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ORIGINAL ARTICLE

The best single-observational and two-observational percentile estimations in the exponentiated Weibull-geometric distribution compared with maximum likelihood and percentile estimations

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Abstract In this research the best single-observation percentile estimation (BSPE) and best two-observation percentile estimation (BTPE), are introduced. Then theses estimators are obtained for probability density function and cumulative distribution function of the exponentiated Weibull-geometric (EWG) with increasing, decreasing, bathtub and unimodal shaped failure rate function. Finally, these estimators are compared with the maximum likelihood (ML) and percentile (PC) estimations using the Monte Carlo simulation and a real data set.

Keywords Single-observational percentile estimation · Two-observational percentile estimation · Maximum likelihood estimation · Percentile estimation · Monte Carlo simulation

1 Introduction

The estimation of probability density function (PDF) and cumulative density function (CDF) of several lifetime distributions using the maximum likelihood (ML), uniformly minimum variance unbiased (UMVU), percentile (PC), least squares (LS) and weighted least squares (WLS) estimators have been obtained and compared by researchers. A number of papers have been attempted to estimate the lifetime distribution parameters, for instance the estimation of pdf and cdf of the Pareto distribution by Dixit and Jabbari Nooghabi (2010), exponentiated Pareto distribution by Jabbari Nooghabi and Jabbari Nooghabi (2010), exponentiated Gumbel distribution by Bagheri et al. (2013b), generalized Rayliegh distribution by Alizadeh et al. (2013) and generalized Poisson-exponential distribution by Bagheri et al. (2013a). Note that Menon (1963) and Zanakis and Mann (1982) estimated the parameters of Weibull distribution by best single-observation percentile estimation (BSPE) and best two-observation percentile estimation (BTPE), but in this research the PDF and CDF of the Exponentiated Weibull-Geometric (EWG) which is originally introduced by Mahmoudi and Shiran (2012) are obtained by BSPE and BTPE methods for One or Two known parameters and compared with the corresponding estimations found by PC and MLE procedures.

According to the structure in this paper, in Sects. 2 and 3, the BEPE, PCE, MLE and BTPE, PCE, MLE are obtained respectively. By using the Monte Carlo simulations, estimators were compared in Sect. 4, and the results for real data are provided in Sect. 5.

2 Calculating estimations when only one parameter is unknown

Let $X_1, \ldots, X_n$ is a random sample with ordinal statistics of $Y_1, \ldots, Y_n$, of a distribution with the following probability density and cumulative distribution functions:

$$f(x;\alpha,\beta,\gamma,\theta) = \frac{\alpha\beta^{\gamma}\gamma(1-\theta)x^{\gamma-1}e^{-(\beta x)^{\gamma}}\left(1-e^{-(\beta x)^{\gamma}}\right)^{\alpha-1}}{\left[1-\theta\left(1-e^{-(\beta x)^{\gamma}}\right)^{\alpha}\right]^{2}} \tag{1}$$

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$$F(x;\alpha,\beta,\gamma,\theta)=\frac{(1-\theta)\left(1-e^{-(\beta x)^{\gamma}}\right)^{\alpha}}{1-\theta\left(1-e^{-(\beta x)^{\gamma}}\right)^{\alpha}} \tag{2}$$

Such that $x>0,\alpha>0,\beta>0,\gamma>0,0<\theta<1$. In this section, assuming that parameters β,γ,θ are known and parameter α is unknown, the BSPE, PCE and MLE of α are obtained.

2.1 Estimation of the BSP

If Y_k is the p-th percentile $(0<p<1)$ of distribution (2), then

$$p=F(Y_k;\alpha,\beta,\gamma,\theta)=\frac{(1-\theta)\left(1-e^{-(\beta Y_k)^{\gamma}}\right)^{\alpha}}{1-\theta\left(1-e^{-(\beta Y_k)^{\gamma}}\right)^{\alpha}}$$

where $k=[np]$, if np is an integer, otherwise, $k=[np]+1$ where $[np]$ is the greatest integer smaller than np. Therefore, a single-observation percentile estimation of α which is shown by α^* is as follows:

$$\alpha^*=\frac{\log\frac{p}{1-\theta(1-p)}}{\log\left(1-e^{(-\beta Y_k)^{\gamma}}\right)}=\frac{\log[-\log(1-p^*)]}{\log Z_k} \tag{3}$$

