The linear codes of *t*-designs held in the Reed-Muller and Simplex codes

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Abstract

A fascinating topic of combinatorics is t-designs, which have a very long history. The incidence matrix of a t-design generates a linear code over GF(q) for any prime power q, which is called the linear code of the t-design over GF(q). On the other hand, some linear codes hold t-designs for some $t \ge 1$. The purpose of this paper is to study the linear codes of some t-designs held in the Reed-Muller and Simplex codes. Some general theory for the linear codes of t-designs held in linear codes is presented. Open problems are also presented.

Keywords: Cyclc code, linear code, Reed-Muller code, *t*-design.

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1. Introduction

Let \mathcal{P} be a set of $v \ge 1$ elements, and let \mathcal{B} be a set of k-subsets of \mathcal{P} , where k is a positive integer with $1 \le k \le v$. Let t be a positive integer with $t \le k$. The pair $\mathbb{D} = (\mathcal{P}, \mathcal{B})$ is called a t- (v, k, λ) design, or simply t-design, if every t-subset of \mathcal{P} is contained in exactly λ elements of \mathcal{B} . The elements of \mathcal{P} are called points, and those of \mathcal{B} are referred to as blocks. We usually use t to denote the number of blocks in t to design is called simple if t does not contain repeated blocks. In this paper, we consider only simple t-designs. A t-design is called symmetric if t is clear that t-designs with t is t always exist. Such t-designs are trivial. In this paper, we consider only t-designs with t is t in this paper, we consider only t-designs with t is t in this paper.

1.1. The codes of designs

Let $\mathbb{D} = (\mathcal{P}, \mathcal{B})$ be a t- (v, k, λ) design with $b \geq 1$ blocks. The points of \mathcal{P} are usually indexed with p_1, p_2, \cdots, p_v , and the blocks of \mathcal{B} are normally denoted by B_1, B_2, \cdots, B_b . The *incidence matrix* $M_{\mathbb{D}} = (m_{ij})$ of \mathbb{D} is a $b \times v$ matrix where $m_{ij} = 1$ if p_j is on B_i and $m_{ij} = 0$ otherwise. The binary matrix $M_{\mathbb{D}}$ is viewed as a matrix over GF(q) for any prime power q, and its row vectors span a linear code of length v over GF(q), which is denoted by $C_q(\mathbb{D})$ and called the *code* of \mathbb{D} over GF(q). It is clear that the code $C_q(\mathbb{D})$ depends on the labelling of the points of \mathbb{D} , but is unique up to coordinate permutations.

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1.2. The support designs of linear codes

We assume that the reader is familiar with the basics of linear codes and cyclic codes, and proceed to introduce the support designs of linear codes directly. Let C be a $[v, \kappa, d]$ linear code over GF(q). Let $A_i := A_i(C)$, which denotes the number of codewords with Hamming weight i in C, where $0 \le i \le v$. The sequence (A_0, A_1, \cdots, A_v) is called the *weight distribution* of C, and $\sum_{i=0}^{v} A_i z^i$ is referred to as the *weight enumerator* of C. For each k with $A_k \ne 0$, let \mathcal{B}_k denote the set of the supports of all codewords with Hamming weight k in C, where the coordinates of a codeword are indexed by (p_1, \ldots, p_v) . Let $\mathcal{P} = \{p_1, \ldots, p_v\}$. The pair $(\mathcal{P}, \mathcal{B}_k)$ may be a t- (v, k, λ) design for some positive integer λ , which is called a *support design* of the code, and is denoted by $\mathbb{D}_k(C)$. In such a case, we say that the code C holds a t- (v, k, λ) design. Throughout this paper, we denote the dual code of C by \mathbb{C}^\perp , and the extended code of C by $\overline{\mathbb{C}}$.

1.3. The objectives of this paper

While most linear codes over finite fields do not hold t-designs, some linear codes do hold t-designs for $t \ge 1$. Studying the linear codes of t-designs has been a topic of research for a long time [1, 3, 2, 5, 8, 15, 17, 24, 25, 26].

Let q_1 be a power of a prime p. Our starting point is a linear code C_1 over a finite field $GF(q_1)$, which holds a t-design $\mathbb{D}_k(C_1)$, our objective is to study the classical linear code $C_2 = C_{q_2}(\mathbb{D}_k(C_1))$ over a finite field $GF(q_2)$, and hope that the new code C_2 has interesting parameters and properties. This idea is depicted as follows:

Original code
$$C_1$$
 over $GF(q_1) \Rightarrow A$ t -design $\mathbb{D}_k(C_1)$ held in $C_1 \Rightarrow New \text{ code } C_2 := C_{q_2}(\mathbb{D}_k(C_1))$.

It may happen that $C_2 = C_1$, but they are different in many cases. Note that a linear code C_1 may hold exponentially many t-designs. We may obtain exponentially many new codes $C_2 = C_{q_2}(\mathbb{D}_k(C_1))$ from the original code C_1 . Although the finite field $GF(q_2)$ has many choices for the given C_1 and q_1 , we will restrict ourself to the case $q_2 = p$ for simplicity in most parts of this paper. It is well known that the code $C_p(\mathbb{D})$ of a t- (v,k,λ) design \mathbb{D} has dimension less than v-1 only if p divides $\lambda_1 - \lambda_2$, where λ_i denotes the number of blocks of \mathbb{D} that contain i points (i = 1,2) (cf. [14], [26, Theorem 1.86].)

In this paper, we will consider several families of linear codes C over GF(q) and some of the designs $\mathbb{D}_k(C)$ held in C, and will determine the parameters of the linear code $C_p(\mathbb{D}_k(C))$ for some designs $\mathbb{D}_k(C)$ held in C. This is doable in the case that q=2, but is a hard problem for q>2. In the binary case, we will present some general theory for codes $C_2(\mathbb{D}_k(C))$. The objective of this paper is to study the linear codes of some known t-designs held in the generalised Reed-Muller codes and the Simplex codes. Some general theory for the linear codes of t-designs held in linear codes is presented. Open problems on this topic will also be presented.

2. Auxiliary results

2.1. Designs from linear codes via the Assmus-Mattson Theorem

The following theorem, developed by Assumus and Mattson, shows that the pair $(\mathcal{P}, \mathcal{B}_k)$ defined by a linear code is a *t*-design under certain conditions.

Theorem 1 (Assmus-Mattson Theorem). ([2], [15, p. 303]) Let C be a [v,k,d] code over GF(q). Let d^{\perp} denote the minimum distance of C^{\perp} . Let w be the largest integer satisfying $w \leq v$ and

$$w - \left\lfloor \frac{w + q - 2}{q - 1} \right\rfloor < d.$$

Define w^{\perp} analogously using d^{\perp} . Let $(A_i)_{i=0}^{v}$ and $(A_i^{\perp})_{i=0}^{v}$ denote the weight distribution of C and C^{\perp} , respectively. Fix a positive integer t with t < d, and let s be the number of i with $A_i^{\perp} \neq 0$ for $1 \leq i \leq v - t$. Suppose $s \leq d - t$. Then

- the codewords of weight i in C hold a t-design provided $A_i \neq 0$ and $d \leq i \leq w$, and
- the codewords of weight i in C^{\perp} hold a t-design provided $A_i^{\perp} \neq 0$ and $d^{\perp} \leq i \leq \min\{v t, w^{\perp}\}.$

The Assmus-Mattson Theorem is a very useful tool for constructing *t*-designs from linear codes, and has been recently employed to construct infinitely many 2-designs and 3-designs.

2.2. Designs from linear codes via the automorphism group

In this section, we introduce the automorphism approach to obtaining *t*-designs from linear codes. To this end, we have to define the automorphism group of linear codes. We will also present some basic results about this approach.

The set of coordinate permutations that map a code C to itself forms a group, which is referred to as the *permutation automorphism group* of C and denoted by PAut(C). If C is a code of length n, then PAut(C) is a subgroup of the *symmetric group* Sym_n .

A monomial matrix over GF(q) is a square matrix having exactly one nonzero element of GF(q) in each row and column. A monomial matrix M can be written either in the form DP or the form PD_1 , where D and D_1 are diagonal matrices and P is a permutation matrix.

The set of monomial matrices that map C to itself forms the group MAut(C), which is called the *monomial automorphism group* of C. Clearly, we have

$$PAut(C) \subseteq MAut(C)$$
.

The *automorphism group* of C, denoted by Aut(C), is the set of maps of the form $M\gamma$, where M is a monomial matrix and γ is a field automorphism, that map C to itself. In the binary case, PAut(C), MAut(C) and Aut(C) are the same. If q is a prime, MAut(C) and Aut(C) are identical. In general, we have

$$PAut(C)\subseteq MAut(C)\subseteq Aut(C).$$

By definition, every element in $\operatorname{Aut}(C)$ is of the form $DP\gamma$, where D is a diagonal matrix, P is a permutation matrix, and γ is an automorphism of $\operatorname{GF}(q)$. The automorphism group $\operatorname{Aut}(C)$ is said to be t-transitive if for every pair of t-element ordered sets of coordinates, there is an element $DP\gamma$ of the automorphism group $\operatorname{Aut}(C)$ such that its permutation part P sends the first set to the second set.

The next theorem gives another sufficient condition for a linear code to hold t-designs [15, p. 308].

Theorem 2. Let C be a linear code of length n over GF(q) where Aut(C) is t-transitive. Then the codewords of any weight i > t of C hold a t-design.

2.3. Relations between $C_q(\mathbb{D})$ and $C_q(\mathbb{D}^c)$

Let \mathbb{D} be a t- (v, k, λ) design. Then its complement \mathbb{D}^c is a t- $(v, v - k, \lambda^c)$ design, where

$$\lambda^{c} = \lambda \frac{\binom{v-t}{k}}{\binom{v-t}{k-t}}.$$

Since \mathbb{D} and \mathbb{D}^c are complementary, the two codes $C_q(\mathbb{D})$ and $C_q(\mathbb{D}^c)$ should be related. Below we present a few relations between the two codes. We assume that the columns of both incidence matrices are indexed by the points in the same order.

Theorem 3. Let notation be the same as before. Let $\bar{1}$ denote the all-one vector.

