$$
\begin{aligned}
W^{(1)}= & \frac{a+b_{1}}{2} y^{5}+\left(-12 a+37 b_{1}+\frac{b_{2}-b}{2}\right) y^{9} \\
& +\left(138 a+691 b_{1}+14 b_{2}+11 b+\frac{b_{3}+c}{2}\right) y^{13} \\
& +\left(-1012 a-19335 b_{1}-9 b_{3}-10 c-\frac{675 b_{2}+231 b+d}{2}\right) y^{17}+\cdots .
\end{aligned}
$$

Since $b_{2}-b$ is even, $d$ must be even. The result follows.
It is unknown whether there is a singly even self-dual $[106,53,18]$ code or not (see [11, Table I]).

### 3.5 Singly even self-dual [130, 65,22$]$ codes with shadows of minimum weight at least 5

By a method similar to that given in Section 2, the possible weight enumerators of singly even self-dual $[130,65,22]$ codes with shadows of minimum weight at least 5 and the shadows are determined as follows:

$$
\begin{aligned}
W_{130}^{C}= & +(388700+2 e) y^{22}+(4791150+128 d-2 e) y^{24} \\
& +(81082890+8192 c-896 d-42 e) y^{26} \\
& +(1200197180+524288 b-106496 c+512 d+42 e) y^{28} \\
& +(14196225992-33554432 a-9961472 b+532480 c \\
& +10752 d+420 e) y^{30}+\cdots, \\
W_{130}^{S}= & a y^{5}+(-30 a+b) y^{9}+(435 a-28 b-c) y^{13} \\
& +(-4060 a+378 b+26 c+d) y^{17} \\
& +(27405 a-3276 b-325 c-24 d-e) y^{21}+\cdots,
\end{aligned}
$$

respectively, where $a, b, c, d, e$ are integers.
Proposition 4. If there is a singly even self-dual [130,65,22] code with weight enumerator $W_{130}^{C}$, then e must be an even integer.

Proof. Let $C$ be a singly even self-dual $[130,65,22]$ code with weight enumerator $W_{130}^{C}$. From (2), we obtain

$$
\begin{aligned}
W^{(1)}-W^{(3)}= & b_{0} y+\left(162 b_{0}+b_{1}\right) y^{5}+\left(11186 b_{0}+116 b_{1}+b_{2}\right) y^{9} \\
& +\left(394498 b_{0}+5081 b_{1}+70 b_{2}+b_{3}\right) y^{13} \\
& +\left(4628826 b_{0}+65936 b_{1}+1092 b_{2}+24 b_{3}+b_{4}\right) y^{17} \\
& +\left(-226397710 b_{0}-2983519 b_{1}-43758 b_{2}-781 b_{3}-22 b_{4}\right) y^{21} \\
& +\cdots .
\end{aligned}
$$

Since the shadow has minimum weight at least 5 , we have $b_{0}=0$. Hence, we
have

$$
\begin{aligned}
W^{(1)}= & \frac{a+b_{1}}{2} y^{5}+\left(-15 a+58 b_{1}+\frac{b_{2}}{2}+\frac{b}{2}\right) y^{9} \\
& +\left(\frac{435 a}{2}+\frac{5081 b_{1}}{2}+35 b_{2}+\frac{b_{3}}{2}-14 b-\frac{c}{2}\right) y^{13} \\
& +\left(-2030 a+32968 b_{1}+546 b_{2}+12 b_{3}+\frac{b_{4}}{2}+189 b+13 c+\frac{d}{2}\right) y^{17} \\
& +\left(\frac{27405 a}{2}-\frac{2983519 b_{1}}{2}-21879 b_{2}-\frac{781 b_{3}}{2}-11 b_{4}-1638 b\right. \\
& \left.\quad-\frac{325 c}{2}-12 d-\frac{e}{2}\right) y^{21}+\cdots .
\end{aligned}
$$

From the coefficients of $y^{13}$ and $y^{21}, e$ must be even. The result follows.
It is unknown whether there is a singly even self-dual [130, 65, 22] code or not (see [11, Table I]).

