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Choice correspondences for public goods

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Abstract. We consider collective choice problems where a group of agents has to decide on the location of a public facility in a Euclidean space. A well-known solution for such problems is the coordinatewise median of the reported votes and additional fixed ballots. Instead of adding ballots, we extend the median solution by allowing set-valued outcomes. This especially applies for location problems with an even number of agents.

1 Introduction

Restrictions to the domain of single-peaked preferences have frequently been studied for public good models. Here we consider such restrictions for preferences on a Euclidean space. The public issue(s) to decide on are represented by points in a Euclidean space, leaving many interpretations open: locations for a public facility, budgetary constrained investment divisions among several public projects, bundles of public goods. We are interested in the following class of single-peaked preference relations: every agent has an individual best point and his preferences decline according to the distance to this best point. Because agents might weigh coordinates differently, we assume that preferences are induced by separable-quadratic distance functions. A (collective) choice function assigns to each tuple of reported preference relations a single-valued outcome, a compromise point. A central property in this paper is strategy-proofness. Strategy-proofness requires that no agent can benefit by lying about his true preference relation. Well-known strategy-proof choice

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functions are median choice functions: for the one-dimensional case, at each preference profile the median of the reported best points is chosen, and for higher dimensions this method is extended by taking the median coordinatewise.

Median voting schemes were first analyzed by Black (1948) for single-peaked preferences. Moulin (1980) characterized all strategy-proof choice functions for one-dimensional Euclidean spaces when preferences are single-peaked by median choice functions for which additional fixed ballots are allowed. Berga and Serizawa (2000) consider maximal domains for rules that in addition to strategy-proofness satisfy the so-called no vetoer condition. Extensions of Moulin's (1980) results to higher-dimensional Euclidean spaces are provided by Barberà et al. (1993) and Border and Jordan (1983). Similarly to Border and Jordan (1983), we focus on the class of separable-quadratic preferences. For generalized median choice functions as considered by Moulin (1980), Barberà et al. (1993), and Border and Jordan (1983) compromise points depend on the actual position of the fixed ballots. If there is no external reason that determines the position of these ballots, then solving the location problem by generalized medians is as difficult as the original problem itself. The determination of the fixed ballots, which should then be left to the agents, is another collective choice problem. Moreover, for an even number of agents a problem for median choice functions arises when additional fixed ballots are avoided. In that case, it is not clear how the median should be defined. If there is no unique median of all reported best points, then one possibility to define the median is to choose both "median points". In this paper, we define the median as the closed interval between the two median points and consider so called choice correspondences, which assign sets of compromise points to every preference profile. For instance, the one-dimensional median choice correspondence yields the closed interval between the two median points, and for arbitrary dimensions the coordinatewise median correspondence yields the Cartesian product of these one-dimensional choice correspondences.

In this article, we study the well-known condition of strategy-proofness for choice correspondences. Similarly as before, strategy-proofness requires that lying is not profitable for any agent. For single-valued choice functions this profitability is determined by an agent's individual preference relation. For set-valued choice correspondences this profitability is no longer straightforward. To extend strategy-proofness for choice functions to choice correspondences it is necessary to extend the "pointwise" preference relations to "setwise" preference relations, i.e., preference relations on the power set of compromise points. Assuming the existence of best and worst points in two sets A and B, A is weakly preferred to B if the best point in B is not better than that of A and the worst point in A is not worse than that of B. This type of relation on sets can be seen as an extension of approaches studied by Kannai and Peleg (1984), Bossert (1989, finite subsets), and Nehring and Puppe (1996, compact subsets). Different approaches can be found in Ching and Zhou (1997), Kelly (1977), and Nitzan and Pattanaik (1984). By virtue of these extended preferences, strategy-proofness is reformulated: unilateral deviations should yield sets of compromise points that are comparable and non-profitable with respect to the extended preferences on subsets. As these preference relations are not complete, comparability is an essential condition.

Beside strategy-proofness we consider unanimity and two so-called newcomer conditions. Unanimity means that if all agents report the same preference relation, then the compromise point equals the unanimous best point. The newcomer conditions restrain the possible influence of a new agent on the set of compromise points of a fixed coalition N; these are the tie-breaking and the non-decisive newcomer condition. If a choice correspondence satisfies the tie-breaking newcomer condition, then an additional agent, who reports a preference relation with best point in the set of compromise points of coalition N, causes a reduction of this set of compromise points to his best point: he breaks the tie. The non-decisive newcomer condition requires that if an additional agent reports a preference relation with best point outside the set of compromise points of coalition N, then after joining this coalition the set of compromise points never reduces to his best point only. Alternatively we consider the weak non-decisive newcomer condition where we replace the set of compromise points with its convex hull: if an additional agent reports a preference relation with best point outside the convex hull of the set of compromise points of coalition N, then after joining this coalition the set of compromise points never reduces to his best point only. In this latter condition, the convex hull of the set of compromise points can be seen as an extended set of compromise points. The coordinatewise median correspondence is the only collective choice correspondence satisfying strategy-proofness, unanimity, the tie-breaking and the non-decisive newcomer condition. A stronger characterization result is obtained for the weak non-decisive newcomer condition: a collective choice correspondence satisfies strategy-proofness, unanimity, the tie-breaking and the weak non-decisive newcomer condition if and only if the convex hull of each compromise set it assigns equals the set of coordinatewise median compromise points.

The organization of the paper is as follows. In Sect. 2 we introduce location problems and the "classical" median choice functions. The restricted definition of the (coordinatewise) median choice function to odd numbers of agents leads to the setting for our further discussion. In Sect. 3 we skip the restriction on the number of agents for the median choice function by switching to set-valued choice correspondences. Now, the original model is adapted, preferences for sets are described and the four central conditions, strategy-proofness, unanimity, the tie-breaking and the (weak) non-decisive newcomer condition, are introduced. In Sect. 4 we show that these four conditions imply a monotonicity property and a weak form of Pareto optimality. In Sect. 5 we characterize the set of all choice correspondences that satisfy strategy-proofness, unanimity, the tie-breaking and the weak non-decisive newcomer condition. Furthermore, we show that the coordinatewise median correspondence is the only choice correspondence that satisfies strategy-proofness, unanimity, the tie-breaking and the non-decisive newcomer condition. Sect. 6 provides some comments on these results. First, we show that extending the set of preferences by non-separable single-peaked preferences yields incompatibility of the four central properties. Then, we discuss other extensions of preferences to subsets of the Euclidean space in relation to strategy-proofness. Finally, in Sect. 7, we discuss two closely related articles: Border and Jordan (1983) and Ching and Zhou (1997).

2 Choice functions

A *location problem* is given by a finite set of agents N who have to agree on a location, or *compromise point*, in some Euclidean space E. Here we assume that $E = \mathbb{R}^M$ where $M = \{1, \dots, m\}$.[1] Each agent $i \in N$ is equipped with a *separable-quadratic preference relation* $p(i)$ over E. It is well-known that this boils down to the following: for each agent $i \in N$ there exists a *weight vector* $\delta(i) = (\delta(i)_1, \dots, \delta(i)_m) \in \mathbb{R}^M_{++}$ and a *best point* $b(i) \in E$ such that for all $x, y \in E$, agent i *weakly prefers* x *to* y if and only if

$$\sum_{j \in M} \delta(i)_j (x_j - b(i)_j)^2 \leq \sum_{j \in M} \delta(i)_j (y_j - b(i)_j)^2.$$

In the sequel we normalize the weight vector to length one. Note that every separable-quadratic preference relation is completely determined by a pair $(\delta(i), b(i))$ where $\delta(i) \in \mathbb{R}^M_{++}$, $\sum_{j \in M} \delta(i)_j^2 = 1$, and $b(i) \in E$. Let $\mathscr{S}$ denote the *set of all separable-quadratic preference relations*. For each agent $i \in N$, we identify the preference relation $p(i) \in \mathscr{S}$ with its characteristic pair $(\delta(i), b(i))$ and write $p(i) = (\delta(i), b(i)) \in \mathscr{S}$. If at preference relation $p(i) = (\delta(i), b(i))$ agent i weakly prefers $x \in E$ to $y \in E$, then we denote this by $x \succsim_{p(i)} y$. Equivalently, we write $x \succsim_{(\delta(i), b(i))} y$. Strict preference is denoted by $x \succ_{p(i)} y$, *i.e.*, $x \succsim_{p(i)} y$ and not $y \succsim_{p(i)} x$, and indifference by $x \sim_{p(i)} y$, *i.e.*, $x \succsim_{p(i)} y$ and $y \succsim_{p(i)} x$. Equivalently, we write $x \succ_{(\delta(i), b(i))} y$ and $x \sim_{(\delta(i), b(i))} y$.

It is easy to check that all separable-quadratic preference relations are *single-peaked, i.e.*, for $p(i) \in \mathscr{S}$ there exists a best point, or *peak*, $b(i) \in E$ such that for all $x \in E$, $x \neq b(i)$, and all $0 < \lambda < 1$, $b(i) \succ_{p(i)} \lambda b(i) + (1 - \lambda) x \succ_{p(i)} x$. A geometric implication of $p(i) \in \mathscr{S}$ being separable-quadratic is that the corresponding indifference sets are ellipsoids around the best point $b(i)$ with main diagonals parallel to the coordinate axes. The closer these ellipsoids are to $b(i)$ the better the points on it are.

By $\mathscr{S}^N$ we denote the *set of all (preference) profiles* $p = \langle p(i) \rangle_{i \in N}$ such that for all $i \in N$, $p(i) \in \mathscr{S}$. A *(collective) choice function* φ is a function that assigns to every profile $p \in \mathscr{S}^N$ a point $\varphi(p)$ in E. This point is called the *compromise point*. A choice function that only depends on the peaks of the preference profiles and disregards the underlying preference relations satisfies

[1] By $\mathbb{R}$ we denote the set of real numbers, $\mathbb{R}_+ \equiv \{x \in \mathbb{R} \mid x \geq 0\}$, and $\mathbb{R}_{++} \equiv \{x \in \mathbb{R} \mid x > 0\}$. By $\mathbb{R}^M$ we denote the Cartesian product of $|M|$ copies of $\mathbb{R}$, indexed by the elements of M; $\mathbb{R}^M_+$ and $\mathbb{R}^M_{++}$ are defined similarly.

peak-onliness.[2] Choice functions that satisfy *peak-onliness* are called *voting schemes*.

Note that the agents' preference relations are private information. Still we would like to find the compromise point on basis of "correct information". Therefore, the first property of choice functions we are interested in is *strategy-proofness*: no agent ever benefits from misrepresenting his preference relation.[3] Before we formulate its definition, we introduce some notation.

Let $p, q \in \mathscr{S}^N$ and $\emptyset \neq S \subseteq N$. The *restriction of profile p to S* is denoted by $p_S \in \mathscr{S}^S$. We write $p =_S q$ if $p_S = q_S$. For finite sets X, Y, $|X|$ denotes the *cardinality of X* and $X \setminus Y \equiv \{x \in X \mid x \notin Y\}$.

Strategy-proofness. A choice function φ is *strategy-proof* if for all $i \in N$ and all profiles $p, q \in \mathscr{S}^N$ with $p =_{N \setminus \{i\}} q$,

$$\varphi(p) \succsim_{p(i)} \varphi(q).$$

For one-dimensional location problems with an odd number of agents it is easy to see that taking the median of all reported points is a well-defined and *strategy-proof* voting scheme; see for example Moulin (1980). The definition of the median we state here is well-defined for arbitrary finite numbers of agents and coincides with the "classical" median whenever it exists. We define the *median* of a finite set $V \subset \mathbb{R}$ by

$$\operatorname{med}(V) \equiv \left\{ x \in \mathbb{R} \mid |\{v \in V \mid v \leq x\}| \geq \frac{|V|}{2} \text{ and } |\{v \in V \mid v \geq x\}| \geq \frac{|V|}{2} \right\}.$$

If $|V|$ is odd, then $\operatorname{med}(V)$ is a singleton. If $|V|$ is even, then $\operatorname{med}(V)$ is either a singleton or a closed interval.

For higher dimensional location problems in E with an odd number of agents, applying the median coordinatewise yields the following well-defined choice function.

The coordinatewise median choice function. Let N be such that $|N|$ is odd. For all $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathscr{S}^N$, the *coordinatewise median choice function* is defined by

$$\varphi_{\text{med}}(p) = x \text{ such that for all } j \in M,\ x_j = \operatorname{med}(\{b(i)_j | i \in N\}).$$

Note that the coordinatewise median choice function is *strategy-proof*; see for example Border and Jordan (1983). Because the coordinatewise median choice function only depends on the individual best points, we also refer to it as to the *coordinatewise median voting scheme*.