Such that $p^*=1-e^{-\frac{p}{1-\theta(1-p)}}$ and $Z_k=1-e^{(-\beta Y_k)^{\gamma}}$. According to Dubey (1967, p. 122), α^* has an asymptotic normal distribution with mean of α and variance of

$$Var(\alpha^*)=\frac{\alpha^2p^*}{n(1-p^*)\log^2(1-p^*)\log^2[-\log(1-p^*)]}=\frac{\alpha^2(1-e^{-q})}{nq^2e^{-q}\log^2 q}$$

where $q=\frac{p}{1-\theta(1-p)}$. Now q is determined in a way that $Var(\alpha^*)$ is minimum, which in this case, solves the equation

$$q\log q-2(1+\log q)(1+e^{-q})=0$$

By an iterative method, $q=0.1189$ and finally, the optimal p is obtained by the following relation.

$$p=\frac{0.1189(1-\theta)}{1-0.1189\theta}$$

Therefore, the BSPE of α as shown by $\hat{\alpha}_{BSPE}$ is determined as follows:

$$\hat{\alpha}_{BSPE}=\frac{\log[0.1189(1-\theta)/(1-0.1189\theta)]}{\log\left(1-e^{-(\beta Y_k)^{\gamma}}\right)}$$

Such that, $k=\left[n\frac{0.1189(1-\theta)}{1-0.1189\theta}\right]$ or $k=1+\left[n\frac{0.1189(1-\theta)}{1-0.1189\theta}\right]$.

Thus, the BSPE of functions (1) and (2) are obtained by the following relation, respectively.

$$\hat{f}_{BSPE}(x;\alpha,\beta,\gamma,\theta)=\frac{\hat{\alpha}_{BSPE}\beta^{\gamma}\gamma(1-\theta)x^{\gamma-1}e^{-(\beta x)^{\gamma}}\left(1-e^{-(\beta x)^{\gamma}}\right)^{(\hat{\alpha}_{BSPE})-1}}{\left[1-\theta\left(1-e^{-(\beta x)^{\gamma}}\right)^{\hat{\alpha}_{BSPE}}\right]^2}$$

$$\hat{F}_{BSPE}(x;\alpha,\beta,\gamma,\theta)=\frac{(1-\theta)\left(1-e^{-(\beta x)^{\gamma}}\right)^{\hat{\alpha}_{BSPE}}}{1-\theta\left(1-e^{-(\beta x)^{\gamma}}\right)^{\hat{\alpha}_{BSPE}}}$$

2.2 PCE

Let $X_1,\ldots,X_n$ is a random sample distribution with CDF given in (2) with order statistics of $Y_1,\ldots,Y_n$, and p_i is the percentile of Y_i, then, $F(Y_i,\alpha,\beta,\gamma,\theta)=p_i$ or

$$\log\frac{p_i}{[1-\theta(1-p_i)]}=\alpha\log\left(1-e^{-(\beta Y_i)^{\gamma}}\right)$$

The PCE of α which is shown by $\hat{\alpha}_{PCE}$ is obtained by the minimization of

$$\sum_{i=1}^{n}\left[\alpha\log\left(1-e^{-(\beta Y_i)^{\gamma}}\right)-\log\frac{p_i}{[1-\theta(1-p_i)]}\right]^2$$

with respect to α. ($p_i=\frac{i}{n+1}$), so

$$\hat{\alpha}_{PCE}=\sum_{i=1}^{n}\log\frac{p_i\left(1-e^{-(\beta Y_i)^{\gamma}}\right)}{[1-\theta(1-p_i)]}\Bigg/\sum_{i=1}^{n}\left(1-e^{-(\beta Y_i)^{\gamma}}\right)^2$$

Therefore, the PCEs of functions (1) and (2) are obtained as follows:

$$\hat{f}_{PCE}(x;\alpha,\beta,\gamma,\theta)=\frac{\hat{\alpha}_{PCE}\beta^{\gamma}\gamma(1-\theta)x^{\gamma-1}e^{-(\beta x)^{\gamma}}\left(1-e^{-(\beta x)^{\gamma}}\right)^{(\hat{\alpha}_{PCE})-1}}{\left[1-\theta\left(1-e^{-(\beta x)^{\gamma}}\right)^{\hat{\alpha}_{PCE}}\right]^2}$$

$$\hat{F}_{PCE}(x;\alpha,\beta,\gamma,\theta)=\frac{(1-\theta)\left(1-e^{-(\beta x)^{\gamma}}\right)^{\hat{\alpha}_{PCE}}}{1-\theta\left(1-e^{-(\beta x)^{\gamma}}\right)^{\hat{\alpha}_{PCE}}}$$

For more details about the PCE method, see Kao (1958, 1959) and Johnson et al. (1994). Mean square error (MSE) of percentile estimations of functions (1) and (2) is calculated by Monte Carlo simulation method of the sample mean.