- If $\overline{\mathbf{1}} \in \mathsf{C}_q(\mathbb{D})$ and $\overline{\mathbf{1}} \not\in \mathsf{C}_q(\mathbb{D}^c)$, then $\mathsf{C}_q(\mathbb{D}) \supseteq \mathsf{C}_q(\mathbb{D}^c)$ and $\dim(\mathsf{C}_q(\mathbb{D})) = \dim(\mathsf{C}_q(\mathbb{D}^c)) + 1$.
- If $\bar{\mathbf{1}} \in \mathsf{C}_q(\mathbb{D}^c)$ and $\bar{\mathbf{1}} \not\in \mathsf{C}_q(\mathbb{D})$, then $\mathsf{C}_q(\mathbb{D}^c) \supseteq \mathsf{C}_q(\mathbb{D})$ and $\dim(\mathsf{C}_q(\mathbb{D}^c)) = \dim(\mathsf{C}_q(\mathbb{D})) + 1$.
- If $\overline{\mathbf{1}} \in \mathsf{C}_q(\mathbb{D}) \cap \mathsf{C}_q(\mathbb{D}^c)$, then $\mathsf{C}_q(\mathbb{D}^c) = \mathsf{C}_q(\mathbb{D})$.
- If $\overline{\mathbf{1}} \not\in \mathsf{C}_q(\mathbb{D}) \cup \mathsf{C}_q(\mathbb{D}^c)$, then $\mathsf{C}_q(\mathbb{D}) \not\subseteq \mathsf{C}_q(\mathbb{D}^c)$ and $\mathsf{C}_q(\mathbb{D}^c) \not\subseteq \mathsf{C}_q(\mathbb{D})$. In addition,

$$\mathsf{C}_q(\mathbb{D})\cap\mathsf{C}_q(\mathbb{D}^c)=\left\{\sum_i b_i(\mathbf{\bar{1}}-\mathbf{g}_i):b_i\in\mathsf{GF}(q),\;\sum_i b_i=0
ight\},$$

where \mathbf{g}_i is the *i*-th row vector in the incidence matrix of \mathbb{D} .

Proof. By definition, $\bar{1} - g_1, \dots, \bar{1} - g_b$ are the rows of the incidence matrix of \mathbb{D}^c .

Assume that $\bar{\mathbf{1}} \in \mathsf{C}_q(\mathbb{D})$ and $\bar{\mathbf{1}} \not\in \mathsf{C}_q(\mathbb{D}^c)$. Then $\bar{\mathbf{1}} - \mathbf{g}_1, \cdots, \bar{\mathbf{1}} - \mathbf{g}_b$ are codewords of $\mathsf{C}_q(\mathbb{D})$ It then follows that $\mathsf{C}_q(\mathbb{D}) \supseteq \mathsf{C}_q(\mathbb{D}^c)$. Clearly, $\bar{\mathbf{1}}, \bar{\mathbf{1}} - \mathbf{g}_1, \cdots, \bar{\mathbf{1}} - \mathbf{g}_b$ generate $\mathbf{g}_1, \cdots, \mathbf{g}_b$. Since $\bar{\mathbf{1}} \not\in \mathsf{C}_q(\mathbb{D}^c)$, $\dim(\mathsf{C}_q(\mathbb{D})) = \dim(\mathsf{C}_q(\mathbb{D}^c)) + 1$.

The conclusion of the second part is symmetric to that of the first part. The conclusion of the third part follows from the proof of the first conclusion.

Finally, we prove the conclusions of the fourth part. On the contrary, suppose that $C_q(\mathbb{D}) \subseteq C_q(\mathbb{D}^c)$. Then $\mathbf{g}_1 \in C_q(\mathbb{D}^c)$, But $\overline{\mathbf{1}} - \mathbf{g}_1$ is also a codeword of $C_q(\mathbb{D}^c)$. Consequently $\overline{\mathbf{1}} = \mathbf{g}_1 + (\overline{\mathbf{1}} - \mathbf{g}_1)$ is a codeword of $C_q(\mathbb{D}^c)$, which is contrary to the assumption. Consequently, $C_q(\mathbb{D}) \not\subseteq C_q(\mathbb{D}^c)$. By symmetry, $C_q(\mathbb{D}^c) \not\subseteq C_q(\mathbb{D})$.

Let $\mathbf{g} \in \mathsf{C}_q(\mathbb{D}) \cap \mathsf{C}_q(\mathbb{D}^c)$. Since $\mathbf{g} \in \mathsf{C}_q(\mathbb{D})$, there are $c_i \in \mathsf{GF}(q)$ such that $\mathbf{g} = \sum_i c_i \mathbf{g}_i$. Similarly, there are $a_i \in \mathsf{GF}(q)$ such that $\mathbf{g} = \sum_i a_i (\mathbf{1} - \mathbf{g}_i)$. As a result,

$$\mathbf{g} = \sum_{i} c_i \mathbf{g}_i = \sum_{i} a_i (\mathbf{\bar{1}} - \mathbf{g}_i).$$

We then deduce that

$$(\sum_{i} a_i)\overline{\mathbf{1}} = \sum_{i} (c_i + a_i)\mathbf{g}_i \in \mathsf{C}_q(\mathbb{D}).$$

By assumption, $\bar{\mathbf{1}} \not\in C_q(\mathbb{D})$. It then follows that $\sum_i a_i = 0$. We then deduce that

$$\mathsf{C}_q(\mathbb{D})\cap\mathsf{C}_q(\mathbb{D}^c)\subseteq\left\{\sum_i a_i(\mathbf{1}-\mathbf{g}_i):a_i\in\mathsf{GF}(q),\;\sum_i a_i=0\right\}.$$

On the other hand, it is easily seen that

$$\mathsf{C}_q(\mathbb{D})\cap\mathsf{C}_q(\mathbb{D}^c)\supseteq\left\{\sum_i a_i(\mathbf{\bar{1}}-\mathbf{g}_i):a_i\in\mathsf{GF}(q),\;\sum_i a_i=0
ight\}.$$

The desired equality of the two sets finally follows. This completes the proof of this theorem. \Box

Theorem 3 is a refined and slightly extended result of the fact $C_q(\mathbb{D})+GF(q)\bar{\mathbf{1}}=C_q(\mathbb{D}^c)+GF(q)\bar{\mathbf{1}}$ pointed out in [1, p. 46]. It will play a vital role in this paper. It says that in the first three cases the two codes $C_q(\mathbb{D})$ and $C_q(\mathbb{D}^c)$ are closely related. Sometimes, it may be very hard to study $C_q(\mathbb{D})$ directly, but it may be possible to investigate $C_q(\mathbb{D}^c)$. One can then get information on $C_q(\mathbb{D}^c)$ from information on $C_q(\mathbb{D}^c)$. This is a key idea employed in this paper. To make use of this idea, we first need to know if $\bar{\mathbf{1}} \in C_q(\mathbb{D})$ or $\bar{\mathbf{1}} \in C_q(\mathbb{D}^c)$. This could be a hard problem itself. For instance, it took ten years to settle this problem for the binary linear codes of a class of symmetric designs [22]. In the last case (i.e., $\bar{\mathbf{1}} \not\in C_q(\mathbb{D}) \cup C_q(\mathbb{D}^c)$, the two codes $C_q(\mathbb{D})$ and $C_q(\mathbb{D}^c)$ are loosely related.

Theorem 4. Let q be a power of a prime p. Let \mathbb{D} be a t- (v,k,λ) design with $t \geq 2$. Put

$$\lambda_1 = \lambda \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}.$$

If $\lambda_1 \not\equiv 0 \pmod{p}$, then the all-one vector $\overline{\mathbf{1}}$ is a codeword in $C_q(\mathbb{D})$.

Proof. It is known that \mathbb{D} is also a 1- (v,k,λ_1) design. Consequently, every point is incident with λ_1 blocks. It then follows that the sum over GF(q) of the row vectors of the incidence matrix of \mathbb{D} is

$$(\lambda_1, \lambda_1, ..., \lambda_1) = (\lambda_1 \mod p)\overline{\mathbf{1}},$$

which is a codeword in $C_q(\mathbb{D})$. Therefore, $\bar{\mathbf{1}} \in C_q(\mathbb{D})$.

Theorem 4 will be employed in this paper shortly, and it is quite useful. We inform that the condition $\lambda_1 \not\equiv 0 \pmod{p}$ is not necessary for $\overline{\mathbf{1}}$ being a codeword of $C_q(\mathbb{D})$.

2.4. Relations between $C_p(\mathbb{D})$ and $C_q(\mathbb{D})$

Let $q = p^s$, where $s \ge 2$ and p is a prime. Let $\mathbb D$ be a t- (v,k,λ) design. In this section, we document some relations between $C_p(\mathbb D)$ and $C_q(\mathbb D)$.

Theorem 5. Let $q = p^s$, where $s \ge 2$. Let \mathbb{D} be a t- (v,k,λ) design. Then $C_p(\mathbb{D})$ is the subfield subcode over GF(p) of $C_q(\mathbb{D})$. Further,

$$\mathsf{C}_p(\mathbb{D})^{\perp} = \mathsf{Tr}(\mathsf{C}_q(\mathbb{D})^{\perp}),$$

where $\mathrm{Tr}(\mathsf{C}_q(\mathbb{D})^{\perp})$ denotes the trace code of $\mathsf{C}_q(\mathbb{D})^{\perp}$.

Proof. Let $\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_b$ be the row vectors in the incidence matrix of \mathbb{D} . Let α be a generator of $GF(q)^*$. Let $u_i = \sum_{j=0}^{s-1} u_{ij} \alpha^j \in GF(q)$ for all $1 \le i \le b$, where all $u_{ij} \in GF(p)$. We have then

$$\sum_{i=1}^b u_i \mathbf{m}_i = \sum_{i=1}^b \left(\sum_{j=0}^{s-1} u_{ij} \alpha^j \right) \mathbf{m}_i = \sum_{j=0}^{s-1} \left(\sum_{i=1}^b u_{ij} \mathbf{m}_i \right) \alpha^j.$$

Since each $\mathbf{m}_i \in GF(p)^{\nu}$, $\sum_{i=1}^b u_{ij}\mathbf{m}_i$ is a vector in $GF(p)^{\nu}$ for each j. It then follows that $\sum_{i=1}^b u_i\mathbf{m}_i \in GF(p)^{\nu}$ if and only if

$$\sum_{i=1}^{b} u_{ij} \mathbf{m}_{i} = \overline{\mathbf{0}}, \text{ for all } 1 \le j \le s - 1.$$

$$\tag{1}$$

If the system of equations in (1) holds, then

$$\sum_{i=1}^b u_i \mathbf{m}_i = \sum_{i=1}^b u_{i0} \mathbf{m}_i.$$

Consequently, $C_p(\mathbb{D})$ is the subfield subcode over GF(p) of $C_q(\mathbb{D})$. The last desired result then follows from Delsarte's theorem [7].