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[2] A choice function φ is *peak-only* if for all $p = \langle \delta(i), b(i) \rangle_{i \in N}$, $p' = \langle \delta'(i), b'(i) \rangle_{i \in N}$ $\in \mathscr{S}^N$ such that for all $i \in N$, $b(i) = b'(i)$,

$$\varphi(p) = \varphi(p').$$

[3] In game theoretical terms, a choice function is *strategy-proof* if in the direct revelation game form it is a weakly dominant strategy for each agent to announce his true preference relation.

3 Choice correspondences

Since later on we also admit variations of the set of agents, we consider a set of *potential agents* denoted by $\mathbb{N}$. By $\mathcal{N}$ we denote the class of non-empty and finite subsets of $\mathbb{N}$. We call $N \in \mathcal{N}$ a *coalition*. A *(collective) choice correspondence* ψ is a function that assigns to every coalition $N \in \mathcal{N}$ and every profile $p \in \mathcal{S}^N$ a subset $\psi^N(p)$ of E. We call $\psi^N(p)$ the *set of compromise points*.[4] Choice correspondences that satisfy *peak-onliness* are called *voting correspondences*.

Because the set of compromise points assigned by a choice correspondence might generically be empty, we formulate the following property.

Nonemptiness. A choice correspondence ψ is *nonempty* if for all $N \in \mathcal{N}$ and all $p \in \mathcal{S}^N$,

$$\psi^N(p) \neq \emptyset.$$

For $(\delta, b) \in \mathcal{S}$, $(\delta, b)^N$ denotes the *unanimous profile* $\langle(\delta, b), \ldots, (\delta, b)\rangle \in \mathcal{S}^N$. Next, we focus on choice correspondences that assign to each unanimous profile the "unanimous best point".

Unanimity. A choice correspondence ψ is *unanimous* if for all $N \in \mathcal{N}$ and all $(\delta, b) \in \mathcal{S}$,

$$\psi^N((\delta, b)^N) = \{b\}.$$

The next conditions we introduce describe the influence of an additional agent, a "newcomer", on the set of compromise points. Let ψ be a choice correspondence, N be a coalition, and p be a profile in $\mathcal{S}^N$. Then we can interpret the set of compromise points $\psi^N(p)$ as the set of alternatives among which coalition N is unable to make any further restriction. In some sense, according to N, all compromise points in $\psi^N(p)$ are equally good. Now, consider a *newcomer* $k \notin N$ who joins N at profile p and reports preference relation $p(k) \in \mathcal{S}$ with best point $b(k)$ contained in the set of compromise points $\psi^N(p)$. Then agent k is in favor of point $b(k)$ whereas N is indifferent between all compromise points in $\psi^N(p)$. If the choice correspondence satisfies the *tie-breaking newcomer condition*, then agent k breaks the tie in favor of his best point.

Let $N \in \mathcal{N}$, $k \notin N$, $p \in \mathcal{S}^N$, and $p(k) = (\delta(k), b(k)) \in \mathcal{S}$. Then, $\langle p, p(k)\rangle \in \mathcal{S}^{N \cup \{k\}}$ denotes the profile where each agent $i \in N$ reports $p(i)$ and agent k reports $p(k)$. With some abuse of notation we write $\psi^{N \cup \{k\}}(p, p(k))$ instead of $\psi^{N \cup \{k\}}(\langle p, p(k)\rangle)$.

[4] For convenience, we will identify any choice function φ on $\mathcal{S}^N$ with a (N-voting) correspondence φ^N by identifying compromise points with singleton compromise sets: for all $p \in \mathcal{S}^N$, $\varphi^N(p) = \{\varphi(p)\}$.

Tie-breaking newcomer condition. A choice correspondence ψ is *tie-breaking for newcomers* if for all $N \in \mathcal{N}$, all $p \in \mathcal{S}^N$, and all $k \notin N$ with $p(k) = (\delta(k), b(k))$,

$$b(k) \in \psi^N(p) \quad \text{implies} \quad \psi^{N \cup \{k\}}(p, p(k)) = \{b(k)\}.$$

Young (1974, 1975) introduced a stronger condition to characterize score rules, called consistency, which implies the *tie-breaking newcomer condition*. Young's consistency notion requires the following. Let $N, N' \in \mathcal{N}$ be disjoint, $p \in \mathcal{S}^N$, and $p' \in \mathcal{S}^{N'}$. If $\psi^N(p) \cap \psi^{N'}(p') \neq \emptyset$, then $\psi^{N \cup N'}(p, p') = \psi^N(p) \cap \psi^{N'}(p')$.

The *non-decisive newcomer condition* discussed next limits the decisiveness of newcomers and implies non-dictatorship. Consider the situation described above with one difference, namely that the newcomer $k \notin N$ who joins N at profile p reports a preference relation $p(k) \in \mathcal{S}$ with best point $b(k)$ *not* contained in the set of compromise points $\psi^N(p)$. Then the *non-decisive newcomer condition* requires that the set of compromise points is not reduced to the best point $b(k)$ of agent k's reported preference relation. Hence, the newcomer k is not decisive.

Non-decisive newcomer condition. A choice correspondence ψ is *non-decisive for newcomers* if for all $N \in \mathcal{N}$, all $p \in \mathcal{S}^N$, and all $k \notin N$ with $p(k) = (\delta(k), b(k))$,

$$b(k) \notin \psi^N(p) \quad \text{implies} \quad \psi^{N \cup \{k\}}(p, p(k)) \neq \{b(k)\}.$$

A weakening of the *non-decisive newcomer condition* is the following *weak non-decisive newcomer condition* where a newcomer $k \notin N$ who joins N at profile p and reports a preference relation $p(k) \in \mathcal{S}$ with best point $b(k)$ *not* contained in the convex hull $\operatorname{conv}(\psi^N(p))$ of the set of compromise points $\psi^N(p)$ is not decisive. Here, we interpret $\operatorname{conv}(\psi^N(p))$ as an extended set of compromise points.

Weak non-decisive newcomer condition. A choice correspondence ψ is *weakly non-decisive for newcomers* if for all $N \in \mathcal{N}$, all $p \in \mathcal{S}^N$, and all $k \notin N$ with $p(k) = (\delta(k), b(k))$,

$$b(k) \notin \operatorname{conv}(\psi^N(p)) \quad \text{implies} \quad \psi^{N \cup \{k\}}(p, p(k)) \neq \{b(k)\}.$$

In the sequel we often use the contrapositive of the latter implication:

$$\psi^{N \cup \{k\}}(p, p(k)) = \{b(k)\} \quad \text{implies} \quad b(k) \in \operatorname{conv}(\psi^N(p)).$$

It is easy to check that the *non-decisive newcomer condition* implies the *weak non-decisive newcomer condition*. Note that for convex valued choice correspondences the *weak non-decisive newcomer condition* coincides with the *non-decisive newcomer condition*.

Finally, we extend *strategy-proofness*, already defined for choice functions, to correspondences. Again, *strategy-proofness* should guarantee that no agent ever benefits from misrepresenting his preferences. Given a choice function, it is clear that an agent can only benefit from lying if he strictly prefers the compromise point when lying to the compromise point when telling the truth. Given a choice correspondences, agents have to compare sets of compromise points in order to evaluate possible benefits from lying. Hence, in order to define *strategy-proofness*, we need to extend the agents' preference relations on E to preference relations on the power set of E, denoted by 2^E. Several of these extensions are studied in the literature (see our references later on). Here, we focus on one of these possible extensions. However, as we will explain in Sect. 6, this choice is not arbitrary and our results depend very much on this specific extension.

Preferences on subsets of E. For all $X, Y \in 2^E$ and all $(\delta, b) \in \mathscr{S}$,

$$X \succsim_{(\delta,b)} Y$$

if and only if

(i) for all $x \in X$ there exist $y \in Y$ such that $x \succsim_{(\delta,b)} y$ and
(ii) for all $y \in Y$ there exist $x \in X$ such that $x \succsim_{(\delta,b)} y$.

In words, we say that an agent with best point b and weight vector δ *weakly prefers X to Y* if for every point in Y there exists a point in X which is at least as good for him and for each point in X there is a point in Y which is at least as bad for him. If X and Y are compact sets of compromise points, then an agent with best point b and weight vector δ weakly prefers his best points in X to his best points in Y. Furthermore, he weakly prefers his worst points in X to his worst points in Y.[5] Hence, for points $x, y \in E$ it follows that $x \succsim_{(\delta,b)} y$ if and only if $\{x\} \succsim_{(\delta,b)} \{y\}$. In this sense the relation "$\succsim_{(\delta,b)}$ on E" is extended to a relation "$\succsim_{(\delta,b)}$ on 2^E".

The extension of a preference relation on a set to its power set we propose here generalizes extensions analyzed in Kannai and Peleg (1984), Barberà et al. (1984), Bossert (1989), and Nehring and Puppe (1996). Kannai and Peleg (1984) discuss an extension of a linear order on a finite set by comparisons of best and worst points. Bossert (1989) provided an axiomatic characterization of this extension. For finite subsets of E our extension coincides with those discussed in Kannai and Peleg (1984), Barberà et al. (1984), and Bossert (1989). Nehring and Puppe (1996) study an extension of preferences on E to compact subsets of E. Their extension coincides with our $\succsim_{(\delta,b)}$ on these compact subsets.

It is easy to see that any preference relation $\succsim_{(\delta,b)}$ on 2^E, defined by $(\delta, b) \in \mathscr{S}$, is reflexive but need not be complete (*e.g.,* for the closed intervals $X = [0,3]$, $Y = [1,2]$, neither $X \succsim_{(\delta,0)} Y$ nor $Y \succsim_{(\delta,0)} X$). Furthermore, by the definition of $\succsim_{(\delta,b)}$ on E and transitivity of $\leq$, transitivity of $\succsim_{(\delta,b)}$ on 2^E follows easily.

Now, similarly as before, we can formalize *strategy-proofness* for choice correspondences.

[5] For compact sets, best and worst points with respect to (δ, b) do exist.

Strategy-proofness. A choice correspondence ψ is *strategy-proof* if for all $N \in \mathscr{N}$, all $i \in N$, and all $p, q \in \mathscr{S}^N$ with $p =_{N\setminus\{i\}} q$,

$$\psi^N(p) \gtrsim_{p(i)} \psi^N(q).$$

Note that our notion of *strategy-proofness* also contains the requirement that the sets of compromise points before and after any unilateral deviation are comparable. The following lemma is an immediate consequence of *strategy-proofness* and *unanimity*.

Lemma 1. *Let the choice correspondence ψ satisfy unanimity and strategy-proof-ness. Then ψ is nonempty.*

Proof. Let $N \in \mathscr{N}$ and assume, without loss of generality, that $N = \{1, \dots, n\}$. Let $p = (\delta, b)^N$ be a unanimous profile. By *unanimity*, $\psi^N(p) \neq \emptyset$. We have to show that for all $q \in \mathscr{S}^N$, $\psi^N(q) \neq \emptyset$. Let $q = \langle q(1), \dots, q(n) \rangle$ and for $l \in \{0, \dots, n\}$, $q^l \equiv \langle q(1), \dots, q(l), p(l+1), \dots, p(n) \rangle$. Note that $q^0 = p$ and $q^n = q$. For $l = 1$ it follows that $p =_{N\setminus\{1\}} q^1$. Hence, by *strategy-proofness*, $\psi^N(p) \gtrsim_{p(1)} \psi^N(q^1)$. Thus, for all $x^0 \in \psi^N(p)$ there exists $x^1 \in \psi^N(q^1)$ (such that $x^0 \gtrsim_{p(1)} x^1$).[6] Since $q^1 =_{N\setminus\{2\}} q^2$, by *strategy-proofness*, for $x^1 \in \psi^N(q^1)$ there exists $x^2 \in \psi^N(q^2)$. Similarly, for each $l \in \{3, \dots, n\}$ there exists $x^l \in \psi^N(q^l)$. Hence, $\psi^N(q^n) = \psi^N(q) \neq \emptyset$. □

In situations where all agents of a subcoalition report the same preference profile, *strategy-proofness* can be adapted as follows. *Intermediate strategy-proofness* requires that unanimous subcoalitions cannot gain by strategic behavior.

Intermediate strategy-proofness.[7] A choice correspondence ψ is *intermediate strategy-proof* if for all $N \in \mathscr{N}$, all $S \subseteq N$, all $(\delta, b) \in \mathscr{S}$, and all $p, q \in \mathscr{S}^N$ with $p =_{N\setminus S} q$ and $p_S = (\delta, b)^S$,

$$\psi^N(p) \gtrsim_{(\delta, b)} \psi^N(q).$$

The following lemma corresponds to a result for voting schemes; see Peters et al. (1992), Lemma 2.4.

Lemma 2. *A choice correspondence ψ is strategy-proof if and only if it is intermediate strategy-proof.*

Proof. By definition, *intermediate strategy-proofness* implies *strategy-proofness*.