## 4 Construction of singly even self-dual codes with minimal shadow

The following method for constructing singly even self-dual codes was given in [15]. Let $C$ be a doubly even self-dual code of length $8 t$. Let $x$ be a vector of odd weight. Let $C^{0}$ denote the subcode of $C$ consisting of all codewords which are orthogonal to $x$. Then there are cosets $C^{1}, C^{2}, C^{3}$ of $C^{0}$ such that $C^{0 \perp}=C^{0} \cup C^{1} \cup C^{2} \cup C^{3}$, where $C=C^{0} \cup C^{2}$ and $x+C=C^{1} \cup C^{3}$. Then

$$
\begin{equation*}
C(x)=\left(0,0, C^{0}\right) \cup\left(1,1, C^{2}\right) \cup\left(1,0, C^{1}\right) \cup\left(0,1, C^{3}\right) \tag{3}
\end{equation*}
$$

is a singly even self-dual code of length $8 t+2$. Using this method, a singly even self-dual code with minimal shadow was constructed in [15] for the parameters $[42,21,8]$ and $[58,29,10]$. This may be generalized as follows.

Theorem 5. Let $C$ be an extremal doubly even self-dual code of length $8 t$ with covering radius $R$. Then there is a vector $x$ of weight $2\left\lfloor\frac{R+1}{2}\right\rfloor-1$ such that $C(x)$ in (3) is a singly even self-dual $\left[8 t+2,4 t+1, \min \left\{4\left\lfloor\frac{t}{3}\right\rfloor+4,2\left\lfloor\frac{R+1}{2}\right\rfloor\right\}\right]$ code with minimal shadow.

Proof. Since there is a coset of minimum weight $R$, there is a coset of minimum weight $2\left\lfloor\frac{R+1}{2}\right\rfloor-1$ (see [1, Fact 4]). We denote the coset by $x+C$, where $x$ has weight $2\left\lfloor\frac{R+1}{2}\right\rfloor-1$. Then the code $C(x)$ in (3) is a self-dual code of length $8 t+2$ [15]. The minimum weight of $C^{0} \cup C^{2}$ is $4\left\lfloor\frac{t}{3}\right\rfloor+4$. The minimum weight of $C^{1} \cup C^{3}$ is $2\left\lfloor\frac{R+1}{2}\right\rfloor-1$. Hence, $C(x)$ has minimum weight $\min \left\{4\left\lfloor\frac{t}{3}\right\rfloor+4,2\left\lfloor\frac{R+1}{2}\right\rfloor\right\}$.

It remains to show that $C(x)$ has shadow of minimum weight 1 . Without loss of generality, we may assume that $x \in C^{1}$. Let $v$ be a vector of $C^{1}$. Then $v$ is written as $x+c$, where $c \in C^{0}$. Since $c \cdot x=0$, we obtain

$$
\begin{equation*}
\mathrm{wt}(x+c) \equiv \mathrm{wt}(x) \quad(\bmod 4) \tag{4}
\end{equation*}
$$

Let $w$ be a vector of $C^{3}$. Then $w$ is written as $x+c+c^{\prime}$, where $c \in C^{0}$ and $c^{\prime} \in C^{2}$. From (4) and $(x+c) \cdot c^{\prime}=1$, we obtain

$$
\mathrm{wt}\left(x+c+c^{\prime}\right) \equiv \mathrm{wt}(x)+2 \quad(\bmod 4)
$$

Suppose that $\mathrm{wt}(x) \equiv 1(\bmod 4)(\operatorname{resp} . \mathrm{wt}(x) \equiv 3(\bmod 4))$. Then $\left(0,0, C^{0}\right) \cup$ $\left(0,1, C^{3}\right)$ (resp. $\left.\left(0,0, C^{0}\right) \cup\left(1,0, C^{1}\right)\right)$ is the doubly even subcode of $C(x)$. In addition, the vector $(1,0, \ldots, 0)$ (resp. $(0,1,0, \ldots, 0)$ ) is orthogonal to any vector of the doubly even subcode. This shows that the shadow has minimum weight 1.

We concentrate on singly even self-dual $[24 k+10,12 k+5,4 k+2]$ codes with minimal shadow. There is no extremal singly even self-dual code of length $24 k+10$ with minimal shadow for any nonnegative integer $k$ 5]. Hence, we have the following proposition.

Proposition 6. If there is an extremal doubly even self-dual code of length $24 k+8$ with covering radius $R \geq 4 k+1$, then there is a singly even self-dual $[24 k+10,12 k+5,4 k+2]$ codes with minimal shadow.