Theorem 6. Let $q = p^s$, where $s \ge 2$. Let $\mathbb D$ be a t- (v,k,λ) design. Then $C_p(\mathbb D) = \mathrm{Tr}(C_q(\mathbb D))$.

Proof. Let $\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_b$ be the row vectors in the incidence matrix of \mathbb{D} . Note that each $\mathbf{m}_i \in \mathrm{GF}(q)^{\nu}$ and each codeword of $\mathsf{C}_q(\mathbb{D})$ can be expressed as $\sum_{i=1}^b u_i \mathbf{m}_i$, where $u_i \in \mathrm{GF}(q)$. We have

$$\operatorname{Tr}\left(\sum_{i=1}^{b} u_i \mathbf{m}_i\right) = \sum_{i=1}^{b} \operatorname{Tr}(u_i) \mathbf{m}_i.$$

When u_i ranges over the elements in GF(q), $Tr(u_i)$ ranges over each element of GF(p) eaxctly p^{s-1} times. The desired conclusion then follows.

Theorem 7. Let $q = p^s$, where $s \ge 2$. Let \mathbb{D} be a t- (v,k,λ) design. Then $\dim_{GF(p)}(\mathsf{C}_p(\mathbb{D})) = \dim_{GF(q)}(\mathsf{C}_q(\mathbb{D}))$ and $d(\mathsf{C}_p(\mathbb{D})) = d(\mathsf{C}_q(\mathbb{D}))$, where $d(\mathsf{C})$ denotes the minimum distance of the code C .

Proof. Let $M_{\mathbb{D}} = (m_{ij})$ be the incidence matrix of \mathbb{D} . Since $\dim_{\mathrm{GF}(p)}(\mathsf{C}_p(\mathbb{D}))$, $\dim_{\mathrm{GF}(q)}(\mathsf{C}_q(\mathbb{D}))$ both equal to the rank of the matrix $M_{\mathbb{D}}$, then $\dim_{\mathrm{GF}(p)}(\mathsf{C}_p(\mathbb{D})) = \dim_{\mathrm{GF}(q)}(\mathsf{C}_q(\mathbb{D}))$.

By Theorem 5, $d(C_p(\mathbb{D})) \geq d(C_q(\mathbb{D}))$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_s\}$ be a basis of GF(q) over GF(p). Let $\mathbf{c} = (c_1, \dots, c_v)$ be any nonzero codeword in $C_q(\mathbb{D})$. Then, there are $\alpha_1, \dots, \alpha_b \in GF(q)$ such that $c_j = \sum_{i=1}^b m_{ij}\alpha_i$. Let $\alpha_i = \sum_{t=1}^s a_{it}\mathbf{e}_t$, where $a_{it} \in GF(p)$. Then $c_j = \sum_{i=1}^b \sum_{t=1}^s m_{ij}a_{it}\mathbf{e}_t$ and $\mathbf{c} = \sum_{t=1}^s \mathbf{e}_t\mathbf{c}^t$, where $\mathbf{c}^t = (\sum_{i=1}^b \sum_{t=1}^s m_{i1}a_{it}, \dots, \sum_{i=1}^b \sum_{t=1}^s m_{iv}a_{it}) \in C_p(\mathbb{D})$. There is a t_0 such that $\mathbf{c}^{t_0} \neq \mathbf{0}$, as $\mathbf{c} \neq \mathbf{0}$. Then $\mathrm{wt}(\mathbf{c}) \geq \mathrm{wt}(\mathbf{c}^{t_0}) \geq d(C_p(\mathbb{D}))$. Thus $d(C_q(\mathbb{D})) \geq d(C_p(\mathbb{D}))$. This completes the proof.

Theorem 7 explains why we restrict ourself to $C_p(\mathbb{D})$ rather than treating $C_q(\mathbb{D})$ in this paper, though the two codes have different weight distributions.

2.5. A general result about $C_q(\mathbb{D})^{\perp}$

The next result is useful [1, p. 54], and will be used later in this paper.

Theorem 8. Let $\mathbb{D} = (\mathcal{P}, \mathcal{B})$ be a 2- (v, k, λ) design with k < v. If $C_q(\mathbb{D}) \neq GF(q)^v$, then the minimum weight of $C_q(\mathbb{D})^{\perp}$ is at least

$$\frac{v-1}{k-1} + 1$$
.

The lower bound in Theorem 8 is not tight in general, but reasonably good in some special cases.

3. The binary case

In this section, we present some fundamental results about binary codes and their designs, which do not hold in general for nonbinary codes.

Theorem 9. Let C be an [n,k,d] binary code which holds designs. Let $\mathbb{D}_i(C)$ denote the support design of the codewords of weight i in C, where the point set is the set of coordinates, i.e., $\{0,1,\ldots,n-1\}$. Let $C_2(\mathbb{D}_i(C))$ denote the binary code of the design $\mathbb{D}_i(C)$, where the point set is the ordered set $\{0,1,\ldots,n-1\}$. Then the following statements are true.

- 1. $C_2(\mathbb{D}_i(C))$ is a subcode of C and they are equal if and only if the codewords of weight i span C.
- 2. $Aut(C) \leq Aut(\mathbb{D}_i(C))$, i.e., the former is a subgroup of the latter.
- 3. If the codewords of weight i or the codewords of weight i and the all-one vector $\overline{\mathbf{1}}$ generate C, then $Aut(C) = Aut(\mathbb{D}_i(C))$.

Proof. Notice that C is a binary linear code. By definition, each row of the incidence matrix of the design $\mathbb{D}_2(C)$ is a codeword of C. Consequently, $C_2(\mathbb{D}_i(C))$ is a subcode of C. The desired first conclusion then follows.

Recall that the automorphism group $\operatorname{Aut}(C)$ of a binary linear code C is its permutation automorphism group $\operatorname{PAut}(C)$. Any $\sigma \in \operatorname{Aut}(C)$ is clearly a permutation of the coordinates of the codewords in C that fixes C, and is thus a permutation of the point set and block set of $\mathbb{D}_i(C)$. This proves the conclusion of the second part.

We now prove the conclusion of the third part. Note that any permutation of $\{0,1,\ldots,n-1\}$ fixes the all-one vector $\overline{\bf 1}$. By assumption, every codeword of C is a linear combination of the rows of the incidence matrix of $\mathbb{D}_i(C)$ and the all-one vector. Then by assumption, any $\sigma \in \operatorname{Aut}(\mathbb{D}_i(C))$ is an element of $\operatorname{Aut}(C)$. Therefore, $\operatorname{Aut}(\mathbb{D}_i(C)) \leqslant \operatorname{Aut}(C)$. The desired third conclusion then follows from that of the second part.

The proof of Theorem 9 showed that $C_2(\mathbb{D}_i(C))$ is a subcode of the original code C. Regarding these two codes, we have the following comments:

1. C and $C_2(\mathbb{D}_i(C))$ have the same length, but may have different dimensions and minimum distances. In general,

$$\dim(\mathsf{C}_2(\mathbb{D}_i(\mathsf{C}))) \leq \dim(\mathsf{C}) \text{ and } d(\mathsf{C}_2(\mathbb{D}_i(\mathsf{C}))) \geq d(\mathsf{C}),$$

where d(C) denotes the minimum distance of C.

2. Let C be an [n,k,d] binary code. Then $C_2(\mathbb{D}_d(C))$ has parameters [n,k',d] with $k' \leq k$. When k' < k, $C_2(\mathbb{D}_d(C))$ is not as good as the original code C. However, the dual code $C_2(\mathbb{D}_i(C))^{\perp}$ may be better then C^{\perp} , as it may happen that

$$\dim(\mathsf{C}_2(\mathbb{D}_d(\mathsf{C}))^{\perp}) > \dim(\mathsf{C}^{\perp}) \text{ and } d(\mathsf{C}_2(\mathbb{D}_d(\mathsf{C}))^{\perp}) = d(\mathsf{C}^{\perp}).$$

3. Let C be an [n,k,d] binary code. Let i be an integer such that d < i < n and $\mathbb{D}_i(\mathsf{C})$ is a 2-design. Then $\mathsf{C}_2(\mathbb{D}_i(\mathsf{C}))$ has parameters [n,k',d'] with $k' \le k$ and $d' \ge d$. The code $\mathsf{C}_2(\mathbb{D}_i(\mathsf{C}))$ could be optimal and thus interesting. The following Example 10 justifies this claim.

Hence, the code $C_2(\mathbb{D}_i(C))$ or its dual could be interesting in many cases.