In the remainder of the proof we show that *strategy-proofness* implies *intermediate strategy-proofness*. Let $N \in \mathscr{N}$ and $S \subseteq N$. Assume, without loss of generality, that $N = \{1, \dots, n\}$ and $S = \{1, \dots, s\}$. Furthermore, let $(\delta, b) \in \mathscr{S}$, $p, q \in \mathscr{S}^N$ with $p =_{N\setminus S} q$ and $p_S = (\delta, b)^S$. We prove that $\psi^N(p) \gtrsim_{(\delta, b)} \psi^N(q)$.

[6] Note that in fact we are only interested in the existence of such an $x^1 \in \psi^N(q^1)$.
[7] See Peters et al. (1992) for *intermediate strategy-proofness* of voting schemes.

For $l \in \{0, \ldots, s\}$, $q^l \equiv \langle q(1), \ldots, q(l), p(l+1), \ldots, p(n) \rangle$. Note that $q^0 = p$ and $q^s = q$. For $l \in \{0, \ldots, s-1\}$ it follows that $q^l =_{N \setminus \{l+1\}} q^{l+1}$. Hence, by *strategy-proofness*, $\psi^N(q^l) \succsim_{(\delta,b)} \psi^N(q^{l+1})$. Thus, $\psi^N(p) = \psi^N(q^0) \succsim_{(\delta,b)} \psi^N(q^1) \succsim_{(\delta,b)} \cdots \succsim_{(\delta,b)} \psi^N(q^s) = \psi^N(q)$. So, by the transitivity of $\succsim_{(\delta,b)}$, $\psi^N(p) \succsim_{(\delta,b)} \psi^N(q)$. □

A choice correspondence that satisfies all conditions for choice correspondences we introduced in this section is the coordinatewise median correspondence. It assigns to each preference profile the Cartesian product of the coordinatewise medians.

The coordinatewise median correspondence. For all $N \in \mathcal{N}$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathcal{S}^N$ the *coordinatewise median correspondence* ψ_{med} is defined by

$$\psi^N_{\text{med}}(p) = \bigotimes_{j \in M} \text{med}(\{b(i)_j \mid i \in N\}).$$

Note that although the coordinatewise median correspondence is defined as choice correspondence, we can also interpret it as a voting correspondence that only depend on the peaks of the reported profiles.

Lemma 3. *The coordinatewise median correspondence ψ_{med} satisfies unanimity, strategy-proofness, the tie-breaking and the non-decisive newcomer condition.*

The straightforward proof of Lemma 3 is left to the reader.

4 Monotonicity and coordinatewise Pareto optimality

In this section we show that choice correspondences that are *unanimous, strategy-proof, tie-breaking* and *weakly non-decisive for newcomers* satisfy a *monotonicity* condition and a weak form of *Pareto optimality*. Furthermore, we show that the correspondence that associates with each set of compromise points the smallest closed set of compromise points inherits *unanimity, strategy-proofness*, the *tie-breaking* and the *weak non-decisive newcomer condition* from the original correspondence.

First, we discuss *monotonicity*. The monotonicity condition we introduce here resembles the well-known strong positive association introduced by Muller and Satterthwaite (1977). Loosely speaking, *monotonicity* requires the following. Consider $p, q \in \mathcal{S}^N$ and $x \in E$. If $\psi^N(p) = \{x\}$ and for all agents i in N, x "improves" by going from profile p to profile q, then $\psi^N(q) = \{x\}$. In order to formalize *monotonicity*, we introduce some notation.

Let $i \in N$, $p(i) = (\delta(i), b(i)) \in \mathcal{S}$, and $x \in E$. The *weak upper contour set of $p(i)$ at x*, denoted by $\bar{C}(x, p(i))$, equals the set of all $y \in E$ such that $y \succsim_{p(i)} x$. So, $\bar{C}(x, p(i))$ is the ellipsoid, with centre $b(i)$ and weight vector $\delta(i)$ through x, plus its interior:

$$\bar{C}(x, p(i)) = \left\{ y \in E \mid \sum_{j \in M} \delta(i)_j (y_j - b(i)_j)^2 \leq \sum_{j \in M} \delta(i)_j (x_j - b(i)_j)^2 \right\}.$$

The *strict upper contour set of* $p(i)$ *at* x, denoted by $C^\circ(x, p(i))$, equals the set of all $y \in E$ such that $y \succ_{p(i)} x$. So, $C^\circ(x, p(i))$ is the interior of $\bar{C}(x, p(i))$:

$$C^\circ(x, p(i)) = \left\{ y \in E \mid \sum_{j \in M} \delta(i)_j (y_j - b(i)_j)^2 < \sum_{j \in M} \delta(i)_j (x_j - b(i)_j)^2 \right\}.$$

For a finite set V we define *the box of* V by

$$\mathrm{box}(V) \equiv \left\{ y \in E \mid \text{for all } j \in M, \min_{v \in V} v_j \leq y_j \leq \max_{v \in V} v_j \right\}.$$

Now consider $\bar{b}(i) \in \mathrm{box}(\{b(i), x\})$ such that for all $j \in M$, $\bar{b}(i)_j = x_j$ only if $b(i)_j = x_j$. For each such $\bar{b}(i)$ there is a vector of weights $\bar{\delta}(i)$ such that at x the elliptic boundaries of the weak upper contour sets of $p(i) = (\delta(i), b(i))$ and $\bar{p}(i) = (\bar{\delta}(i), \bar{b}(i))$ are tangent to each other and $\bar{C}(x, \bar{p}(i)) \subseteq \bar{C}(x, p(i))$. So, if $b(i) \neq \bar{b}(i)$, then $\bar{C}(x, \bar{p}(i)) \backslash \{x\} \subseteq C^\circ(x, p(i))$. Note that going from $p(i)$ to $\bar{p}(i)$ location x *improves, i.e.*, for all $y \in E$, $x \succsim_{p(i)} y$ implies $x \succsim_{\bar{p}(i)} y$.

Summarizing, we say that $\bar{p}(i) = (\bar{\delta}(i), \bar{b}(i)) \in \mathscr{S}$ is an x-*improvement of* $p(i) = (\delta(i), b(i)) \in \mathscr{S}$ if

- $\bar{b}(i) \in \mathrm{box}(\{b(i), x\})$ is such that for all $j \in M$, $\bar{b}(i)_j = x_j$ only if $b(i)_j = x_j$ and
- $\bar{\delta}(i)$ is a weight vector such that the ellipsoid with centre $\bar{b}(i)$ and weight vector $\bar{\delta}(i)$ is tangent at x to the ellipsoid with centre $b(i)$ and weight vector $\delta(i)$.

Monotonicity. A choice correspondence ψ is *monotonic* if for all $N \in \mathscr{N}$, all $x \in E$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N}$, $\bar{p} = \langle \bar{\delta}(i), \bar{b}(i) \rangle_{i \in N} \in \mathscr{S}^N$ such that for all $i \in N$, either $p(i) = \bar{p}(i)$ or $\bar{p}(i)$ is an x-improvement of $p(i)$,

$$\psi^N(p) = \{x\} \quad \text{implies} \quad \psi^N(\bar{p}) = \{x\}. \tag{1}$$

Lemma 4. *Let the choice correspondence ψ satisfy strategy-proofness, unanimity, the tie-breaking and the weak non-decisive newcomer condition. Then,*

(i) *ψ is monotonic.*
(ii) *for all $x \in E$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N}$, $\bar{p} = \langle \bar{\delta}(i), \bar{b}(i) \rangle_{i \in N} \in \mathscr{S}^N$ such that for all $i \in N$, either $p(i) = \bar{p}(i)$ or $\bar{p}(i)$ is an x-improvement of $p(i)$,*

$$x \in \psi^N(p) \quad \textit{implies} \quad x \in \mathrm{conv}(\psi^N(\bar{p})).$$

Proof. Let $N \in \mathscr{N}$, $x \in E$, and $p = \langle \delta(i), b(i) \rangle_{i \in N}$, $\bar{p} = \langle \bar{\delta}(i), \bar{b}(i) \rangle_{i \in N} \in \mathscr{S}^N$ be such that for all $i \in N$, either $p(i) = \bar{p}(i)$ or $\bar{p}(i)$ is an x-improvement of $p(i)$.

(i) We have to prove that $\psi^N(p) = \{x\}$ implies $\psi^N(\bar{p}) = \{x\}$.

Let $\psi^N(p) = \{x\}$. We assume, without loss of generality, that for some

$i \in N$, $p =_{N \setminus \{i\}} \bar{p}$. If $p(i) = \bar{p}(i)$, then $p = \bar{p}$ and we are done. Let $p(i) \neq \bar{p}(i)$. Since $\bar{p}(i)$ is an x-improvement of $p(i)$, $\bar{b}(i) \neq b(i)$ and $\bar{\delta}(i) \neq \delta(i)$. Hence,

$$\bar{C}(x, \bar{p}(i)) \setminus \{x\} \subseteq C^\circ(x, p(i)). \tag{2}$$

Now, by *strategy-proofness*, $\psi^N(p) \succsim_{p(i)} \psi^N(\bar{p})$ and $\psi^N(\bar{p}) \succsim_{\bar{p}(i)} \psi^N(p)$. Since $\psi^N(p) = \{x\}$, for all $y \in \psi^N(\bar{p})$,

$$x \succsim_{p(i)} y \quad \text{and} \quad y \succsim_{\bar{p}(i)} x. \tag{3}$$

By (3), $\psi^N(\bar{p}) \cap C^\circ(x, p(i)) = \varnothing$ and $\psi^N(\bar{p}) \subseteq \bar{C}(x, \bar{p}(i))$. Hence, by *non-emptiness* (Lemma 1) and (2), $\psi^N(\bar{p}) = \{x\}$.

(ii) We have to prove that $x \in \psi^N(p)$ implies $x \in \operatorname{conv}(\psi^N(\bar{p}))$.

Let $x \in \psi^N(p)$. Let $k \notin N$ and $p(k) = (\delta(k), x) \in \mathscr{S}$. Since ψ is *tie-breaking for newcomers*, $\psi^{N \cup \{k\}}(p, p(k)) = \{x\}$. So, by *monotonicity*, $\psi^{N \cup \{k\}}(\bar{p}, p(k)) = \{x\}$. Hence, by *weak non-decisiveness for newcomers*, $x \in \operatorname{conv}(\psi^N(\bar{p}))$. □

Next, we introduce a necessary condition for *Pareto optimality*.[8] This weaker form of Pareto optimality, called *coordinatewise Pareto optimality*, requires that every coordinate of a compromise point is bounded from below (and above) by some best point belonging to an individual preference relation. Hence, in each coordinate, we have *Pareto optimality*.

Coordinatewise Pareto optimality. A choice correspondence ψ is *coordinatewise Pareto optimal* if for all $N \in \mathscr{N}$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathscr{S}^N$,

$$\psi^N(p) \subseteq \operatorname{box}(\{b(i) \mid i \in N\}).$$

Border and Jordan (1983) and Peters et al. (1992) show that there exist no *strategy-proof, anonymous*, and *Pareto optimal* choice functions when the number of agents is even. For choice correspondences this incompatibility does not hold. The coordinatewise median correspondence is *Pareto optimal* in terms of the extended preferences even if the number of agents is even.[9]

Now we prove that a choice correspondence ψ satisfying *strategy-proofness, unanimity*, and both *newcomer conditions* is *coordinatewise Pareto optimal*. The proof is by induction on the number of different preference relations at a profile. For $p \in \mathscr{S}^N$ we denote this number by $\mu(p) \equiv |\{p(i) \mid i \in N\}|$.

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[8] As usual, a set of compromise points is *Pareto optimal* if there exists no other set of compromise points such that all agents are weakly better off and at least one agent is strictly better off.

Both, the choice of the preferences and the choice for set-valued outcomes makes it difficult to find a (simple) description of all *Pareto optimal* sets of compromise points. Sufficient conditions for *Pareto optimality* strongly depend on the weight vectors of the individual preferences.

[9] Note that a set that is *Pareto optimal* in terms of the extended preferences might contain compromise points that are itself not *Pareto optimal* in terms of the original ("pointwise") preferences on E.

Lemma 5. *Let the choice correspondence ψ satisfy strategy-proofness, unanimity, the tie-breaking and the weak non-decisive newcomer condition. Then ψ is coordinatewise Pareto optimal.*

Proof. Since ψ is *nonempty* (Lemma 1), for all $p \in \mathscr{S}^N$, $\psi(p) \neq \emptyset$. By induction on $\kappa \in \mathbb{N}$ we prove that for all $N \in \mathcal{N}$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathscr{S}^N$ with $\mu(p) = \kappa$,

$$\psi^N(p) \subseteq \mathrm{box}(\{b(i) \mid i \in N\}).$$

Induction basis. $\kappa = 1$.
If $\mu(p) = 1$, then p is a unanimous profile. We are done by *unanimity*.