2.3 MLE

According to a random sample of $X_1,\ldots,X_n$ of distribution with the probability density function (1), the MLE of the parameter α, i.e. $\hat{\alpha}_{MLE}$ is obtained by:

$$\frac{n}{\alpha}+\sum_{i=1}^{n}\frac{\left[1+\theta\left(1-e^{-(\beta x_i)^{\gamma}}\right)^{\alpha}\right]\log\left(1-e^{-(\beta x_i)^{\gamma}}\right)}{1-\theta\left(1-e^{-(\beta x_i)^{\gamma}}\right)^{\alpha}}=0$$

where replacing $\hat{\alpha}_{MLE}$ by α in relations (1) and (2), The MLE of the probability density and cumulative distribution functions of EWG distribution can be obtained. Moreover, by Monte Carlo simulation method of the sample mean, the mean square error (MSE) of the MLE of functions (1) and (2) could be found.

3 Calculating estimators when two parameters are unknown

In this section, a random sample of size n from the pdf given in (1) is considered. We assume that the parameters γ and β are unknown, and parameters α and θ are known,. Then the BTPE, PCE and MLE of γ and β, for the pdf (1) and cdf (2) are obtained.

3.1 BTPE

Suppose $X_1,\ldots,X_n$ is a random sample of distribution with cdf (2) with ordinal statistics of $Y_1,\ldots,Y_n$, and p_i is the percentile of Y_i, then, $F(Y_i,\alpha,\beta,\gamma,\theta)=p_i$ or

$$\gamma(\log\beta+\log Y_i)=\log\left\{-\log\left[1-\left(\frac{p_i}{1-\theta+\theta p_i}\right)^{\frac{1}{\alpha}}\right]\right\} \tag{4}$$

such that for two real values of p_1 and p_2 $(0<p_1<p_2<1)$ and with the help of relation (4), a two-observational percentile estimation of γ which is shown by γ^* can be obtained as follows:

$$\gamma^*=\frac{\log\left\{-\log\left[1-\left(\frac{p_1}{1-\theta+\theta p_1}\right)^{\frac{1}{\alpha}}\right]\right\}-\log\left\{-\log\left[1-\left(\frac{p_2}{1-\theta+\theta p_2}\right)^{\frac{1}{\alpha}}\right]\right\}}{\log Y_{k_1}-\log Y_{k_2}}$$
$$=\frac{\log\left[-\log\left(1-p_1^*\right)\right]-\log\left[-\log\left(1-p_2^*\right)\right]}{\log Y_{k_1}-\log Y_{k_2}}=\frac{k}{\log Y_{k_1}-\log Y_{k_2}}$$

where

$$k=\log\left[-\log\left(1-p_1^*\right)\right]-\log\left[-\log\left(1-p_2^*\right)\right]$$

and for $i=1,2$, $k_i=[np_i]$ or $k_i=[np_i]+1$ and

$$p_i^*=\left(\frac{p_i}{1-\theta+\theta p_i}\right)^{\frac{1}{\alpha}} \tag{5}$$

According to Dubey (1967, p. 122), γ^* has an asymptotic normal distribution with a mean of γ and variance of

$$Var(\gamma^*)=\frac{\gamma^2}{nk^2}\left[\frac{p_1^*}{\left(1-p_1^*\right)\log^2\left(1-p_1^*\right)}+\frac{p_2^*}{\left(1-p_2^*\right)\log^2\left(1-p_2^*\right)}-\frac{2p_1^*p_2^*}{\left(1-p_1^*\right)\left(1-p_2^*\right)\log\left(1-p_1^*\right)\log\left(1-p_2^*\right)}\right]$$

Now, p_1^* and p_2^* should be determined in a way that $Var(\gamma^*)$ is minimized where, according to Dubey (1967, p. 122), $p_1^*=0.16730679$ and $p_2^*=0.97366352$. Therefore, calculating p_1 and p_2 with the help of (5), the BTPE of γ which is shown by $\hat{\gamma}_{BTPE}$ is obtained as follows:

$$\hat{\gamma}_{BTPE}=\frac{\log\left\{-\log\left[1-\left(\frac{q_1}{1-\theta+\theta q_1}\right)^{\frac{1}{\alpha}}\right]\right\}-\log\left\{-\log\left[1-\left(\frac{q_2}{1-\theta+\theta q_2}\right)^{\frac{1}{\alpha}}\right]\right\}}{\log Y_{k_1}-\log Y_{k_2}}$$

where

$$q_1=\frac{(1-\theta)(0.16730679)^{\alpha}}{1-\theta(0.16730679)^{\alpha}},$$
$$q_2=\frac{(1-\theta)(0.97366352)^{\alpha}}{1-\theta(0.97366352)^{\alpha}}$$

In addition, for p_1 and p_2 $(0<p_1<p_2<1)$, with the help of (3), a TPE of β which is shown by β^* is obtained as follows:

$$\beta^*=\exp(w_1\log Y_{k1}+w_2\log Y_{k2})$$

where $w_1=T_2/(T_1-T_2)$, $w_1+w_2=-1$ and

$$T_i=\log\left\{-\log\left[1-\left(\frac{p_i}{1-\theta+\theta p_i}\right)^{\frac{1}{\alpha}}\right]\right\},\quad i=1,2$$

