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RESEARCH ARTICLE

New Hermite–Hadamard and Jensen Inequalities for Log-*h*-Convex Fuzzy-Interval-Valued Functions

Muhammad Bilal Khan[1] · Lazim Abdullah[2] · Muhammad Aslam Noor[1] · Khalida Inayat Noor[1]

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Abstract
In the preset study, we introduce the new class of convex fuzzy-interval-valued functions which is called log-h-convex fuzzy-interval-valued functions (log-h-convex FIVFs) by means of fuzzy order relation. We have also investigated some properties of log-h-convex FIVFs. Using this class, we present Jensen and Hermite–Hadamard inequalities (HH-inequalities). Moreover, some useful examples are presented to verify HH-inequalities for log-h-convex FIVFs. Several new and known special results are also discussed which can be viewed as an application of this concept.

Keywords Fuzzy-interval-valued functions · Log-h-convex · Hermite–Hadamard inequality · Hemite–Hadamard–Fejér inequality · Jensen's inequality

1 Introduction

The theory of convexity in pure and applied sciences has become a rich source of inspiration. In several branches of mathematical and engineering sciences, this theory had not only inspired new and profound results, but also offers a coherent and general basis for studying a wide range of problems. Many new notions of convexity have been developed and investigated for convex functions and convex sets. Various integral inequalities for convex functions and their variant forms are being constructed using unique and imaginative concepts and methodologies. Every function is convex if and only if it fulfills the HH-inequality, which is a type of integral inequality. Hermite presented this inequality, which was independently introduced by Hadamard, see [24, 25, 31]. It can be expressed in the following way:

Let $\Psi : K \to \mathbb{R}$ be a convex function on a convex set K and $u, v \in K$ with $u \leq v$. Then,

$$\Psi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \Psi(x)\mathrm{d}x \leq \frac{\Psi(u)+\Psi(v)}{2}. \tag{1}$$

Fejér [21] introduced HH-Fejér inequality which is major generalizations of HH-inequality. It can be expressed as follows:

Let $\Psi : K \to \mathbb{R}$ be a convex function on a convex set K and $u, v \in K$ with $u \leq v$ and $\nabla : [u, v] \to \mathbb{R}, \nabla(x) \geq 0$, symmetric with respect to $\frac{u+v}{2}$, then

$$\Psi\left(\frac{u+v}{2}\right) \leq \frac{1}{\int_u^v \nabla(x)\mathrm{d}x}\int_u^v \Psi(x)\nabla(x)\mathrm{d}x \leq \frac{\Psi(u)+\Psi(v)}{2}\int_u^v \nabla(x)\mathrm{d}x, \tag{2}$$

If $\nabla(x) = 1$, then (1) is obtained from (2). Similarly, several inequalities can be obtained from (2) by taking different values of symmetric function $\nabla(x)$.

✉ Lazim Abdullah
lazim_m@umt.edu.my

Muhammad Bilal Khan
bilal42742@gmail.com

Muhammad Aslam Noor
noormaslam@gmail.com

Khalida Inayat Noor
khalidanoor@hotmail.com

1 Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan

2 Management Science Research Group, Faculty of Ocean Engineering Technology and Informatics, University Malaysia Terengganu, Terengganu, Malaysia

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It is commonly known that log-convex functions play a significant role in convex theory since they allow us to derive more precise inequalities than convex functions.

Noor et al. [35] presented following HH-inequality for log-h-convex functions.

Let $\Psi : K \to \mathbb{R}$ be a convex function on a convex set K and $\mathcal{u}, \nu \in K$ with $\mathcal{u} \le \nu$. Then,

$$\Psi\left(\frac{\mathcal{u}+\nu}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} \le \exp\left[\frac{1}{\nu-\mathcal{u}}\int_{\mathcal{u}}^{\nu} \ln \Psi(\varkappa)\mathrm{d}\varkappa\right] \le [\Psi(\mathcal{u})\Psi(\nu)]^{\int_0^1 h(\xi)\mathrm{d}\xi}. \tag{3}$$

If Ψ is concave, then (3) is reversed.

Some writers recently studied the different classes of log-convex and generalized log-convex functions see [16–20, 36, 38, 43, 46] and the references therein.

One of these inequalities for convex functions is the Jensen inequality [1, 27], which may be written as follows.

Let $\omega_j \in [0, 1], u_j \in [\mathcal{u}, \nu], (j = 1, 2, 3, \dots k, k \ge 2)$ and Ψ be a convex function, then

$$\Psi\left(\sum_{j=1}^{k} \omega_j z_j\right) \le \left(\sum_{j=1}^{k} \omega_j \Psi\left(z_j\right)\right), \tag{4}$$

with $\sum_{j=1}^{k} \omega_j = 1$. If Ψ is concave, then (4) is reversed.

Moore [32] explored the fundamental principles of interval analysis, and Kulish and Miranker [29] looked into the fundamental properties and defined the partial order relation between intervals. Recently, Guo et al. [23] proposed the definition of log-*h*-convex interval-valued functions (in short, log-*h*-convex-IVF) and proved the following HH-inequality for log-*h*-convex IVFs:

Let $\Psi : [\mathcal{u}, \nu] \subset \mathbb{R} \to \mathcal{K}_C^+$ be a log-*h*-convex-IVF given by $\Psi(\varkappa) = \left[\Psi_*(\varkappa), \Psi^*(\varkappa)\right]$ for all $\varkappa \in [\mathcal{u}, \nu]$. If Ψ is Riemann integrable, then

$$\Psi\left(\frac{\mathcal{u}+\nu}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} \supseteq \exp\left[\frac{1}{\nu-\mathcal{u}}\int_{\mathcal{u}}^{\nu} \ln \Psi(\varkappa)\mathrm{d}\varkappa\right] \supseteq [\Psi(\mathcal{u})\Psi(\nu)]^{\int_0^1 h(\xi)\mathrm{d}\xi}. \tag{5}$$

We encourage readers to go more into the literature on generalized convex functions and HH-inequality, particularly [2, 4–6, 8, 9, 13, 20, 40, 49–55] and the references therein.

The theory of fuzzy sets and systems has progressed in a variety of ways from its introduction five decades ago, as seen in [48]. As a result, it is useful in the study of a variety of problems in pure mathematics and applied sciences, such as operation research, computer science, management sciences, artificial intelligence, control engineering, and decision sciences [26, 28, 30].

The concepts of convexity and generalized convexity are very crucial in fuzzy optimization. The definition of fuzzy mapping was firstly introduced by Chang and Zadeh [10]. Since then, fuzzy mapping has been widely researched by several scholars. In 1992, Nanda and Kar [33] suggested an idea of convex fuzzy mapping, showing that a fuzzy mapping is convex if and only if a convex set is its epigraph. By considering the definition of ordering suggested by Goetschael-Voxman [22], Yan-Xu [47] addressed the convexity and quasicovexity of fuzzy mappings. The class of fuzzy preinvex functions and fuzzy log-preinvex was presented by Noor [34], and some properties of fuzzy preinvex fuzzy functions were obtained. For fuzzy mapping of one variable, Syau [41] introduced the concepts of pseudoconvexity, invexity and pseudoinvexity by using the concept of differentiability and the results provided by Goetschel and Voxman [22]. A new notion of noncovex fuzzy mapping, which is known as B-vex fuzzy mapping, was proposed and explored by Syau [42]. The application to convex fuzzy programing was considered by Wang and Wu [44] by defining the fuzzy subdifferential of a fuzzy mapping. Wu-Xu [45] presented the notions of fuzzy pseudoconvex, fuzzy pseudoinvex, fuzzy invex and fuzzy preinvex mapping from "*n*" dimensional Euclidean space to the set of fuzzy numbers depending upon the Wang-Wu [44] definitions of differentiability of fuzzy mapping. Moreover, several properties were investigated. Refer to [3, 7, 22, 26, 28, 30] for more studies on convexity and nonconvexity for fuzzy mappings.

There are some integrals that deal with FIVFs, with FIVFs as the integrands. For example, Oseuna-Gomez et al. [37] and Costa et al. [12] built Jensen's integral inequality for FIVFs. Costa and Floures used the same approach to show Minkowski and Beckenbach's inequalities, where the integrands are FIVFs. Inspired by [11, 12, 23, 37], we generalize integral inequality (1), (2), and (3) by constructing fuzzy-interval integral inequality for convex fuzzy-interval-valued functions (convex FIVF), where the integrands are convex FIVFs, utilizing this notion on fuzzy-interval space.

Motivated and inspired by the above literature, we consider new class of convex FIVFs, which is called log-*h*-convex FIVFs. By using this class, we discuss integral inequalities (2) and (3) by constructing fuzzy-interval integral inequalities, which are known as fuzzy-interval HH-integral inequality and HH-Fejér integral inequality. For log-*h*-convex FIVFs, some Jensen inequalities are also introduced.

2 Preliminaries

In this section, we recall some basic preliminary notions, definitions and results. With the help of these results, some new basic definitions and results are also discussed.

We begin by recalling the basic notations and definitions. We define interval as,

$$\left[\omega_*, \omega^*\right] = \left\{\varkappa \in \mathbb{R} : \omega_* \leq \varkappa \leq \omega^* \text{ and } \omega_*, \omega^* \in \mathbb{R}\right\},$$

where $\omega_* \leq \omega^*$.