Example 10. Let m be a positive integer. For each $(a,b,h) \in GF(2^m) \times GF(2^{2m}) \times GF(2)$, define a Boolean function from $GF(2^{2m})$ to GF(2) by

$$f_{(a,b,h)}(x) = \operatorname{Tr}_{m/1} \left[a \operatorname{Tr}_{2m/m} \left(ux^{1+2^{m-1}} \right) \right] + \operatorname{Tr}_{2m/1}(bx) + h,$$

where $\operatorname{Tr}_{j/i}$ is the trace function from $\operatorname{GF}(2^j)$ to $\operatorname{GF}(2^i)$ and $u \in \operatorname{GF}(2^{2m}) \setminus \operatorname{GF}(2^m)$. Define a linear code

$$\mathsf{C}(m) = \left\{ (f_{(a,b,h)}(x))_{x \in \mathsf{GF}(2^{2m})} : a \in \mathsf{GF}(2^m), \ b \in \mathsf{GF}(2^{2m}), \ h \in \mathsf{GF}(2) \right\}.$$

It is shown in [10] that C_m has parameters $[2^{2m}, 3m+1, 2^{2m-1}-2^{m-1}]$ and weight enumerator

$$1 + (2^{m} - 1)2^{2m}z^{2^{2m-1} - 2^{m-1}} + 2(2^{2m} - 1)z^{2^{2m-1}} + (2^{m} - 1)2^{2m}z^{2^{2m-1} + 2^{m-1}} + z^{2^{2m}}.$$

In addition, we have the following [10]:

1. The codewords of minimum weight of C(m) hold a 2-design $\mathbb{D}_{2^{2m-1}-2^{m-1}}(C(m))$ with parameters

$$2-(2^{2m},2^{2m-1}-2^{m-1},(2^m-1)(2^{2m-2}-2^{m-1})).$$

2. The codewords of weight $2^{2m-1} + 2^{m-1}$ of C(m) hold a 2-design $\mathbb{D}_{2^{2m-1}+2^{m-1}}(C(m))$ with parameters

$$2-(2^{2m},2^{2m-1}+2^{m-1},(2^m-1)(2^{2m-2}+2^{m-1})).$$

3. The codewords of weight 2^{2m-1} of C(m) hold a 2-design $\mathbb{D}_{2^{2m-1}}(C(m))$ with parameters $2-(2^{2m},2^{2m-1},2^{2m-1}-1)$.

which is actually a 3-design.

It is proved in [10] that the minimum weight codewords generate C(m). It then follows from Theorem 9 that $C_2(\mathbb{D}_{2^{2m-1}-2^{m-1}}(C(m))) = C(m)$. It is easily seen that $C_2(\mathbb{D}_{2^{2m-1}}(C(m)))$ is the first-order Reed-Muller code, which is optimal. This demonstrates that studying the binary code $C_2(D_i(C))$ for some i could be interesting.

Theorem 11. Let \mathbb{D} be a design. Then $\operatorname{Aut}(\mathbb{D}) \leqslant \operatorname{Aut}(\mathsf{C}_2(\mathbb{D}))$.

The proof of this theorem is straightforward. The equality in Theorem 11 may be valid in some special cases. The following theorem follows from Theorem 11 and the second part of Theorem 9.

Theorem 12. Let C be an [n,k,d] binary code which holds designs. Let $\mathbb{D}_i(C)$ denote the support design of the codewords of weight i in C, where the point set is the set of coordinates, i.e., $\{0,1,\ldots,n-1\}$. Let $C_2(\mathbb{D}_i(C))$ denote the binary code of the design $\mathbb{D}_i(C)$, where the point set is the ordered set $\{0,1,\ldots,n-1\}$. Then

$$\operatorname{Aut}(\mathsf{C}) \leqslant \operatorname{Aut}(\mathsf{C}_2(\mathbb{D}_i(\mathsf{C}))).$$

4. The code of the design held in the Simplex code

Our task in this section is to study the code of the design held in the Simplex code. To this end, we have to introduce some known results about the codes of the designs in the projective geometry PG(m-1,q) and the projective Reed-Muller codes in Section 4.1, as they are needed in Section 4.2. Hence, Section 4.1 below is not meant to be a survey, but a recall of some auxiliary results needed in Section 4.2.

4.1. The codes of the designs in the projective geometry PG(m-1,q)

The points of the *projective space* (also called *projective geometry*) PG(m-1,q) are all the 1-dimensional subspaces of the vector space $GF(q)^m$; the lines are the 2-dimensional subspaces of $GF(q)^m$, the planes are the 3-dimensional subspaces of $GF(q)^m$, and the hyperplanes are the (m-1)-dimensional subspaces of $GF(q)^m$; and incidence is the set-theoretic inclusion. The elements of the projective space PG(m-1,q) are the points, lines, planes, ..., and the hyperplanes. But the space $GF(q)^m$ is not an element of PG(m-1,q), as it contains every other subspace and thus plays no role. The *projective dimension* of an element in PG(m-1,q) is one less than that of the corresponding element in the vector space $GF(q)^m$. The *d*-flats in the projective geometry PG(m-1,q) form a 2-design, which is documented below and is well known in the literature [4].

Theorem 13. Let \mathcal{B} denote the set of all d-flats in $\operatorname{PG}(m-1,q)$, and \mathcal{P} the point set of $\operatorname{PG}(m-1,q)$, and the incidence relation I is the containment relation. Then the triple $\operatorname{PG}_d(m-1,q) := (\mathcal{P},\mathcal{B},I)$ is a 2- (v,k,λ) design, where

$$v = \frac{q^m - 1}{q - 1}, \ k = \frac{q^{d+1} - 1}{q - 1}, \ \lambda = \begin{bmatrix} m - 2 \\ d - 1 \end{bmatrix}_q.$$

In addition, the number of blocks in this design is

$$b = \begin{bmatrix} m \\ d+1 \end{bmatrix}_q.$$

In particular, $PG_1(m-1,q)$ is a Steiner system $S(2,q+1,(q^m-1)/(q-1))$, and $PG_{m-2}(m-1,q)$ is a symmetric design with parameters

$$2 - \left(\frac{q^m - 1}{q - 1}, \frac{q^{m-1} - 1}{q - 1}, \frac{q^{m-2} - 1}{q - 1}\right)$$

for $m \geq 3$.

Let q be a prime power and let $m \ge 2$. A point of the projective geometry PG(m-1, GF(q)) is given in homogeneous coordinates by $(x_0, x_1, ..., x_{m-1})$ where all x_i are in GF(q) and are not all zero; each point has q-1 coordinate representations, since $(ax_0, ax_1, ..., ax_{m-1})$ and $(x_0, x_1, ..., x_{m-1})$ yield the same 1-dimensional subspace of $GF(q)^m$ for any non-zero $a \in GF(q)$.

For an integer $r \ge 0$, let PP(r, m-1, q) denote the linear subspace of $GF(q)[x_0, x_1, \dots, x_{m-1}]$ that is spanned by all monomial $x_0^{i_0} x_1^{i_1} \cdots x_{m-1}^{i_{m-1}}$ satisfying the following two conditions:

•
$$\sum_{j=0}^{m-1} i_j \equiv 0 \pmod{q-1}$$
,

•
$$0 < \sum_{j=0}^{m-1} i_j \le r(q-1)$$
.

Each $a \in GF(q)$ is viewed as the constant function $f_a(x_0, x_1, \dots, x_{m-1}) \equiv a$.

Let $\{\mathbf{x}^1, \dots, \mathbf{x}^N\}$ be the set of projective points in PG(m-1,q), where $N = \frac{q^m-1}{q-1}$. Then, the *rth* order projective generalized Reed-Muller code PRM(r, m-1, q) of length $\frac{q^m-1}{q-1}$ is defined by

$$PRM(r, m-1, q) = \left\{ \left(f(\mathbf{x}^1), \dots, f(\mathbf{x}^N) \right) : f \in PP(r, m-1, q) \cup GF(q) \right\}.$$

When $r \ge 1$, let PRM*(r, m-1, q) be the subcode of PRM(r, m-1, q) defined by

$$PRM^*(r, m-1, q) = \{(f(\mathbf{x}^1), \dots, f(\mathbf{x}^N)) : f \in PP(r, m-1, q)\}.$$

Thus, $PRM^*(r, m-1, q)$ is a subcode of PRM(r, m-1, q). For the minimum weight and the dual of the projective generalized Reed-Muller code, we have the following [3].

Theorem 14. Let $0 \le r \le m-1$. Then, the minimal weight of PRM(r, m-1, q) is $\frac{q^{m-r}-1}{a-1}$ and

$$PRM(r, m-1, q)^{\perp} = PRM^*(m-1-r, m-1, q).$$

Let p be a prime. Then the relation between the codes $C_p(PG_{r-1}(m-1,p))$ of the designs of projective geometries over GF(p) and the projective generalized Reed-Muller codes over GF(p) is given as follows [3].

Theorem 15. Let m be a positive integer, p a prime, and $1 \le r \le m$.

- (i) The code $C_p(PG_{r-1}(m-1,p))$ from the design of points and projective (r-1)-dimensional subspaces of the projective geometry PG(m-1,p) is the same as PRM(m-r,m-1,p) up to a permutation of coordinates.
- (ii) $C_p(PG_{r-1}(m-1,p))$ has minimum weight $\frac{p^r-1}{p-1}$ and the minimum-weight vectors are the multiples of the characteristic vectors of the blocks.
- (iii) The dual code $C_p(PG_{r-1}(m-1,p))^{\perp}$ is the same as $PRM^*(r-1,m-1,p)$ up to a permutation of coordinates and has minimum weight at least $\frac{p^{m-r+1}-1}{p-1}+1$. (iv) The dimension of the code $C_p(PG_{r-1}(m-1,p))$ is

$$\frac{p^m-1}{p-1} - \sum_{i=0}^{r-2} (-1)^i \binom{(r-1-i)(p-1)-1}{i} \binom{m-r+(r-1-i)p}{m-1-i}.$$

To obtain the codes of the designs coming from projective spaces over GF(q) with $q = p^s$, we need to restrict the codes PRM(m-r, m-1, q) to subfield subcodes. Let C be a linear code over GF(q). The set $C_{q/p}$ of vectors in C, all of whose coordinates lie in GF(p), is called the subfield subcode of C over GF(p). Denote by $PRM_{q/p}(m-r, m-1, q)$ the subfield subcode of the projective generalized Reed-Muller code PRM(m-r, m-1, q). Then the relation between the codes $C_p(PG_{r-1}(m-1,q))$ of the designs of projective geometries over GF(q) and the subfield subcode $PRM_{q/p}(m-r, m-1, q)$ of the projective generalized Reed-Muller code is given as follows [3].

Theorem 16. Let m be any positive integer, $q = p^s$ where p is a prime, and let $1 \le r \le m$.