The bordered double circulant extremal doubly even self-dual [80, 40, 16] code $B_{80,4}$ in [8] has generator matrix

$$
\left(\begin{array}{ccccc} 
& 0 & 1 & \cdots & 1 \\
& 1 & & & \\
I_{40} & \vdots & & R & \\
& 1 & & &
\end{array}\right)
$$

where $I_{40}$ is the identity matrix of order 40 and $R$ is the $39 \times 39$ circulant matrix with first row
(1111000000100101111101011101001101100011).

It was shown in [13] that $B_{80,4}$ has covering radius 13, where a coset of minimum weight 13 is given by $x_{80}+B_{80,4}$ and $x_{80}$ has the following support:

$$
\{2,5,8,11,14,17,20,23,26,29,32,35,38\}
$$

We denote the code $B_{80,4}\left(x_{80}\right)$ by $C_{82}$.
Proposition 7. The code $C_{82}$ is a singly even self-dual [82, 41, 14] code with minimal shadow.

For $k \geq 4$, only the extended quadratic residue code $Q R_{104}$ of length 104 is the known extremal doubly even self-dual code of length $24 k+8$. It is not known whether $Q R_{104}$ has covering radius $R \geq 17$. Our computer search failed to find a coset of weight $\geq 17$ in $Q R_{104}$.

## 5 New singly even self-dual [82, 41, 14] codes

In this section, we continue a search to find singly even self-dual $[82,41,14]$ codes with weight enumerator for which no singly even self-dual code was previously known to exist.

Two self-dual codes $C$ and $C^{\prime}$ of length $n$ are said to be neighbors if $\operatorname{dim}\left(C \cap C^{\prime}\right)=n / 2-1$. Any self-dual code of length $n$ can be reached from any other by taking successive neighbors (see [6]). By considering self-dual neighbors of $C_{82}$, we found 50 singly even self-dual [82,41,14] codes $N_{82, i}$ $(i=1,2, \ldots, 50)$ with weight enumerator for which no singly even self-dual code was previously known to exist. These codes are constructed as

$$
\left\langle\left(C_{82} \cap\langle x\rangle^{\perp}\right), x\right\rangle,
$$

where the $\operatorname{supports} \operatorname{supp}(x)$ of $x$ are listed in Table 1 . The weight enumerators $W$ and the values $(\alpha, \beta)$ are also listed in the table.

Combined with the known result in [7], the results in the previous section and this section show the following:

Proposition 8. There is a singly even self-dual [82, 41, 14] code with shadow of minimum weight $s$ for $s \in\{1,5,9,13\}$.