We write len $\left[\omega_*, \omega^*\right] = \omega^* - \omega_*$. If len $\left[\omega_*, \omega^*\right] = 0$, then $\left[\omega_*, \omega^*\right]$ is called degenerate. In this article, all intervals will be non-degenerate intervals. The collection of all closed and bounded intervals of $\mathbb{R}$ is denoted and defined as $K_C = \left\{[\omega_*, \omega^*] : \omega_*, \omega^* \in \mathbb{R} \text{ and } \omega_* \leq \omega^*\right\}$. If $\omega_* \geq 0$, then $\left[\omega_*, \omega^*\right]$ is called positive interval. The set of all positive interval is denoted by K_C^+ and defined as $K_C^+ = \left\{\left[\omega_*, \omega^*\right] : \left[\omega_*, \omega^*\right] \in K_C \text{ and } \omega_* \geq 0\right\}$.

We'll now look at some of the properties of intervals using arithmetic operations. Let $\left[\varrho_*, \varrho^*\right], \left[\jmath_*, \jmath^*\right] \in \mathcal{K}_C$ and $\rho \in \mathbb{R}$, then we have

$$\left[\varrho_*, \varrho^*\right] + \left[\jmath_*, \jmath^*\right] = \left[\varrho_* + \jmath_*, \varrho^* + \jmath^*\right],$$

$$\left[\varrho_*, \varrho^*\right] \times \left[\jmath_*, \jmath^*\right] = \begin{bmatrix} \min\left\{\varrho_*\jmath_*, \varrho^*\jmath_*, \varrho_*\jmath^*, \varrho^*\jmath^*\right\}, \\ \max\left\{\varrho_*\jmath_*, \varrho^*\jmath_*, \varrho_*\jmath^*, \varrho^*\jmath^*\right\} \end{bmatrix}$$

$$\rho.\left[\varrho_*, \varrho^*\right] = \begin{cases} \left[\rho\varrho_*, \rho\varrho^*\right] & \text{if } \rho \geq 0, \\ \left[\rho\varrho^*, \rho\varrho_*\right] & \text{if } \rho < 0. \end{cases}$$

For $\left[\varrho_*, \varrho^*\right], \left[\jmath_*, \jmath^*\right] \in K_C$, the inclusion "$\subseteq$" is defined by $\left[\varrho_*, \varrho^*\right] \subseteq \left[\jmath_*, \jmath^*\right]$ if and only if $\jmath_* \leq \varrho_*, \varrho^* \leq \jmath^*$.

Remark 2.1 The relation "$\leq_I$" defined on $\mathcal{K}_C$ by $\left[\varrho_*, \varrho^*\right] \leq_I \left[\jmath_*, \jmath^*\right]$ if and only if $\varrho_* \leq \jmath_*, \varrho^* \leq \jmath^*$, for all $\left[\varrho_*, \varrho^*\right], \left[\jmath_*, \jmath^*\right] \in \mathcal{K}_C$, it is an order relation, see [29]. For given $\left[\varrho_*, \varrho^*\right], \left[\jmath_*, \jmath^*\right] \in \mathcal{K}_C$, we say that $\left[\varrho_*, \varrho^*\right] \leq_I \left[\jmath_*, \jmath^*\right]$ if and only if $\varrho_* \leq \jmath_*, \varrho^* \leq \jmath^*$ or $\varrho_* \leq \jmath_*, \varrho^* < \jmath^*$.

Moore [32] initially proposed the concept of Riemann integral for IVF, which is defined as follows:

Theorem 2.2 [32] *If* $\Psi : [u, v] \subset \mathbb{R} \to \mathcal{K}_C$ *is an IVF on such that* $\Psi(\varkappa) = \left[\Psi_*, \Psi^*\right]$. *Then,* Ψ *is Riemann integrable over* $[u, v]$ *if and only if* Ψ_* *and* Ψ^* *both are Riemann integrable over* $[u, v]$ *such that*

$$(\mathrm{IR})\int_u^v \Psi(\varkappa)\mathrm{d}\varkappa = \left[(R)\int_u^v \Psi_*(\varkappa)\mathrm{d}\varkappa, (R)\int_u^v \Psi^*(\varkappa)\mathrm{d}\varkappa\right]. \tag{6}$$

Let $\mathbb{R}$ be the set of real numbers. A mapping $\zeta : \mathbb{R} \to [0, 1]$ called the membership function distinguishes a fuzzy subset set $\mathscr{A}$ of $\mathbb{R}$. This representation is found to be acceptable in this study. $\mathbb{F}(\mathbb{R})$ also stand for the collection of all fuzzy subsets of $\mathbb{R}$.

A real fuzzy interval ζ is a fuzzy set in $\mathbb{R}$ with the following properties:

(1) ζ is normal, i.e., there exists $\varkappa \in \mathbb{R}$ such that $\zeta(\varkappa) = 1$;
(2) ζ is upper semi continuous, i.e., for given $\varkappa \in \mathbb{R}$, for every $\varkappa \in \mathbb{R}$ there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\zeta(\varkappa) - \zeta(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\varkappa - y| < \delta$.
(3) ζ is fuzzy convex, i.e., $\zeta((1-\xi)\varkappa + \xi y) \geq min(\zeta(\varkappa), \zeta(y))$, $\forall$ $\varkappa, y \in \mathbb{R}$ and $\xi \in [0, 1]$;
(4) ζ is compactly supported, i.e., $cl\{\varkappa \in \mathbb{R} | \zeta(\varkappa) \rangle 0\}$ is compact.

The collection of all real fuzzy intervals is denoted by $\mathbb{F}_0$.

Let $\zeta \in \mathbb{F}_0$ be real fuzzy interval, if and only if, β-levels $[\zeta]^\beta$ is a nonempty compact convex set of $\mathbb{R}$. This is represented by

$$[\zeta]^\beta = \{\varkappa \in \mathbb{R} | \zeta(\varkappa) \geq \beta\},$$

from these definitions, we have

$$[\zeta]^\beta = \left[\zeta_*(\beta), \zeta^*(\beta)\right],$$

where

$$\zeta_*(\beta) = \inf\{\varkappa \in \mathbb{R} | \zeta(\varkappa) \geq \beta\},$$
$$\zeta^*(\beta) = \sup\{\varkappa \in \mathbb{R} | \zeta(\varkappa) \geq \beta\}.$$

Thus, a real fuzzy interval ζ can be identified by a parametrized triples

$$\left\{\left(\zeta_*(\beta), \zeta^*(\beta), \beta\right) : \beta \in [0, 1]\right\}.$$

These two end point functions $\zeta_*(\beta)$ and $\zeta^*(\beta)$ are used to characterize a real fuzzy interval as a result.

Proposition 2.3 [11] *Let* $\zeta, \Theta \in \mathbb{F}_0$. *Then, fuzzy order relation* "$\preccurlyeq$" *given on* $\mathbb{F}_0$ *by* $\zeta \preccurlyeq \Theta$ *if and only if* $[\zeta]^\beta \leq_I [\Theta]^\beta$ *for all* $\beta \in (0, 1]$, *it is partial order relation.*

We'll now look at some of the properties of fuzzy intervals using arithmetic operations. Let $\zeta, \Theta \in \mathbb{F}_0$ and $\rho \in \mathbb{R}$, then we have

$$\left[\zeta \tilde{+} \Theta\right]^\beta = [\zeta]^\beta + [\Theta]^\beta, \tag{7}$$

$$[\zeta \tilde{\times} \Theta]^\beta = [\zeta]^\beta \times [\Theta]^\beta, \tag{8}$$

$$[\rho.\zeta]^\beta = \rho.[\zeta]^\beta \tag{9}$$

For $\psi \in \mathbb{F}_0$ such that $\zeta = \Theta \tilde{+} \psi$, we have the existence of the Hukuhara difference of ζ and Θ, which we call the

H-difference of ζ and Θ and denoted by $\zeta \tilde{-} \Theta$. If H-difference exists, then

$$(\psi)^*(\beta) = (\zeta \tilde{-} \Theta)^*(\beta) = \zeta^*(\beta) - \Theta^*(\beta),$$
$$(\psi)_*(\beta) = (\zeta \tilde{-} \Theta)_*(\beta) = \zeta_*(\beta) - \Theta_*(\beta). \tag{10}$$

Theorem 2.4 [14, 39] *The space $\mathbb{F}_0$ dealing with a supremum metric, i.e., for $\psi, \Theta \in \mathbb{F}_0$.*

$$D(\psi, \Theta) = \sup_{0 \le \beta \le 1} H\big([\zeta]^\beta, [\Theta]^\beta\big),$$

it is a complete metric space, where H denotes the well-known Hausdorff metric on space of intervals.

Definition 2.5 [11] A fuzzy-interval-valued map $\Psi : K \subset \mathbb{R} \to \mathbb{F}_0$ is called FIVF. For each $\beta \in (0, 1]$, whose β-levels define the family of IVFs $\Psi_\beta : K \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\Psi_\beta(\varkappa) = \big[\Psi_*(\varkappa, \beta), \Psi^*(\varkappa, \beta)\big]$ for all $\varkappa \in K$. Here, for each $\beta \in (0, 1]$, the end point real functions $\Psi_*(., \beta), \Psi^*(., \beta) : K \to \mathbb{R}$ are called lower and upper functions of Ψ.

