Knowledge reduction of dynamic covering decision information systems with varying attribute values

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Abstract. Knowledge reduction of dynamic covering information systems involves with the time in practical situations. In this paper, we provide incremental approaches to computing the type-1 and type-2 characteristic matrices of dynamic coverings because of varying attribute values. Then we present incremental algorithms of constructing the second and sixth approximations of sets by using characteristic matrices. We employ experimental results to illustrate that the incremental approaches are effective to calculate approximations of sets in dynamic covering information systems. Finally, we perform knowledge reduction of dynamic covering information systems with the incremental approaches.

Keywords: Boolean matrice; Characteristic matrice; Dynamic covering approximation space; Dynamic covering information system; Rough set

1 Introduction

Covering approximation spaces, as generalizations of classical approximation spaces based on equivalence relations, have attracted more attentions, and a great deal of approximation operators have been proposed for knowledge reduction of covering approximation spaces. Nowadays, covering-based rough set theory [20–23, 27, 29, 30, 34–37] are being enriched with the development of computer sciences and related theories.

To our best knowledge, there exist many lower and upper approximation operators for covering approximation spaces, and their basic properties are investigated concretely by researchers. Especially, Wang et al. [26] studied the second and sixth lower and upper approximation operators of covering approximation spaces and proposed effective approaches to computing the second and sixth lower and upper approximations of sets by using characteristic matrices. In practice, dynamic covering approximation spaces are variations of the time. For example, two specialists A and B decided the quality of five cars $U = \{A, B, C, D, E\}$ as follows: $good = \{A, C\}, middle = \{C, E\}, bad = \{B, D, E\}, and (U, C)$ is a covering approximation space, where $C = \{good, middle, bad\}$. With the variation of the time, the specialists find that the quality of *C* is very bad, and (U, C) is revised into dynamic covering approximation space

 (U, \mathcal{C}^*) , where $\mathcal{C}^* = \{good^*, middle^*, bad^*\}$, $good^* = \{A\}$, $middle^* = \{E\}$ and $bad^* = \{B, C, D, E\}$. Accordingly, the characteristic matrice of \mathcal{C} changes into that of \mathcal{C}^* . Since it is time-consuming to compute the characteristic matrice in large-scale covering approximation space, it costs more time to construct the characteristic matrice of large-scale dynamic coverings for computing approximations of sets. Until now, Lang et al. [4,5] presented incremental approaches to computing approximations of sets in dynamic covering approximation spaces, in which object sets are variations of the time. But little attention has been paid to dynamic covering approximation spaces, in which elements of coverings are variations of the time. Therefore, it is of interest to study how to compute approximations of sets in dynamic covering approximation spaces when varying attribute values.

Many researchers [1–3,6–19,24,25,28,29,31–33] have investigated knowledge reduction of dynamic information systems with incremental approaches. For example, when coarsening and refining attribute values and varying attribute sets, Chen et al. [1–3] constructed approximations of sets which provides an effective approach to knowledge reduction of dynamic information systems. Based on characteristic relations, Li, Ruan and Song [9] extended rough sets for incrementally updating decision rules which handles dynamic maintenance of decision rules in data mining. Liu et al. [11–13] presented incremental approaches for knowledge reduction of dynamic information systems and dynamic incomplete information systems. From the perspective of knowledge engineering and neighborhood systems-based rough sets, Yang, Zhang, Dou and Yang [28] studied the neighborhood system for knowledge reduction of incomplete informations, positive, boundary and negative regions in composite information systems. Illustrated by existing researches, the incremental approaches are effective to conduct knowledge reduction of dynamic information spaces and knowledge reduction of dynamic information spaces and knowledge reduction of dynamic information spaces and knowledge reduction of dynamic information systems.

The purpose of this paper is to further study knowledge reduction of dynamic covering information systems when varying attribute values. First, we investigate structures of the characteristic matrices of dynamic covering approximation spaces when varying attribute values and present incremental approaches to computing characteristic matrices of dynamic coverings. We employ several examples to illustrate that the process of calculating the characteristic matrices is simplified greatly by utilizing the incremental approaches. Second, we provide incremental algorithms for constructing the characteristic matrices-based approximations of sets in dynamic covering approximation spaces when varying attribute values. We also compare the time complexities of the incremental algorithms with those of non-incremental algorithms. Third, we perform experiments on ten dynamic covering approximation spaces generated randomly. The experimental results illustrate that the proposed approached are effective to calculate approximations of sets with respect to the variation of attribute values. We also employ examples to show that how to conduct knowledge reduction of dynamic covering information systems with the incremental approaches.

The rest of this paper is organized as follows: Section 2 briefly reviews the basic concepts of coveringbased rough set theory. In Section 3, we introduce incremental approaches to computing the characteristic matrices of dynamic coverings when varying attribute values. Section 4 presents non-incremental and incremental algorithms of calculating the second and fifth lower and upper approximations of sets by using the characteristic matrices. Section 5 performs experiments to show that the incremental approaches are effective to compute approximations of sets in dynamic covering approximation spaces. Section 6 is devoted to knowledge reduction of dynamic covering information systems with the incremental approaches. We conclude the paper in Section 7.

2 **Preliminaries**

A brief summary of related concepts in covering-based rough sets is given in this section.

Let U be a finite universe of discourse, and \mathscr{C} is a family of subsets of U. If none of elements of \mathscr{C} is empty and $\bigcup \{C | C \in \mathscr{C}\} = U$, then \mathscr{C} is referred to as a covering of U. In addition, (U, \mathscr{C}) is called a covering approximation space if \mathscr{C} is a covering of U.

Definition 2.1 [26] Let $U = \{x_1, x_2, ..., x_n\}$ be a finite universe, and $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ a covering of U. For any $X \subseteq U$, the second, fifth and sixth upper and lower approximations of X with respect to \mathcal{C} , respectively, are defined as follows:

(1) The second upper and lower approximations of X:

$$SH_{\mathscr{C}}(X) = \bigcup \{ C \in \mathscr{C} \mid C \cap X \neq \emptyset \}, \quad SL_{\mathscr{C}}(X) = [SH_{\mathscr{C}}(X^c)]^c;$$

(2) *The fifth upper and lower approximations of X:*

$$IH_{\mathscr{C}}(X) = \{ x \in U \mid N(x) \cap X \neq \emptyset \}, \quad IL_{\mathscr{C}}(X) = \{ x \in U \mid N(x) \subseteq X \};$$

(3) The sixth upper and lower approximations of X:

$$XH_{\mathscr{C}}(X) = \bigcup \{N(x) \mid N(x) \cap X \neq \emptyset\}, \quad XL_{\mathscr{C}}(X) = \bigcup \{N(x) \mid N(x) \subseteq X\}.$$

Definition 2.2 [26] Let $\mathscr{C} = \{C_1, ..., C_m\}$ be a family of subsets of a finite set $U = \{x_1, ..., x_n\}$. We define $M_{\mathscr{C}} = (a_{ij})_{n \times m}$, where $a_{ij} = \begin{cases} 1, & x_i \in C_j, \\ 0, & x_i \notin C_j. \end{cases}$

Definition 2.3 [26] Let $U = \{x_1, ..., x_n\}$, $A \subseteq U$. We define the characteristic function as $X_A = \begin{bmatrix} a_1 & a_2 & . & . & a_n \end{bmatrix}^T$, where $a_i = \begin{cases} 1, & x_i \in A, \\ 0, & x_i \notin A. \end{cases}$ $i = 1, \dots, n$.