Induction hypothesis. For all $N \in \mathcal{N}$ and all $\bar{p} = \langle \bar{\delta}(i), \bar{b}(i) \rangle_{i \in N} \in \mathscr{S}^N$,

$$\mu(\bar{p}) \leq \kappa \quad \text{implies} \quad \psi^N(\bar{p}) \subseteq \mathrm{box}(\{\bar{b}(i) \mid i \in N\}). \tag{4}$$

Induction step. $\kappa \to \kappa + 1$.
Let $N \in \mathcal{N}$ and $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathscr{S}^N$ be such that $\mu(p) = \kappa + 1$. Suppose, by contradiction, that $\psi^N(p) \nsubseteq \mathrm{box}(\{b(i) \mid i \in N\})$. Since $\kappa \geq 1$ it follows that $\mu(p) \geq 2$. Let $T_1, T_2, \ldots, T_{\kappa+1}$ be the partition of N in subcoalitions such that in each subcoalition T_ℓ, $\ell \in \{1, \ldots, \kappa+1\}$, all members have the same preference relation at p. Because $\psi^N(p) \nsubseteq \mathrm{box}(\{b(i) \mid i \in N\})$, there exists $x \in \psi^N(p)$ and $j \in M$ such that either for all $i \in N$, $x_j < b(i)_j$ or for all $i \in N$, $x_j > b(i)_j$. We assume, without loss of generality, that for all $i \in N$, $x_j < b(i)_j$ and $T_1 = \{1, 2, \ldots, t\}$.

Let $y \in C^\circ(x, p(1))$ such that for all $m \in M \backslash \{j\}$, $x_m = y_m$ and $x_j < y_j < \min_{i \in N} b(i)_j$. Because $C^\circ(x, p(1))$ is an open set there is an open ε-neighborhood around y, say $N_\varepsilon(y) = \{z \in E \mid \sum_{j \in M} (z_j - y_j)^2 < \varepsilon^2\}$, such that $N_\varepsilon(y) \subseteq C^\circ(x, p(1))$. Let $\bar{\varepsilon} = \dfrac{\varepsilon}{|M|}$. Then for all $i \in N$, there exists $\bar{b}(i) \in \mathrm{box}(\{x, b(i)\}) \cap N_{\bar{\varepsilon}}(y)$ such that for all $l \in M$, $\bar{b}(i)_l = x_l$ only if $b(i)_l = x_l$. Note that $\mathrm{box}(\{\bar{b}(i) \mid i \in N\}) \subseteq N_\varepsilon(y)$.

For all $\ell \in \{1, \ldots, \kappa+1\}$ and all $i_1, i_2 \in T_\ell$, $b(i_1) = b(i_2)$. Hence, without loss of generality, for all $\ell \in \{1, \ldots, \kappa+1\}$ and all $i_1, i_2 \in T_\ell$, $\bar{b}(i_1) = \bar{b}(i_2)$. Now consider $\bar{p} \in \mathscr{S}^N$ such that for all $i \in T_1$, $\bar{p}(i) = p(i)$ and for all $i \in N \backslash T_1$, $\bar{p}(i) = (\bar{\delta}(i), \bar{b}(i))$ such that either $\bar{p}(i) = p(i)$ or $\bar{p}(i)$ is an x-improvement of $p(i)$. Since $x \in \psi^N(p)$, by Lemma 4 (ii), $x \in \mathrm{conv}(\psi^N(\bar{p}))$. Thus, by $x \notin C^\circ(x, p(1))$ and the convexity of $C^\circ(x, p(1))$ it follows that there exists $\bar{x} \in \psi^N(\bar{p})$ such that $\bar{x} \notin C^\circ(x, p(1))$.

Let $s \in T_2$, which exists because $\mu(p) \geq 2$. Consider $q \in \mathscr{S}^N$ such that for all $i \in T_1$ and $s \in T_2$, $q(i) = \bar{p}(s)$ and $q =_{N \backslash T_1} \bar{p}$. Note that by construction, $\mu(\bar{p}) \leq \kappa + 1$ and $\mu(q) \leq \kappa$. Therefore, by the induction hypothesis (4), $\psi^N(q) \subseteq \mathrm{box}(\{\bar{b}(i) \mid i \in N \backslash T_1\})$. Thus, $\psi^N(q) \subseteq N_\varepsilon(y) \subseteq C^\circ(x, p(1))$. Hence, for all $z \in \psi^N(q)$, $z \succ_{p(1)} \bar{x}$.

Next, recall that for all $i \in T_1$, $\bar{p}(i) = p(1)$, $q(i) = \bar{p}(s)$, and $\bar{p} =_{N \backslash T_1} q$. So, by *intermediate strategy-proofness*, $\psi^N(\bar{p}) \succsim_{p(1)} \psi^N(q)$. Thus, there exists $z \in \psi^N(q)$ such that $\bar{x} \succsim_{p(1)} z$. This yields the desired contradiction. □

Finally, we define the closure $\bar{\psi}$ of any correspondence ψ. Note that for any subset X of E, the set $\bar{X}$ denotes the closure of X (with respect to the standard Euclidean topology). Thus, $\bar{X}$ is the smallest closed set in E that contains X.

Closure of ψ. For any correspondence ψ the corresponding *closure* $\bar{\psi}$ is defined as follows. For all $N \in \mathcal{N}$ and $p \in \mathcal{S}^N$, $\bar{\psi}^N(p) = \overline{\psi^N(p)}$.

Lemma 6. *Let the choice correspondence ψ satisfy strategy-proofness, unanimity, the tie-breaking and the (weak) non-decisive newcomer condition. Then $\bar{\psi}$ also satisfies strategy-proofness, unanimity, the tie-breaking and the (weak) non-decisive newcomer condition.*

Proof. It follows easily that $\bar{\psi}$ satisfies *unanimity* and the *(weak) non-decisive newcomer condition.*

Strategy-proofness. Let $N \in \mathcal{N}$, $i \in N$, and $p, q \in \mathcal{S}^N$ be such that $p =_{N\setminus\{i\}} q$. We have to show that $\bar{\psi}^N(p) \succsim_{p(i)} \bar{\psi}^N(q)$, *i.e.*, (i) for all $x \in \bar{\psi}^N(p)$ there exist $y \in \bar{\psi}^N(q)$ such that $x \succsim_{p(i)} y$ and (ii) for all $y \in \bar{\psi}^N(q)$ there exist $x \in \bar{\psi}^N(p)$ such that $x \succsim_{p(i)} y$.

(i) Let $x \in \bar{\psi}^N(p)$. Because $\bar{\psi}^N(p)$ is the closure of $\psi^N(p)$, there exists a sequence $\{x^l\}_{l\in\mathbb{N}}$ in $\psi^N(p)$ that converges to x. By *strategy-proofness* of ψ, there exists a sequence $\{y^l\}_{l\in\mathbb{N}}$ in $\psi^N(q)$ such that for all $l \in \mathbb{N}$, $x^l \succsim_{p(i)} y^l$. By Lemma 5, ψ^N is *coordinatewise Pareto optimal*. Hence, for $q = \langle\delta(i), b(i)\rangle_{i\in N}$, $\psi^N(q) \subseteq \text{box}(\{b(i) \mid i \in N\})$. Since $\text{box}(\{b(i) \mid i \in N\})$ is a compact set, it is without loss of generality to assume that $\{y^l\}_{l\in\mathbb{N}}$ converges to a point $y \in \bar{\psi}^N(q)$. Since $\succsim_{p(i)}$ is a continuous preference relation, $x^l \succsim_{p(i)} y^l$ for all $l \in \mathbb{N}$ implies $x \succsim_{p(i)} y$. Hence, for all $x \in \bar{\psi}^N(p)$ there exist $y \in \bar{\psi}^N(q)$ such that $x \succsim_{p(i)} y$. The proof of (ii) is similarly.

Tie-Breaking newcomer condition. Let $N \in \mathcal{N}$, $p \in \mathcal{S}^N$, and $k \notin N$ be such that $p(k) = (\delta(k), b(k))$ and $b(k) \in \bar{\psi}^N(p)$. We have to show that $\bar{\psi}^{N\cup\{k\}}(p, p(k)) = \{b(k)\}$.

Because $\bar{\psi}^N(p)$ is the closure of $\psi^N(p)$, there exists a sequence $\{x^l\}_{l\in\mathbb{N}}$ in $\psi^N(p)$ that converges to $b(k)$. For all $l \in \mathbb{N}$, let $p(k)^l = (\delta(k)^l, b(k)^l)$ be such that $b(k)^l = x^l$. By the *tie-breaking newcomer condition* of ψ, for all $l \in \mathbb{N}$, $\psi^{N\cup\{k\}}(p, p(k)^l) = \{x^l\}$. So, for all $l \in \mathbb{N}$, $\bar{\psi}^{N\cup\{k\}}(p, p(k)^l) = \{x^l\}$. Because $\{x^l\}_{l\in\mathbb{N}}$ converges to $b(k)$, it follows by *strategy-proofness* of $\bar{\psi}$ that $\bar{\psi}^{N\cup\{k\}}(p, p(k)) = \{b(k)\}$. □

5 Two characterization results

The main objective of this section is to characterize the class of choice correspondences that satisfy *strategy-proofness, unanimity*, the *tie-breaking* and the *weak non-decisive newcomer condition.* We prove that the convex and closed hull of any choice correspondence that satisfies all properties mentioned above equals the median choice correspondence (Theorem 1). Furthermore, if the *weak non-decisive newcomer condition* is strengthened to the *non-decisive newcomer condition*, then the median choice correspondence is the only correspondence satisfying all the properties (Theorem 2).

Throughout this section we assume that ψ and $\bar{\psi}$ (Lemma 6) are choice correspondence that satisfies all properties mentioned above. Hence, by Lemmas 1, 4, and 5, ψ and $\bar{\psi}$ satisfy *nonemptiness, monotonicity*, and *coordinatewise Pareto optimality*.

Let x be a compromise point of $\bar{\psi}$ at profile $p \in \mathscr{S}^N$ and $j \in M$. First we prove that the number of agents reporting a best point at p with its j^{th} coordinate strictly smaller than that of x is smaller than, or equal to, half of the number of agents at p. A similar result holds for the number of agents reporting a peak with its j^{th} coordinate strictly greater than that of x. From this and *nonemptiness* it is obvious that for any location problem with an odd number of agents the compromise point is the unique coordinatewise median of the reported best points. Applying the two *newcomer conditions* then yields that this holds for the convex hull of the set of compromise points at any location problem with an even number of agents. The proof of the first two steps is by induction on $\mu(p)$, the number of different preference relations reported at a profile p. The induction step when $\mu(p) = 2$ differs from all the other steps, therefore it is treated separately in the following two lemmas.

In Lemma 7 we proof the induction step $\mu(p) = 2$ in the special case where the reported peaks at p are on a line parallel to one of the axis. In Lemma 8 the result is generalized to arbitrary profiles p with $\mu(p) = 2$.

Lemma 7. *Let the choice correspondence ψ satisfy strategy-proofness, unanimity, the tie-breaking and the weak non-decisive newcomer condition. Let $N \in \mathscr{N}$ and $\emptyset \neq S \subseteq N$. Let $j \in M$ and $(\delta, b), (\delta', b') \in \mathscr{S}$ be such that $b \neq b'$ and $b =_{M \setminus \{j\}} b'$. Let $p \in \mathscr{S}^N$ be such that for all $i \in S$, $p(i) = (\delta, b)$ and for all $i \in N \setminus S$, $p(i) = (\delta', b')$. Then, $b \in \bar{\psi}^N(p)$ if and only if $|S| \geq \frac{1}{2}|N|$.*

Proof. The proof is by induction on $|N| = \kappa$.

Induction basis. $\kappa = 1$.
Then, $|N| = 1$ and $p = (\delta, b) \in \mathscr{S}$. We are done by *unanimity*.

Induction hypothesis. Let $\bar{N} \in \mathscr{N}$ be such that $|\bar{N}| \leq \kappa$ and $\emptyset \neq \bar{S} \subseteq \bar{N}$. Let $j \in M$ and $(\bar{\delta}, \bar{b}), (\tilde{\delta}, \tilde{b}) \in \mathscr{S}$ be such that $\bar{b} \neq \tilde{b}$ and $\bar{b} =_{M \setminus \{j\}} \tilde{b}$. Let $\bar{p} \in \mathscr{S}^{\bar{N}}$ be such that for all $i \in \bar{S}$, $\bar{p}(i) = (\bar{\delta}, \bar{b})$ and for all $i \in \bar{N} \setminus \bar{S}$, $\bar{p}(i) = (\tilde{\delta}, \tilde{b})$. Then,

$$\bar{b} \in \bar{\psi}^{\bar{N}}(\bar{p}) \quad \text{if and only if} \quad |\bar{S}| \geq \tfrac{1}{2}|\bar{N}|. \tag{5}$$

Induction step. $\kappa \to \kappa + 1$.
Let $N \in \mathscr{N}$ be such that $|N| = \kappa + 1$. Let $\emptyset \neq S \subseteq N$, $(\delta, b), (\delta', b') \in \mathscr{S}$, and $p \in \mathscr{S}^N$ be as in the lemma. Then we have to prove that $b \in \psi^N(p)$ if and only if $|S| \geq \frac{1}{2}|N|$.