The following conclusions can be drawn from the preceding literature review [11, 14, 28, 32]:

Definition 2.6 Let $\Psi : [\varkappa, \nu] \subset \mathbb{R} \to \mathbb{F}_0$ be a FIVF. Then, fuzzy integral of Ψ over $[\varkappa, \nu]$ denoted by $(FR)\int_u^\nu \Psi(\varkappa)d\varkappa$, and it is given level-wise by

$$\left[(FR)\int_u^\nu \Psi(\varkappa)d\varkappa\right]^\beta = (IR)\int_u^\nu \Psi_\beta(\varkappa)d\varkappa = \left\{\int_u^\nu \Psi(\varkappa, \beta)d\varkappa : \Psi(\varkappa, \beta) \in \mathcal{R}_{([u,\nu],\beta)}\right\}, \tag{11}$$

for all $\beta \in (0, 1]$, where $\mathcal{R}_{([u,\nu],\beta)}$ denotes the collection of Riemannian integrable functions of IVFs. Ψ is FR-integrable over $[u, \nu]$ if $(FR)\int_u^\nu \Psi(\varkappa)d\varkappa \in \mathbb{F}_0$. Note that, if both end point functions are Lebesgue-integrable, then Ψ is fuzzy Aumann-integrable function over $[u, \nu]$, see [28, 32].

Theorem 2.7 *Let $\Psi : [u, \nu] \subset \mathbb{R} \to \mathbb{F}_0$ be a FIVF, whose β-levels define the family of IVFs $\Psi_\beta : [u, \nu] \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\Psi_\beta(\varkappa) = \big[\Psi_*(\varkappa, \beta), \Psi^*(\varkappa, \beta)\big]$ for all $\varkappa \in [u, \nu]$ and for all $\beta \in (0, 1]$. Then, Ψ is FR-integrable over $[u, \nu]$ if and only if $\Psi_*(\varkappa, \beta)$ and $\Psi^*(\varkappa, \beta)$ both are R-integrable over $[u, \nu]$. Moreover, if Ψ is FR-integrable over $[u, \nu]$, then*

$$\left[(FR)\int_u^\nu \Psi(\varkappa)d\varkappa\right]^\beta = \left[(R)\int_u^\nu \Psi_*(\varkappa, \beta)d\varkappa, (R)\int_u^\nu \Psi^*(\varkappa, \beta)d\varkappa\right] = (IR)\int_u^\nu \Psi_\beta(\varkappa)d\varkappa, \tag{12}$$

for all $\beta \in (0, 1]$. For all $\beta \in (0, 1]$, $\mathcal{FR}_{([u,\nu],\beta)}$ denotes the collection of all FR-integrable FIVFs over $[u, \nu]$.

Theorem 2.8 *Let $\Psi : [u, \nu] \subset \mathbb{R} \to \mathbb{F}_0$ be a FIVF, whose β-levels define the family of IVFs $\Psi_\beta : [u, \nu] \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\Psi_\beta(\varkappa) = \big[\Psi_*(\varkappa, \beta), \Psi^*(\varkappa, \beta)\big]$ for all $\varkappa \in [u, \nu]$ and for all $\beta \in (0, 1]$. Then, Ψ is (FR)-integrable over $[u, \nu]$ if and only if $\Psi_*(\varkappa, \beta)$ and $\Psi^*(\varkappa, \beta)$ both are R-integrable over $[u, \nu]$. Moreover, if Ψ is (FR)-integrable over $[u, \nu]$, then*

$$\left[(FR)\int_u^\nu \Psi(\varkappa)d\varkappa\right]^\beta = \left[(R)\int_u^\nu \Psi_*(\varkappa, \beta)d\varkappa, (R)\int_u^\nu \Psi^*(\varkappa, \beta)d\varkappa\right] = (IR)\int_u^\nu \Psi_\beta(\varkappa)d\varkappa, \tag{13}$$

for all $\beta \in (0, 1]$.

Definition 2.9 [38] A function $\Psi : K \to \mathbb{R}$ is said to be log-convex function if

$$\Psi(\xi\varkappa + (1 - \xi)y) \le \Psi(\varkappa)^\xi \times \Psi(y)^{1-\xi}, \tag{14}$$

$\forall \varkappa, y \in K, \xi \in [0, 1]$, where $\Psi(\varkappa) \ge 0$. If (14) is reversed, then Ψ is called log-concave.

Definition 2.10 [35] A function $\Psi : K \to \mathbb{R}$ is said to be log-h-convex function if

$$\Psi(\xi\varkappa + (1 - \xi)y) \le \Psi(\varkappa)^{h(\xi)} \times \Psi(y)^{h(1-\xi)}, \tag{15}$$

$\forall \varkappa, y \in K, \xi \in [0, 1]$, where $\Psi(\varkappa) \ge 0$. If (15) is reversed, then Ψ is called log-h-concave.

A function $h : L \to \mathbb{R}$ is called super multiplicative if for all z, $y \in L$, we have

$$h(zy) \ge h(\varkappa)h(y). \tag{16}$$

If (16) is reversed, then h is known as sub multiplicative. If the equality hold in (16), then h is called multiplicative.

Definition 2.11 Let K be a convex set and $h:[0,1]\subseteq K\to\mathbb{R}^+$ such that $h\not\equiv 0$. Then, FIVF $\Psi: K\to\mathbb{F}_0$ is said to be:

- log-h-convex on K if

$$\Psi(\xi\varkappa+(1-\xi)y)\preccurlyeq\Psi(\varkappa)^{h(\xi)}\tilde{\times}\Psi(y)^{h(1-\xi)}, \tag{17}$$

for all $\varkappa, y\in K,\ \xi\in[0,1]$, where $\Psi(\varkappa)\succcurlyeq\tilde{0}$.

- log-h-concave on K if inequality (17) is reversed.
- Affine log-h-convex on K if

$$\Psi(\xi\varkappa+(1-\xi)y)=\Psi(\varkappa)^{h(\xi)}\tilde{\times}\Psi(y)^{h(1-\xi)}, \tag{18}$$

for all $\varkappa, y\in K, \xi\in[0,1]$, where $\Psi(\varkappa)\succcurlyeq\tilde{0}$.

Remark 2.12 If $h(\xi)=\xi^s$, then (17) reduces to:

$$\Psi(\xi\varkappa+(1-\xi)y)\preccurlyeq\Psi(\varkappa)^{\xi^s}\tilde{\times}\Psi(y)^{(1-\xi)^s}. \tag{19}$$

If $h(\xi)=\xi$, then (17) reduces to:

$$\Psi(\xi\varkappa+(1-\xi)y)\preccurlyeq\Psi(\varkappa)^{\xi}\tilde{\times}\Psi(y)^{1-\xi}. \tag{20}$$

If $h(\xi)\equiv 1$, then (17) reduces to:

$$\Psi(\xi\varkappa+(1-\xi)y)\preccurlyeq\Psi(\varkappa)\tilde{\times}\Psi(y). \tag{21}$$

Note that, Remarks (i) and (iii) are also new ones.

And $\Psi_\beta: K\subset\mathbb{R}\to\mathcal{K}_C^+\subset\mathcal{K}_C$ represent the family of IVFs through β-levels are define by

Theorem 2.13 *Let K be convex set and non-negative real valued function $h:[0,1]\subseteq K\to\mathbb{R}$ such that $h\not\equiv 0$, Let $\Psi: K\to\mathbb{F}_0$ be a FIVF with $\Psi(\varkappa)\succcurlyeq\tilde{0}$, and $\Psi_\beta: K\subset\mathbb{R}\to\mathcal{K}_C^+\subset\mathcal{K}_C$ represent the family of IVFs through β-levels are given by*

$$\Psi_\beta(\varkappa)=\left[\Psi_*(\varkappa,\beta),\Psi^*(\varkappa,\beta)\right], \tag{22}$$

for all $\varkappa\in K$ and for all $\beta\in(0,1]$. Then, Ψ is log-h-convex on K, if and only if, for all $\beta\in(0,1]$, $\Psi_*(\varkappa,\beta)$ and $\Psi^*(\varkappa,\beta)$ are log-h-convex.

Proof Let Ψ is log-h-convex FIVF on K. Then, for all $\varkappa, y\in K$ and $\xi\in(0,1]$, we have

$$\Psi(\xi\varkappa+(1-\xi)y)\preccurlyeq\Psi(\varkappa)^{h(\xi)}\tilde{\times}\Psi(y)^{h(1-\xi)}.$$

Therefore**,** from (22) and Proposition 2.3, we have

$$\begin{aligned}&\left[\Psi_*(\xi\varkappa+(1-\xi)y,\beta),\Psi^*(\xi\varkappa+(1-\xi)y,\beta)\right]\\ &\quad\le_I\left[\Psi_*(\varkappa,\beta)^{h(\xi)},\Psi^*(\varkappa,\beta)^{h(\xi)}\right]\\ &\quad\times\left[\Psi_*(y,\beta)^{h(1-\xi)},\Psi^*(y,\beta)^{h(1-\xi)}\right].\end{aligned} \tag{23}$$

It follows that

$$\Psi_*(\xi\varkappa+(1-\xi)y,\beta)\le\Psi_*(\varkappa,\beta)^{h(\xi)}\Psi_*(y,\beta)^{h(1-\xi)},$$

and

$$\Psi^*(\xi\varkappa+(1-\xi)y,\beta)\le\Psi^*(\varkappa,\beta)^{h(\xi)}\Psi^*(y,\beta)^{h(1-\xi)},$$

hence, the result has been proved.