Definition 2.4 [26] Let $U = \{x_1, x_2, ..., x_n\}$ be a finite universe, $\mathscr{C} = \{C_1, C_2, ..., C_m\}$ a covering of U, and $M_{\mathscr{C}} = (a_{ij})_{n \times m}$ the matrice representation of \mathscr{C} . Then $\Gamma(\mathscr{C}) = M_{\mathscr{C}} \cdot M_{\mathscr{C}}^T = (b_{ij})_{n \times n}$ is called the type-1 characteristic matrix of \mathscr{C} , where $a_{ij} = \begin{cases} 1, & x_i \in C_j; \\ 0, & x_i \notin C_j. \end{cases}$ and $b_{ij} = \bigvee_{k=1}^m (a_{ik} \cdot a_{jk})$.

Definition 2.5 [26] Let $U = \{x_1, x_2, ..., x_n\}$ be a finite universe, $\mathscr{C} = \{C_1, C_2, ..., C_m\}$ a covering of U, and $M_{\mathscr{C}} = (a_{ij})_{n \times m}$ the matrice representation of \mathscr{C} . Then $\prod(\mathscr{C}) = M_{\mathscr{C}} \odot M_{\mathscr{C}}^T = (c_{ij})_{n \times n}$ is called the type-2 characteristic matrice of \mathscr{C} , where $c_{ij} = \bigwedge_{k=1}^m (a_{jk} - a_{ik} + 1)$.

By Definitions 2.4 and 2.5, the second and fifth lower and upper approximation operators are axiomatized equivalently as follows.

Definition 2.6 [26] Let $U = \{x_1, x_2, ..., x_n\}$ be a finite universe, $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ a covering of U, and X_X the characteristic function of X in U. Then

$$X_{SH(X)} = \Gamma(\mathscr{C}) \cdot \mathcal{X}_X, \mathcal{X}_{SL(X)} = \Gamma(\mathscr{C}) \odot \mathcal{X}_X; \mathcal{X}_{IH(X)} = \prod(\mathscr{C}) \cdot \mathcal{X}_X, \quad \mathcal{X}_{IL(X)} = \prod(\mathscr{C}) \odot \mathcal{X}_X.$$

Definition 2.7 [4] Let $(U, \mathcal{D} \cup U/d)$ be a covering decision information system, where $\mathcal{D} = \{\mathscr{C}_i | i \in I\}$, $U/d = \{D_i | i \in J\}$, I and J are indexed sets. We define $\mathscr{P} \subseteq \mathscr{D}$ as the type-1 reduct of $(U, \mathscr{D} \cup U/d)$ if it satisfies

- (1) $\Gamma(\mathscr{D}) \cdot \mathcal{X}_{D_i} = \Gamma(\mathscr{P}) \cdot \mathcal{X}_{D_i}, \Gamma(\mathscr{D}) \odot \mathcal{X}_{D_i} = \Gamma(\mathscr{P}) \odot \mathcal{X}_{D_i}, \forall i \in J;$
- (2) $\Gamma(\mathscr{D}) \cdot X_{D_i} \neq \Gamma(\mathscr{P}') \cdot X_{D_i}, \Gamma(\mathscr{D}) \odot X_{D_i} \neq \Gamma(\mathscr{P}') \odot X_{D_i}, \forall \mathscr{P}' \subset \mathscr{P}.$

Definition 2.8 [4] Let $(U, \mathcal{D} \cup U/d)$ be a covering decision information system, where $\mathcal{D} = \{\mathscr{C}_i | i \in I\}$, $U/d = \{D_i | i \in J\}$, I and J are indexed sets. We define $\mathscr{P} \subseteq \mathscr{D}$ as the type-2 reduct of $(U, \mathscr{D} \cup U/d)$ if it satisfies

- (1) $\prod(\mathscr{D}) \cdot \mathcal{X}_{D_i} = \prod(\mathscr{P}) \cdot \mathcal{X}_{D_i}, \prod(\mathscr{D}) \odot \mathcal{X}_{D_i} = \prod(\mathscr{P}) \odot \mathcal{X}_{D_i}, \forall i \in J;$
- (2) $\prod(\mathscr{D}) \cdot X_{D_i} \neq \prod(\mathscr{P}') \cdot X_{D_i}, \prod(\mathscr{D}) \odot X_{D_i} \neq \prod(\mathscr{P}') \odot X_{D_i}, \forall \mathscr{P}' \subset \mathscr{P}.$

Definition 2.9 [26] Let $A = (a_{ij})_{n \times m}$ and $B = (b_{ij})_{n \times m}$ be two matrices. We define $A + B = (a_{ij} + b_{ij})_{n \times m}$ for $1 \le i \le n, 1 \le j \le m$.

3 Incremental approaches to computing approximations of sets

In this section, we present incremental approaches to computing the second and fifth lower and upper approximations of sets when revising attribute values.

Definition 3.1 (Dynamic covering approximation space) Let (U, \mathcal{C}) and (U, \mathcal{C}^*) be covering approximation spaces, where $U = \{x_1, x_2, ..., x_n\}$, $\mathcal{C} = \{C_1, C_2, ..., C_m\}$, $\mathcal{C}^* = \{C_1^*, C_2^*, ..., C_m^*\}$, and $C_i^* = C_i - \{x_k\}$ or $C_i^* = C_i \cup \{x_k\}$ when revising the attribute value of $x_k \in U$. Then (U, \mathcal{C}^*) is called a dynamic covering approximation space. In addition, \mathcal{C}^* is called a dynamic covering.

In practice, revising attribute values will result in $|\mathcal{C}^*| < |\mathcal{C}|, |\mathcal{C}^*| = |\mathcal{C}|$ and $|\mathcal{C}^*| > |\mathcal{C}|$. In this work, we only discuss the situation that $|\mathcal{C}^*| = |\mathcal{C}|$ when revising attribute values of an object.

Below, we discuss the relationship between $\Gamma(\mathscr{C})$ and $\Gamma(\mathscr{C}^*)$. For convenience, we denote $M_{\mathscr{C}} = (a_{ij})_{n \times m}$, $M_{\mathscr{C}^*} = (b_{ij})_{n \times m}$, $\Gamma(\mathscr{C}) = (c_{ij})_{n \times n}$ and $\Gamma(\mathscr{C}^*) = (d_{ij})_{n \times n}$.

Theorem 3.2 Let (U, \mathcal{C}^*) be a dynamic covering approximation space of (U, \mathcal{C}) , $\Gamma(\mathcal{C})$ and $\Gamma(\mathcal{C}^*)$ the type-1 characteristic matrices of \mathcal{C} and \mathcal{C}^* , respectively. Then

$$\Gamma(\mathscr{C}^*) = \Gamma(\mathscr{C}) + \Delta \Gamma(\mathscr{C})$$

where

$$\Delta\Gamma(\mathscr{C}) = \begin{bmatrix} 0 & 0 & \cdots & d_{1k}^{*} & \cdots & 0 \\ 0 & 0 & \cdots & d_{2k}^{*} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{k1}^{*} & d_{k2}^{*} & \cdots & d_{kk}^{*} & \cdots & d_{kn}^{*} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_{nk}^{*} & \cdots & 0 \end{bmatrix};$$

$$d_{kj}^{*} = d_{jk}^{*} = \begin{bmatrix} b_{k1} & b_{k2} & \cdots & b_{km} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} & b_{2j} & \cdots & b_{mj} \end{bmatrix}^{T} - c_{kj}.$$

Proof. By Definition 2.4, $\Gamma(\mathscr{C})$ and $\Gamma(\mathscr{C}^*)$ are presented as follows:

$$\begin{split} \Gamma(\mathscr{C}) &= M_{\mathscr{C}} \cdot M_{\mathscr{C}}^{T} \\ &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}^{T} \\ &= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}; \\ \Gamma(\mathscr{C}^{*}) &= M_{\mathscr{C}^{*}} \cdot M_{\mathscr{C}^{*}}^{T} \\ &= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & d_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \cdot \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}. \end{split}$$