If part. $|S| \geq \frac{1}{2}|N|$ implies $b \in \bar{\psi}^N(p)$.

Suppose, without loss of generality, that $b_j < b'_j$. Suppose that $b \notin \bar{\psi}^N(p)$. Now, it is sufficient to prove that $|S| < \frac{1}{2}|N|$. By *unanimity* it follows that $S \neq N$. Because $\bar{\psi}^N(p)$ is closed, there exists an open ε-neighborhood $N_\varepsilon(b)$ such that $\bar{\psi}^N(p) \cap N_\varepsilon(b) = \emptyset$. Consider $p^0 \in \mathscr{S}^N$ such that for all $i \in S$, $p^0(i) = (\delta, b)$ and for all $i \in N \backslash S$, $p^0(i) = (\delta', b^0)$ where $b^0 =_{M\backslash\{j\}} b$ and $b_j^0 = b_j + \varepsilon$.

First we prove that $\bar{\psi}^N(p^0) \cap N_\varepsilon(b) = \emptyset$. Suppose, by contradiction, there exists $x \in \bar{\psi}^N(p^0) \cap N_\varepsilon(b)$. By *coordinatewise Pareto optimality* it follows that $b =_{M\backslash\{j\}} x =_{M\backslash\{j\}} b^0$ and $b_j \leq x_j < b_j^0$. Let $b^1 =_{M\backslash\{j\}} b^0$ and $b_j^1 = \min\left\{b'_j, b_j^0 + \dfrac{b_j^0 - x_j}{2}\right\}$ and consider $p^1 \in \mathscr{S}^N$ such that for all $i \in S$, $p^1(i) = (\delta, b)$ and for all $i \in N \backslash S$, $p^1(i) = (\delta', b^1)$. Because of *coordinatewise Pareto optimality* it follows that for all $y \in \bar{\psi}^N(p^1)$,

$$y =_{M\backslash\{j\}} b =_{M\backslash\{j\}} b^1 \quad \text{and} \quad b_j \leq y_j \leq b_j^1. \tag{6}$$

By (6) and *intermediate strategy-proofness* applied to $p^0, p^1 \in \mathscr{S}^N$, there exists $x^1 \in \bar{\psi}^N(p^1)$ such that $|b_j^0 - x_j| \leq |b_j^0 - x_j^1|$ and $x_j^1 \leq x_j$. Hence, $x^1 \in \bar{\psi}^N(p^1) \cap N_\varepsilon(b)$ and $\bar{\psi}^N(p^1) \cap N_\varepsilon(b) \neq \emptyset$. Next, let $b^2 =_{M\backslash\{j\}} b^1$ and $b_j^2 = \min\left\{b'_j, b_j^1 + \dfrac{b_j^1 - x_j^1}{2}\right\}$ and consider $p^2 \in \mathscr{S}^N$ such that for all $i \in S$, $p^2(i) = (\delta, b)$ and for all $i \in N \backslash S$, $p^2(i) = (\delta', b^2)$. Similarly as before it follows that $\bar{\psi}^N(p^2) \cap N_\varepsilon(b) \neq \emptyset$. By repeating this argument we construct a sequence of profiles $\{p^l\}$ that converges to p in finitely many steps. Furthermore, for each profile p^l it follows that $\bar{\psi}^N(p^l) \cap N_\varepsilon(b) \neq \emptyset$. Thus, $\bar{\psi}^N(p) \cap N_\varepsilon(b) \neq \emptyset$. Hence, we have a contradiction.

So, $\bar{\psi}^N(p^0) \cap N_\varepsilon(b) = \emptyset$. Thus, by *nonemptiness* and *coordinatewise Pareto optimality* it follows that $\bar{\psi}^N(p^0) = \{b^0\}$. Because $S \neq N$ there exists $l \in N \backslash S$. Thus, by the *weak non-decisive newcomer condition* we have that $b^0 \in \operatorname{conv}(\bar{\psi}^{N\backslash\{l\}}(p^0_{N\backslash\{l\}}))$. Furthermore, by *coordinatewise Pareto optimality*, $\bar{\psi}^{N\backslash\{l\}}(p^0_{N\backslash\{l\}}) \subseteq \operatorname{box}(\{b, b^0\})$. Hence, $b^0 \in \bar{\psi}^{N\backslash\{l\}}(p^0_{N\backslash\{l\}})$. Applying the induction hypothesis (5) yields $|(N\backslash\{l\})\backslash S| \geq \frac{1}{2}|N\backslash\{l\}|$. Hence, $|S| < \frac{1}{2}|N|$.

Only if part. $b \in \bar{\psi}^N(p)$ implies $|S| \geq \frac{1}{2}|N|$.
Suppose $|S| < \frac{1}{2}|N|$. Then we have to prove that $b \notin \bar{\psi}^N(p)$. Let $T \subsetneq N \backslash S$ be such that $|S| = |T|$. Then $|S \cup T| < |N|$. So, by the induction hypothesis (5), it follows that $b' \in \bar{\psi}^{S \cup T}(p_{S \cup T})$. Hence, by the *tie-breaking newcomer condition*, $\bar{\psi}^N(p) = \{b'\}$ and $b \notin \bar{\psi}^N(p)$. □

Lemma 8. *Let the choice correspondence ψ satisfy strategy-proofness, unanimity, the tie-breaking and the weak non-decisive newcomer condition. Let $N \in \mathscr{N}$ and $S \subseteq N$ be such that $|S| > \frac{1}{2}|N|$. Let $(\delta, b), (\delta', b') \in \mathscr{S}$ and $p \in \mathscr{S}^N$ be such that for all $i \in S$, $p(i) = (\delta, b)$ and for all $i \in N \backslash S$, $p(i) = (\delta', b')$. Then, $\psi^N(p) = \{b\}$.*

Proof. Let $j \in M$. It is sufficient to prove that for all $x \in \psi^N(p)$, $x_j = b_j$.

Case 1. $b'_j = b_j$.
Then we are done by *coordinatewise Pareto optimality*.

Case 2. $b'_j > b_j$.
Let $(\delta'', b'') \in \mathscr{S}$ be such that $b'' =_{M\backslash\{j\}} b$ and $b''_j = b'_j$. Consider $q \in \mathscr{S}^N$ such that for all $i \in S$, $q(i) = (\delta, b)$ and for all $i \in N\backslash S$, $q(i) = (\delta'', b'')$. Because $|S| > \frac{1}{2}|N|$, we conclude by Lemma 7 that $b \in \psi^N(q)$ and $b'' \notin \psi^N(q)$. By *coordinatewise Pareto optimality*, for all $x \in \psi^N(q)$, $x =_{M\backslash\{j\}} b$ and $b_j \le x_j < b''_j$. We prove that in fact $\psi^N(q) = \{b\}$. Assume, by contradiction, that there exists a compromise point $x \in \psi^N(q)$ such that $x_j > b_j$. Consider $p^1 \in \mathscr{S}^N$ such that for all $i \in S$, $p^1(i) = (\delta, b)$ and for all $i \in N\backslash S$, $p^1(i) = (\delta'', x)$. By *intermediate strategy-proofness*, $x \in \psi^N(p^1)$. Since $|N\backslash S| < \frac{1}{2}|N|$, this contradicts Lemma 7. Hence, $\psi^N(q) = \{b\}$.

So, by *intermediate strategy-proofness*, for all $x \in \psi^N(p)$, $x \notin C^\circ(b, (\delta'', b''))$. As this is independent of the choice of weight vectors, by considering a sequence of weight vectors $\{\delta^k\}_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty} \delta^k_l = 0$ for $l \in M\backslash\{j\}$ and $\lim\limits_{k\to\infty} \delta^k_j = 1$, we conclude that for all $x \in \psi^N(p)$, either $x_j \le b_j$ or $x_j \ge 2b'_j - b_j$. Hence, by *coordinatewise Pareto optimality*, for all $x \in \psi^N(p)$, $x_j = b_j$.

Case 3. $b'_j < b_j$. Similarly to Case 2. □

Now we are able to prove that the number of agents who are reporting a peak with its j^{th} coordinate strictly smaller (greater) than that of a compromise point x is bounded by half of the number of agents that are present.

For $j \in M$, $N \in \mathscr{N}$, $p = \langle\delta(i), b(i)\rangle \in \mathscr{S}^N$, and $x \in E$ define

$$L(j, x, p) \equiv \{i \in N \mid b(i)_j < x_j\};$$

$$I(j, x, p) \equiv \{i \in N \mid b(i)_j = x_j\};$$

$$G(j, x, p) \equiv \{i \in N \mid b(i)_j > x_j\}.$$

Lemma 9. *Let the choice correspondence ψ satisfy strategy-proofness, unanimity, the tie-breaking and the weak non-decisive newcomer condition. Let $N \in \mathscr{N}$, $j \in M$, $p = \langle\delta(i), b(i)\rangle_{i\in N} \in \mathscr{S}^N$, and $x \in \bar{\psi}^N(p)$. Then, $|L(j, x, p)| \le \frac{1}{2}|N|$ and $|G(j, x, p)| \le \frac{1}{2}|N|$.*

Proof. Let $N \in \mathscr{N}$, $j \in M$, $p \in \mathscr{S}^N$ with $\mu(p) = \kappa$, and $x \in \bar{\psi}^N(p)$. We prove by induction on $\kappa \in \mathbb{N}$ that $|L(j, x, p)| \le \frac{1}{2}|N|$. The proof of $|G(j, x, p)| \le \frac{1}{2}|N|$ is similar.

Induction basis. $\mu(p) \le 2$.
If $\mu(p) = 1$, then we are done by *unanimity*. If $\mu(p) = 2$, then we are done by Lemma 8.

Induction hypothesis. Let $\bar{N} \in \mathscr{N}$, $\bar{j} \in M$, $\bar{p} \in \mathscr{S}^{\bar{N}}$ such that $\mu(\bar{p}) \le \kappa$, and $\bar{x} \in \bar{\psi}^{\bar{N}}(\bar{p})$. Then,

$$|L(\bar{j},\bar{x},\bar{p})| \leq \tfrac{1}{2}|\bar{N}|. \tag{7}$$

Induction step. $\kappa \to \kappa + 1$.
Let $N \in \mathcal{N}$, $j \in M$, $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathcal{S}^N$ be such that $\mu(p) = \kappa + 1 \geq 3$, and $x \in \bar{\psi}^N(p)$. Suppose, by contradiction, that $|L(j,x,p)| > \frac{1}{2}|N|$. Let $k \notin N$ and $(\delta, x) \in \mathcal{S}$. Note that by the *tie-breaking newcomer condition*, $\bar{\psi}^{N \cup \{k\}}(p, (\delta, x)) = \{x\}$. Let $q \in \mathcal{S}^N$ be such that for all $i \in L(j,x,p)$, $q(i) = p(i)$ and for all $i \in I(j,x,p) \cup G(j,x,p)$, $q(i) = (\delta, x)$. By *strategy-proofness*, $\bar{\psi}^{N \cup \{k\}}(q, (\delta, x)) = \{x\}$. Note that $L(j,x,q) = L(j,x,p)$ and $\mu(q) \leq \mu(p) = \kappa + 1$.

Let $T_1, T_2, \ldots, T_{\kappa+1}$ be the partition of N in subcoalitions such that in each subcoalition T_ℓ, $\ell \in \{1, \ldots, \kappa+1\}$, all members have the same preference relation, say (δ^ℓ, b^ℓ), at q. Suppose, without loss of generality, that $T_{\kappa+1} = I(j,x,p) \cup G(j,x,p)$. Let $y \in C^\circ(x, (\delta^1, b^1))$ be such that $x =_{M \setminus \{j\}} y$ and $x_j > y_j > \max\{b_j^1, b_j^2, \ldots, b_j^\kappa\}$. Because $C^\circ(x, (\delta^1, b^1))$ is open, there exists an open ε-neighborhood around y, say $N_\varepsilon(y)$, such that $N_\varepsilon(y) \subseteq C^\circ(x, (\delta^1, b^1))$. Let $\bar{\varepsilon} = \dfrac{\varepsilon}{|M|}$. Then for all $\ell \in \{2, \ldots \kappa\}$, there exists $\bar{b}^\ell \in \operatorname{box}(\{x, b^\ell\}) \cap N_{\bar{\varepsilon}}(y)$, such that for all $\bar{j} \in M$, $\bar{b}^\ell_{\bar{j}} = x_{\bar{j}}$ only if $b^\ell_{\bar{j}} = x_{\bar{j}}$. Note that $\operatorname{box}(\{\bar{b}^2, \ldots, \bar{b}^\kappa\}) \subseteq N_\varepsilon(y)$.