Conversely, suppose that $\Psi_*(\varkappa,\beta)$ and $\Psi^*(\varkappa,\beta)$ both are log-h-convex functions. Then, from definition and above (23), it follows that $\Psi(\varkappa)$ is log-h-convex FIVF.

Example 2.14 We consider $h(\xi)\equiv\hbar(\hbar\ge 1)$, for $\xi\in[0,1]$ and the FIVF $\Psi:[1,4]\to\mathbb{F}_0$ is given by,

$$\Psi(\varkappa)(\partial)=\begin{cases}\frac{\partial}{e^{z^2}} & \partial\in\left[0,e^{z^2}\right]\\ \frac{2e^{z^2}-\partial}{e^{z^2}} & \partial\in\left(e^{z^2},2e^{z^2}\right]\\ 0 & \text{otherwise,}\end{cases}$$

Then, for each $\beta\in(0,1]$, we have $\Psi_\beta(\varkappa)=\left[\beta e^{z^2},(2-\beta)e^{z^2}\right]$. Since $\Psi_*(\varkappa,\beta)$ and $\Psi^*(\varkappa,\beta)$ are log-h-convex functions for each $\beta\in(0,1]$ then, by Theorem 2.13, $\Psi(\varkappa)$ is log-h-convex FIVF.

Theorem 2.15 *Let K be convex set, non-negative real valued function $h:[0,1]\subseteq K\to\mathbb{R}$ such that $h\not\equiv 0$. Let $\Psi: K\to\mathbb{F}_0$ be a FIVF, and $\Psi_\beta: K\subset\mathbb{R}\to\mathcal{K}_C^+\subset\mathcal{K}_C$ represent the family of IVFs through β-levels are given by*

$\Psi_\beta(\varkappa)=\left[\Psi_*(\varkappa,\beta),\Psi^*(\varkappa,\beta)\right]$,

for all $\varkappa\in K$ and for all $\beta\in(0,1]$. Then, Ψ is log-h-concave on K, if and only if, for all $\beta\in[0,1]$, $\Psi_*(\varkappa,\beta)$ and $\Psi^*(\varkappa,\beta)$ are log-h-concave.

Proof The proof is similar to that of Theorem 6.

Example 2.16 We consider $h(\xi)=\xi$, for $\xi\in[0,1]$ and the FIVFs $\Psi:[u,v]=[0,1]\to\mathbb{F}_0$ is given by,

$$\Psi(\varkappa)(\partial)=\begin{cases}\frac{\partial}{\varkappa}, & \partial\in[0,\varkappa],\\ \frac{2\varkappa-\partial}{\varkappa}, & \partial\in(\varkappa,2\varkappa],\\ 0, & \text{otherwise,}\end{cases}$$

Then, for each $\beta\in[0,1]$, we have

$$\Psi_\beta(\varkappa)=[\beta\varkappa,(2-\beta)\varkappa].$$

Since $\Psi_*(\varkappa,\beta)=\beta z$, and $\Psi^*(\varkappa,\beta)=(2-\beta)z$ log-h-convex functions, for each $\beta\in[0,1]$ then, by Theorem 2.15, $\Psi(\varkappa)$ is log-h-concave FIVF.

3 Main Results

First of all, we prove that the following Hermite–Hadamard-type inequality for log-h-convex FIVF.

Theorem 3.1 *Let $\Psi:[u,v]\to\mathbb{F}_0$ be a log-h-convex FIVF with non-negative real valued function $h:[0,1]\to\mathbb{R}$ and $h\left(\frac{1}{2}\right)\not\equiv 0$, and for all $\beta\in[0,1]$, $\Psi_\beta : K\subset\mathbb{R}\to\mathscr{K}_C^+\subset\mathscr{K}_C$ represent the family of IVFs through β-levels. If $\Psi\in\mathscr{FR}_{([u,v],\beta)}$, then*

$$\Psi\left(\frac{u+v}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}}\preccurlyeq\exp\left[\frac{1}{v-u}(FR)\int_u^v\ln\Psi(\varkappa)d\varkappa\right]\preccurlyeq[\Psi(u)\tilde{\times}\Psi(v)]^{\int_0^1 h(\xi)d\xi}. \tag{24}$$

If Ψ is log-h-concave FIVF then

$$\Psi\left(\frac{u+v}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}}\succcurlyeq\exp\left[\frac{1}{v-u}(FR)\int_u^v\ln\Psi(\varkappa)d\varkappa\right]\succcurlyeq[\Psi(u)\tilde{\times}\Psi(v)]^{\int_0^1 h(\xi)d\xi}.$$

Proof Let $\Psi:[u,v]\to\mathbb{F}_0$, log-$h$-convex FIVF. Then, by Theorem 2.13 and by hypothesis, we have

$$\Psi\left(\frac{u+v}{2}\right)\preccurlyeq[\Psi(\xi u+(1-\xi)v)]^{h\left(\frac{1}{2}\right)}\tilde{\times}[\Psi((1-\xi)u+\xi v)]^{h\left(\frac{1}{2}\right)}.$$

Therefore, for every $\beta\in[0,1]$, we have

$$\begin{aligned}\Psi_*\left(\frac{u+v}{2},\beta\right)&\le\left[\Psi_*(\xi u+(1-\xi)v,\beta)\right]^{h\left(\frac{1}{2}\right)}\\&\quad\times\left[\Psi_*((1-\xi)u+\xi v,\beta)\right]^{h\left(\frac{1}{2}\right)}\\\Psi^*\left(\frac{u+v}{2},\beta\right)&\le\left[\Psi^*(\xi u+(1-\xi)v,\beta)\right]^{h\left(\frac{1}{2}\right)}\\&\quad\times\left[\Psi^*((1-\xi)u+\xi v,\beta)\right]^{h\left(\frac{1}{2}\right)}.\end{aligned} \tag{25}$$

From (25), we have

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}\ln\Psi_*\left(\frac{u+v}{2},\beta\right)&\le\ln\Psi_*(\xi u+(1-\xi)v,\beta)\\&\quad+\ln\Psi_*((1-\xi)u+\xi v,\beta),\\\frac{1}{h\left(\frac{1}{2}\right)}\ln\Psi^*\left(\frac{u+v}{2},\beta\right)&\le\ln\Psi^*(\xi u+(1-\xi)v,\beta)\\&\quad+\ln\Psi^*((1-\xi)u+\xi v,\beta),\end{aligned}$$

Then,

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}\int_0^1\ln\Psi_*\left(\frac{u+v}{2},\beta\right)d\xi&\le\int_0^1\ln\Psi_*(\xi u+(1-\xi)v,\beta)d\xi\\&\quad+\int_0^1\ln\Psi_*((1-\xi)u+\xi v,\beta)d\xi\\\frac{1}{h\left(\frac{1}{2}\right)}\int_0^1\ln\Psi^*\left(\frac{u+v}{2},\beta\right)d\xi&\le\int_0^1\ln\Psi^*(\xi u+(1-\xi)v,\beta)d\xi\\&\quad+\int_0^1\ln\Psi^*((1-\xi)u+\xi v,\beta)d\xi.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\ln\Psi_*\left(\frac{u+v}{2},\beta\right)&\le\frac{1}{v-u}\int_u^v\ln\Psi_*(\varkappa,\beta)d\varkappa,\\\frac{1}{2h\left(\frac{1}{2}\right)}\ln\Psi^*\left(\frac{u+v}{2},\beta\right)&\le\frac{1}{v-u}\int_u^v\ln\Psi^*(\varkappa,\beta)d\varkappa,\end{aligned}$$

which implies that

$$\begin{aligned}\Psi_*\left(\frac{u+v}{2},\beta\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}}&\le\exp\left(\frac{1}{v-u}\int_u^v\ln\Psi_*(\varkappa,\beta)d\varkappa\right),\\\Psi^*\left(\frac{u+v}{2},\beta\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}}&\le\exp\left(\frac{1}{v-u}\int_u^v\ln\Psi^*(\varkappa,\beta)d\varkappa\right),\end{aligned}$$

That is

$$\left[\Psi_*\left(\frac{u+v}{2},\beta\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}}, \Psi^*\left(\frac{u+v}{2},\beta\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}}\right] \leq_I \left[\begin{array}{c}\exp\left(\frac{1}{v-u}\int_u^v \ln \Psi_*(x,\beta)\mathrm{d}x\right), \\ \exp\left(\frac{1}{v-u}\int_u^v \ln \Psi^*(x,\beta)\mathrm{d}x\right)\end{array}\right].$$

Thus,

$$\Psi\left(\frac{u+v}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} \preccurlyeq \exp\left[\frac{1}{v-u}(FR)\int_u^v \ln \Psi(x)\mathrm{d}x\right]. \tag{26}$$

In a similar way as above, we have

$$\exp\left[\frac{1}{v-u}(FR)\int_u^v \ln \Psi(x)\mathrm{d}x\right] \preccurlyeq [\Psi(u)\tilde{\times}\Psi(v)]^{\int_0^1 h(\xi)\mathrm{d}\xi}. \tag{27}$$

Combining (26) and (27), we have

$$\Psi\left(\frac{u+v}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} \preccurlyeq \exp\left[\frac{1}{v-u}(FR)\int_u^v \ln \Psi(x)\mathrm{d}x\right] \preccurlyeq [\Psi(u)\tilde{\times}\Psi(v)]^{\int_0^1 h(\xi)\mathrm{d}\xi}.$$

the required result.