By Definition 2.4, we have $c_{ij} = d_{ij}$ for $i \neq k$, $j \neq k$ since $a_{ij} = b_{ij}$ for $i \neq k$. To compute $\Gamma(\mathscr{C}^*)$ on the basis of $\Gamma(\mathscr{C})$, we only need to compute $(d_{ij})_{(i=k,1\leq j\leq n)}$ and $(d_{ij})_{(1\leq i\leq n,j=k)}$. Since $\Gamma(\mathscr{C}^*)$ is symmetric, we only need to compute $(d_{ij})_{(i=k,1\leq j\leq n)}$. In other words, we need to compute $\Delta\Gamma(\mathscr{C})$, where

$$\Delta\Gamma(\mathscr{C}) = \begin{bmatrix} 0 & 0 & \cdots & d_{1k}^* & \cdots & 0 \\ 0 & 0 & \cdots & d_{2k}^* & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{k1}^* & d_{k2}^* & \cdots & d_{kk}^* & \cdots & d_{kn}^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_{nk}^* & \cdots & 0 \end{bmatrix};$$

$$d_{kj}^* = d_{jk}^* = \begin{bmatrix} b_{k1} & b_{k2} & \cdots & b_{km} \end{bmatrix} \cdot \begin{bmatrix} b_{1j} & b_{2j} & \cdots & b_{mj} \end{bmatrix}^T - c_{kj}$$

Therefore, we have that

$$\Gamma(\mathscr{C}^*) = \Gamma(\mathscr{C}) + \Delta \Gamma(\mathscr{C}).\square$$

The following example is employed to show the process of constructing approximations of sets by Theorem 3.2.

Example 3.3 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C} = \{C_1, C_2, C_3\}$ and $\mathscr{C}^* = \{C_1^*, C_2^*, C_3^*\}$, where $C_1 = \{x_1, x_4\}$, $C_2 = \{x_1, x_2, x_4\}$, $C_3 = \{x_3, x_4\}$, $C_1^* = \{x_1, x_3, x_4\}$, $C_2^* = \{x_1, x_2, x_3, x_4\}$, $C_3^* = \{x_4\}$, and $X = \{x_3, x_4\}$. By Definition 2.4, we first have that

$$\begin{split} \Gamma(\mathscr{C}) &= M_{\mathscr{C}} \cdot M_{\mathscr{C}}^{T} \\ &= (c_{ij})_{4 \times 4} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{T} \\ &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \end{split}$$

Second, we denote $\Gamma(\mathscr{C}^*) = (d_{ij})_{4 \times 4}$. By Theorem 3.2, we get that

$$\begin{bmatrix} d_{31}^* & d_{32}^* & d_{33}^* & d_{34}^* \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \cdot M_{\mathscr{C}^*}^T - \begin{bmatrix} c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix};$$
$$\begin{bmatrix} d_{13}^* & d_{23}^* & d_{33}^* & d_{43}^* \end{bmatrix} = \begin{bmatrix} d_{31}^* & d_{32}^* & d_{33}^* & d_{34}^* \end{bmatrix}.$$

By Theorem 3.2, we have that

$$\Delta\Gamma(\mathscr{C}) = \begin{bmatrix} 0 & 0 & d_{13}^* & 0 \\ 0 & 0 & d_{23}^* & 0 \\ d_{31}^* & d_{32}^* & d_{33}^* & d_{34}^* \\ 0 & 0 & d_{43}^* & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, we obtain that

By Definition 2.6, we have that

Therefore, $SH(X) = \{x_1, x_2, x_3, x_4\}$ and $SL(X) = \emptyset$.

In Example 3.3, we only need to compute $\Delta\Gamma(\mathscr{C})$ by Theorem 3.2. But there is a need to compute all elements in $\Gamma(\mathscr{C}^*)$ by Definition 2.4. Therefore, the computing time of the incremental algorithm is less than the non-incremental algorithm.

Subsequently, we discuss the construction of $\prod(\mathscr{C}^*)$ based on $\prod(\mathscr{C})$. For convenience, we denote $\prod(\mathscr{C}) = (e_{ij})_{n \times n}$ and $\prod(\mathscr{C}^*) = (f_{ij})_{n \times n}$.

Theorem 3.4 Let (U, \mathcal{C}^*) be a dynamic covering approximation space of (U, \mathcal{C}) , $\prod(\mathcal{C})$ and $\prod(\mathcal{C}^*)$ the type-2 characteristic matrice of \mathcal{C} and \mathcal{C}^* , respectively. Then

$$\prod(\mathscr{C}^*) = \prod(\mathscr{C}) + \Delta \prod(\mathscr{C})$$

where

$$\Delta \prod(\mathscr{C}) = \begin{bmatrix} 0 & 0 & \cdots & f_{1k}^* & \cdots & 0 \\ 0 & 0 & \cdots & f_{2k}^* & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{k1}^* & f_{k2}^* & \cdots & f_{kk}^* & \cdots & f_{kn}^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f_{nk}^* & \cdots & 0 \end{bmatrix};$$

$$\begin{bmatrix} f_{k1}^* & f_{k2}^* & \cdots & f_{kn}^* \end{bmatrix} = \begin{bmatrix} b_{k1} & b_{k2} & \cdots & b_{km} \end{bmatrix} \odot M_{\mathscr{C}^*}^T - \begin{bmatrix} e_{k1} & e_{k2} & \cdots & e_{kn} \end{bmatrix};$$

$$\begin{bmatrix} f_{1k}^* & f_{2k}^* & \cdots & f_{nk}^* \end{bmatrix}^T = M_{\mathscr{C}^*} \odot \begin{bmatrix} b_{1k} & b_{2k} & \cdots & b_{mk} \end{bmatrix}^T - \begin{bmatrix} e_{1k} & e_{2k} & \cdots & e_{nk} \end{bmatrix}.$$

Proof. By Definition 2.5, $\prod(\mathscr{C})$ and $\prod(\mathscr{C}^*)$ are presented as follows:

$$\prod(\mathscr{C}) = M_{\mathscr{C}} \odot M_{\mathscr{C}}^{T}$$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \odot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}^{T}$$

$$= \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1n} \\ e_{21} & e_{22} & \cdots & e_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ e_{n1} & e_{n2} & \cdots & e_{nn} \end{bmatrix};$$

$$\prod(\mathscr{C}^{*}) = M_{\mathscr{C}^{*}} \odot M_{\mathscr{C}^{*}}^{T}$$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \odot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \circ$$

By Definition 2.5, we have $e_{ij} = f_{ij}$ for $i \neq k$, $j \neq k$ since $a_{ij} = b_{ij}$ for $i \neq k$. To compute $\prod(\mathscr{C}^*)$ on the basis of $\prod(\mathscr{C})$, we only need to compute $(f_{ij})_{(i=k,1\leq j\leq n)}$ and $(f_{ij})_{(1\leq i\leq n,j=k)}$. In other words, we need to compute $\Delta \prod(\mathscr{C})$, where

$$\Delta \prod(\mathscr{C}) = \begin{bmatrix} 0 & 0 & \cdots & f_{1k}^* & \cdots & 0 \\ 0 & 0 & \cdots & f_{2k}^* & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{k1}^* & f_{k2} & \cdots & f_{kn}^* \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & f_{nk}^* & \cdots & 0 \end{bmatrix};$$

$$\begin{bmatrix} f_{k1}^* & f_{k2}^* & \cdots & f_{kn}^* \end{bmatrix} = \begin{bmatrix} b_{k1} & b_{k2} & \cdots & b_{km} \end{bmatrix} \odot M_{\mathscr{C}^*}^T - \begin{bmatrix} e_{k1} & e_{k2} & \cdots & e_{kn} \end{bmatrix};$$

$$\begin{bmatrix} f_{1k}^* & f_{2k}^* & \cdots & f_{nk}^* \end{bmatrix}^T = M_{\mathscr{C}^*} \odot \begin{bmatrix} b_{1k} & b_{2k} & \cdots & b_{mk} \end{bmatrix}^T - \begin{bmatrix} e_{1k} & e_{2k} & \cdots & e_{nk} \end{bmatrix}.$$

Therefore, we have that

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$$\prod(\mathcal{C}^*) = \prod(\mathcal{C}) + \Delta \prod(\mathcal{C}).\square$$

The following example is employed to show the process of constructing approximations of sets by Theorem 3.4.