- (i) The code $C_p(PG_{r-1}(m-1,q))$ from the design of points and projective (r-1)-dimensional subspaces of the projective geometry PG(m-1,q) is the same as $PRM_{q/p}(m-r,m-1,q)$ up to a permutation of coordinates.
- (ii) $C_p(PG_{r-1}(m-1,q))$ has minimum weight $\frac{q^r-1}{q-1}$ and the minimum-weight vectors are the multiples of the characteristic vectors of the blocks.
 - (iii) The dual code $C_p(PG_{r-1}(m-1,q))^{\perp}$ has minimum weight at least $\frac{q^{m-r+1}-1}{q-1}+1$.
 - (iv) The dimension of the code $C_p(PG_{m-2}(m-1,q))$ is

$$\binom{p+m-2}{m-1}^s+1.$$

Serre has proved in [23] the following inequality, conjectured by Tsfasman:

Theorem 17. Let $m \ge 2$ and f be a nonzero homogeneous polynomial in $GF(q)[x_0, x_1, \dots, x_{m-1}]$ with $\deg(f) \le q+1$. Let $N_f = |\{\mathbf{x} \in PG(m-1,q) : f(\mathbf{x}) = 0\}|$. Then

$$N_f \le \deg(f)q^{m-2} + \frac{q^{m-2}-1}{q-1}.$$

Moreover, if $\deg(f) \le q$, the upper bound is attained only if the set $\{\mathbf{x} \in \mathrm{PG}(m-1,q) : f(\mathbf{x}) = 0\}$ is a union of $\deg(f)$ hyperplanes whose intersection contains a subspace of codimension 2.

Taking deg(f) = q - 1, we have the following result,

Theorem 18. Let $m \ge 2$. Then $PRM^*(1, m-1, q)$ has minimum weight $2q^{m-2}$.

Combining Theorems 14 and 15, we have

$$PRM^*(1, m-1, p) = C_p(PG_1(m-1, p))^{\perp} = PRM(m-2, m-1, p)^{\perp},$$

where the equalities mean the equivalence of codes. By definition, $PRM^*(1, m-1, p)$ is a subcode of PRM(1, m-1, p).

4.2. The code of the design held in the Simplex code

We view $GF(q^m)$ as an *m*-dimensional vector space over GF(q). Let α be a generator of $GF(q^m)^*$. Then

$$\mathcal{P} = \{1, \alpha, \alpha^2, ..., \alpha^{\nu-1}\} = GF(q^m)^*/GF(q)^*$$

is the set of points in the projective geometry PG(m-1,q), where $v=(q^m-1)/(q-1)$.

By the definition α and ν , it is easily seen that

$$\left\{ (\operatorname{Tr}(a\alpha^{i}))_{i=0}^{\nu-1} : a \in \operatorname{GF}(q^{m}) \right\}$$
 (2)

is the Simplex code whose dual is the Hamming code. Clearly, the weight enumerator of the Simplex code is given by

$$1 + (q^m - 1)z^{q^{m-1}}. (3)$$

By the Assmus-Mattson theorem, the codewords of weight q^{m-1} in the Simplex code form a design $\mathbb D$ with the following parameters

$$2 - \left(\frac{q^m - 1}{q - 1}, \ q^{m - 1}, \ (q - 1)q^{m - 2}\right). \tag{4}$$

Our objective in this section is to study the code $C_q(\mathbb{D})$. Note that the design \mathbb{D} is not a geometric design in the projective geometry PG(m-1,q). Hence, we are not able to apply Theorem 15 directly, but we will make use of it indirectly. To this end, we need to do some preparations.

Lemma 19. The complementary design \mathbb{D}^c of \mathbb{D} is the geometric design $PG_{m-2}(m-1,q)$ with parameters

$$2 - \left(\frac{q^m - 1}{q - 1}, \frac{q^{m-1} - 1}{q - 1}, \frac{q^{m-2} - 1}{q - 1}\right). \tag{5}$$

Proof. We use the trace expression of the Simplex code given in (2), and index the coordinates of the code with the elements in $GF(q^m)$. Let

$$\mathbf{c}_a = (\operatorname{Tr}(a\alpha^i))_{i=0}^{v-1}$$

where $a \neq 0$. Then the complement suppt(\mathbf{c}_a)^c of the support suppt(\mathbf{c}_a) of the codeword \mathbf{c}_a is given by

$$\operatorname{suppt}(\mathbf{c}_a)^c = \{\alpha^i : 0 \le i \le v - 1 \text{ and } \operatorname{Tr}(a\alpha^i) = 0\},$$

which is a hyperplane in PG(m-1,q). On the other hand, every hyperplane in PG(m-1,q) is of this form and corresponds to such codeword in $PRM^*(1,m-1,q)$. The desired conclusion then follows.

The following lemma will play an important role in proving the main result of this section.

Lemma 20. The code $C_p(\mathbb{D})^{\perp}$ contains the all-one vector $\overline{\mathbf{1}}$ and the code $C_p(\mathbb{D})$ does not contain the all-one vector $\overline{\mathbf{1}}$.

Proof. Since each row in the incidence matrix of the design \mathbb{D} has Hamming weight q^{m-1} , the all-one vector $\overline{\mathbf{1}}$ of length $v = (q^m - 1)/(q - 1)$ is orthogonal to all rows in the incidence matrix. As a result, $\overline{\mathbf{1}} \in \mathsf{C}_p(\mathbb{D})^{\perp}$. Note that the inner product of $\overline{\mathbf{1}}$ and itself is $v \pmod{q} = 1$. It then follows that $\overline{\mathbf{1}} \not\in \mathsf{C}_p(\mathbb{D})$.

The main result of this section is the following.

Theorem 21. The code $C_p(\mathbb{D})$ of the design \mathbb{D} has parameters

$$\left[\frac{q^m-1}{q-1}, \, \binom{p+m-2}{m-1}^s, \, d\right],\,$$

where

$$d \ge 2q^{m-2}. (6)$$

Moreover, if q = p, $d = 2q^{m-2}$.

Proof. We first prove that the all-one vector $\bar{\mathbf{1}}$ is a codeword of $C_p(PG_{m-2}(m-1,q))$. Note that the number of blocks of the design $PG_{m-2}(m-1,q)$ containing a point of the design is

$$\lambda_1 = \frac{q^{m-2}-1}{q-1} \frac{\binom{\frac{q(q^{m-1}-1)}{q-1}}{1}}{\binom{\frac{q(q^{m-1}-1)}{q-1}}{1}} = \frac{q^{m-1}-1}{q-1}.$$

Hence, $\lambda_1 \pmod{q} = 1$. Consequently, the sum of the row vectors over GF(q) of the incidence matrix of the design $PG_{m-2}(m-1,q)$ is

$$(\lambda_1,...,\lambda_1) = (1,...,1) = \overline{\mathbf{1}} \in C_p(PG_{m-2}(m-1,q)).$$

We then deduce from Theorem 3 and Lemma 20 that

$$\mathsf{C}_p(\mathbb{D}) \subset \mathsf{C}_p(\mathsf{PG}_{m-2}(m-1,q)) \text{ and } \dim(\mathsf{C}_p(\mathbb{D})) = \dim(\mathsf{C}_p(\mathsf{PG}_{m-2}(m-1,q))) - 1.$$

Note that

$$\mathsf{C}_p(\mathbb{D}) \subseteq \mathsf{PRM}^*_{q/p}(1, m-1, q) \subset \mathsf{PRM}_{q/p}(1, m-1, q) = \mathsf{C}_p(\mathsf{PG}_{m-2}(m-1, q)),$$

where $\operatorname{PRM}_{q/p}^*(1, m-1, q)$ is the subfield subcode of the code $\operatorname{PRM}^*(1, m-1, q)$. Noticing that $\dim(\mathsf{C}_p(\mathbb{D})) - \dim(\mathsf{C}_p(\operatorname{PG}_{m-2}(m-1,q))) = 1$, one has $\mathsf{C}_p(\mathbb{D}) = \operatorname{PRM}_{q/p}^*(1, m-1, q)$. Then, the desired conclusions follow from Theorems 16 and 18.

Note that the lower bound on the minimum distance d given in (6) is the minimum distance of the code $C_p(\operatorname{PG}_{m-2}(m-1,q))$. Although the difference of the dimensions of $C_p(\operatorname{PG}_{m-2}(m-1,q))$ and $C_p(\mathbb{D})$ is only one, the difference between their minimum distances could be very large for $q \geq 3$. Table 1 documents the parameters of the two codes in some cases. When m=2, both code are MDS and optimal. When (q,m)=(3,3), the code $C_p(\mathbb{D})$ has parameters [13,6,6] and is optimal. Note that the code $C_p(\mathbb{D})$ is much better than $C_p(\operatorname{PG}_{m-2}(m-1,q))$ in many cases in terms of error correcting capability.

Table 1: The parameters of $C_p(\mathbb{D})$ and $C_p(PG_{m-2}(m-1,q))$

	1	p() p(-m 2()1
(q,m)	$C_p(\mathbb{D})$	$C_p(PG_{m-2}(m-1,q))$
(3,2)	[4,3,2]	[4,4,1]
(3,3)	[13, 6, 6]	[13, 7, 4]
(3,4)	[40, 10, 18]	[40, 11, 13]
(3,5)	[121, 15, 54]	[121, 16, 40]
(4,2)	[5,4,2]	[5,5,1]
(4,3)	[21, 9, 8]	[21, 10, 5]
(4,4)	[85, 16, 32]	[85, 17, 21]
(5,2)	[6,5,2]	[6,6,1]
(5,3)	[31, 15, 10]	[31, 16, 6]

In fact, experimental data strongly supports the following conjecture.

Conjecture 22. Let \mathbb{D} be defined as before. The minimum distance of the code $C_p(\mathbb{D})$ equals $2q^{m-2}$.

Theorem 23. Let \mathbb{D} be defined as before. The dual code $C_p(\mathbb{D})^{\perp}$ has parameters

$$\left[\frac{q^m-1}{q-1}, \frac{q^m-1}{q-1} - \binom{p+m-2}{m-1}^s, d^{\perp}\right],$$

where $d^{\perp} \geq 3$. Moreover, if q = p, $d^{\perp} = p + 1$.

Proof. The dimension of the code $C_p(\mathbb{D})^{\perp}$ follows from Theorem 23. Note that the design \mathbb{D} has parameters

$$2 - \left(\frac{q^m - 1}{q - 1}, \ q^{m - 1}, \ (q - 1)q^{m - 2}\right). \tag{7}$$

The desired lower bound on d^{\perp} follows from Theorem 8.