According to Dubey (1967, p. 122), β^* has an asymptotic normal distribution with a mean of β and variance of

$$Var(\beta^*)=\frac{\beta^2}{n\gamma^2k^2}\left\{r_1^*\left(\frac{k-\log k_1}{k_1}\right)\left[\frac{k-\log k_1}{k_1}+\frac{2\log k_1}{k_2}\right]+\frac{r_2^*\log^2 k_1}{k_2^2}\right\}$$

where

$$k=\log\left\{-\log\left[1-\left(\frac{p_1}{1-\theta+\theta p_1}\right)^{\frac{1}{\alpha}}\right]\right\}-\log\left\{-\log\left[1-\left(\frac{p_2}{1-\theta+\theta p_2}\right)^{\frac{1}{\alpha}}\right]\right\}$$

And for $i=1,2$

$$r_i=\left(\frac{p_i}{1-\theta+\theta p_i}\right)^{\frac{1}{\alpha}},r_i^*=\frac{r_i}{1-r_i},k_i=-\log(1-r_i) \tag{6}$$

Now, r_1 and r_2 should be determined in a way that $Var(\beta^*)$ is minimized where, according to Dubey (1967, p. 122), $r_1=0.39777$ and $r_2=0.82111$. Therefore, calculating p_1 and p_2 with the help of (6), the BTPE of β which is shown by $\hat{\beta}_{BTPE}$ is obtained as follows:

$$\hat{\beta}_{BTPE} = \exp(\hat{w}_1 \log Y_{k1} + \hat{w}_2 \log Y_{k2})$$

where

$$\hat{w}_1 = \frac{\log\left\{-\log\left[1-\left(\frac{r_2^{**}}{1-\theta+\theta r_2^{**}}\right)^{\frac{1}{\alpha}}\right]\right\}}{\log\left\{-\log\left[1-\left(\frac{r_1^{**}}{1-\theta+\theta r_1^{**}}\right)^{\frac{1}{\alpha}}\right]\right\}-\log\left\{-\log\left[1-\left(\frac{r_2^{**}}{1-\theta+\theta r_2^{**}}\right)^{\frac{1}{\alpha}}\right]\right\}}$$

$$\hat{w}_2 = \frac{\log\left\{-\log\left[1-\left(\frac{r_1^{**}}{1-\theta+\theta r_1^{**}}\right)^{\frac{1}{\alpha}}\right]\right\}}{\log\left\{-\log\left[1-\left(\frac{r_1^{**}}{1-\theta+\theta r_1^{**}}\right)^{\frac{1}{\alpha}}\right]\right\}-\log\left\{-\log\left[1-\left(\frac{r_2^{**}}{1-\theta+\theta r_2^{**}}\right)^{\frac{1}{\alpha}}\right]\right\}}$$

and

$$r_1^{**} = \frac{(1-\theta)(0.39777)^{\alpha}}{1-\theta(0.39777)^{\alpha}}, \quad r_2^{**} = \frac{(1-\theta)(0.82111)^{\alpha}}{1-\theta(0.82111)^{\alpha}}$$

where replacing $\hat{\gamma}_{BTPE}$ and $\hat{\beta}_{BTPE}$ in relations (1) and (2), the BTPE, for the pdf (1) and cdf (2), and MSE of these estimators can be achieved.

3.2 PCE

Let $X_1, \ldots, X_n$ is a random sample of distribution with cdf (2) with ordinal statistics of $Y_1, \ldots, Y_n$, and p_i is the percentile of Y_i, then, $F(Y_i, \alpha, \beta, \gamma, \theta) = p_i$ or

$$\gamma\log(\beta Y_i) = \log\left\{-\log\left[1-\left(\frac{p_i}{1-\theta+\theta p_i}\right)^{\frac{1}{\alpha}}\right]\right\}$$

Percentile estimations of γ and β which are shown by $\hat{\gamma}_{PCE}$ and $\hat{\beta}_{PCE}$, respectively, are obtained by minimizing

$$\left[\left[\sum_{i=1}^{n}\gamma\log(\beta Y_i)-\log\left\{-\log\left[1-\left(\frac{p_i}{1-\theta+\theta p_i}\right)^{\frac{1}{\alpha}}\right]\right\}^2\right]\right]$$

with respect to γ and β, i.e. by considering the following equations and the Newton–Raphson numerical method are obtained.

$$\gamma\sum_{i=1}^{n}[\log(\beta Y_i)]^2-\sum_{i=1}^{n}\log(\beta Y_i)\log\left\{-\log\left[1-\left(\frac{p_i}{1-\theta+\theta p_i}\right)^{\frac{1}{\alpha}}\right]\right\}=0$$

$$n\gamma\log\beta+\gamma\sum_{i=1}^{n}\log Y_i - \sum_{i=1}^{n}\log\left\{-\log\left[1-\left(\frac{p_i}{1-\theta+\theta p_i}\right)^{\frac{1}{\alpha}}\right]\right\}=0$$

Replacing $\hat{\gamma}_{MLE}$ and $\hat{\beta}_{BPTE}$ by γ and β in relations (1) and (2), the PCE of pdf and cdf of EWG distribution, and MSE of these estimators are obtained.