Remark 3.2 If $h(\xi) = \xi^s$, then (24) reduces to:

$$\Psi\left(\frac{u+v}{2}\right)^{2^{s-1}} \preccurlyeq \exp\left[\frac{1}{v-u}(FR)\int_u^v \ln \Psi(x)\mathrm{d}x\right] \preccurlyeq [\Psi(u)\tilde{\times}\Psi(v)]^{\frac{1}{s+1}}.$$

If $h(\xi) = \xi$, then (24) reduces to:

$$\Psi\left(\frac{u+v}{2}\right) \preccurlyeq \exp\left[\frac{1}{v-u}(FR)\int_u^v \ln \Psi(x)\mathrm{d}x\right] \preccurlyeq \sqrt{\Psi(u)\tilde{\times}\Psi(v)}.$$

If $h(\xi) \equiv 1$, then (24) reduces to:

$$\Psi\left(\frac{u+v}{2}\right)^{\frac{1}{2}} \preccurlyeq \exp\left[\frac{1}{v-u}(FR)\int_u^v \ln \Psi(x)\mathrm{d}x\right] \preccurlyeq \Psi(u)\tilde{\times}\Psi(v).$$

If $\Psi_*(u,\beta) = \Psi^*(v,\beta)$ with $\beta = 1$, then (24) reduces to, see [35]:

$$\Psi\left(\frac{u+v}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} \leq \exp\left[\frac{1}{v-u}(R)\int_u^v \ln \Psi(x)\mathrm{d}x\right] \leq [\Psi(u)\tilde{\times}\Psi(v)]^{\int_0^1 h(\xi)\mathrm{d}\xi}.$$

If $\Psi_*(u,\beta) = \Psi^*(v,\beta)$ with $\beta = 1$ and $h(\xi) = \xi^s$, then (24) reduces to, see [35]:

$$\Psi\left(\frac{u+v}{2}\right)^{2^{s-1}} \leq \exp\left[\frac{1}{v-u}(R)\int_u^v \ln \Psi(x)\mathrm{d}x\right] \leq [\Psi(u)\tilde{\times}\Psi(v)]^{\frac{1}{s+1}}.$$

If $\Psi_*(u,\beta) = \Psi^*(v,\beta)$ with $\beta = 1$ and $h(\xi) = \xi$, then (24) reduces to, see [15]:

$$\Psi\left(\frac{u+v}{2}\right) \leq \exp\left[\frac{1}{v-u}(R)\int_u^v \ln \Psi(x)\mathrm{d}x\right] \leq \sqrt{\Psi(u)\tilde{\times}\Psi(v)}.$$

If $\Psi_*(u,\beta) = \Psi^*(v,\beta)$ with $\beta = 1$ and $h(\xi) \equiv 1$ then (24) reduces to, see [35]:

$$\Psi\left(\frac{u+v}{2}\right)^{\frac{1}{2}} \leq \exp\left[\frac{1}{v-u}(R)\int_u^v \ln \Psi(x)\mathrm{d}x\right] \leq \Psi(u)\tilde{\times}\Psi(v).$$

Note that, Remarks (i)–(iii) are also new ones.

Example 3.3 We consider $h(\xi) = \xi$, for $\xi \in [0,1]$, and the FIVF $\Psi : [u,v] = [1,4] \to \mathbb{F}_0$ is given by, $\Psi_\beta(x) = \left[\beta e^{x^2}, (2-\beta)e^{x^2}\right]$, as in Example 2.14, then $\Psi(x)$ is log-h-convex FIVF. Since, $\Psi_*(x,\beta) = \beta e^{x^2}$ and $\Psi^*(x,\beta) = (2-\beta)e^{x^2}$ then, we have

$$\Psi_*\left(\frac{u+v}{2},\beta\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} = \left[\beta e^{\left(\frac{5}{2}\right)^2}\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}} = \beta e^{\frac{25}{4}},$$

$$\exp\left(\frac{1}{v-u}\int_u^v \ln \Psi_*(x,\beta)\mathrm{d}x\right) = \exp\left(\frac{1}{3}\int_1^4 \ln\left(\beta e^{x^2}\right)\mathrm{d}x\right) = e^{\ln(\beta)+7},$$

$$[\Psi_*(u,\beta)\times\Psi_*(v,\beta)]^{\int_0^1 h(\xi)\mathrm{d}\xi} = \left[(\beta e)\left(4\beta e^{16}\right)\right]^{\frac{1}{2}} = 2\beta e^{\frac{17}{2}},$$

for all $\beta \in [0,1]$. That means

$$\beta e^{\frac{25}{4}} \leq e^{\ln(\beta)+7} \leq 2\beta e^{\frac{17}{2}}.$$

Now, we find the following inequality

$$\Psi^*\left(\frac{u+v}{2}, \beta\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} \leq \exp\left[\frac{1}{v-u}\int_u^v \ln \Psi^*(x, \beta)\mathrm{d}x\right] \leq \left[\Psi^*(u, \beta)+\Psi^*(v, \beta)\right]^{\int_0^1 h(\xi)\mathrm{d}\xi}.$$

for all $\beta \in [0, 1]$, such that

$$\Psi^*\left(\frac{u+v}{2}, \beta\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} = \left[(2-\beta)e^{\left(\frac{5}{2}\right)^2}\right]^{\frac{1}{2h\left(\frac{1}{2}\right)}} = (2-\beta)e^{\frac{25}{4}},$$

$$\exp\left(\frac{1}{v-u}\int_u^v \ln \Psi^*(x, \beta)\mathrm{d}x\right) = \exp\left(\frac{1}{3}\int_1^4 \ln\left((2-\beta)e^{z^2}\right)\mathrm{d}x\right) = e^{\ln(2-\beta)+7},$$

$$\left[\Psi^*(u, \beta)\times\Psi^*(v, \beta)\right]^{\int_0^1 h(\xi)\mathrm{d}\xi} = \left[(2-\beta)e.4(2-\beta)e^{16}\right]^{\frac{1}{2}} = 2(2-\beta)e^{\frac{17}{2}}.$$

From which, it follows that

$$(2-\beta)e^{\frac{25}{4}} \leq e^{ln(2-\beta)+7} \leq 2(2-\beta)e^{\frac{17}{2}},$$

that is

$$\left[\beta e^{\frac{25}{4}}, (2-\beta)e^{\frac{25}{4}}\right] \leq_I \left[e^{ln(\beta)+7}, e^{ln(2-\beta)+7}\right] \leq_I \left[2\beta e^{\frac{17}{2}}, 2(2-\beta)e^{\frac{17}{2}}\right],$$

for all $\beta \in [0, 1]$. Hence,

$$\Psi\left(\frac{u+v}{2}\right)^{\frac{1}{2h\left(\frac{1}{2}\right)}} \preccurlyeq \exp\left[\frac{1}{v-u}(FR)\int_u^v \Psi(x)\mathrm{d}x\right] \preccurlyeq [\Psi(u)\tilde{\times}\Psi(v)]^{\int_0^1 h(\xi)\mathrm{d}\xi}.$$

Now, we discuss second and first HH-Fejér inequality for log-h-convex FIVF, respectively.