Example 3.5 Let $U = \{x_1, x_2, x_3, x_4\}$, $\mathscr{C} = \{C_1, C_2, C_3\}$ and $\mathscr{C}^* = \{C_1^*, C_2^*, C_3^*\}$, where $C_1 = \{x_1, x_4\}$, $C_2 = \{x_1, x_2, x_4\}$, $C_3 = \{x_3, x_4\}$, $C_1^* = \{x_1, x_3, x_4\}$, $C_2^* = \{x_1, x_2, x_3, x_4\}$, $C_3^* = \{x_4\}$, and $X = \{x_3, x_4\}$. By Definition 2.5, we first have that

$$\begin{split} \square(\mathscr{C}) &= M_{\mathscr{C}} \odot M_{\mathscr{C}}^{T} \\ &= (e_{ij})_{4 \times 4} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{T} \\ &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Second, we denote $\prod(\mathscr{C}^*) = (f_{ij})_{4\times 4}$. By Theorem 3.4, we get that

$$\begin{bmatrix} f_{31}^{*} & f_{32}^{*} & f_{33}^{*} & f_{34}^{*} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \odot M_{\mathscr{C}^{*}}^{T} - \begin{bmatrix} e_{31} & e_{32} & e_{33} & e_{34} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}^{T} - \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix};$$
$$\begin{bmatrix} f_{13}^{*} & f_{23}^{*} & f_{33}^{*} & f_{43}^{*} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{T} - \begin{bmatrix} e_{13} & e_{23} & e_{33} & e_{43} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}^{T} - \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}^{T} - \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}^{T} - \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

By Theorem 3.4, we have that

$$\Delta \prod(\mathscr{C}) = \begin{bmatrix} 0 & 0 & f_{13}^* & 0 \\ 0 & 0 & f_{23}^* & 0 \\ f_{31}^* & f_{32}^* & f_{33}^* & f_{34}^* \\ 0 & 0 & f_{43}^* & .0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, we obtain that

$$\begin{array}{rcl}
\left[(\mathscr{C}^*) &=& \prod (\mathscr{C}) + \Delta \prod (\mathscr{C}) \\
&=& \left[\begin{array}{rrrr} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right] + \left[\begin{array}{rrrr} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \\
&=& \left[\begin{array}{rrrr} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right].
\end{array}$$

By Definition 2.6, we have that

$$\begin{aligned} \mathcal{X}_{SH(X)} &= \prod (\mathscr{C}^*) \cdot \mathcal{X}_X; \\ &= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T; \\ \mathcal{X}_{SL(X)} &= \prod (\mathscr{C}^*) \odot \mathcal{X}_X \\ &= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T. \end{aligned}$$

Therefore, $SH(X) = \{x_1, x_2, x_3, x_4\}$ and $SL(X) = \{x_4\}$.

In Example 3.5, we only need to $\Delta \prod(\mathscr{C})$ by Theorem 3.4. But there is a need to compute all elements in $\prod(\mathscr{C}^*)$ by Definition 2.5. Therefore, the computing time of the incremental algorithm is less than the non-incremental algorithm.

4 Non-incremental and incremental algorithms of computing approximations of sets with varying attribute values

In this section, we present non-incremental and incremental algorithms of computing the second and sixth lower and upper approximations of sets with varying attribute values.

In Algorithm 4.1, the time complexity of Step 3 is $O(mn^2)$; the time complexity of step 4 is $O(2n^2)$. The total time complexity is $O((m + 2)n^2)$. In Algorithm 4.2, the time complexity of Step 4 is O(nm); the time complexity of Step 6 is O(n); the time complexity of Step 7 is O(n); the time complexity of Step 8 is $O(2n^2)$. The total time complexity is $O(2n^2 + nm + 2n)$. Furthermore, $O((m + 2)n^2)$ is the time complexity of the non-incremental algorithm. Thus the incremental algorithm is more effective than the non-incremental algorithm.

Algorithm 4.1: Non-incremental algorithm of computing the second lower and upper approximations of sets(NIS)

Input: (U, \mathscr{C}^*) and $X \subseteq U$. Output: $\mathcal{X}_{SH(X)}$ and $\mathcal{X}_{SL(X)}$. 1 begin 2 Construct $M_{\mathscr{C}^*}$ based on \mathscr{C}^* ; 3 Compute $\Gamma(\mathscr{C}^*) = M_{\mathscr{C}} \cdot M_{\mathscr{C}^*}^T$; 4 Obtain $\mathcal{X}_{SH(X)} = \Gamma(\mathscr{C}^*) \cdot \mathcal{X}_X$ and $\mathcal{X}_{SL(X)} = \Gamma(\mathscr{C}^*) \odot \mathcal{X}_X$. 5 end

Algorithm 4.2: Incremental algorithm of computing the second lower and upper approximations of sets(IS)

Input: 1. $(U, \mathscr{C}), \Gamma(\mathscr{C}), (U, \mathscr{C}^*), X \subseteq U$. **Output**: $X_{SH(X)}$ and $X_{SL(X)}$. begin 1 2 Construct $M^*_{\mathscr{C}} = (b_{ij})_{n \times m}$ based on \mathscr{C}^* ; Denote $row_k = [b_{k1}, b_{k2}, ..., b_{km}];$ 3 Compute $\Delta row_k = row_k \cdot M_{\mathscr{C}^*}^T$; 4 Let $\Gamma(\mathscr{C}^*)=\Gamma(\mathscr{C});$ 5 Set *k*th row of $\Gamma(\mathscr{C}^*)$ as Δrow_k ; 6 Set *k*th col of $\Gamma(\mathscr{C}^*)$ as $(\Delta row_k)^T$; 7 Obtain $\mathcal{X}_{SH(X)} = \Gamma(\mathscr{C}^*) \cdot \mathcal{X}_X$ and $\mathcal{X}_{SL(X)} = \Gamma(\mathscr{C}^*) \odot \mathcal{X}_X$. 8 9 end

Algorithm 4.3: Non-incremental algorithm of computing the sixth lower and upper approximations of sets(NIX)

Input: (U, \mathcal{C}^*) and $X \subseteq U$. Output: $\mathcal{X}_{XH(X)}$ and $\mathcal{X}_{XL(X)}$. 1 begin 2 Construct $M_{\mathcal{C}^*}$ based on \mathcal{C}^* ; 3 Compute $\prod(\mathcal{C}^*) = M_{\mathcal{C}^*} \odot M_{\mathcal{C}^*}^T$; 4 Obtain $\mathcal{X}_{XH(X)} = \prod(\mathcal{C}^*) \cdot \mathcal{X}_X$ and $\mathcal{X}_{XL(X)} = \prod(\mathcal{C}^*) \odot \mathcal{X}_X$. 5 end

In Algorithm 4.3, the time complexity of Step 3 is $O(mn^2)$, the time complexity of Step 4 is $O(n^2)$. The total time complexity is $O((m + 2)n^2)$. In Algorithm 4.4, the time complexity of Step 4 is O(nm); the time complexity of Step 6 is O(nm); the time complexity of Step 8 is O(n); the time complexity of Step 9 is O(n); the time complexity of Step 10 is $O(2n^2)$. The total time complexity is $O(2n^2 + 2nm + 2n)$. Furthermore, $O((m + 2)n^2)$ is the time complexity of the non-incremental algorithm. Thus the incremental algorithm is more effective than the non-incremental algorithm.