3.3 MLE

In this section, according to a random sample of $X_1, \ldots, X_n$ from a distribution with pdf (1), the MLE of the parameters of γ and β which are shown by $\hat{\gamma}_{MLE}$ and $\hat{\beta}_{MLE}$, respectively, are obtained by the help of a set of equations

$$\frac{n}{\gamma}+\sum_{i=1}^{n}[1-(\beta x_i)^{\gamma}]\log(\beta x_i) + (\alpha-1)\sum_{i=1}^{n}\frac{(\beta x_i)^{\gamma}\log(\beta x_i)e^{-(\beta x_i)^{\gamma}}}{1-e^{-(\beta x_i)^{\gamma}}} + 2\alpha\theta\sum_{i=1}^{n}\frac{(\beta x_i)^{\gamma}\log(\beta x_i)e^{-(\beta x_i)^{\gamma}}\left(1-e^{-(\beta x_i)^{\gamma}}\right)^{\alpha-1}}{\left[1-\theta\left(1-e^{-(\beta x_i)^{\gamma}}\right)^{\alpha}\right]^2} = 0$$

$$n+\beta^{\gamma}\left\{\sum_{i=1}^{n}x_i^{\gamma}+(\alpha-1)\sum_{i=1}^{n}\frac{x_i^{\gamma}e^{-(\beta x_i)^{\gamma}}}{1-e^{-(\beta x_i)^{\gamma}}} + 2\alpha\theta\sum_{i=1}^{n}\frac{x_i^{\gamma}e^{-(\beta x_i)^{\gamma}}\left(1-e^{-(\beta x_i)^{\gamma}}\right)^{\alpha-1}}{\left[1-\theta\left(1-e^{-(\beta x_i)^{\gamma}}\right)^{\alpha}\right]^2}\right\}=0$$

and the Newton–Raphson numerical method. By replacing the γ and β by $\hat{\gamma}_{MLE}$ and $\hat{\beta}_{BPTE}$ in relations (1) and (2), the MLE of pdf and cdf of EWG distribution, and MSE of these estimators can be found.

4 Numerical experiments

In this section, a Monte Carlo simulation and a numerical example are presented to illustrate all the estimation methods described in the preceding sections.

4.1 Simulation studies

In this section, in the first step, using

$$X = \frac{1}{\beta}\left\{-\log\left[1-\left(\frac{U}{1-\theta(1-U)}\right)^{\frac{1}{\alpha}}\right]\right\}^{\frac{1}{\gamma}}$$

where U has uniformly distribution in the interval (0,1), and for $\alpha = 1.5, 2, 4$, $\beta = 0.25, 1.5, 3, 3.5$, $\gamma = 2, 3, 4, 4.5$ and $\theta = 0.2, 0.5, 0.6, 0.8$ random samples are generated as $n = 100, 200, \ldots, 500$. In the second step, the BSPE, PCE, and MLE of parameter α discussd in Sect. 2, and the BTPE, PCE, and MLE of parameters γ and β given in Sect. 3 are obtained. In the third step, the mean square error of estimations of functions (1) and (2) is calculated. Steps 1 to 3 were repeated 5000 times and the mean of MSE is obtained from 5000 times repetition was found. The optimal estimator is that one with a smallest Mean MSE. Comparing the results of simulations studies in

Table 1 A parameter α estimation and estimate the average mean square error (AM) of function (2) the second part of the estimation methods based on simulation results for different values $(\alpha, \beta, \gamma, \theta)$ of EWG distribution