Now consider $\bar{q} \in \mathcal{S}^N$ such that $\bar{q} =_{T_1 \cup T_{\kappa+1}} q$ and for $\ell \in \{2, \ldots, \kappa\}$, $i \in T_\ell$, $\bar{q}(i) = (\bar{\delta}^\ell, \bar{b}^\ell)$ such that either $\bar{q}(i) = q(i)$ or $\bar{q}(i)$ is an x-improvement of $q(i)$. Now, by *monotonicity*, $\bar{\psi}^N(\bar{q}, (\delta, x)) = \{x\}$. Note that $\mu(\bar{q}) \leq \mu(q) \leq \kappa + 1$ and $L(j,x,\bar{q}) = L(j,x,q) = L(j,x,p)$. Thus, $|L(j,x,\bar{q})| > \frac{1}{2}|N|$.

Since $\bar{\psi}^N(\bar{q}, (\delta, x)) = \{x\}$ it follows by the *non-decisive newcomer condition* that $x \in \bar{\psi}^N(\bar{q})$. Note that $x \notin C^\circ(x, (\delta^1, b^1))$.

If $\mu(\bar{q}) = 2$, then $L(j,x,\bar{q}) = T_1$. So, $|T_1| > \frac{1}{2}|N|$ and by Lemma 8, $\bar{\psi}^N(\bar{q}) = \{b^1\}$. This contradicts $x \in \bar{\psi}^N(\bar{q})$. Hence, $\mu(\bar{q}) \geq 3$.

Finally, consider $\tilde{q} \in \mathcal{S}^N$ such that $\tilde{q} =_{N \setminus T_1} \bar{q}$ and for all $i \in T_1$, $\tilde{q}(i) = (\bar{\delta}^2, \bar{b}^2)$. Then, $\mu(\tilde{q}) = \mu(\bar{q}) - 1 = \kappa$ and $L(j,x,\tilde{q}) = L(j,x,\bar{q})$. So, $|L(j,x,\tilde{q})| > \frac{1}{2}|N|$. By the induction hypothesis (7), $x \notin \psi^N(\tilde{q})$ and $\psi^N(\tilde{q}) \subseteq \operatorname{box}\{\bar{b}^2, \ldots, \bar{b}^\kappa\}$. Hence, $\psi^N(\tilde{q}) \subseteq N_\varepsilon(y) \subseteq C^\circ(x, (\delta^1, b^1))$. Thus, for all $z \in \psi^N(\tilde{q})$, $z \succ_{(\delta^1, b^1)} x$.

Next, recall that for all $i \in T_1$, $\bar{q}(i) = (\delta^1, b^1)$ and $\bar{q} =_{N \setminus T_1} \tilde{q}$. So, by *intermediate strategy-proofness*, $\psi^N(\bar{q}) \succsim_{(\delta^1, b^1)} \psi^N(\tilde{q})$. This implies that there exists $z \in \psi^N(\tilde{q})$ such that $x \succsim_{(\delta^1, b^1)} z$. This yields the desired contradiction. □

Theorem 1. *A choice correspondence ψ satisfies strategy-proofness, unanimity, the tie-breaking and the weak non-decisive newcomer condition if and only if for all $N \in \mathcal{N}$ and $p \in \mathcal{S}^N$,* $\operatorname{conv} \psi^N(p) = \psi^N_{\text{med}}(p)$.

Proof. Note that for all $N \in \mathcal{N}$ and $p \in \mathcal{S}^N$ such that $\operatorname{conv} \psi^N(p) = \psi^N_{\text{med}}(p)$ it follows that for all $i \in N$, $\psi^N(p) \sim_{p(i)} \psi^N_{\text{med}}(p)$. Hence, by Lemma 3, any choice correspondence ψ such that for all $N \in \mathcal{N}$ and $p \in \mathcal{S}^N$, $\operatorname{conv} \psi^N(p) = \psi^N_{\text{med}}(p)$ satisfies the properties mentioned in the theorem.

Next, let ψ be a choice correspondence that satisfies the properties named in the theorem. We have to show that for all $N \in \mathcal{N}$ and $p \in \mathcal{S}^N$, $\operatorname{conv}\psi^N(p) = \psi^N_{\text{med}}(p)$.

By Lemma 9, for all $j \in M$ and all $x \in \bar{\psi}^N(p)$, $|L(j,x,p)| \le \frac{1}{2}|N|$ and $|G(j,x,p)| \le \frac{1}{2}|N|$. Thus, for all $j \in M$ and $x \in \bar{\psi}^N(p)$, $|L(j,x,p) \cup I(j,x,p)| \ge \frac{1}{2}|N|$ and $|G(j,x,p) \cup I(j,x,p)| \ge \frac{1}{2}|N|$. So, $\bar{\psi}^N(p) \subseteq \psi^N_{\text{med}}(p)$. Therefore, $\operatorname{conv}\psi^N(p) \subseteq \operatorname{conv}\bar{\psi}^N(p) \subseteq \psi^N_{\text{med}}(p)$.

If $|N|$ is odd, then $\psi^N_{\text{med}}(p)$ is a singleton. Thus, by *nonemptiness*, $\operatorname{conv}\psi^N(p) = \psi^N(p) = \psi^N_{\text{med}}(p)$. Note that $\psi^N_{\text{med}}(p)$ is the Cartesian product of m closed intervals $[a_j, b_j]$, $a_j \le b_j$, and that for any extreme point $x = (x_1, \dots, x_m)$ of $\psi^N_{\text{med}}(p)$: for all $j \in M$, either $x_j = a_j$ or $x_j = b_j$. Let $|N|$ be even. In order to prove $\operatorname{conv}\psi^N(p) \supseteq \psi^N_{\text{med}}(p)$ it is sufficient to prove that every extreme point of $\psi^N_{\text{med}}(p)$ is a compromise point in $\psi(p)$. Let x be an extreme point of $\psi^N_{\text{med}}(p)$. Let $k \notin N$ and $(\delta, x) \in \mathcal{S}$. By the *tie-breaking newcomer condition*, $\psi^{N\cup\{k\}}_{\text{med}}(p,(\delta,x)) = \{x\}$. Since $|N \cup \{k\}|$ is odd, $\psi^{N\cup\{k\}} \cdot (p,(\delta,x)) = \psi^{N\cup\{k\}}_{\text{med}}(p,(\delta,x)) = \{x\}$. Thus, by the *weak non-decisive newcomer condition*, $x \in \operatorname{conv}\psi^N(p)$. Since, $\psi^N(p) \subseteq \psi^N_{\text{med}}(p)$ it follows that $x \in \psi^N(p)$. □

Remark 1. *Note that ψ_{med} is a voting correspondence. So, by Theorem 1 it follows that the four characterizing conditions imply peak-onliness.*

The following example demonstrates that the *non-decisive newcomer condition* is not implied by *strategy-proofness, unanimity*, the *tie-breaking* and the *weak non-decisive newcomer condition.*

Example 1. For simplicity, assume that $E = \mathbb{R}$. Then the choice correspondence $\tilde{\psi}$ is defined as follows. For all $N \in N$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N} \in S^N$,

$$\tilde{\psi}^N(p) = \psi^N_{\text{med}}(p) \cap (\{b(i) \mid i \in N\}).$$

Is is easy to check that $\tilde{\psi}$ satisfies *strategy-proofness, unanimity*, the *tie-breaking* and the *weak non-decisive newcomer condition*, but not the *non-decisive newcomer condition.*

Theorem 2. *A choice correspondence ψ satisfies strategy-proofness, unanimity, the tie-breaking and the non-decisive newcomer condition if and only if ψ is the coordinatewise median correspondence ψ_{med}.*

Loosely speaking, the difference between Theorem 1 and 2 is that compromise sets assigned by choice correspondences satisfying the conditions in Theorem 1 can have "holes": as long as the convex hull of each compromise set is equal to the compromise set assigned by the coordinatewise median correspondence all properties will be satisfied (and all agents are in fact indifferent between the compromise set with the hole(s) and the convex hull that equals the coordinatewise median correspondence compromise set).

Proof. By Lemma 3, the coordinatewise median correspondence ψ_{med} satisfies the properties mentioned in the theorem.

Next, let ψ be a choice correspondence that satisfies the properties named in the theorem. Then, by Theorem 1, for all $N \in \mathcal{N}$ and $p \in \mathcal{S}^N$, $\operatorname{conv} \psi^N(p) = \psi^N_{\text{med}}(p)$. Hence, $\psi^N(p) \subseteq \psi^N_{\text{med}}(p)$.

If $|N|$ is odd, then, by Theorem 1, it follows that $\operatorname{conv} \psi^N(p) = \psi^N_{\text{med}}(p)$. As $\psi^N_{\text{med}}(p)$ is a singleton and $\psi^N(p)$ is a nonempty subset of $\operatorname{conv} \psi^N(p)$ it follows that $\psi^N(p) = \psi^N_{\text{med}}(p)$. Let $|N|$ be even. Then it is sufficient to prove that $\psi^N_{\text{med}}(p) \subseteq \psi^N(p)$. Let $x \in \psi^N_{\text{med}}(p)$. Let $k \notin N$ and $(\delta, x) \in \mathcal{S}$. By the *tie-breaking newcomer condition*, $\psi^{N \cup \{k\}}_{\text{med}}(p, (\delta, x)) = \{x\}$. Since $|N \cup \{k\}|$ is odd, $\psi^{N \cup \{k\}}(p, (\delta, x)) = \psi^{N \cup \{k\}}_{\text{med}}(p, (\delta, x)) = \{x\}$. Thus, by the *non-decisive newcomer condition*, $x \in \psi^N(p)$. Hence, $\psi^N_{\text{med}}(p) \subseteq \psi^N(p)$. □

The following list of examples shows that *unanimity, strategy-proofness*, the *tie-breaking newcomer condition*, and the *non-decisive newcomer condition* are logically independent from each other.

Example 2. The *constant-zero voting correspondence* ψ_0 is defined as follows. For all $N \in \mathcal{N}$ and all $p \in \mathcal{S}^N$,

$$\psi^N_0(p) = \{0\}.$$

The constant zero voting correspondence ψ_0 satisfies *strategy-proofness*, the *tie-breaking* and the *non-decisive newcomer condition*, but it is not *unanimous*. ◇

Example 3. The *mean voting correspondence* ψ_{mean} is defined as follows. For all $N \in \mathcal{N}$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathcal{S}^N$,

$$\psi^N_{\text{mean}}(p) = \left\{ \frac{1}{|N|} \sum_{i \in N} b(i) \right\}.$$

The mean voting correspondence ψ_{mean} satisfies *unanimity*, the *tie-breaking* and the *non-decisive newcomer condition*, but it is not *strategy-proof*. ◇

Example 4. The *box voting correspondence* ψ_{box} is defined as follows. For all $N \in \mathcal{N}$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathcal{S}^N$,

$$\psi^N_{\text{box}}(p) = \operatorname{box}(\{b(i) \mid i \in N\}).$$

The voting correspondence ψ_{box} satisfies *unanimity, strategy-proofness,* and the *non-decisive newcomer condition*, but it does not satisfy the *tie-breaking newcomer condition*. ◇

As we will also discuss in Sect. 7, the results presented here and those of Border and Jordan (1983) are logically independent. The following example not only shows the independence of the non-decisive newcomer property, it also proves that *unanimity, strategy-proofness*, and the *tie-breaking newcomer condition* do not imply *peak-onliness*. In Border and Jordan (1983) *peak-onliness* is implied by *unanimity, strategy-proofness,* and *single-valuedness*. So, the example shows that *unanimity, strategy-proofness*, and *set-valuedness* are weaker than the properties studied by Border and Jordan (1983).

Example 5. The *choice correspondence* Φ is defined as follows. For all $N \in \mathcal{N}$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathcal{S}^N$,

$$\Phi^N(p) = \begin{cases} \mathrm{conv}(\{b(i) \mid i \in N\}) & \text{if } 1 \in N, |N| = 2, \text{ and} \\ & \text{for all } j, j' \in M, \delta(1)_j = \delta(1)_{j'} \\ \psi^N_{\mathrm{med}}(p) & \text{otherwise.} \end{cases}$$

It is obvious that Φ does neither satisfy *peak-onliness*, nor the *weak non-decisive newcomer condition*. Furthermore, it is straightforward to prove that Φ satisfies *unanimity* and the *tie-breaking newcomer condition*. To see that Φ is *strategy-proof* notice that for each agent $i \in N$, the best and worst points of $\psi^N_{\mathrm{med}}(p)$ are on the boundary of $\psi^N_{\mathrm{med}}(p)$. ◇

6 Robustness of the results

We consider the robustness of our results with respect to changes in the set of admissible preference relations. This is done in two ways. First, we determine a maximal domain of single-peaked preferences for Theorems 1 and 2. Then we discuss the extension of preferences over compromise points to preferences over sets of compromise points.