Algorithm 4.4: Incremental algorithm of computing the sixth lower and upper approximations of sets(IX)

Input: $(U, \mathscr{C}), \overline{\prod(\mathscr{C}), (U, \mathscr{C}^*)} \text{ and } X \subseteq U.$ **Output**: $X_{XH(X)}$ and $X_{XL(X)}$. 1 begin Construct $M^*_{\mathscr{C}} = (b_{ij})_{n \times m}$ based on \mathscr{C}^* ; 2 3 Denote $row_k = [b_{k1}, b_{k2}, ..., b_{km}];$ Compute $\Delta row_k = row_k \odot M_{\mathscr{L}^*}^T$; 4 Denote $col_k = [b_{1k}, b_{2k}, ..., b_{mk}]^T$; 5 Compute $\Delta col_k = M_{\mathscr{C}^*} \odot col_k$; 6 Let $\prod(\mathscr{C}^*) = \prod(\mathscr{C});$ 7 Set *k*th row of $\prod (\mathscr{C}^*)$ as Δrow_k ; 8 Set *k*th col of $\Pi(\mathscr{C}^*)$ as Δcol_k ; 9 Obtain $\mathcal{X}_{XH(X)} = \prod (\mathscr{C}^*) \cdot \mathcal{X}_X$ and $\mathcal{X}_{XL(X)} = \prod (\mathscr{C}^*) \odot \mathcal{X}_X$. 10 11 end

5 Experimental analysis

In this section, we perform the series of experiments to validate the effectiveness of Algorithms 4.2 and 4.4 for computing approximations in dynamic covering approximation spaces when varying attribute values.

5.1 Experimental environment

Since transforming information systems into covering approximation spaces takes a great deal of time, and the main objective of this work is to illustrate the efficiency of the Algorithms 4.2 and 4.4 in computing approximations of sets. To evaluate the performance of Algorithms 4.2 and 4.4, we generated ten covering approximation spaces (U_i, \mathcal{C}_i) for the experiment, where i, j = 1, 2, 3, ..., 10. We outline all these covering approximation spaces in Table 1, where $|U_i|$ denotes the number of objects in U_i and $|\mathcal{C}_i|$ is the cardinality of \mathcal{C}_i .

All computations were conducted on a PC with a Inter(R) Core(TM) i5-4200M CPU @ 2.50 GHZ and 4 GB memory, running 64-bit Windows 7 Service Pack 1. The software used was 64-bit Matlab R2013b. Details of the hardware and software are given in Table 2.

5.2 Experimental results

5.2.1 Computational times in dynamic covering approximation spaces

In this subsection, we apply Algorithms 4.1-4.4 to the covering approximation space (U_i, \mathcal{C}_i) , where i = 1, 2, 3, ..., 10, and compare the computing times by using Algorithms 4.1 and 4.3 with those of Algorithms 4.2 and 4.4, respectively.

First, we calculate $\Gamma(\mathscr{C}_i)$ and $\prod(\mathscr{C}_i)$ by Definitions 2.4 and 2.5, respectively. We also obtain the dynamic covering approximation space (U_i, \mathscr{C}_i^*) when revising attribute values of x_k , where and $C_i^* =$

No.	Name	$ U_i $	$ \mathscr{C}_i $
1	(U_1, \mathscr{C}_1)	2000	100
2	(U_2, \mathscr{C}_2)	4000	200
3	(U_3, \mathscr{C}_3)	6000	300
4	(U_4, \mathscr{C}_4)	8000	400
5	(U_5, \mathscr{C}_5)	10000	500
6	(U_6, \mathscr{C}_6)	12000	600
7	(U_7, \mathscr{C}_7)	14000	700
8	(U_8, \mathscr{C}_8)	16000	800
9	(U_9, \mathscr{C}_9)	18000	900
10	$(U_{10}, \mathscr{C}_{10})$	20000	1000

Table 1: Covering approximation spaces.

Table 2: The experimental environment.

No.	Name	Model	Parameters
1	CPU	Inter(R) Core(TM) i5-4200M	2.50 GHZ
2	Memory	ADAT DDR3	4G
3	Hard disk	SATA	1T
4	System	Windows 7	64bit
5	Platform	Matlab R2013b	64bit

 $C_j \cup \{x_k\}$ or C_j , where $C_j^* \in \mathscr{C}_i^*$ and $C_j \in \mathscr{C}_i$. Subsequently, we get $\Gamma(\mathscr{C}_i^*)$ and $\prod(\mathscr{C}_i^*)$ by Algorithms 4.1 and 4.3, respectively.

Second, we calculate SH(X), SL(X), XH(X) and XL(X) based on $\Gamma(\mathscr{C}_i^*)$ and $\prod(\mathscr{C}_i^*)$ for $X \subseteq U_i$, respectively. The time of computing SH(X), SL(X), XH(X) and XL(X) is shown in Tables 3-12. Concretely, *NIS* and *NIX* stands for the time of constructing the second and sixth lower and upper approximations of sets by Algorithms 4.1 and 4.3 in Tables 3-12. Additionally, we obtain $\Gamma(\mathscr{C}_i^*)$ and $\prod(\mathscr{C}_i^*)$ by Algorithms 4.2 and 4.4, respectively. Then the time of computing SH(X), SL(X), XH(X) and XL(X) for $X \subseteq U_i$ is shown in Tables 3-12. Concretely, *IS* and *IX* stands for the time of computing the second and sixth lower and upper approximations of sets by Algorithms 4.2 and 4.4 in Tables 3-12.

Third, we conduct all experiments ten times and show the results in Tables 3-12 and Figures 1-10. We see all algorithms are stable to compute approximations of sets in all experiments. Concretely, we observe that the computing times by using the same algorithm are almost the same in Tables 3-12. Consequently, we see that the times of computing approximations of sets by using incremental algorithms are much smaller than those of the non-incremental algorithms. In Figures 1-10, we also observe that the computing times of Algorithms 4.2 and 4.4 are far less than those of Algorithms 4.1 and 4.3, respectively. Therefore, the incremental algorithms are more effective to construct approximations of sets in the dynamic covering approximation space (U_i, \mathcal{C}_i^*) , where i = 1, 2, ..., 10.

Remark: In Tables 3-12, the measure of time is in seconds; \overline{t} indicates the average time of ten experiments; In Figures 1-10, *i* stands for the experimental number in *X* Axis; In Figure 11, *i* refers to as the covering approximation space (U_i, \mathcal{C}_i) in *X* Axis; In Figures 1-11, *i* is the computing time in *Y* Axis.

Table 3: Computational times using Algorithms 4.1-4.4 in (U_1, \mathcal{C}_1) .