$(\alpha, \beta, \gamma, \theta)$	n	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{PCE}$	$\hat{\alpha}_{BPSE}$	$AM(\hat{F}_{MLE})$	$AM(\hat{F}_{PCE})$	$AM(\hat{F}_{BPSE})$
(2, 3, 4.5, 0.6)	100	0.72882	3.18157	2.07177	0.043671	0.009191	0.000345
	200	0.99002	3.17151	2.02393	0.044508	0.007156	0.000175
	300	0.61051	2.80171	2.04363	0.043832	0.006584	0.000112
	400	0.84104	2.51407	2.04793	0.043577	0.006370	0.000086
	500	0.72915	2.79879	2.00003	0.043471	0.006579	0.000061
(4, 1.5, 3, 0.8)	100	1.77625	1.16928	4.45912	0.063122	0.021581	0.000207
	200	1.22447	7.56527	3.94682	0.056789	0.014676	0.000112
	300	1.46708	8.34581	4.04442	0.055679	0.014432	0.000073
	400	0.89767	5.67570	3.87779	0.056952	0.014692	0.000054
	500	1.12513	7.47266	3.90375	0.055481	0.012984	0.000042
(1.5, 0.25, 2, 0.2)	100	0.73831	1.85071	1.75892	0.035794	0.002485	0.000602
	200	0.74349	1.78249	1.67786	0.035266	0.001221	0.000292
	300	0.65627	1.61530	1.52227	0.035939	0.001031	0.000196
	400	0.63688	1.75294	1.56881	0.035619	0.000965	0.000147
	500	0.74216	1.70336	1.58335	0.035433	0.000911	0.000119
(2, 3.5, 4, 0.5)	100	0.60068	2.37787	1.89153	0.041364	0.006139	0.000422
	200	0.79001	2.55847	1.93271	0.040709	0.005613	0.000207
	300	0.70204	2.50779	1.95416	0.040697	0.004734	0.000136
	400	0.87352	2.54511	1.98047	0.040889	0.004315	0.000103
	500	0.85687	2.71180	2.04269	0.040301	0.004426	0.000079

Table 2 A parameter α estimation and estimate the average mean square error (AM) of function (1) the second part of the estimation methods based on simulation results for different values $(\alpha, \beta, \gamma, \theta)$ of EWG distribution

$(\alpha, \beta, \gamma, \theta)$	n	$\hat{\alpha}_{MLE}$	$\hat{\alpha}_{PCE}$	$\hat{\alpha}_{BPSE}$	$AM(\hat{f}_{MLE})$	$AM(\hat{f}_{PCE})$	$AM(\hat{f}_{BPSE})$
(2, 3, 4.5, 0.6)	100	0.72882	3.18157	2.07177	1.06530	2.11166	0.128081
	200	0.99002	3.17151	2.02393	1.04294	2.01515	0.118559
	300	0.61051	2.80171	2.04363	1.05483	2.01050	0.114069
	400	0.84104	2.51407	2.04793	1.04966	1.99166	0.111187
	500	0.72915	2.79879	2.00003	1.03595	2.00381	0.108477
(4, 1.5, 3, 0.8)	100	1.77625	1.16928	4.45912	0.05557	0.122678	0.004562
	200	1.22447	7.56527	3.94682	0.05570	0.109756	0.003513
	300	1.46708	8.34581	4.04442	0.05501	0.109418	0.003061
	400	0.89767	5.67570	3.87779	0.05504	0.109172	0.002886
	500	1.12513	7.47266	3.90375	0.05509	0.106254	0.002731
(1.5, 0.25, 2, 0.2)	100	0.73831	1.85071	1.75892	0.00116	0.001049	0.000526
	200	0.74349	1.78249	1.67786	0.00113	0.000801	0.000480
	300	0.65627	1.61530	1.52227	0.00112	0.000819	0.000474
	400	0.63688	1.75294	1.56881	0.00111	0.000818	0.000469
	500	0.74216	1.70336	1.58335	0.00111	0.000819	0.000465
(2, 3.5, 4, 0.5)	100	0.60068	2.37787	1.89153	2.38466	1.29901	0.193829
	200	0.79001	2.55847	1.93271	2.40927	1.28597	0.178659
	300	0.70204	2.50779	1.95416	2.30379	1.27271	0.167009
	400	0.87352	2.54511	1.98047	2.25761	1.26465	0.168395
	500	0.85687	2.71180	2.04269	2.27999	1.25984	0.162167

Tables 1, 2, 3, 4 show that the BSPE and the BTPE are the best. On the other hand based on a 1000 random samples simulated from the EWG distribution, Fig. 1 show the graphs of estimations of the pdf (1) for the estimation methods of the third section which is given in Table 5, which represents the superiority of the BTPE toward other estimates.