First, we consider greater sets of admissible preferences. Let $\mathcal{D}$ be a set of single-peaked, strictly convex preferences (*i.e.*, all weak upper contour sets are strictly convex) such that $\mathcal{S} \subseteq \mathcal{D}$. We prove that there exists a choice correspondence that satisfies *strategy-proofness, unanimity*, the *tie-breaking* and the *weak non-decisive newcomer condition* if and only if all preferences $\succsim$ in $\mathcal{D}$ have the *box property*[10], *i.e.*, for all $x, y \in E$ and $\succsim$ with best point, or peak, $b \in E$, $y \in \mathrm{box}(x, b)$ implies $y \succsim x$. Note that many single-peaked preference relations have this property, *e.g.*, all separable-quadratic preference relations and all preference relations that are based on one of the following $L_1, L_2, \ldots, L_\infty$ norms: for $k \in \mathbb{N}$, x is weakly preferred to y with respect to L_k, $x \succsim_{L_k} y$, if and only if

$$\left(\sum_{j \in M} |x_j - b_j|^k \right)^{1/k} \leq \left(\sum_{j \in M} |y_j - b_j|^k \right)^{1/k}.$$

Of course, there are also single-peaked preferences which do not have the box property. Take for instance preference relations with ellipsoid indifference sets such that the main diagonals are not parallel to the axes of E.

The box property is essential for having choice correspondences satisfying the four properties in Theorems 1 or 2. The addition of one single-peaked preference to $\mathcal{S}$ that does not have the box property causes non-existence of these choice correspondences.

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[10] See also Peters et al. (1991).

Theorem 3.[11]

(i) *There exists a choice correspondence* ψ *that satisfies strategy-proofness, unanimity, the tie-breaking and the weak non-decisive newcomer condition if and only if all preferences in* $\mathscr{D}$ *have the box property.*[12]

(ii) *If all preference relations in* $\mathscr{D}$ *have the box property, then for any choice correspondence* ψ *that satisfies strategy-proofness, unanimity, the tie-breaking and the weak non-decisive newcomer condition the following holds.*

For all $N \in \mathscr{N}$ *and* $p \in \mathscr{D}^N$, $\operatorname{conv} \psi^N(p) = \psi^N_{\text{med}}(p)$.

Proof.

(i) If part. If all preferences in $\mathscr{D}$ have the box property, then there exists a choice correspondence ψ satisfying *strategy-proofness, unanimity*, the *tie-breaking* and the *weak non-decisive newcomer condition*.

It is straightforward to prove that ψ_{med} is *strategy-proof*, if all preference relations have the box property. Furthermore, ψ_{med} satisfies *unanimity*, the *tie-breaking* and the *weak non-decisive newcomer condition.*

(i) Only if part. If there exists a choice correspondence ψ satisfying *strategy-proofness, unanimity*, the *tie-breaking* and the *weak non-decisive newcomer condition*, then all preference relations in $\mathscr{D}$ have the box property.

Suppose, by contradiction, that ψ satisfies all properties named above and there exists a preference relation $q(1) \in \mathscr{D}$ that does not have the box property. Then there exists $x, y \in E$ such that $y \in \operatorname{box}(\{x, b\})$ and $x \succ_{q(1)} y$. Call such a pair $(x, y) \in E \times E$ a *box violation at* $q(1)$. Let $\sigma(x, b)$ denote the number of coordinates on which x and b differ. Note that because $q(1)$ is single-peaked, $\sigma(x, b) \neq 1$ (otherwise $y = \lambda x + (1 - \lambda)b$ for some $\lambda \in (0, 1)$ and by single-peakedness, $y \succ_{q(1)} x$). Hence, $\sigma(x, b) \geq 2$.

We prove that there exists a box violation $(\bar{x}, \bar{y}) \in E \times E$ at $q(1)$ such that $\sigma(\bar{x}, b) < \sigma(x, b)$. Applying this result iteratively yields a box violation, say $(\tilde{x}, \tilde{y})$, such that $\sigma(\tilde{x}, b) \leq 1$. Hence, we have a contradiction and are done.

Consider the line through x and y, denoted by $\operatorname{line}(x, y) \equiv \{x + \lambda(x - y) \mid \lambda \in \mathbb{R}\}$. There exists a point $\bar{y} \in \operatorname{line}(x, y)$ on the boundary of $\operatorname{box}(\{x, b\})$ and a coordinate j such that $\bar{y}_j = b_j$ and $\bar{y}_j \neq x_j$. Since $x \succ_{q(1)} y$, strict convexity implies that

$$x \succ_{q(1)} \bar{y}. \tag{8}$$

Next, let $N = \{1, 2, 3\}$, $(\delta, x), (\bar{\delta}, \bar{y}) \in \mathscr{S}$, and $p = \langle(\delta, x), (\bar{\delta}, \bar{y})\rangle \in \mathscr{S}^{\{2,3\}}$. Denote by $\tilde{\psi}$ the restriction of ψ to $\bigcup_{\tilde{N} \in \mathscr{N}} \mathscr{S}^{\tilde{N}}$. Clearly, $\tilde{\psi}$ satisfies *strategy-*

[11] A referee suggested that it may be possible to drop the requirement of single-peakedness in Theorem 3 by using a similar argument as Berga and Serizawa (2000), Corollary 2. One of the necessary adjustments of Theorem 3 would be to extend the median correspondence to deal with strictly increasing preferences. We leave it an open problem whether or not this can be done.
[12] Replace $\mathscr{S}$ by $\mathscr{D}$ in the previous definitions of conditions.

proofness, unanimity, the *tie-breaking* and the *weak non-decisive newcomer condition*. Hence, by Theorem 1, for all $\tilde{N} \in \mathcal{N}$ and $p \in \mathcal{S}^{\tilde{N}}$, $\operatorname{conv} \tilde{\psi}^{\tilde{N}}(p) = \tilde{\psi}^{\tilde{N}}_{\text{med}}(p)$. Let $\tilde{\delta}$ be such that $(\tilde{\delta}, b) \in \mathcal{S}$. Since $\psi^N((\tilde{\delta}, b), (\delta, x), (\bar{\delta}, b)) = \tilde{\psi}^N((\tilde{\delta}, b), (\delta, x), (\bar{\delta}, b)) = \{b\}$, by *strategy-proofness*, for all $z \in \psi^N(q(1), (\delta, x), (\bar{\delta}, b))$, $z \succsim_{q(1)} b$. Because b is the peak of $q(1)$, $\psi^N(q(1), (\delta, x), (\bar{\delta}, b)) = \{b\}$. From this and *strategy-proofness* it follows that for all $z \in \psi^N(q(1), (\delta, x), (\bar{\delta}, \bar{y})) = \psi^N(q(1), p)$,

$$z \succsim_{(\bar{\delta}, \bar{y})} b. \tag{9}$$

Let $z \in \psi^N(q(1), p)$. By *strategy-proofness*, there exists $z' \in \psi^N((\tilde{\delta}, z), p)$ such that $z' \succsim_{(\tilde{\delta}, z)} z$. Since z is the best point of $(\tilde{\delta}, z)$, we conclude that $z \in \psi^N(q(1), p)$ implies $z \in \psi^N((\tilde{\delta}, z), p) = \bar{\psi}^N((\tilde{\delta}, z), p) = \psi^N_{\text{med}}((\tilde{\delta}, z), p) \subseteq \text{box}(\{x, \bar{y}\})$. So, $\psi^N(q(1), p) \subseteq \text{box}(\{x, \bar{y}\})$. Since $q(1)$ is strictly convex, according to $q(1)$ there is at most one best point, say $\bar{x}$, in $\text{box}(\{x, \bar{y}\})$. Note that $z \in \text{box}(\{x, \bar{y}\})$ implies $\psi^N((\tilde{\delta}, z), p) = \psi^N_{\text{med}}((\tilde{\delta}, z), p) = \{z\}$. It follows by *strategy-proofness* that for all $z \in \text{box}(\{x, \bar{y}\})$ and all $z' \in \psi^N(q(1), p)$, $z' \succsim_{q(1)} z$. So, the best point $\bar{x}$ of $q(1)$ in $\text{box}(\{x, \bar{y}\})$ exists and $\psi^N(q(1), p) = \{\bar{x}\}$. Hence, by (9), $\bar{x} \succsim_{(\bar{\delta}, \bar{y})} b$. As this holds for all $\bar{\delta}$ such that $(\bar{\delta}, \bar{y}) \in \mathcal{S}$ it follows that $\bar{x}_j = \bar{y}_j = b_j$. Thus,

$$\sigma(\bar{x}, b) < \sigma(x, b). \tag{10}$$

Because $\bar{x} \in \text{box}(\{x, \bar{y}\})$ and $\bar{y} \in \text{box}(\{x, b\})$, it follows that

$$\bar{y} \in \text{box}(\{\bar{x}, b\}). \tag{11}$$

Because by (8), $x \succ_{q(1)} \bar{y}$ and $\bar{x}$ is the best point of $q(1)$ in $\text{box}(\{x, \bar{y}\})$, it follows that

$$\bar{x} \succ_{q(1)} \bar{y}. \tag{12}$$

Now, (10), (11), and (12) imply that $(\bar{x}, \bar{y})$ is a box violation such that $\sigma(\bar{x}, b) < \sigma(x, b)$. This completes the proof of (i).

(ii) Assume that all preferences in $\mathcal{D}$ have the box property and that ψ satisfies all properties named above. We have to prove that for all $N \in \mathcal{N}$ and $p \in \mathcal{D}^N$, $\operatorname{conv} \psi^N(p) = \psi^N_{\text{med}}(p)$. The proof is by induction on the number of agents that report a preference relation in $\mathcal{D} \backslash \mathcal{S}$. For $p \in \mathcal{D}^N$ we denote this number by $\lambda(p) \equiv |\{i \in N \mid p(i) \in \mathcal{D} \backslash \mathcal{S}\}|$.

Induction basis. $\lambda(p) = 0$.
If $\lambda(p) = 0$, then we are done by Theorem 1.

Induction hypothesis. Let $\bar{N} \in \mathcal{N}$, $\bar{p} \in \mathcal{D}^{\bar{N}}$ such that $\lambda(\bar{p}) \leq \kappa$. Then,

$$\operatorname{conv} \psi^{\bar{N}}(\bar{p}) = \psi^{\bar{N}}_{\text{med}}(\bar{p}). \tag{13}$$

Induction step. $\kappa \to \kappa + 1$.
Let $p \in \mathcal{D}^N$ be such that $\lambda(p) = \kappa + 1$. We have to show that $\operatorname{conv} \psi^N(p) = \psi^N_{\text{med}}(p)$. Assume, without loss of generality, that $1 \in N$ and $p(1) \notin \mathcal{S}$ with

peak point $b(1)$. Without loss of generality, we assume that for all $y \in \psi^N_{\text{med}}(p)$ and all $j \in M$, $b(1)_j \leq y_j$.

First, we prove that $\psi^N(p) \subseteq \psi^N_{\text{med}}(p)$.

Case 1. $|N|$ is odd. Let $\{z\} = \psi^N_{\text{med}}(p)$.

Assume, by contradiction, that there exists $x \in \psi^N(p)$ such that $x \neq z$. Consider $q \in \mathscr{D}^N$ such that $q =_{N\setminus\{1\}} p$ and $q(1) = (\delta(1), b(1)) \in \mathscr{S}$. Note that $\lambda(q) = \lambda(p) - 1 = \kappa$. Thus, by the induction hypothesis (13), $\psi^N(q) = \psi^N_{\text{med}}(q) = \{z\}$. By *strategy-proofness*, $z \succsim_{q(1)} x$ and $x \succsim_{p(1)} z$. Since $\delta(1)$ was chosen arbitrarily and $q(1)$ is strictly convex, there must exist $j \in M$ such that $x_j < b(1)_j \leq z_j$. Next, consider $q' \in \mathscr{D}^N$ such that $q' =_{N\setminus\{1\}} q$ and $q'(1) = (\delta(1), x) \in \mathscr{S}$. By *strategy-proofness*, $x \in \psi^N(q')$. By the induction hypothesis (13), $\psi^N(q') = \psi^N_{\text{med}}(q')$. Furthermore, by construction, for all $z' \in \psi^N_{\text{med}}(q')$, $z'_j = z_j$. Hence, in contradiction to our assumption, $x_j = z_j$.