Theorem 3.4 *Let $\Psi : [u, v] \to \mathbb{F}_0$ be a log-h-convex FIVF with $u < v$ and $h : [0, 1] \to \mathbb{R}^+$, for all $\beta \in [0, 1]$, $\Psi_\beta : K \subset \mathbb{R} \to \mathcal{K}_C^+ \subset \mathcal{K}_C$ represent the family of IVFs through β-levels. If $\Psi \in \mathcal{FR}_{([u,v],\beta)}$ and $\nabla : [u, v] \to \mathbb{R}, \nabla(x) \geq 0$, symmetric with respect to $\frac{u+v}{2}$, then*

$$\frac{1}{v-u}(FR)\int_u^v [\ln \Psi(x)]\nabla(x)\mathrm{d}x \preccurlyeq \ln[\Psi(u)\tilde{\times}\Psi(v)]\int_0^1 h(\xi)\nabla((1-\xi)u+\xi v)\mathrm{d}\xi. \tag{28}$$

If Ψ is log-h-concave FIVF then

$$\frac{1}{v-u}(FR)\int_u^v [\ln \Psi(x)]\nabla(x)\mathrm{d}x \succcurlyeq \ln[\Psi(u)\tilde{\times}\Psi(v)]\int_0^1 h(\xi)\nabla((1-\xi)u+\xi v)\mathrm{d}\xi.$$

Proof Let Ψ be a log-h-convex FIVF. Then, by Theorem 2.13, for each $\beta \in [0, 1]$, we have

$$\begin{aligned}&[\ln \Psi_*(\xi u+(1-\xi)v, \beta)]\nabla(\xi u+(1-\xi)v)\\ &\quad\leq \begin{pmatrix} h(\xi)\ln \Psi_*(u, \beta)+ \\ h(1-\xi)\ln \Psi_*(v, \beta)\end{pmatrix}\nabla(\xi u+(1-\xi)v),\\ &[\ln \Psi^*(\xi u+(1-\xi)v, \beta)]\nabla(\xi u+(1-\xi)v)\\ &\quad\leq \begin{pmatrix} h(\xi)\ln \Psi^*(u, \beta)+ \\ h(1-\xi)\ln \Psi^*(v, \beta)\end{pmatrix}\nabla(\xi u+(1-\xi)v).\end{aligned} \tag{29}$$

And

$$\begin{aligned}&\left[\ln \Psi_*((1-\xi)u+\xi v, \beta)\right]\nabla((1-\xi)u+\xi v)\\ &\quad\leq \begin{pmatrix} h(1-\xi)\ln \Psi_*(u, \beta)+ \\ h(\xi)\ln \Psi_*(v, \beta)\end{pmatrix}\nabla((1-\xi)u+\xi v),\\ &\left[\ln \Psi^*((1-\xi)u+\xi v, \beta)\right]\nabla((1-\xi)u+\xi v)\\ &\quad\leq \begin{pmatrix} h(1-\xi)\ln \Psi^*(u, \beta)+ \\ h(\xi)\ln \Psi^*(v, \beta)\end{pmatrix}\nabla((1-\xi)u+\xi v).\end{aligned} \tag{30}$$

After adding (29) and (30), and integrating over [0, 1], we get

$$\int_0^1 \left[\ln \Psi_*(\xi u + (1-\xi)v, \beta)\right] \nabla(\xi u + (1-\xi)v) d\xi$$

$$+ \int_0^1 \ln \Psi_*((1-\xi)u + \xi v, \beta) \nabla((1-\xi)u + \xi v) d\xi$$

$$\leq \int_0^1 \left[\begin{array}{c} \ln \Psi_*(u,\beta) \left\{ \begin{array}{c} h(\xi)\nabla(\xi u + (1-\xi)v) \\ +h(1-\xi)\nabla((1-\xi)u + \xi v) \end{array} \right\} \\ + \ln \Psi_*(v,\beta) \left\{ \begin{array}{c} h(1-\xi)\nabla(\xi u + (1-\xi)v) \\ +h(\xi)\nabla((1-\xi)u + \xi v) \end{array} \right\} \end{array} \right] d\xi,$$

$$\int_0^1 \left[\ln \Psi^*((1-\xi)u + \xi v, \beta)\right] \nabla((1-\xi)u + \xi v) d\xi$$

$$+ \int_0^1 \ln \Psi^*(\xi u + (1-\xi)v, \beta) \nabla(\xi u + (1-\xi)v) d\xi$$

$$\leq \int_0^1 \left[\begin{array}{c} \ln \Psi^*(u,\beta) \left\{ \begin{array}{c} h(\xi)\nabla(\xi u + (1-\xi)v) \\ +h(1-\xi)\nabla((1-\xi)u + \xi v) \end{array} \right\} \\ + \ln \Psi^*(v,\beta) \left\{ \begin{array}{c} h(1-\xi)\nabla(\xi u + (1-\xi)v) \\ +h(\xi)\nabla((1-\xi)u + \xi v) \end{array} \right\} \end{array} \right] d\xi.$$

$$= 2\ln \Psi_*(u,\beta) \int_0^1 h(\xi)\nabla(\xi u + (1-\xi)v)\, d\xi$$

$$+ 2\ln \Psi_*(v,\beta) \int_0^1 h(\xi)\nabla((1-\xi)u + \xi v)\, d\xi,$$

$$= 2\ln \Psi^*(u,\beta) \int_0^1 h(\xi)\nabla(\xi u + (1-\xi)v)\, d\xi$$

$$+ 2\ln \Psi^*(v,\beta) \int_0^1 h(\xi)\nabla((1-\xi)u + \xi v)\, d\xi.$$

Since ∇ is symmetric, then

$$=2\ln \left[\begin{array}{c} \Psi_*(u,\beta) \\ \times\Psi_*(v,\beta) \end{array} \right] \int_0^1 h(\xi)\nabla\left(\begin{array}{c} (1-\xi)u \\ +\xi v \end{array} \right) d\xi$$

$$=2\ln \left[\begin{array}{c} \Psi^*(u,\beta) \\ \times\Psi^*(v,\beta) \end{array} \right] \int_0^1 h(\xi)\nabla\left(\begin{array}{c} (1-\xi)u \\ +\xi v \end{array} \right) d\xi. \tag{31}$$

Since

$$\int_0^1 \left[\ln \Psi_*(\xi u + (1-\xi)v, \beta)\right] \nabla(\xi u + (1-\xi)v) d\xi$$

$$= \int_0^1 \left[\ln \Psi_*((1-\xi)u + \xi v, \beta)\right] \nabla\left(\begin{array}{c} (1-\xi)u+ \\ \xi v \end{array} \right) d\xi$$

$$= \frac{1}{v-u} \int_u^v \left[\ln \Psi_*(x,\beta)\right] \nabla(x) dx \tag{32}$$

$$\int_0^1 \left[\ln \Psi^*((1-\xi)u + \xi v, \beta)\right] \nabla((1-\xi)u + \xi v) d\xi$$

$$= \int_0^1 \left[\ln \Psi^*(\xi u + (1-\xi)v, \beta)\right] \nabla\left(\begin{array}{c} \xi u+ \\ (1-\xi)v \end{array} \right) d\xi$$

$$= \frac{1}{v-u} \int_u^v \left[\ln \Psi^*(x,\beta)\right] \nabla(x) dx.$$

From (31) and (32), we have

$$\frac{1}{v-u} \int_u^v \left[\ln \Psi_*(x,\beta)\right] \nabla(x) dx$$

$$\leq \ln\left[\Psi_*(u,\beta) \times \Psi_*(v,\beta)\right] \int_0^1 h(\xi)\nabla\left(\begin{array}{c} (1-\xi)u \\ +\xi v \end{array} \right) d\xi,$$

$$\frac{1}{v-u} \int_u^v \left[\ln \Psi^*(x,\beta)\right] \nabla(x) dx$$

$$\leq \ln\left[\Psi^*(u,\beta) \times \Psi^*(v,\beta)\right] \int_0^1 h(\xi)\nabla\left(\begin{array}{c} (1-\xi)u \\ +\xi v \end{array} \right) d\xi,$$

that is

$$\left[\begin{array}{c} \frac{1}{v-u} \int_u^v \left[\ln \Psi_*(x,\beta)\right] \nabla(x) dx, \\ \frac{1}{v-u} \int_u^v [\ln \Psi^*(x,\beta)] \nabla(x) dx \end{array} \right]$$

$$\leq_I \left[\ln[\Psi_*(u,\beta) \times \Psi_*(v,\beta)],\ \ln[\Psi^*(u,\beta) \times \Psi^*(v,\beta)]\right] \int_0^1 h(\xi)\nabla((1-\xi)u + \xi v)\, d\xi,$$

hence

$$\frac{1}{\nu-\mathcal{u}}(FR)\int_{\mathcal{u}}^{\nu}[\ln\Psi(\varkappa)]\nabla(\varkappa)\mathrm{d}\varkappa$$

$$\preccurlyeq \ln\left[\Psi(\mathcal{u})\tilde{\times}\Psi(\nu)\right]\int_{0}^{1}h(\xi)\nabla((1-\xi)\mathcal{u}+\xi\nu)\mathrm{d}\xi.$$

This concludes the proof.