Table 3 parameters γ and β estimation and estimate the average mean square error (AM) of function (2) the second part of the estimation methods based on simulation results for different values $(\alpha, \beta, \gamma, \theta)$ of EWG distribution

$(\alpha, \beta, \gamma, \theta)$	n	$\hat{\gamma}_{BTPE}$	$\hat{\beta}_{BTPE}$	$\hat{\gamma}_{PCE}$	$\hat{\beta}_{PCE}$	$\hat{\gamma}_{MLE}$	$\hat{\beta}_{PCE}$	$AM(\hat{F}_{MLE})$	$AM(\hat{F}_{PCE})$	$AM(\hat{F}_{BTPE})$
(2, 3, 4.5, 0.6)	100	4.8367	1.0079	1.4786	2.2954	1.9731	3.6372	0.15101	0.13021	0.03613
	200	4.6467	1.0102	1.4728	2.2611	1.9642	3.6005	0.15825	0.07952	0.03390
	300	4.5511	1.0110	1.4551	2.2551	1.9634	3.5936	0.15983	0.08319	0.03276
	400	4.6769	1.0109	1.4498	2.2295	1.9586	3.5841	0.15672	0.08829	0.03338
	500	4.6228	1.0108	1.4518	2.2351	1.9594	3.5835	0.15933	0.08519	0.03356
(4, 1.5, 3, 0.8)	100	4.8612	1.1072	2.4553	0.8936	0.3301	1.1672	0.05273	0.07924	0.03038
	200	4.2922	1.1492	2.4476	0.8332	0.3299	1.1652	0.03054	0.05521	0.02974
	300	4.0555	1.1412	2.4444	0.7956	0.3295	1.1652	0.02117	0.03104	0.01774
	400	3.8799	1.1577	2.4409	0.7693	0.3292	1.1656	0.02989	0.04122	0.01507
	500	3.7940	1.1510	2.4426	0.7542	0.3293	1.1651	0.02296	0.10221	0.00343
(1.5, 0.25, 2, 0.2)	100	2.0932	1.5782	0.1531	1.6962	2.5424	0.2540	0.22450	0.09921	0.00272
	200	2.0281	1.5801	0.1093	3.0855	2.5072	0.2534	0.22421	0.09616	0.00240
	300	2.0289	1.5799	0.0902	6.4183	2.4900	0.2531	0.22577	0.09368	0.00213
	400	2.0029	1.5769	0.0783	24.016	2.4718	0.2526	0.2258	0.09434	0.00215
	500	2.0118	1.5771	0.0696	11.233	2.4811	0.2529	0.22661	0.09229	0.01925
(2, 3.5, 4, 0.5)	100	4.1289	1.0064	0.9515	2.7688	2.3723	3.8177	0.18144	0.07928	0.01769
	200	4.1988	1.0113	0.9424	2.7571	2.3536	3.8049	0.17706	0.10130	0.01735
	300	4.0719	1.0101	0.9372	2.7964	2.3449	3.7993	0.18281	0.08311	0.01692
	400	4.1220	1.0111	0.9341	2.8412	2.3407	3.7961	0.17867	0.08771	0.01699
	500	4.0517	1.0105	0.9313	2.7955	2.3429	3.7972	0.18344	0.09134	0.01726

Table 4 parameters α and β estimation and estimate the average mean square error (AM) of function (1) the second part of the estimation methods based on simulation results for different values $(\alpha, \beta, \gamma, \theta)$ of EWG distribution

$(\alpha, \beta, \gamma, \theta)$	n	$\hat{\gamma}_{BTPE}$	$\hat{\beta}_{BTPE}$	$\hat{\gamma}_{PCE}$	$\hat{\beta}_{PCE}$	$\hat{\gamma}_{MLE}$	$\hat{\beta}_{PCE}$	$AM(\hat{f}_{MLE})$	$AM(\hat{f}_{PCE})$	$AM(\hat{f}_{BTPE})$
(2, 3, 4.5, 0.6)	100	4.8367	1.0079	1.4786	2.2954	1.9731	3.6372	0.77035	4.34874	0.48280
	200	4.6467	1.0102	1.4728	2.2611	1.9642	3.6005	0.79027	4.46875	0.43818
	300	4.5511	1.0110	1.4551	2.2551	1.9634	3.5936	0.78236	4.48392	0.46429
	400	4.6769	1.0109	1.4498	2.2295	1.9586	3.5841	0.78222	4.42212	0.46307
	500	4.6228	1.0108	1.4518	2.2351	1.9594	3.5835	0.77847	4.47561	0.48618
(4, 1.5, 3, 0.8)	100	4.8612	1.1072	2.4553	0.8936	0.3301	1.1672	0.02392	0.00994	0.00931
	200	4.2922	1.1492	2.4476	0.8332	0.3299	1.1652	0.02798	0.01020	0.00859
	300	4.0555	1.1412	2.4444	0.7956	0.3295	1.1652	0.02561	0.01051	0.00859
	400	3.8799	1.1577	2.4409	0.7693	0.3292	1.1656	0.02709	0.01077	0.00793
	500	3.7940	1.1510	2.4426	0.7542	0.3293	1.1651	0.02778	0.01095	0.00795
(1.5, 0.25, 2, 0.2)	100	2.0932	1.5782	0.1531	1.6962	2.5424	0.2540	3.47621	0.01412	0.00817
	200	2.0281	1.5801	0.1093	3.0855	2.5072	0.2534	3.14124	0.00890	0.00808
	300	2.0289	1.5799	0.0902	6.4183	2.4900	0.2531	2.65151	0.00786	0.00745
	400	2.0029	1.5769	0.0783	24.016	2.4718	0.2526	2.53194	0.00866	0.00816
	500	2.0118	1.5771	0.0696	11.233	2.4811	0.2529	2.28138	0.00862	0.00798
(2, 3.5, 4, 0.5)	100	4.1289	1.0064	0.9515	2.7688	2.3723	3.8177	1.87422	0.80433	0.12703
	200	4.1988	1.0113	0.9424	2.7571	2.3536	3.8049	1.89349	0.70437	0.00981
	300	4.0719	1.0101	0.9372	2.7964	2.3449	3.7993	1.84600	0.60702	0.06714
	400	4.1220	1.0111	0.9341	2.8412	2.3407	3.7961	1.87497	0.62749	0.95869
	500	4.0517	1.0105	0.9313	2.7955	2.3429	3.7972	1.89357	0.65273	0.92011