Table 1: Singly even self-dual $[82,41,14]$ neighbors $N_{82, i}$

| Code | $\operatorname{supp}(x)$ | W | ( $\alpha, \beta$ ) |
| :---: | :---: | :---: | :---: |
| $N_{82,1}$ | $\{2,7,10,14,47,51,54,56,58,59,62,64,72,79\}$ | $W_{82,2}^{C}$ | (18, -750) |
| $N_{82,2}$ | $\{2,7,12,13,14,42,47,56,57,59,61,71,73,79\}$ | $W_{82,3}^{C}$ | $(1,-650)$ |
| $N_{82,3}$ | $\{6,9,11,42,44,47,51,56,59,61,75,77,78,79\}$ | $W_{82,3}^{C}$ | $(1,-668)$ |
| $N_{82,4}$ | $\{5,9,45,49,55,59,61,63,66,70,71,72,75,81\}$ | $W_{82,3}^{C}$ | $(1,-680)$ |
| $N_{82,5}$ | $\{2,3,9,13,14,39,40,47,49,56,57,64,77,82\}$ | $W_{82,3}^{C}$ | $(1,-682)$ |
| $N_{82,6}$ | $\{2,3,9,11,12,45,46,49,53,64,72,75,77,80\}$ | $W_{82,3}^{C}$ | $(1,-686)$ |
| $N_{82,7}$ | $\{5,43,46,49,50,51,63,65,66,71,72,73,77,81\}$ | $W_{82,3}^{C}$ | $(1,-688)$ |
| $N_{82,8}$ | $\{3,4,5,7,9,45,48,55,56,58,61,66,73,77\}$ | $W_{82,3}^{C}$ | $(1,-692)$ |
| $N_{82,9}$ | $\{3,7,40,46,49,52,54,57,58,59,72,74,75,79\}$ | $W_{82,3}^{C}$ | $(1,-694)$ |
| $N_{82,10}$ | $\{3,11,14,44,45,46,49,51,59,71,72,76,77,81\}$ | $W_{82,3}^{C}$ | $(1,-696)$ |
| $N_{82,11}$ | $\{6,7,10,12,46,51,53,55,58,70,71,73,78,82\}$ | $W_{82,3}^{C}$ | $(1,-698)$ |
| $N_{82,12}$ | $\{5,8,47,51,52,57,61,66,67,71,72,74,79,80\}$ | $W_{82,3}^{C}$ | $(1,-700)$ |
| $N_{82,13}$ | $\{2,3,7,8,9,11,40,44,49,52,55,63,77,82\}$ | $W_{82,3}^{C}$ | $(1,-702)$ |
| $N_{82,14}$ | $\{11,12,45,46,49,50,52,55,60,62,66,70,71,81\}$ | $W_{82,3}^{C}$ | $(1,-704)$ |
| $N_{82,15}$ | $\{3,44,45,46,58,60,62,64,65,67,68,73,74,77\}$ | $W_{82,3}^{C}$ | $(1,-712)$ |
| $N_{82,16}$ | $\{2,4,10,43,45,46,49,54,64,66,76,78,80,81\}$ | $W_{82,3}^{C}$ | $(1,-722)$ |
| $N_{82,17}$ | $\{2,4,9,10,45,56,57,59,63,64,67,68,70,76\}$ |  | $(1,-738)$ |
| $N_{82,18}$ | $\{3,6,9,10,40,47,53,54,55,68,73,76,80,81\}$ |  | $(1,-748)$ |
| $N_{82,19}$ | $\{2,11,13,37,47,51,52,55,70,77,78,79,80,82\}$ |  | $(2,-672)$ |
| $N_{82,20}$ | $\{3,9,11,47,49,59,60,62,67,68,74,76,81,82\}$ |  | $(2,-720)$ |
| $N_{82,21}$ | $\{4,8,9,40,48,49,52,54,55,66,67,68,73,81\}$ | $W_{82,3}^{C}$ | $(2,-732)$ |
| $N_{82,22}$ | $\{5,6,8,11,44,45,53,56,57,61,62,64,65,66\}$ | $W_{82,3}^{C}$ | $(2,-734)$ |
| $N_{82,23}$ | $\{4,7,8,9,46,57,58,61,63,68,71,73,78,81\}$ | $W_{82,3}^{C}$ | $(0,-640)$ |
| $N_{82,24}$ | $\{2,3,5,10,40,44,57,58,60,63,65,71,76,79\}$ | $W_{82,3}^{C}$ | $(0,-650)$ |
| $N_{82,25}$ | $\{2,5,6,8,50,51,58,63,64,66,67,71,73,81\}$ | $W_{82,3}^{C}$ | $(0,-660)$ |
| $N_{82,26}$ | $\{2,3,9,46,54,56,59,60,61,62,67,76,78,82\}$ | $W_{82,3}^{C}$ | (0, -662) |
| $N_{82,27}$ | $\{4,5,38,40,48,53,56,57,62,64,66,69,71,76\}$ | $W_{82,3}^{C}$ | $(0,-664)$ |
| $N_{82,28}$ | $\{3,7,8,10,39,50,51,62,66,67,70,73,77,82\}$ | $W_{82,3}^{C}$ | $(0,-668)$ |
| $N_{82,29}$ | $\{2,43,45,46,50,51,52,53,61,69,72,74,77,81\}$ | $W_{82,3}^{C}$ | $(0,-672)$ |
| $N_{82,30}$ | $\{6,7,9,40,58,61,63,70,73,77,79,80,81,82\}$ | $W_{82,3}^{C}$ | $(0,-676)$ |
| $N_{82,31}$ | $\{3,4,5,7,43,45,48,50,54,59,64,70,71,81\}$ | $W_{82,3}^{C}$ | $(0,-678)$ |
| $N_{82,32}$ | $\{6,11,50,53,54,56,59,61,64,68,69,72,74,76\}$ | $W_{82,3}^{C}$ | $(0,-680)$ |
| $N_{82,33}$ | $\{8,11,12,35,49,50,53,56,57,58,62,72,77,82\}$ | $W_{82,3}^{C}$ | $(0,-684)$ |
| $N_{82,34}$ | $\{5,11,46,56,57,58,60,62,63,64,65,70,71,79\}$ | $W_{82,3}^{C}$ | $(0,-686)$ |
| $N_{82,35}$ | $\{10,11,13,14,52,54,60,64,70,71,72,76,77,80\}$ | $W_{82,3}^{C}$ | $(0,-688)$ |
| $N_{82,36}$ | $\{5,9,45,49,56,57,61,62,63,64,67,70,75,81\}$ | $W_{82,3}^{C}$ | $(0,-690)$ |
| $N_{82,37}$ | $\{2,6,8,9,44,45,48,56,66,68,75,77,80,81\}$ | $W_{82,3}^{C}$ | $(0,-692)$ |
| $N_{82,38}$ | $\{4,8,10,42,44,54,58,60,63,65,68,77,79,80\}$ | $W_{82,3}^{C}$ | $(0,-694)$ |
| $N_{82,39}$ | $\{3,9,43,44,49,50,51,52,55,61,65,71,75,81\}$ | $W_{82,3}^{C}$ | $(0,-696)$ |
| $N_{82,40}$ | $\{6,7,13,42,44,49,50,52,54,55,57,63,72,74\}$ | $W_{82,3}^{C}$ | $(0,-698)$ |