Algorithm	1	2	3	4	5	6	7	8	9	10	\overline{t}
NIS	0.4578	0.4213	0.4279	0.4223	0.4271	0.4236	0.4235	0.4263	0.4236	0.4273	0.4281
NIX	0.4681	0.4671	0.4636	0.4646	0.4668	0.4651	0.4651	0.4681	0.4668	0.4720	0.4667
IS	0.0044	0.0026	0.0033	0.0040	0.0029	0.0028	0.0031	0.0030	0.0028	0.0028	0.0032
IX	0.0351	0.0339	0.0333	0.0339	0.0340	0.0334	0.0340	0.0335	0.0338	0.0333	0.0338

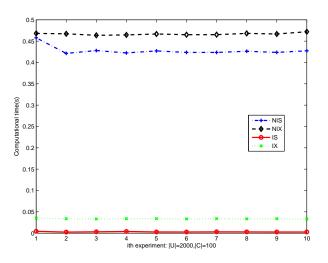


Figure 1: Computational times using Algorithms 4.1-4.4 in (U_1, \mathscr{C}_1) .

Algorithm	1	2	3	4	5	6	7	8	9	10	t
NIS	1.8902	1.8452	1.8610	1.8203	1.8179	1.8257	1.8223	1.8224	1.8294	1.8189	1.8353
NIX	2.0389	2.0437	2.0314	2.0237	2.0378	2.0331	2.0531	2.0565	2.0583	2.0641	2.0440
IS	0.0091	0.0118	0.0102	0.0100	0.0098	0.0127	0.0110	0.0099	0.0099	0.0096	0.0104
IX	0.2035	0.2018	0.2013	0.2018	0.2034	0.1992	0.2018	0.2006	0.1987	0.2035	0.2016

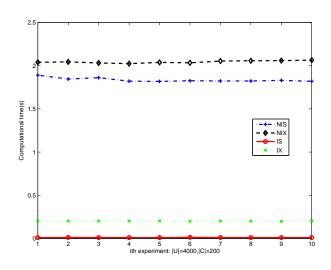


Figure 2: Computational times using Algorithms 4.1-4.4 in (U_2, \mathscr{C}_2) .

Table 5: Computational times using Algorithms 4.1-4.4 in (U_3, \mathcal{C}_3) .													
Algorithm	1	2	3	4	5	6	7	8	9	10	\overline{t}		
NIS	4.2030	4.1889	4.1905	4.1457	4.1446	4.1681	4.1518	4.1765	4.2310	4.1604	4.1760		
NIX	4.6993	4.7126	4.6838	4.6895	4.6941	4.7000	4.7025	4.6711	4.7039	4.6939	4.6951		
IS	0.0177	0.0210	0.0211	0.0199	0.0199	0.0199	0.0199	0.0205	0.0200	0.0197	0.0200		
IX	0.5259	0.5059	0.5076	0.5056	0.5089	0.5055	0.5106	0.5080	0.5059	0.5078	0.5092		

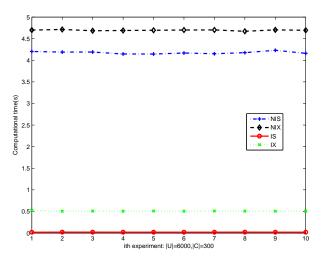


Figure 3: Computational times using Algorithms 4.1-4.4 in (U_3, \mathscr{C}_3) .

Table 6: Computational times using Algorithms 4.1-4.4 in (U_4, \mathscr{C}_4) .

Algorithm			•		5	0 0	7		9	10	t
NIS	7.5968	7.5550	7.7428	7.6536	7.6756	7.7031	7.6304	7.6051	7.6118	7.7013	7.6475
NIX	8.6892	8.7967	8.8918	9.0384	8.7810	8.7764	8.6300	9.2821	8.6324	8.6121	8.8130
IS	0.0428	0.0338	0.0350	0.0394	0.0378	0.0386	0.0345	0.0345	0.0346	0.0348	0.0366
IX	0.9813	0.9681	0.9694	0.9677	0.9669	0.9731	0.9654	0.9683	0.9648	0.9685	0.9694

Table 7: Computational times using Algorithms 4.1-4.4 in (U_5, \mathscr{C}_5) .

Algorithm	1	2	3	4	5	6	7	8	9	10	t
NIS	12.0856	11.9662	11.9944	11.9200	11.9992	11.9683	11.9321	11.9008	11.8811	11.8839	11.9532
NIX	13.8290	13.6560	13.7430	13.7308	13.6831	13.6816	13.7970	13.6794	13.8141	13.7338	13.7348
IS	0.0675	0.0530	0.0549	0.0537	0.0551	0.0537	0.0536	0.0523	0.0535	0.0540	0.0551
IX	1.6266	1.6193	1.6163	1.6138	1.6189	1.6057	1.6230	1.6213	1.6172	1.6211	1.6183

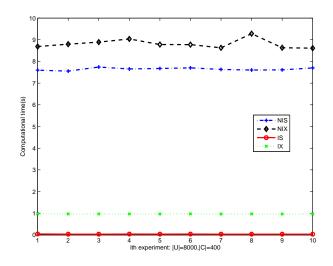


Figure 4: Computational times using Algorithms 4.1-4.4 in (U_4, \mathscr{C}_4) .

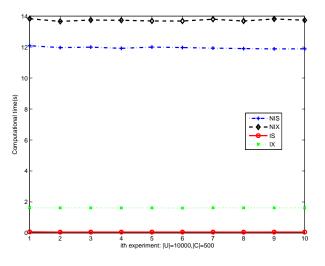


Figure 5: Computational times using Algorithms 4.1-4.4 in (U_5, \mathscr{C}_5) .

Table 8: Computational times using Algorithms 4.1-4.4 in (U_6, \mathscr{C}_6) .												
Algorithm	1	2	3	4	5	6	7	8	9	10	\overline{t}	
NIS	17.8842	17.8858	18.0800	17.6753	17.5945	17.5710	17.7019	18.2036	17.5415	17.9582	17.8096	
NIX	20.1684	20.1404	20.0242	20.0022	20.0277	20.0598	20.0897	20.2560	21.6223	22.1194	20.4510	
IS	0.0977	0.0748	0.0746	0.0744	0.0803	0.0727	0.0753	0.0735	0.0738	0.0723	0.0770	
IX	2.4011	2.3671	2.4204	2.3771	2.3679	2.3662	2.3737	2.3644	2.3614	2.3692	2.3769	

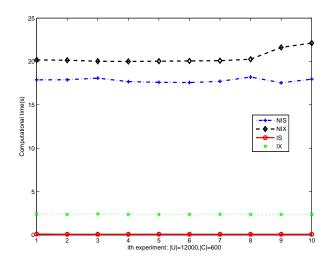


Figure 6: Computational times using Algorithms 4.1-4.4 in (U_6, \mathscr{C}_6) .

Algorithm	n 1	2	3	4	5	6	7	8	9	10	t
NIS	24.2936	24.3201	24.4603	25.2946	24.4922	24.5153	24.3296	25.0792	24.6210	24.2059	24.5612
NIX	27.9154	28.2049	28.2523	28.2664	28.7698	28.2559	28.1121	28.4234	28.6467	29.2779	28.4125
IS	0.1071	0.1014	0.1017	0.0996	0.1015	0.1018	0.1025	0.1007	0.1020	0.1009	0.1019
IX	3.4572	3.3194	3.3070	3.3030	3.2899	3.3109	3.2777	3.2753	3.2790	3.2758	3.3095

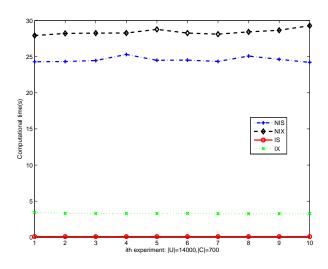


Figure 7: Computational times using Algorithms 4.1-4.4 in (U_7, \mathcal{C}_7) .