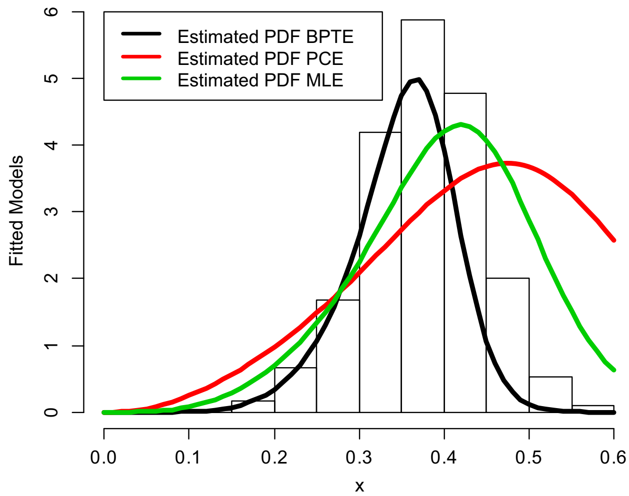

Fig. 1 The graphs of estimations BTPE, PCE and MLE of the pdf (1)

Table 5 Estimate of parameters and corresponding log-liklihood

	Estimate of γ	Estimate of β	Log-liklihood
BTPE	3.2728787	3.7482284	− 789.0882
PCE	2.9071186	2.8885803	− 797.1267
MLE	2.7120806	3.5376112	− 805.6754

4.2 Application with real data set

In this section the BSPE, BTPE, PCE and MLE of pdf and cdf for the EWG distribution are computed and compared for a real data. The data is the waiting times (in minutes) of 100 bank customers collected by Ghitany et al. (2008) presented in the ‘Appendix’. For known parameters $\beta = 5.16$, $\gamma = 0.55$, $\theta = 0.95$ based on MLE method, Table 6 shows the average (AV) and corresponding mean square error (MSE) of the BSPE, PCE, MLE of pdf (1), cdf (2). Comparing theses results show that the BSPE provides better fit to waiting time data.

Also, For known parameters $\alpha = 2.11$, $\theta = 0.85$ based on MLE method, Table 7 shows the average (AV) and corresponding mean square error (MSE) of the BSPE, PCE, MLE of pdf (1), cdf (2). Comparing theses results show that the BTPE provides better fit to waiting time data.

5 Conclusion

In this research, the pdf and the cdf of the four-parameter EWG distribution were determined using several methods. To do this task, we first assume for an unknown parameter the BSPE, PCE and MLE of these functions are obtained. Then for two unknown parameters the BTPE, PCE and MLE of these functions are found. Then Using the Monte Carlo simulation and real data set, it was shown that the BSPE and BTPE are better than the other estimators.

Table 6 Estimate the average (AV) estimation and corresponding mean square error of pdf (1) and cdf (2)

Method	AV (f)	MSE (f)	AV (F)	MSE (F)
BSPE	0.06834893	0.05809714	0.8869903	0.2006903
MLE	0.07025214	0.07020127	0.7384733	0.3131502
PCE	0.05329464	0.07918471	0.6097365	0.3685202

Table 7 Estimate the average (AV) estimation and corresponding mean square error of pdf (1) and cdf (2)

Method	AV (f)	MSE (f)	AV (F)	MSE (F)
BTPE	0.01335107	0.00393291	0.1659099	0.0798941
MLE	0.03688957	0.08424045	0.9484732	0.1200633
PCE	0.03862066	0.12865875	0.9691315	0.1256156

Appendix

The waiting times (in minutes) of 100 bank customers

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

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