Case 2. $|N|$ is even.

Assume, by contradiction, that there exists $x \in \psi^N(p) \setminus \psi^N_{\text{med}}(p)$. Let $k \notin N$ and $(\delta, x) \in \mathscr{S}$. By the *tie-breaking newcomer condition*, $\psi^{N\cup\{k\}}(p, (\delta, x)) = \{x\}$. Since $|N \cup \{k\}|$ is odd, by Case 1, $\psi^{N\cup\{k\}}(p, (\delta, x)) = \psi^{N\cup\{k\}}_{\text{med}}(p, (\delta, x))$. Thus, by the *weak non-decisive newcomer condition*, $x \in \operatorname{conv} \psi^N_{\text{med}}(p) = \psi^N_{\text{med}}(p)$. This is in contradiction to the assumption that $x \in \psi^N(p) \setminus \psi^N_{\text{med}}(p)$.

By Cases 1 and 2, for all $p \in \mathscr{D}^N$ such that $\lambda(p) = \kappa + 1$, $\psi^N(p) \subseteq \psi^N_{\text{med}}(p)$.

The proof that for all $p \in \mathscr{D}^N$ such that $\lambda(p) = \kappa + 1$, $\psi^N(p) \supseteq \psi^N_{\text{med}}(p)$ is similar to the proof of the same statement at the end of the proof of Theorem 1. □

As mentioned before the results presented here very much depend on the actual extension of preferences over compromise points to preferences over sets of compromise points. Other extensions differing from those discussed here might either lead to impossibilities, if the extension implies a stronger notion of *strategy-proofness*, or they might lead to indeterminability, if the extension implies a weaker notion of *strategy-proofness*. We will explain both cases by the following examples.

First, we discuss two extensions that yield stronger notions of *strategy-proofness* and incompatibility with *unanimity*, the *tie-breaking* and the *non-decisive newcomer condition*.

Example 6. Consider the following extension of $\succsim_{(\delta,b)}$ to 2^E. For all $(\delta, b) \in \mathscr{S}$ and all $X, Y \in 2^E$, $X \overset{1}{\trianglerighteq}_{(\delta,b)} Y$ if and only if for all $x \in X$ and all $y \in Y$,

$$\sum_{j\in M} \delta(i)_j (x_j - b(i)_j)^2 \leq \sum_{j\in M} \delta(i)_j (y_j - b(i)_j)^2.$$

Then for an agent with preference $\overset{1}{\trianglerighteq}_{(\delta,b)}$, $X \overset{1}{\trianglerighteq}_{(\delta,b)} Y$ means that all points in X are at least as good as those in Y. Assuming nonemptiness, $X \overset{1}{\trianglerighteq}_{(\delta,b)} Y$ implies $X \succsim_{(\delta,b)} Y$. Hence, under $\overset{1}{\trianglerighteq}_{(\delta,b)}$ even more sets are incomparable than under $\succsim_{(\delta,b)}$, and the notion of $\overset{1}{\trianglerighteq}_{(\delta,b)}$ leads to a stronger strategy-proofness condition. If a choice correspondence is *strategy-proof with respect to* $\overset{1}{\trianglerighteq}_{(\delta,b)}$, then after any unilateral deviation of an agent from his true preference relation either the set of compromise points does not change or any point in the original set of compromise points is (weakly) better than any point in the set of compromise points after the deviation. Obviously the coordinatewise median correspondence is not *strategy-proof with respect to* $\overset{1}{\trianglerighteq}_{(\delta,b)}$. Hence, *strategy-proofness with respect to* $\overset{1}{\trianglerighteq}_{(\delta,b)}$ is not compatible with *unanimity*, the *tie-breaking* and the *non-decisive newcomer condition.* ◇

Example 7. Otten et al. (1995) consider the following extension of $\succsim_{(\delta,b)}$ to $2^E_{(\delta,b)}$. For all $(\delta,b) \in \mathscr{S}$ and all $X, Y \in 2^E$, $X \overset{2}{\trianglerighteq}_{(\delta,b)} Y$ if and only if

(i) for all $x \in X \backslash Y$ and all $y \in Y$, $x \succsim_{(\delta,b)} y$ and
(ii) for all $x \in X$ and all $y \in Y \backslash X$, $x \succsim_{(\delta,b)} y$.

Similarly as in the previous example, $\overset{2}{\trianglerighteq}_{(\delta,b)}$ leads to a stronger strategy-proofness condition which is incompatible with *unanimity*, the *tie-breaking* and the *non-decisive newcomer condition.* ◇

The next examples show that if the preference extensions allow for a high percentage of indifferent pairs, then *strategy-proofness* yields little discrimination.

Example 8. Consider the following extension of $\succsim_{(\delta,b)}$ to 2^E. For all $(\delta,b) \in \mathscr{S}$ and all $X, Y \in 2^E$, $X \overset{3}{\trianglerighteq}_{(\delta,b)} Y$ if and only if

(i) there are $x \in X$ and $y \in Y$ such that $x \succ_{(\delta,b)} y$ or
(ii) for all $x \in X$ and all $y \in Y$, $x \succsim_{(\delta,b)} y$.

Kelly (1977) uses this type of preference relations to derive an impossibility result. Note that *strategy-proofness with respect to* $\overset{3}{\trianglerighteq}_{(\delta,b)}$ is a "weak requirement". ◇

Example 9. Consider the following extension of $\succsim_{(\delta,b)}$ to 2^E. For all $(\delta,b) \in \mathscr{S}$ and all $X, Y \in 2^E$, $X \overset{4}{\trianglerighteq}_{(\delta,b)} Y$ if and only if

(i) $X \overset{3}{\trianglerighteq}_{(\delta,b)} Y$ or
(ii) either $|X| \geq 3$ or $|Y| \geq 3$.

Barberà (1977) derives an impossibility result based on *strategy-proofness with respect to* $\overset{4}{\trianglerighteq}_{(\delta,b)}$. ◇

Obviously $\succsim_{(\delta,b)}$ is "contained" in $\overset{3}{\trianglerighteq}_{(\delta,b)}$ and the latter is contained in $\overset{4}{\trianglerighteq}_{(\delta,b)}$. As ψ_{med} is *strategy-proof with respect to* $\succsim_{(\delta,b)}$ it follows that it is also *strategy-proof with respect to* $\overset{3}{\trianglerighteq}_{(\delta,b)}$ and $\overset{4}{\trianglerighteq}_{(\delta,b)}$. However, there are more choice correspondences that are *strategy-proof* with respect to the latter extensions. Note that with respect to $\overset{3}{\trianglerighteq}_{(\delta,b)}$ and $\overset{4}{\trianglerighteq}_{(\delta,b)}$ the empty set is indifferent to any other set. Next, we introduce two choice correspondences that satisfy *strategy-proofness with respect to* $\overset{3}{\trianglerighteq}_{(\delta,b)}$ and $\overset{4}{\trianglerighteq}_{(\delta,b)}$, *unanimity*, the *tie-breaking* and the *non-decisive newcomer condition*.

Example 10. Let E be the Euclidean plane of dimension 2, $E_+ \equiv \{(x,y) \in E \mid x \geq 0\}$, and $E_- \equiv \{(x,y) \in E \mid x \leq 0\}$. Define the choice correspondence $\psi_{1/2}$ as follows. For all $N \in \mathcal{N}$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathcal{S}^N$,

$$\psi^N_{1/2}(p) = \begin{cases} \psi_{\mathrm{med}}(p) & \text{if } \{b(i) \mid i \in N\} \subseteq E_+ \text{ or } \{b(i) \mid i \in N\} \subseteq E_-, \\ \varnothing & \text{otherwise.} \end{cases}$$

It is straightforward to prove that this correspondence satisfies *strategy-proofness with respect to* $\overset{3}{\trianglerighteq}_{(\delta,b)}$ and $\overset{4}{\trianglerighteq}_{(\delta,b)}$, *unanimity*, the *tie-breaking* and the *non-decisive newcomer condition*. Since $\psi_{1/2}(p)$ may be empty, it is not *strategy-proof with respect to* $\succsim_{(\delta,b)}$. The choice correspondence $\psi_{1/2}$ can be extended to higher dimensional E by taking any partition of E into two convex parts. $\diamondsuit$

Example 11. A choice correspondence that satisfies all conditions mentioned above, but that it is not a subsolution of ψ_{med} is defined as follows. Let $a, b \in E$. By square($\{a,b\}$) we denote the union of $\{a,b\}$ and the interior of box($\{a,b\}$). If $a = b$, then square$(a,b) = \{a\} = \{b\}$. For all $N \in \mathcal{N}$ and all $p = \langle \delta(i), b(i) \rangle_{i \in N} \in \mathcal{S}^N$,

$$\psi^N_{\mathrm{square}}(p) = \begin{cases} \{b\} & \text{if for all } i,k \in N, b(i) = b(k) = \{b\}, \\ \bigcap\{\mathrm{square}(\{b(i), b(k)\}) \mid i,k \in N, b(i) \neq b(k)\} & \text{otherwise.} \end{cases}$$

It is straightforward to prove that ψ^N_{square} satisfies *unanimity* and the two *newcomer conditions. Strategy-proofness with respect to* $\overset{4}{\trianglerighteq}_{(\delta,b)}$ follows from the following observations. The set of compromise points $\psi^N_{\mathrm{square}}(p)$ is convex. Hence, its cardinality is either zero, one, or infinity. In case of cardinality one, either all agents have an unanimous best point or there is an agent l with best point $b(l)$ on the unique compromise point. Only he is able to change this singleton set of compromise points by a unilateral deviation either to the empty set or to another singleton set. Obviously, he cannot gain by doing so. *Strategy-proofness with respect to* $\overset{3}{\trianglerighteq}_{(\delta,b)}$ is also straightforward, although cumbersome to prove. $\diamondsuit$

7 Concluding discussion

Border and Jordan (1983) consider the location problem as described in Sect. 2: a group of agents has to choose exactly one compromise point in a higher dimensional Euclidean space based on the agents' separable-quadratic preferences on this space. One of the results for this model is a characterization of median choice functions with additional fixed ballots, or generalized median choice functions, by *unanimity* and *strategy-proofness*. One of the byproducts of the characterization is that *unanimity* and *strategy-proofness* imply *peak-onliness*.

Border and Jordan's (1983) result for choice rules and our result for choice correspondences seem to be very similar. Indeed by taking appropriate additional fixed ballots, the corresponding generalized median choice function yields a corner point of the set of compromise points assigned by the coordinatewise median correspondence. So, for special fixed ballots, generalized median choice functions are strict subcorrespondences of the coordinatewise median correspondence. But at other sets of ballots, especially non-infinite ones, non-median compromise points may be determined by generalized median choice functions, *i.e.*, such points are not in the set of compromise points of the coordinatewise median correspondence. This shows that the results presented here and those in Borda and Jordan (1983) are no simple consequences of each other.

Due to the possibility of set-valued outcomes, *strategy-proofness* for choice correspondences is significantly weaker than for choice rules. The following will illustrate this. For choice rules, *unanimity* and *strategy-proofness* imply *peak-onliness*. In Sect. 6, Example 5 shows that a similar result does not hold for choice correspondences even if additionally the *tie-breaking newcomer condition* is imposed. Because generalized median choice functions do not satisfy the *non-decisive newcomer condition*[13] and because of the relatively weakness of *strategy-proofness* it seems rather difficult to utilize the results of Border and Jordan (1983) in our setting. We did not succeed in this. Though globally their proof structure resembles ours, the various (local) steps are proved quite differently.

We end the comparison of both models by stressing one similarity: the chosen subset of single-peaked preferences. In both papers the indifference sets are ellipsoids with main diagonals parallel to the axes of the Euclidean space. Allowing for preferences with elliptical indifference curves such that the main diagonals are not parallel to these axes, yields in Borda and Jordan (1983) setting dictatorship and in our model an impossibility; see Sect. 6. So, restricting the set of admissible preferences to the set of separable-quadratic preferences is essential in both models.

Finally, we briefly discuss a recent article by Ching and Zhou (1997). They consider a more general choice model with an arbitrary set of alternatives and a general domain of preferences. Similar to our approach, Ching and Zhou (1997) focus on choice correspondences rather than choice functions. The *strategy-proofness* condition they consider is based on the extension of preferences to the powerset of the set of compromise points we discuss in Example 7. For this more general model, Ching and Zhou (1997) prove two Gibbard-Satterthwaite results, one for general preferences and one for continuous preferences. As already mentioned in Sect. 6, the *strategy-proofness* condition

[13] Note that only specific generalized median choice functions are defined for the variable population model in a straightforward way; *e.g.*, coordinatewise staus quo choice functions.

at hand is rather strong. Therefore, it is not a surprise that we obtain a similar incompatibility result (see Example 7) in our model with single-peaked preferences.

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