When q = p, from the proof of Theorem 21, $C_p(\mathbb{D}) = PRM^*(1, m-1, p)$. By Theorem 14, one has $C_p(\mathbb{D})^{\perp} = PRM(m-2,q)$ and the minimum distance of the code $C_p(\mathbb{D})^{\perp}$ equals p+1 by Theorem 15.

In fact, experimental data strongly supports the following conjecture.

Conjecture 24. Let \mathbb{D} be defined as before. The minimum distance of the code $C_p(\mathbb{D})^{\perp}$ equals q+1.

5. Linear codes from the *t*-designs held in the generalised Reed-Muller codes

Our task in this section is to study the linear codes from the t-designs held in the generalised Reed-Muller codes. To this end, we have to introduce some known results about the codes of the designs in the affine geometry AG(m,q) and the generalised Reed-Muller codes in Section 5.1, as they are needed in Section 5.2. Hence, Section 5.1 below is not meant to be a survey, but a recall of some auxiliary results needed in Section 5.2.

5.1. The codes of the designs in the affine geometry AG(m,q)

The affine geometry AG(m,q), where the points are the vectors in the vector space $GF(q)^m$, the lines are the cosets of all the one-dimensional subspaces, the planes are the cosets of the two-dimensional subspaces, the *i*-flats are the cosets of the *i*-dimensional subspaces, and the hyperplanes are the cosets of the (m-1)-dimensional subspaces of $GF(q)^m$. The *d*-flats of $GF(q)^m$ can be employed to construct 2-designs.

Theorem 25. [4] Let \mathcal{B} denote the sets of all d-flats in $GF(q)^m$, and \mathcal{P} the set of all vectors in $GF(q)^m$, and I the containment relation. Then the triple $AG_d(m,q) := (\mathcal{P},\mathcal{B},I)$ is 2- (v,k,λ) design, where

$$v = q^m, \ k = q^d, \ \lambda = \begin{bmatrix} m-1 \\ d-1 \end{bmatrix}_q,$$

and the Gaussian coefficients are defined by

$${n \brack i}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-i+1} - 1)}{(q^i - 1)(q^{i-1} - 1) \cdots (q - 1)}.$$

In addition, the number of blocks in this design is

$$b = q^{m-d} \begin{bmatrix} m \\ d \end{bmatrix}_q.$$

In particular, $AG_1(m,q)$ is a Steiner system $S(2,q,q^m)$. When $d \ge 2$, $AG_d(m,2)$ is a 3-design. In particular, $AG_2(m,2)$ is a Steiner system $S(3,4,2^m)$.

To study the code of the design $AG_r(m,q)$, we need to define a cyclic code. Let q be a prime power as before. For any integer $j = \sum_{i=0}^{m-1} j_i q^i$, where $0 \le j_i \le q-1$ for all $0 \le i \le m-1$ and m is a positive integer, we define

$$wt_{q}(j) = \sum_{i=0}^{m-1} j_{i}, \tag{8}$$

where the sum is taken over the ring of integers, and is called the q-weight of j.

Let $t \ge 0$ be an integer with $t = a(q-1) + b \le m(q-1)$, where $0 \le b \le q-1$. We define a cyclic code M^t over GF(q) with length $q^m - 1$ and defining set

$$\{i: 1 \le i \le q^m - 1, \ \text{wt}_q(i) < t\}.$$

Let $\overline{M^t}$ denote the extended code of M^t . The following theorem in the case that q is a prime was proved in [3]. It is also true for q being any prime power.

Theorem 26. [3] Let $0 \le r \le m$. The code $\overline{\mathsf{M}^t}$ over $\mathsf{GF}(q)$ has length q^m , dimension

$$|\{i: 0 \le i \le q^m-1, \; \operatorname{wt}_q(i) \le m(q-1)-t\}|$$

and minimum weight $(b+1)q^a$, where t = a(q-1) + b, $0 \le a \le m-1$, $0 \le b < q-1$ and $(a,b) \ne (0,0)$.

The next result will be used later.

Theorem 27. [3] Let $0 \le r \le m$. The code $C_q(AG_r(m,q))$ of the design $AG_r(m,q)$ of points and r-flats of the affine geometry AG(m,q) is the code $\overline{M^{r(q-1)}}$ with minimum weight q^r and dimension

$$|\{i: 0 \le i \le q^m - 1, \text{ wt}_q(i) \le (m - r)(q - 1)\}|.$$

As corollaries of Theorem 27, we have the next two results.

Corollary 28. [13, 14] The code $C_q(AG_{m-1}(m,q))$ of the geometric design $AG_{m-1}(m,q)$ of points and (m-1)-flats of the affine geometry AG(m,q) has length q^m , minimum weight q^{m-1} and dimension $\binom{m+p-1}{m}^s$, where $q=p^s$.

Corollary 29. [13, 14] The code $C_q(AG_1(m,q))$ of the geometric design $AG_1(m,q)$ of points and lines of the affine geometry AG(m,q) has length q^m , minimum weight q. The dimension of the code is $q^m - {m+q-2 \choose m}$ if q is a prime.

In particular, the code $C_3(AG_1(m,3))$ of the Steiner triple system of points and lines of AG(m,3) has parameters $[3^m, 3^m - 1 - m, 3]$.

5.2. Linear codes from the t-designs held in the generalised Reed-Muller codes

Let ℓ be a positive integer with $1 \le \ell < (q-1)m$. The ℓ -th order *punctured generalized Reed-Muller code* $\mathcal{R}_q(\ell,m)^*$ over GF(q) is the cyclic code of length $n=q^m-1$ with generator polynomial

$$g(x) = \sum_{\substack{1 \le j \le n-1 \\ \text{wt}_q(j) < (q-1)m-\ell}} (x - \alpha^j), \tag{9}$$

where α is a generator of $GF(q^m)^*$. Since $\operatorname{wt}_q(j)$ is a constant function on each q-cyclotomic coset modulo $n = q^m - 1$, g(x) is a polynomial over GF(q).

The parameters of the punctured generalized Reed-Muller code $\mathcal{R}_q(\ell,m)^*$ are known and summarized in the next theorem.

Theorem 30. [3] For any ℓ with $0 \le \ell < (q-1)m$, $\mathcal{R}_q(\ell,m)^*$ is a cyclic code over GF(q) with length $n = q^m - 1$, dimension

$$\kappa = \sum_{i=0}^{\ell} \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} \binom{i-jq+m-1}{i-jq}$$

and minimum weight $d = (q - \ell_0)q^{m-\ell_1 - 1} - 1$, where $\ell = \ell_1(q - 1) + \ell_0$ and $0 \le \ell_0 < q - 1$.

The following is also well known in the literature and will be needed later.

Theorem 31. [9] It is also know that $\mathcal{R}_q(1,m)^*$ has parameters $[q^m-1,m+1,(q-1)q^{m-1}-1]$ and weight enumerator

$$1 + (q-1)(q^m-1)z^{(q-1)q^{m-1}-1} + (q^m-1)z^{(q-1)q^{m-1}} + (q-1)z^{q^m-1}$$

The dual code $(\mathcal{R}_q(1,m)^*)^{\perp}$ has parameters $[q^m-1,q^m-m-2,d^{\perp}]$, where $d^{\perp}=4$ if q=2, and $d^{\perp}=3$ if $q\geq 3$.

For $0 \le \ell < m(q-1)$, the code $(\mathcal{R}_q(\ell,m)^*)^{\perp}$ is the cyclic code with generator polynomial

$$g^{\perp}(x) = \sum_{\substack{0 \le j \le n-1 \\ \text{wt}_{\alpha}(j) < \ell}} (x - \alpha^j), \tag{10}$$

where α is a generator of $GF(q^m)^*$. In addition,

$$(\mathcal{R}_q(\ell,m)^*)^{\perp} = (\mathrm{GF}(q)\mathbf{\bar{1}})^{\perp} \cap \mathcal{R}_q(m(q-1)-1-\ell,m)^*$$

where $\bar{\mathbf{1}}$ is the all-one vector in $GF(q)^n$ and $GF(q)\bar{\mathbf{1}}$ denotes the code over GF(q) with length n generated by $\bar{\mathbf{1}}$.

The parameters of the dual of the punctured generalized Reed-Muller code are summarized as follows [1]. For $0 \le \ell < m(q-1)$, the code $(\mathcal{R}_q(\ell,m)^*)^{\perp}$ has length $n=q^m-1$, dimension

$$\kappa = n - \sum_{i=0}^{\ell} \sum_{j=0}^{m} (-1)^{j} {m \choose j} {i-jq+m-1 \choose i-jq},$$

and minimum weight

$$d \ge (q - \ell_0')q^{m-\ell_1'-1},\tag{11}$$

where $m(q-1) - 1 - \ell = \ell'_1(q-1) + \ell'_0$ and $0 \le \ell'_0 < q - 1$.

The generalized Reed-Muller code $\mathcal{R}_q(\ell,m)$ is defined to be the extended code of $\mathcal{R}_q(\ell,m)^*$, and its parameters are given below [3]. Let $0 \le \ell < q(m-1)$. Then the generalized Reed-Muller code $\mathcal{R}_q(\ell,m)$ has length $n=q^m$, dimension

$$\kappa = \sum_{i=0}^{\ell} \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{i-jq+m-1}{i-jq},$$

and minimum weight

$$d = (q - \ell_0)q^{m-\ell_1 - 1}.$$

where $\ell = \ell_1(q-1) + \ell_0$ and $0 \le \ell_0 < q-1$.

The following is a well known result [3] and will be needed shortly.