Theorem 3.5 *Let* $\Psi:[\mathcal{u},\nu]\to\mathbb{F}_0$ *be a* $\log -h$*-convex FIVF with* $\mathcal{u}<\nu$ *and* $h:[0,1]\to\mathbb{R}^{+}$, *for all* $\beta\in[0,1]$, $\Psi_\beta: K\subset\mathbb{R}\to\mathcal{K}_C^{+}\subset\mathcal{K}_C$ *represent the family of IVFs through* β*-levels. If* $\Psi\in\mathcal{FR}_{([\mathcal{u},\nu],\beta)}$ *and* $\nabla:[\mathcal{u},\nu]\to\mathbb{R}$, $\nabla(\varkappa)\geq 0$, *symmetric with respect to* $\frac{\mathcal{u}+\nu}{2}$, *and* $\int_{\mathcal{u}}^{\nu}\nabla(\varkappa)\mathrm{d}\varkappa>0$, *then*

$$\ln\Psi\left(\frac{\mathcal{u}+\nu}{2}\right)\preccurlyeq\frac{2h\left(\frac{1}{2}\right)}{\int_{\mathcal{u}}^{\nu}\nabla(\varkappa)\mathrm{d}\varkappa}(FR)\int_{\mathcal{u}}^{\nu}[\ln\Psi(\varkappa)]\nabla(\varkappa)\mathrm{d}\varkappa. \quad (33)$$

If Ψ *is log-h-concave FIVF then*

$$\ln\Psi\left(\frac{\mathcal{u}+\nu}{2}\right)\succcurlyeq\frac{2h\left(\frac{1}{2}\right)}{\int_{\mathcal{u}}^{\nu}\nabla(\varkappa)d\varkappa}(FR)\int_{\mathcal{u}}^{\nu}[\ln\Psi(\varkappa)]\nabla(\varkappa)\mathrm{d}\varkappa.$$

Proof Since Ψ is a log-h-convex, then by Theorem 2.13, for $\beta\in[0,1]$, we have

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\ln\Psi_*\left(\frac{\mathcal{u}+\nu}{2},\beta\right)\leq\ln\Psi_*(\xi\mathcal{u}+(1-\xi)\nu,\beta)\\
&\quad+\ln\Psi_*((1-\xi)\mathcal{u}+\xi\nu,\beta)\\
&\frac{1}{h\left(\frac{1}{2}\right)}\ln\Psi^*\left(\frac{\mathcal{u}+\nu}{2},\beta\right)\leq\ln\Psi^*(\xi\mathcal{u}+(1-\xi)\nu,\beta)\\
&\quad+\ln\Psi^*((1-\xi)\mathcal{u}+\xi\nu,\beta),
\end{aligned}\quad (34)$$

By multiplying (34) by $\nabla((1-\xi)\mathcal{u}+\xi\nu)=\nabla(\xi\mathcal{u}+(1-\xi)\nu)$ and integrate it by ξ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)}\left[\ln\Psi_*\left(\frac{\mathcal{u}+\nu}{2},\beta\right)\right]\int_0^1\nabla((1-\xi)\mathcal{u}+\xi\nu)\mathrm{d}\xi\\
&\leq\int_0^1\left[\ln\Psi_*(\xi\mathcal{u}+(1-\xi)\nu,\beta)\right]\nabla\left(\begin{array}{c}\xi\mathcal{u}+\\(1-\xi)\nu\end{array}\right)\mathrm{d}\xi\\
&+\int_0^1\left[\ln\Psi_*((1-\xi)\mathcal{u}+\xi\nu,\beta)\right]\nabla((1-\xi)\mathcal{u}+\xi\nu)\mathrm{d}\xi,\\
&\frac{1}{h\left(\frac{1}{2}\right)}\left[\ln\Psi^*\left(\frac{\mathcal{u}+\nu}{2},\beta\right)\right]\int_0^1\nabla((1-\xi)\mathcal{u}+\xi\nu)\mathrm{d}\xi\\
&\leq\int_0^1\left[\ln\Psi^*(\xi\mathcal{u}+(1-\xi)\nu,\beta)\right]\nabla\left(\begin{array}{c}\xi\mathcal{u}+\\(1-\xi)\nu\end{array}\right)\mathrm{d}\xi\\
&+\int_0^1\left[\ln\Psi^*((1-\xi)\mathcal{u}+\xi\nu,\beta)\right]\nabla((1-\xi)\mathcal{u}+\xi\nu)\mathrm{d}\xi,
\end{aligned}\quad (35)$$

Since

$$\begin{aligned}
&\int_0^1\left[\ln\Psi_*(\xi\mathcal{u}+(1-\xi)\nu,\beta)\right]\nabla(\xi\mathcal{u}+(1-\xi)\nu)\mathrm{d}\xi\\
&=\int_0^1\left[\ln\Psi_*((1-\xi)\mathcal{u}+\xi\nu,\beta)\right]\nabla\left(\begin{array}{c}(1-\xi)\mathcal{u}\\+\xi\nu\end{array}\right)\mathrm{d}\xi\\
&=\frac{1}{\nu-\mathcal{u}}\int_{\mathcal{u}}^{\nu}\left[\ln\Psi_*(\varkappa,\beta)\right]\nabla(\varkappa)\mathrm{d}\varkappa,\\
&\int_0^1\left[\ln\Psi^*(\xi\mathcal{u}+(1-\xi)\nu,\beta)\right]\nabla(\xi\mathcal{u}+(1-\xi)\nu)\mathrm{d}\xi\\
&=\int_0^1\left[\ln\Psi^*((1-\xi)\mathcal{u}+\xi\nu,\beta)\right]\nabla\left(\begin{array}{c}(1-\xi)\mathcal{u}\\+\xi\nu\end{array}\right)\mathrm{d}\xi\\
&=\frac{1}{\nu-\mathcal{u}}\int_{\mathcal{u}}^{\nu}\left[\ln\Psi^*(\varkappa,\beta)\right]\nabla(\varkappa)\mathrm{d}\varkappa,
\end{aligned}\quad (36)$$

From (35) and (36), we have

$$\ln \Psi_*\left(\frac{\mathscr{u}+\nu}{2}, \beta\right) \leq \frac{2h\left(\frac{1}{2}\right)}{\int_{\mathscr{u}}^{\nu} \nabla(\varkappa)\mathrm{d}\varkappa} \int_{\mathscr{u}}^{\nu} \left[\ln \Psi_*(\varkappa, \beta)\right] \nabla(\varkappa)\mathrm{d}\varkappa,$$

$$\ln \Psi^*\left(\frac{\mathscr{u}+\nu}{2}, \beta\right) \leq \frac{2h\left(\frac{1}{2}\right)}{\int_{\mathscr{u}}^{\nu} \nabla(\varkappa)\mathrm{d}\varkappa} \int_{\mathscr{u}}^{\nu} \left[\ln \Psi^*(\varkappa, \beta)\right] \nabla(\varkappa)\mathrm{d}\varkappa.$$

From which, we have

$$\left[\ln \Psi_*\left(\frac{\mathscr{u}+\nu}{2}, \beta\right), \ \ln \Psi^*\left(\frac{\mathscr{u}+\nu}{2}, \beta\right)\right]$$
$$\leq_I \frac{2h\left(\frac{1}{2}\right)}{\int_{\mathscr{u}}^{\nu} \nabla(\varkappa)\mathrm{d}\varkappa} \begin{bmatrix} \int_{\mathscr{u}}^{\nu} \left[\ln \Psi_*(\varkappa, \beta)\right] \nabla(\varkappa)\mathrm{d}\varkappa, \\ \int_{\mathscr{u}}^{\nu} [\ln \Psi^*(\varkappa, \beta)] \nabla(\varkappa)\mathrm{d}\varkappa \end{bmatrix},$$

that is

$$\ln \Psi\left(\frac{\mathscr{u}+\nu}{2}\right) \preccurlyeq \frac{2h\left(\frac{1}{2}\right)}{\int_{\mathscr{u}}^{\nu} \nabla(\varkappa)\mathrm{d}\varkappa} (FR) \int_{\mathscr{u}}^{\nu} [\ln \Psi(\varkappa)] \nabla(\varkappa)\mathrm{d}\varkappa,$$

then we complete the proof.

Remark 3.6 If $h(\xi) = \xi^s$ with $s \in (0, 1)$, then from Theorems 3.4 and 3.5, we get result for log-s-convex FIVFs.

If $h(\xi) = \xi$ then from Theorems 3.4 and 3.5, we obtain result for log-convex FIVFs.

If $\Psi_*(\mathscr{u}, \beta) = \Psi^*(\mathscr{u}, \beta)$ with $\beta = 1$, then from Theorems 3.4 and 3.5, we obtain HH-Fejér inequality for log-h-convex function.

Example 3.7 We consider $h(\xi) = \xi$, for $\xi \in [0, 1]$ and the FIVFs $\Psi : [\mathscr{u}, \nu] = [1, 8] \to \mathbb{F}_0$ is given by,

$$\Psi(\varkappa)(\partial) = \begin{cases} \frac{\partial}{2\varkappa}, & \partial \in [0, 2\varkappa], \\ \frac{4\varkappa-\partial}{2\varkappa}, & \partial \in (2\varkappa, 4\varkappa], \\ 0, & \text{otherwise}, \end{cases}$$