5.2.2 The relationship between computational times and the cardinalities of object sets and coverings

In Figure 11, the average times of the incremental and non-incremental algorithms rise monotonically with the increase of the cardinalities of object sets and coverings. We also see that the incremental algorithms perform always faster than the non-incremental algorithms in all experiments, and the average times of the incremental algorithms are much smaller than those of the non-incremental algorithms. Moreover, the speed-up ratios of times by using the non-incremental algorithms are higher than the incremental algorithms with the increasing cardinalities of object sets and coverings. Especially, we observe that there exists little influence of the cardinalities of object sets and coverings on computing the second lower and upper approximations of sets by using Algorithm 4.2.

All experimental results demonstrate that Algorithms 4.2 and 4.4 are more effective to computing the second and sixth lower and upper approximations of sets in dynamic covering approximation spaces. In the future, we will improve the effectiveness of Algorithms 4.2 and 4.4 and test them on large-scale dynamic covering approximation spaces.

6 Attribute reduction of dynamic covering decision information systems

In this section, we employ examples to illustrate that how to compute type-1 and type-2 reducts of covering decision information systems.

Example 6.1 Let $(U, \mathcal{D} \cup U/d)$ be a covering decision information system, where $\mathcal{D} = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}$, $\mathscr{C}_1 = \{\{x_1, x_2, x_3, x_4\}, \{x_5\}\}, \mathscr{C}_2 = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}\}, \mathscr{C}_3 = \{\{x_1, x_2, x_5\}, \{x_3, x_4\}\}, \mathscr{C}_4 = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5\}\}, U/d = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}\}$. By Definitions 2.4 and 2.5, we obtain

Table 10: Computational times using Algorithms 4.1-4.4 in (U_8, \mathcal{C}_8) .

Algorithm	1	2	3	4	5	6	7	8	9	10	\overline{t}
NIS	33.2714	33.3024	33.2390	33.2370	33.3127	33.3602	33.3527	33.2599	33.4496	33.3485	33.3133
NIX	39.0763	39.0729	39.1256	39.1677	39.1382	39.5114	39.2732	38.9632	39.1487	38.8493	39.1327
IS	0.1267	0.1243	0.1293	0.1242	0.1248	0.1239	0.1259	0.1234	0.1226	0.1284	0.1254
IX	6.1013	5.3888	5.3412	5.3710	5.2641	5.3158	5.3229	5.3422	5.2858	5.4398	5.4173

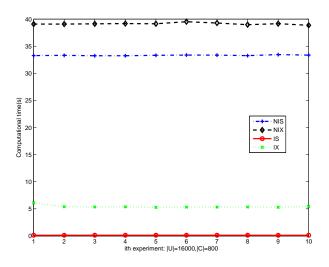


Figure 8: Computational times using Algorithms 4.1-4.4 in (U_8, \mathcal{C}_8) .

Algorithm	n 1	2	3	4	5	6	7	8	9	10	t
NIS	44.2060	43.5990	43.2590	44.3375	43.9165	43.6185	44.3864	44.4667	44.2301	45.2159	44.1236
NIX	50.1711	50.8559	50.4446	49.7286	50.6871	50.3282	50.5291	49.5770	50.0544	50.3550	50.2731
IS	0.2048	0.1611	0.1628	0.1620	0.1607	0.1607	0.1605	0.1612	0.1615	0.1615	0.1657
IX	6.1794	5.8323	5.8586	5.7428	5.8902	5.8318	5.8949	5.7688	5.7606	5.8051	5.8564

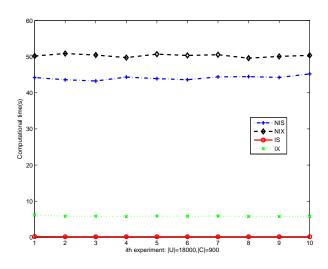


Figure 9: Computational times using Algorithms 4.1-4.4 in (U_9, \mathcal{C}_9) .

Table 12: Computational times using Algorithms 5.1-5.8 in (U_1^*, \mathcal{C}_1^*) , where $ U_1^* = 625$ and $ \mathcal{C}_1^* = 20$.											
Algorithm	1	2	3	4	5	6	7	8	9	10	\overline{t}
NIS	55.6793	55.8107	55.6728	55.9174	55.5917	58.1981	59.1824	56.0537	55.7757	55.5664	56.3448
NIX	64.8043	65.7104	65.2075	64.5169	64.7856	64.7118	65.0349	64.4148	64.7802	64.3155	64.8282
IS	0.2716	0.1941	0.1944	0.1924	0.1938	0.1956	0.1936	0.1917	0.1947	0.1948	0.2017
IX	8.3148	7.6287	7.3082	7.9581	7.2058	7.4084	7.1585	7.2874	7.1620	7.2413	7.4673

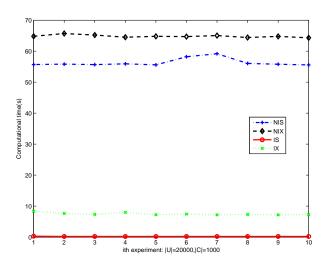


Figure 10: Computational times using Algorithms 4.1-4.4 in $(U_{10}, \mathcal{C}_{10})$.

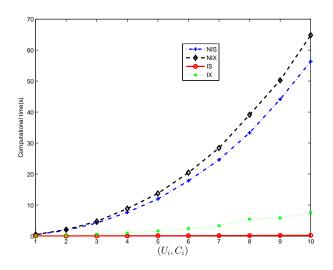


Figure 11: Computation times using Algorithms 4.1-4.4.

By Definition 2.6, we have the second and sixth lower and upper approximations of decision classes as follows:

$$\begin{split} \mathcal{X}_{SH(D_1)} &= \ \Gamma(\mathscr{D}) \cdot \mathcal{X}_{D_1} \\ &= \ \left[\ 1 & 1 & 1 & 1 & 1 \ \right]; \\ \mathcal{X}_{SL(D_1)} &= \ \Gamma(\mathscr{D}) \odot \mathcal{X}_{D_1} \\ &= \ \left[\ 0 & 0 & 0 & 0 & 0 \ \right]; \\ \mathcal{X}_{SH(D_2)} &= \ \Gamma(\mathscr{D}) \cdot \mathcal{X}_{D_2} \\ &= \ \left[\ 1 & 1 & 1 & 1 & 1 \ \right]; \\ \mathcal{X}_{SL(D_2)} &= \ \Gamma(\mathscr{D}) \odot \mathcal{X}_{D_2} \\ &= \ \left[\ 0 & 0 & 0 & 0 & 0 \ \right]; \\ \mathcal{X}_{XH(D_1)} &= \ \Gamma(\mathscr{D}) \cdot \mathcal{X}_{D_1} \\ &= \ \left[\ 1 & 1 & 0 & 0 & 0 \ \right]; \\ \mathcal{X}_{XL(D_1)} &= \ \Gamma(\mathscr{D}) \odot \mathcal{X}_{D_1} \\ &= \ \left[\ 1 & 1 & 0 & 0 & 0 \ \right]; \\ \mathcal{X}_{XL(D_1)} &= \ \Gamma(\mathscr{D}) \odot \mathcal{X}_{D_2} \\ &= \ \left[\ 0 & 0 & 1 & 1 & 1 \ \right]; \\ \mathcal{X}_{XL(D_2)} &= \ \Gamma(\mathscr{D}) \circ \mathcal{X}_{D_2} \\ &= \ \left[\ 0 & 0 & 1 & 1 & 1 \ \right]; \\ \mathcal{X}_{XL(D_2)} &= \ \Gamma(\mathscr{D}) \odot \mathcal{X}_{D_2} \\ &= \ \left[\ 0 & 0 & 1 & 1 & 1 \ \right]; \end{split}$$