Theorem 32. Let $0 \le \ell < q(m-1)$ and $\ell = \ell_1(q-1) + \ell_0$, where $0 \le \ell_0 < q-1$. The total number $A_{(q-\ell_0)q^{m-\ell_1-1}}$ of minimum weight codewords in $\mathcal{R}_q(\ell,m)$ is given by

$$A_{(q-\ell_0)q^{m-\ell_1-1}} = (q-1)\frac{q^{\ell_1}(q^m-1)(q^{m-1}-1)\cdots(q^{\ell_1+1}-1)}{(q^{m-\ell_1}-1)(q^{m-\ell_1-1}-1)\cdots(q-1)}N_{\ell_0},$$

where

$$N_{\ell_0} = \begin{cases} 1 & \text{if } \ell_0 = 0, \\ \binom{q}{\ell_0} \frac{q^{m-\ell_1} - 1}{q - 1} & \text{if } 0 < \ell_0 < q - 1. \end{cases}$$

The generalized Reed-Muller codes $\mathcal{R}_q(\ell,m)$ can also be defined with a multivariate polynomial approach. The reader is referred to [3, Section 5.4] for details. For $\ell < (q-1)m$, it was shown in [3] that

$$\mathcal{R}_q(\ell,m)^{\perp} = \mathcal{R}_q(m(q-1)-1-\ell,m).$$

The general affine group $GA_1(GF(q))$ is defined by

$$GA_1(GF(q)) = \{ax + b : a \in GF(q)^*, b \in GF(q)\},\$$

which acts on GF(q) doubly transitively [9, Section 1.7]. A linear code C of length q is said to be affine-invariant if $GA_1(GF(q))$ fixes C [6]. For affine-invariant codes we use the elements of GF(q) to index the coordinates of their codewords.

Let ℓ be a positive integer with $1 \leq \ell < (q-1)m$, and let q be a prime. Then $\mathcal{R}_q(\ell,m)$ is affine-invariant, and the automorphism group $\operatorname{Aut}(\mathcal{R}_q(\ell,m))$ is doubly transitive. These are well known facts about the generalized Reed-Muller codes $\mathcal{R}_q(\ell,m)$. The results in the next two theorems are also well known (see [1] or [9]) and follow from Theorems 2 and 32.

Theorem 33. Let ℓ be a positive integer with $1 \le \ell < (q-1)m$. Then the supports of the codewords of weight i > 0 in $\mathcal{R}_q(\ell, m)$ form a 2-design, provided that $A_i \ne 0$.

Theorem 34. Let $0 \le \ell < q(m-1)$ and $\ell = \ell_1(q-1) + \ell_0$, where $0 \le \ell_0 < q-1$. The supports of minimum weight codewords in $\mathcal{R}_q(\ell,m)$ form a 2- $(q^m,(q-\ell_0)q^{m-\ell_1-1},\lambda)$ design, where

$$\lambda = \frac{A_{(q-\ell_0)q^{m-\ell_1-1}}}{q-1} \frac{\binom{(q-\ell_0)q^{m-\ell_1-1}}{2}}{\binom{q^m}{2}}$$

and $A_{(q-\ell_0)q^{m-\ell_1-1}}$ was given in Theorem 32.

Note that $\mathcal{R}_q(\ell,m)$ does not hold 3-designs when q > 2. It is known that $\mathcal{R}_q(1,m)$ has parameters $[q^m, 1+m, (q-1)q^{m-1}]$ and weight enumerator

$$1 + q(q^{m} - 1)z^{(q-1)q^{m-1}} + (q-1)z^{q^{m}}. (12)$$

Furthermore, the supports of all minimum weight codewords in $\mathcal{R}_q(1,m)$ form a 2- $(q^m, (q-1)q^{m-1}, (q-1)q^{m-1}-1)$ design [9].

We are now ready to present another result of this paper in the following theorem.

Theorem 35. Let $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m))$ denote the 2-design formed by the codewords of weight $(q-1)q^{m-1}$ in $\mathcal{R}_q(1,m)$. Then $\mathsf{C}_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))$ has parameters

$$\left[q^m, \, \binom{p+m-1}{m}^s, \, q^{m-1}\right],$$

where $q = p^s$.

Proof. Note that each codeword of weight $(q-1)q^{m-1}$ in $\mathcal{R}_q(1,m)$ can be written as

$$\mathbf{c}_{(a,b)} = (\operatorname{Tr}(ax) + b)_{x \in \operatorname{GF}(q^m)}, \ a \in \operatorname{GF}(q^m)^*, \ b \in \operatorname{GF}(q).$$

We index the coordinates of the code $\mathcal{R}_q(1,m)$ with the elements of $GF(q^m)$. Then the support of the codeword $\mathbf{c}_{(a,b)}$ is given by

$$\operatorname{suppt}(\mathbf{c}_{(a,b)}) = \{x \in \operatorname{GF}(q^m) : \operatorname{Tr}(ax) + b \neq 0\}.$$

The complement of suppt($\mathbf{c}_{(a,b)}$) with respect to $GF(q^m)$ is given by

$$\operatorname{suppt}(\mathbf{c}_{(a,b)})^c = \{x \in \operatorname{GF}(q^m) : \operatorname{Tr}(ax) + b = 0\},\$$

which is an (m-1)-flat in $GF(q^m)$ when $GF(q^m)$ is viewed as an m-dimensional vector space over GF(q). Consequently, the complementary design $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m))^c$ of $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m))$ is the design $AG_{m-1}(m,q)$ of points and (m-1)-flats of the affine geometry AG(m,q).

We now prove that the all-one vector $\overline{\bf 1}$ is a codeword in both $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))$ and $C_p(AG_{m-1}(m,q))$. It is well known that the design $AG_{m-1}(m,q)$ has parameters

$$2 - \left(q^m, q^{m-1}, \frac{q^{m-1}-1}{q-1}\right).$$

Therefore each point is incident with the following number of blocks:

$$\lambda_1^c = \frac{q^{m-1} - 1}{q - 1} \frac{\binom{q^m - 1}{1}}{\binom{q^{m-1} - 1}{1}} = \frac{q^m - 1}{q - 1}.$$

It then follows that the sum over GF(q) of the row vectors of the incidence matrix of the design $AG_{m-1}(m,q)$ is

$$(\lambda_1^c, \lambda_1^c, ..., \lambda_1^c) = (1, 1, \cdots, 1) = \overline{1},$$

which is a codeword in $C_p(AG_{m-1}(m,q))$.

Since $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m))$ is a 2- $(q^m,q^m-q^{m-1},(q-1)q^{m-1}-1)$ design, every point of the design $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m))$ is incident with the following number of blocks:

$$\lambda_1 = ((q-1)q^{m-1} - 1) \frac{\binom{q^m - 1}{1}}{\binom{q^m - q^{m-1} - 1}{1}} = q^m - 1$$

We then deduce that the sum over GF(q) of the row vectors of the incidence matrix of the design $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m))$ is

$$(\lambda_1, \lambda_1, ..., \lambda_1) = (-1, -1, ..., -1) = -\overline{1},$$

which is a codeword in $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))$. Consequently, $\overline{\mathbf{1}} \in C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))$. It then follows from Theorem 3 that $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))$ is equal to $C_p(AG_{m-1}(m,q))$. The desired conclusion then follows from Corollary 28.

When q=2, it is easily seen that $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))$ equals $\mathcal{R}_q(1,m)$. However, the two codes are very different if q>2. This is obvious from the dimensions of the two codes. Note that the design $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m))$ in Theorem 35 is not a geometric design. But its code over GF(q) is the same as the code over GF(q) of the geometric design $AG_{m-1}(m,q)$. Our contribution is mainly to prove this fact.

Theorem 36. Let $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m))$ denote the 2-design formed by the codewords of weight $(q-1)q^{m-1}$ in $\mathcal{R}_q(1,m)$. Then $\mathsf{C}_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))^\perp$ has parameters

$$\left[q^m, q^m - \binom{p+m-1}{m}^s, d^{\perp}\right],$$

where $q = p^s$, $d^{\perp} \ge q + 2$ if s > 1 and $d^{\perp} = 2p$ if s = 1.

Proof. The dimension of the code $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))^{\perp}$ follows from Theorem 35. Recall that $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m))) = C_p(AG_{m-1}(m,q))$. It then follows from Theorem 8 that $d^{\perp} \geq q+2$. If s=1, it then follows from Theorem 5.7.9 in [3] that $d^{\perp}=2p$.

Notice that $\mathcal{R}_q^*(1,m)$ is a cyclic code and invariant under the general linear group $\mathrm{GL}_m(q)$, which is transitive on $\mathrm{GF}(q^m)^*$. By Theorem 31, $\mathcal{R}_q^*(1,m)$ is a three-weight code. Hence, $\mathcal{R}_q^*(1,m)$) holds two 1-designs. One of them is $\mathbb{D}_{(q-1)q^{m-1}-1}(\mathcal{R}_q^*(1,m))$ with parameters

$$1-\left(q^m-1,\; (q-1)q^{m-1}-1,\; (q-1)q^{m-1}-1\right).$$

The other is the design $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q^*(1,m))$ with parameters

$$1-(q^m-1, (q-1)q^{m-1}, q^{m-1}).$$

By definition, $C_p(\mathbb{D}_{(q-1)q^{m-1}-1}(\mathcal{R}_q^*(1,m)))$ is a punctured code of $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))$. The following result then follows from Theorem 35.

Theorem 37. Let $\mathbb{D}_{(q-1)q^{m-1}-1}(\mathcal{R}_q^*(1,m))$ denote the 1-design formed by the codewords of weight $(q-1)q^{m-1}-1$ in $\mathcal{R}_q^*(1,m)$. Then $\mathsf{C}_p(\mathbb{D}_{(q-1)q^{m-1}-1}(\mathcal{R}_q^*(1,m)))$ has parameters

$$\left[q^{m}-1, \; {p+m-1 \choose m}^{s}, \; q^{m-1}-1\right],$$

where $q = p^s$.

Theorem 38. Let $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q^*(1,m))$ denote the 1-design formed by the codewords of weight $(q-1)q^{m-1}$ in $\mathcal{R}_q^*(1,m)$. Then $\mathsf{C}_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q^*(1,m)))$ has parameters

$$\left[q^m-1, \, \binom{p+m-2}{m-1}^s, \, d\right],\,$$

where $q = p^s$, $d = (q-1)d(\mathsf{C}_p(\mathbb{D})) \ge q^{m-1} - 1$, $\mathsf{C}_p(\mathbb{D})$ is the code of Theorem 21, and $d(\mathsf{C}_p(\mathbb{D}))$ denotes the minimum distance of the code $\mathsf{C}_p(\mathbb{D})$.