Table 1: Singly even self-dual $[82,41,14]$ neighbors $N_{82, i}$ (continued)

| Code | $\operatorname{supp}(x)$ | $W$ | $(\alpha, \beta)$ |
| :---: | :--- | :---: | :---: |
| $N_{82,41}$ | $\{2,4,8,13,45,46,49,51,58,65,66,73,74,80\}$ | $W_{82,3}^{C}$ | $(0,-700)$ |
| $N_{82,42}$ | $\{3,9,12,45,54,55,59,64,66,72,74,75,78,80\}$ | $W_{82,3}^{C}$ | $(0,-706)$ |
| $N_{82,43}$ | $\{2,4,9,10,45,55,56,57,60,64,67,69,72,74\}$ | $W_{82,3}^{C}$ | $(0,-708)$ |
| $N_{82,44}$ | $\{4,9,11,40,45,46,55,57,63,64,65,71,72,74\}$ | $W_{82,3}^{C}$ | $(0,-710)$ |
| $N_{82,45}$ | $\{3,44,45,46,57,60,61,62,63,70,71,74,75,77\}$ | $W_{82,3}^{C}$ | $(0,-712)$ |
| $N_{82,46}$ | $\{7,40,44,45,52,53,55,56,67,68,71,76,79,81\}$ | $W_{82,3}^{C}$ | $(0,-716)$ |
| $N_{82,47}$ | $\{3,5,9,12,42,45,47,51,53,55,60,64,68,75\}$ | $W_{82,3}^{C}$ | $(0,-718)$ |
| $N_{82,48}$ | $\{6,39,44,45,54,60,62,64,65,75,77,78,79,81\}$ | $W_{82,3}^{C}$ | $(0,-720)$ |
| $N_{82,49}$ | $\{2,5,9,43,60,61,62,64,68,71,74,76,80,81\}$ | $W_{82,3}^{C}$ | $(0,-724)$ |
| $N_{82,50}$ | $\{3,7,9,13,43,46,48,49,50,52,58,60,63,81\}$ | $W_{82,3}^{C}$ | $(0,-728)$ |

It remains to determine whether there is a singly even self-dual [82, 41, 14] code with shadow of minimum weight 17 .

At the end of this section, we summarize the current information on the weight enumerators which actually occur for the possible weight enumerators. A singly even self-dual $[82,41,14]$ code with weight enumerator $W_{82,1}^{C}$ is known (see Proposition (7). A singly even self-dual [82, 41, 14] code with weight enumerator $W_{82,2}^{C}$ is known for $(\alpha, \beta)=(18,-750)$ (see Table (1). A singly even self-dual $[82,41,14]$ code with weight enumerator $W_{82,3}^{C}$ is known for

$$
\begin{aligned}
\alpha=0 \text { and } \beta= & -640,-650,-656,-660,-662,-664,-668,-672,-676, \\
& -678,-680,-684,-686,-688,-690,-692,-694,-696, \\
& -698, \\
\alpha=1 \text { and } \beta= & -650,-668,-680,-682,-686,-688,-692,-694,-696, \\
& -698,-700,-702,-704,-712,-722,-738,-748, \\
\alpha=2 \text { and } \beta= & -672,-720,-732,-734
\end{aligned}
$$

(see (1) and Table (1).
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