To construct type-1 and type-2 reducts, we have that

$$\Gamma(\mathscr{D}/\mathscr{C}_{4}) \cdot \mathcal{X}_{D_{1}} = \mathcal{X}_{SH(D_{1})};$$

$$\Gamma(\mathscr{D}/\mathscr{C}_{4}) \odot \mathcal{X}_{D_{1}} = \mathcal{X}_{SL(D_{1})};$$

$$\Gamma(\mathscr{D}/\mathscr{C}_{4}) \cdot \mathcal{X}_{D_{2}} = \mathcal{X}_{SH(D_{2})};$$

$$\Gamma(\mathscr{D}/\mathscr{C}_{4}) \odot \mathcal{X}_{D_{2}} = \mathcal{X}_{SL(D_{2})};$$

$$\prod(\mathscr{D}/\mathscr{C}_{4}) \circ \mathcal{X}_{D_{1}} = \mathcal{X}_{XH(D_{1})};$$

$$\prod(\mathscr{D}/\mathscr{C}_{4}) \circ \mathcal{X}_{D_{1}} = \mathcal{X}_{XL(D_{1})};$$

$$\prod(\mathscr{D}/\mathscr{C}_{4}) \circ \mathcal{X}_{D_{2}} = \mathcal{X}_{XH(D_{2})};$$

$$\prod(\mathscr{D}/\mathscr{C}_{4}) \odot \mathcal{X}_{D_{2}} = \mathcal{X}_{XL(D_{2})};$$

To perform the above process continuously, we have that $\{C_1, C_3\}$ is type-1 and type-2 reducts of $(U, \mathcal{D} \cup U/d)$.

We employ an example to illustrate that how to construct type-1 and type-2 reducts of dynamic covering decision information systems as follows.

Example 6.2 (Continuation of Example 6.1) Let $(U, \mathcal{D}^* \cup U/d)$ be a covering decision information system, where $\mathcal{D}^* = \{\mathscr{C}_1^*, \mathscr{C}_2^*, \mathscr{C}_3^*, \mathscr{C}_4^*\}, \mathscr{C}_1^* = \{\{x_1, x_2, x_3, x_4\}, \{x_5\}\}, \mathscr{C}_2^* = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}\}, \mathscr{C}_3^* = \{\{x_1, x_2, x_3, x_5\}, \{x_4\}\}, \mathscr{C}_4^* = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5\}\}, U/d = \{\{x_1, x_2\}, \{x_3, x_4, x_5\}\}$. By Theorems 3.2 and 3.4, we obtain

By Definition 2.6, we have the second and sixth lower and upper approximations of decision classes as follows:

$$\begin{split} \mathcal{X}_{SH(D_1)} &= \ \Gamma(\mathcal{D}^*) \cdot \mathcal{X}_{D_1} \\ &= \ [\ 1 \ \ 1 \ \ 1 \ \ 1 \ \ 1 \]; \\ \mathcal{X}_{SL(D_1)} &= \ \Gamma(\mathcal{D}^*) \odot \mathcal{X}_{D_1} \\ &= \ [\ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \]; \\ \mathcal{X}_{SH(D_2)} &= \ \Gamma(\mathcal{D}^*) \cdot \mathcal{X}_{D_2} \\ &= \ [\ 1 \ \ 1 \ \ 1 \ \ 1 \]; \\ \mathcal{X}_{SL(D_2)} &= \ \Gamma(\mathcal{D}^*) \odot \mathcal{X}_{D_2} \\ &= \ [\ 0 \ \ 0 \ \ 0 \ \ 0 \]; \\ \mathcal{X}_{XH(D_1)} &= \ \Gamma(\mathcal{D}^*) \cdot \mathcal{X}_{D_1} \\ &= \ [\ 1 \ \ 1 \ \ 0 \ \ 0 \]; \\ \mathcal{X}_{XH(D_1)} &= \ \Gamma(\mathcal{D}^*) \odot \mathcal{X}_{D_1} \\ &= \ [\ 1 \ \ 1 \ \ 0 \ \ 0 \]; \\ \mathcal{X}_{XL(D_1)} &= \ \Gamma(\mathcal{D}^*) \odot \mathcal{X}_{D_1} \\ &= \ [\ 1 \ \ 1 \ \ 0 \ \ 0 \]; \\ \mathcal{X}_{XL(D_2)} &= \ \Gamma(\mathcal{D}^*) \odot \mathcal{X}_{D_2} \\ &= \ [\ 0 \ \ 0 \ \ 1 \ \ 1 \]; \\ \mathcal{X}_{XL(D_2)} &= \ \Gamma(\mathcal{D}^*) \odot \mathcal{X}_{D_2} \\ &= \ [\ 0 \ \ 0 \ \ 1 \ \ 1 \]; \end{split}$$

To construct type-1 and type-2 reducts, we have that

$$\Gamma(\mathscr{D}^*/\mathscr{C}_4^*) \cdot \mathcal{X}_{D_1} = \mathcal{X}_{SH(D_1)};$$

$$\Gamma(\mathscr{D}^*/\mathscr{C}_4^*) \odot \mathcal{X}_{D_1} = \mathcal{X}_{SL(D_1)};$$

$$\Gamma(\mathscr{D}^*/\mathscr{C}_4^*) \circ \mathcal{X}_{D_2} = \mathcal{X}_{SH(D_2)};$$

$$\Gamma(\mathscr{D}^*/\mathscr{C}_4^*) \odot \mathcal{X}_{D_2} = \mathcal{X}_{SL(D_2)};$$

$$\prod(\mathscr{D}^*/\mathscr{C}_4^*) \circ \mathcal{X}_{D_1} = \mathcal{X}_{XH(D_1)};$$

$$\prod(\mathscr{D}^*/\mathscr{C}_4^*) \circ \mathcal{X}_{D_2} = \mathcal{X}_{XL(D_1)};$$

$$\prod(\mathscr{D}^*/\mathscr{C}_4^*) \circ \mathcal{X}_{D_2} = \mathcal{X}_{XH(D_2)};$$

$$\prod(\mathscr{D}^*/\mathscr{C}_4^*) \circ \mathcal{X}_{D_2} = \mathcal{X}_{XH(D_2)};$$

To perform the above process continuously, we have that $\{\mathscr{C}_1^*, \mathscr{C}_3^*\}$ is a type-1 reduct of $(U, \mathscr{D}^* \cup U/d)$, and $\{\mathscr{C}_1^*, \mathscr{C}_2^*, \mathscr{C}_3^*\}$ is a type-2 reduct of $(U, \mathscr{D}^* \cup U/d)$.

7 Conclusions

Knowledge reduction of covering information systems have attracted more attention of researchers. In this paper, we have introduced incremental approaches to computing the characteristic matrices of dynamic coverings when revising attribute values. We have presented the non-incremental and incremental algorithms for computing the second and sixth lower and upper approximations of sets and compared the computational complexities of the non-incremental algorithms with those of incremental algorithms. We have tested the incremental algorithms on dynamic covering approximation spaces. Experimental results have been employed to illustrate that the incremental approaches are effective to compute approximations of sets in dynamic covering approximation spaces. We have demonstrated that how to conduct knowledge reduction of dynamic covering information systems with the incremental approaches.

In practical situations, there exist many types of dynamic covering information systems and dynamic covering approximation spaces. In the future, we will introduce more effective approaches to constructing the characteristic matrices of these types of dynamic coverings and perform knowledge reduction of these types of dynamic covering information systems.

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