Proof. It is straightforward to see that $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q^*(1,m))$ is the design held by the supports of codewords of weight $(q-1)q^{m-1}$ in the code

$$C = \{ (Tr_{q^m/q}(ax))_{x \in GF(q^m)^*} : a \in GF(q^m) \},$$

which is equivalent to a concatenation of q-1 copies the first-order projective Reed-Muller code. The desired conclusions then follow from Theorem 21.

Once we determine the minimum weight of the code $C_p(\mathbb{D})$ in Theorem 21, we will be able to determine the minimum weight of $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q^*(1,m)))$, and vice versa.

The following problem is very hard to settle. But we will solve it for a few special cases in the rest of this section.

Open Problem 39. Determine the parameters of $C_p(\mathbb{D}_i(\mathcal{R}_q(\ell,m)))$ for other designs $\mathbb{D}_i(\mathcal{R}_q(\ell,m))$ held in $\mathcal{R}_q(\ell,m)$ for $\ell \geq 2$, and study properties of $C_p(\mathbb{D}_i(\mathcal{R}_q(\ell,m)))$.

The parameters of the designs held in $\mathcal{R}_q(\ell,m)$ are still open. Even the weight distribution of the code $\mathcal{R}_q(\ell,m)$ is open for $\ell \geq 3$ and q > 2. The weight distribution of $\mathcal{R}_q(2,m)$ is known for q > 2 [20]. It may be possible to settle the parameters of $C_p(\mathbb{D}_i(\mathcal{R}_q(2,m)))$ for q > 2 and some i.

A comparison between the parameters of $\mathcal{R}_p(r,m)$ and $C_p(\mathbb{D}_d(\mathcal{R}_p(r,m)))$ is given in Table 2, where d is the minimum distance of $\mathcal{R}_p(r,m)$. In general the parameters of the two codes $\mathcal{R}_p(r,m)$ and $C_p(\mathbb{D}_d(\mathcal{R}_p(r,m)))$ are different. However, in the special case (p,r)=(3,2) we have the following.

able 2. The parameters of $2\varphi(r,m)$ and $C_p(\mathbb{D}_d(2\varphi(r,m)))$				
	(p,m,r)	$\mathcal{R}_p(r,m)$	$C_p(\mathbb{D}_d(\mathcal{R}_p(r,m)))$	
	(3,2,1)	[9,3,6]	[9,6,3]	
	(3,3,1)	[27,4,18]	[27, 10, 9]	
	(3,4,1)	[81,5,54]	[81, 15, 27]	
	(3,3,2)	[27, 10, 9]	[27, 10, 9]	
	(3,4,2)	[81, 15, 27]	[81, 15, 27]	
	(5,2,2)	[25, 6, 15]	[25, 15, 5]	

Table 2: The parameters of $\mathcal{R}_p(r,m)$ and $\mathsf{C}_p(\mathbb{D}_d(\mathcal{R}_p(r,m)))$

Theorem 40. For $m \ge 2$ the two codes $\mathcal{R}_3(2,m)$ and $C_3(\mathbb{D}_{3^{m-1}}(\mathcal{R}_3(2,m)))$ are identical.

[27, 17, 6]

(3,3,3)

Proof. By Theorems 30 and 32, $\mathcal{R}_3(2,m)$ has minimum distance $d = 3^{m-1}$ and dimension $k = 1 + m + \frac{(m+1)m}{2}$. In addition, the total number of minimum weight codewords in $\mathcal{R}_3(2,m)$ is $A_d = 3(3^m - 1)$.

Let Tr denote the trace function from $GF(3^m)$ to GF(3). It is easily seen that the set of all minimum weight codewords in $\mathcal{R}_3(2,m)$ is given by

$$\left\{ \pm \left((\operatorname{Tr}(ax) + b)^2 - 1 \right)_{x \in \operatorname{GF}(q)} : (a, b) \in \operatorname{GF}(3^m) \times \operatorname{GF}(3)^* \right\}.$$

Since $(\operatorname{Tr}(ax) + b)^2 - 1 = 0$ or -1, the code $C_3(\mathbb{D}_{3^{m-1}}(\mathcal{R}_3(2, m)))$ is linearly spanned by the codewords in the following set:

$$\left\{ \left((\operatorname{Tr}(ax) + b)^2 - 1 \right)_{x \in \operatorname{GF}(q)} : (a, b) \in \operatorname{GF}(3^m) \times \operatorname{GF}(3)^* \right\}. \tag{13}$$

[27, 23, 3]

Let $b_1, b_2 \in GF(3^m)$. By (13), we have

$$\left(\operatorname{Tr}((b_1+b_2)x)^2 - \operatorname{Tr}((b_1-b_2)x)^2\right)_{x \in \operatorname{GF}(3^m)} \in \mathsf{C}_3(\mathbb{D}_{3^{m-1}}(\mathcal{R}_3(2,m))),$$

which is the same as

$$(\operatorname{Tr}(b_1x)\operatorname{Tr}(b_2x))_{x\in\operatorname{GF}(3^m)}\in\mathsf{C}_3(\mathbb{D}_{3^{m-1}}(\mathcal{R}_3(2,m))),$$
 (14)

for all $b_1, b_2 \in \operatorname{GF}(3^m)$. Let $b \in \operatorname{GF}(3^m)^*$. By (13), $\left((\operatorname{Tr}(bx) - 1)^2 - 1 \right)_{x \in \operatorname{GF}(q)} \in \operatorname{C}_3(\mathbb{D}_{3^{m-1}}(\mathcal{R}_3(2, m)))$. By (14), $\left(\operatorname{Tr}(bx)^2 \right)_{x \in \operatorname{GF}(q)} \in \operatorname{C}_3(\mathbb{D}_{3^{m-1}}(\mathcal{R}_3(2, m)))$. Note that $\operatorname{Tr}(bx) = ((\operatorname{Tr}(bx) - 1)^2 - 1) - \operatorname{Tr}(bx)^2$. Then

$$(\operatorname{Tr}(bx))_{x \in \operatorname{GF}(q)} \in \mathsf{C}_{3}(\mathbb{D}_{3^{m-1}}(\mathcal{R}_{3}(2,m))). \tag{15}$$

By (13) and (14), we have $(1)_{x \in GF(q)} \in C_3(\mathbb{D}_{3^{m-1}}(\mathcal{R}_3(2,m)))$. Thus, $C_3(\mathbb{D}_{3^{m-1}}(\mathcal{R}_3(2,m)))$ is linearly spanned by the set

$$\left\{ (\operatorname{Tr}(ax)\operatorname{Tr}(bx))_{x \in \operatorname{GF}(q)}, (\operatorname{Tr}(ax))_{x \in \operatorname{GF}(q)}, (1)_{x \in \operatorname{GF}(q)} : a, b \in \operatorname{GF}(3^m) \right\}.$$

It is observed that the linear space spanned by

$$\left\{ \left(\operatorname{Tr}(ax)\operatorname{Tr}(bx) \right)_{x \in \operatorname{GF}(q)}, \left(\operatorname{Tr}(ax) \right)_{x \in \operatorname{GF}(q)}, \left(1 \right)_{x \in \operatorname{GF}(q)} : a, b \in \operatorname{GF}(3^m) \right\}$$

is exactly $\mathcal{R}_3(2,m)$. This completes the proof.

6. Summary and concluding remarks

Using the results on the linear codes of geometric designs and the generalised Reed-Muller codes documented in [1], this paper made the following contributions:

- The results about $C_2(\mathbb{D}_i(C))$, C, $\mathbb{D}_i(C)$, and their automorphism groups for binary linear codes C documented in Section 3.
- The determination of some of the parameters of the linear code $C_p(\mathbb{D})$ documented in Theorem 21, where \mathbb{D} is the design held in a code related to the first-order projective Reed-Muller code.
- The determination of the parameters of the linear code $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))$ documented in Theorem 35 and its dual $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m)))^{\perp}$ documented in Theorem 36, where $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q(1,m))$ is the design supported by the codewords of Hamming weight $(q-1)q^{m-1}$ in the Reed-Muller code $\mathcal{R}_q(1,m)$.
- The determination of the parameters of the ternary code $C_3(\mathbb{D}_{3^{m-1}}(\mathcal{R}_3(2,m)))$ documented in Theorem 40.
- The determination of the parameters of the linear code $C_p(\mathbb{D}_{(q-1)q^{m-1}-1}(\mathcal{R}_q^*(1,m)))$ documented in Theorem 37, where $\mathbb{D}_{(q-1)q^{m-1}-1}(\mathcal{R}_q^*(1,m))$ is the design supported by the codewords of Hamming weight $(q-1)q^{m-1}-1$ in the punctured generalised Reed-Muller code $\mathcal{R}_q^*(1,m)$.
- The determination of the parameters of the linear code $C_p(\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q^*(1,m)))$ documented in Theorem 38, where $\mathbb{D}_{(q-1)q^{m-1}}(\mathcal{R}_q^*(1,m))$ is the design supported by the codewords of Hamming weight $(q-1)q^{m-1}$ in the punctured generalised Reed-Muller code $\mathcal{R}_q^*(1,m)$.

These summarize the new results presented in this paper.

Although the designs considered in this paper are not geometric designs and the linear codes are not geometric codes and Reed-Muller codes, they are closely related to geometric designs and the Reed-Muller codes. Thus, in Sections 4.1 and 5.1 we had to introduce these geometric codes and the Reed-Muller codes as well as their basic properties. This took quite some space.

As observed, it is extremely hard to get information on the code $C_p(\mathbb{D}_i(C))$ for general linear codes over GF(q) for nonbinary codes C holding designs. The reader is cordially invited to settle Conjectures 22 and 24 and Open Problem 39. The rank of *t*-designs, i.e., the dimension of the corresponding codes, may be used to classify *t*-designs of certain type. For example, the rank of Steiner triples was intensively studied and employed for classifying Steiner triple systems [16].

Finally, we point out that the idea of using a linear code C_1 supporting a t-design $\mathbb{D}_w(C)$ to obtain a new linear code $C_q(\mathbb{D}_w(C))$ may produce a bad or good code. Distance-optimal ternary linear codes were obtained in [11] with this method.

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