Then, for each $\beta \in [0, 1]$, we have

$\Psi_\beta(\varkappa) = [2\beta\varkappa, 2(2-\beta)\varkappa].$

Since $\Psi_*(\varkappa, \beta) = 2\beta z, \Psi^*(\varkappa, \beta) = 2(2-\beta)z$ log-h-convex functions, for each $\beta \in [0, 1]$ then, by Theorem 2.13 $\Psi(\varkappa)$ is log-h-convex FIVF. If

$$\nabla(\varkappa) = \begin{cases} \varkappa - 1, \ \partial \in \left[1, \frac{9}{2}\right] \\ 8 - \varkappa, \ \partial \in \left(\frac{9}{2}, 8\right], \end{cases}$$

then, we have

$$\begin{aligned}
\frac{1}{\nu-\mathscr{u}} \int_{\mathscr{u}}^{\nu} \left[\ln \Psi_*(\varkappa, \beta)\right] \nabla(\varkappa)\mathrm{d}\varkappa
&= \frac{1}{7} \int_{1}^{8} \left[\ln \Psi_*(\varkappa, \beta)\right] \nabla(\varkappa)\mathrm{d}\varkappa \\
&= \frac{1}{7} \int_{1}^{\frac{9}{2}} \left[\ln \Psi_*(\varkappa, \beta)\right] \nabla(\varkappa)\mathrm{d}\varkappa \\
&\quad + \frac{1}{7} \int_{\frac{9}{2}}^{8} \ln \Psi_*(\varkappa, \beta) \nabla(\varkappa)\mathrm{d}\varkappa, \\
\frac{1}{\nu-\mathscr{u}} \int_{\mathscr{u}}^{\nu} \left[\ln \Psi^*(\varkappa, \beta)\right] \nabla(\varkappa)\mathrm{d}\varkappa
&= \frac{1}{7} \int_{1}^{8} \left[\ln \Psi^*(\varkappa, \beta)\right] \nabla(\varkappa)\mathrm{d}\varkappa \\
&= \frac{1}{7} \int_{1}^{\frac{9}{2}} \left[\ln \Psi^*(\varkappa, \beta)\right] \nabla(\varkappa)\mathrm{d}\varkappa \\
&\quad + \frac{1}{7} \int_{\frac{9}{2}}^{8} \left[\ln \Psi^*(\varkappa, \beta)\right] \nabla(\varkappa)\mathrm{d}\varkappa, \\
&= \frac{1}{7} \int_{1}^{\frac{9}{2}} [\ln (2\beta\varkappa)](\varkappa - 1)\mathrm{d}\varkappa \\
&\quad + \frac{1}{7} \int_{\frac{9}{2}}^{8} [\ln (2\beta\varkappa)](8 - \varkappa)\mathrm{d}\varkappa \\
&= \frac{7}{4} \ln (2\beta) + \frac{5}{2}, \\
&= \frac{1}{7} \int_{1}^{\frac{9}{2}} [\ln (2(2-\beta)\varkappa)](\varkappa - 1)\mathrm{d}\varkappa \\
&\quad + \frac{1}{7} \int_{\frac{9}{2}}^{8} [\ln (2(2-\beta)\varkappa)](8 - \varkappa)\mathrm{d}\varkappa \\
&= \frac{7}{4} \ln (2(2-\beta)) + \frac{5}{2},
\end{aligned} \tag{37}$$

And

$$\ln\begin{bmatrix}\Psi_*(u,\beta)\\ \times\Psi_*(v,\beta)\end{bmatrix}\int_0^1 h(\xi)\nabla(u+\xi\partial(v,u))\,d\xi$$

$$\ln\begin{bmatrix}\Psi^*(u,\beta)\\ \times\Psi^*(v,\beta)\end{bmatrix}\int_0^1 h(\xi)\nabla(u+\xi\partial(v,u))\,d\xi$$

$$\begin{aligned}&=\left[\ln\left(32\beta^2\right)\right]\left[\int_0^{\frac{1}{2}}7\xi^2 d\varkappa+\int_{\frac{1}{2}}^1\xi(7-7\xi)d\xi\right]\\&=\frac{7}{8}\left[\ln\left(32\beta^2\right)\right]\\&=\left[\ln\left(32(2-\beta)^2\right)\right]\left[\int_0^{\frac{1}{2}}7\xi^2 d\varkappa+\int_{\frac{1}{2}}^1\xi(7-7\xi)d\xi\right]\\&=\frac{7}{8}\left[\ln\left(32(2-\beta)^2\right)\right]\end{aligned} \tag{38}$$

From (37) and (38), we have

$$\left[\frac{7}{4}\ln(2\beta)+\frac{5}{2},\frac{7}{4}\ln(2(2-\beta))+\frac{5}{2}\right]$$
$$\leq_I\left[\frac{7}{8}\left[\ln\left(32\beta^2\right)\right],\frac{7}{8}\left[\ln\left(32(2-\beta)^2\right)\right]\right],$$

for all $\beta\in(0,1]$.

Hence, Theorem 3.4 is verified.

For Theorem 3.5, we have

$$\begin{aligned}\ln\Psi_*\left(\frac{u+v}{2},\beta\right)&=\ln(9\beta),\\ \ln\Psi^*\left(\frac{u+v}{2},\beta\right)&=\ln(9(2-\beta)),\end{aligned} \tag{39}$$

$$\int_u^v\nabla(\varkappa)d\varkappa=\int_1^{\frac{9}{2}}(\varkappa-1)d\varkappa+\int_{\frac{9}{2}}^8(8-\varkappa)d\varkappa=\frac{49}{4},$$

$$\begin{aligned}&\frac{2h\left(\frac{1}{2}\right)}{\int_u^v\nabla(\varkappa)d\varkappa}\int_u^v\left[\ln\Psi_*(\varkappa,\beta)\right]\nabla(\varkappa)d\varkappa\\&=\ln(2\beta)+1.54\\&\frac{2h\left(\frac{1}{2}\right)}{\int_u^v\nabla(\varkappa)d\varkappa}\int_u^v\left[\ln\Psi^*(\varkappa,\beta)\right]\nabla(\varkappa)d\varkappa\\&=\ln(2(2-\beta))+1.54\end{aligned} \tag{40}$$

From (39) and (40), we have

$$[\ln(9\beta),\ \ln(9(2-\beta))]$$
$$\leq_I[\ln(2\beta)+1.54,\ln(2(2-\beta))+1.54].$$

Hence, Theorem 3.5 is verified.

Now, we prove the Jensen's inequality for log-h-convex FIVF.

Theorem 3.8 *Let $\omega_j\in\mathbb{R}^+$, $u_j\in[u,v]$, $(j=1,2,3,\ldots k,k\geq 2)$ and $\Psi:[u,v]\to\mathbb{F}_0$ be a log h-convex FIVF with non-negative real valued function $h:[0,1]\to\mathbb{R}$, for all $\beta\in[0,1]$, $\Psi_\beta:K\subset\mathbb{R}\to\mathcal{K}_C^+$ represent the family of IVFs through β-levels. If h is multiplicative function then,*

$$\Psi\left(\frac{1}{W_k}\sum_{j=1}^k\omega_j z_j\right)\preccurlyeq\prod_{j=1}^k\left[\Psi\left(z_j\right)\right]^{h\left(\frac{\omega_j}{W_k}\right)}, \tag{41}$$

If Ψ is log-h-concave FIVF then

$$\Psi\left(\frac{1}{W_k}\sum_{j=1}^k\omega_j z_j\right)\succcurlyeq\prod_{j=1}^k\left[\Psi\left(z_j\right)\right]^{h\left(\frac{\omega_j}{W_k}\right)},$$

where $W_k=\sum_{j=1}^k\omega_j$. If Ψ is log-h-concave then, (41) is reversed.

Proof When $k=2$, (41) is true. Consider (16) is true for $k=n-1$, then by Theorem 2.13, we have

$$\Psi\left(\frac{1}{W_{n-1}}\sum_{j=1}^{n-1}\omega_j z_j\right)\preccurlyeq\prod_{j=1}^{n-1}\left[\Psi\left(z_j\right)\right]^{h\left(\frac{\omega_j}{W_{n-1}}\right)},$$

Now, let us prove that (41) holds for $k=n$.

$$\Psi\left(\frac{1}{W_n}\sum_{j=1}^n\omega_j z_j\right)$$
$$=\Psi\left(\begin{array}{c}\frac{1}{W_{n-2}}\sum_{j=1}^{n-2}\omega_j z_j+\frac{\omega_{n-1}+\omega_n}{W_n}\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}z_{n-1}\right)\\+\frac{\omega_n}{\omega_{n-1}+\omega_n}z_n\end{array}\right).$$

Therefore, for every $\beta\in[0,1]$, we have

$$\begin{aligned}
&\Psi_*\left(\frac{1}{W_n}\sum_{j=1}^{n}\omega_j z_j,\ \beta\right)\\
&\Psi^*\left(\frac{1}{W_n}\sum_{j=1}^{n}\omega_j z_j,\ \beta\right)\\
&\le \Psi_*\left(\begin{array}{c}\frac{1}{W_n}\sum_{j=1}^{n-2}\omega_j z_j + \frac{\omega_{n-1}+\omega_n}{W_n}\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}z_{n-1}\right)\\ +\frac{\omega_n}{\omega_{n-1}+\omega_n}z_n, \beta\end{array}\right),\\
&\le \Psi^*\left(\begin{array}{c}\frac{1}{W_n}\sum_{j=1}^{n-2}\omega_j z_j + \frac{\omega_{n-1}+\omega_n}{W_n}\left(\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}z_{n-1}\right)\\ +\frac{\omega_n}{\omega_{n-1}+\omega_n}z_n, \beta\end{array}\right),\\
&\le \prod_{j=1}^{n-2}\left[\Psi_*\left(z_j,\beta\right)\right]^{h\left(\frac{\omega_j}{W_n}\right)}\\
&\quad\times\left[\Psi_*\left(\begin{array}{c}\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}z_{n-1}\\ +\frac{\omega_n}{\omega_{n-1}+\omega_n}z_n, \beta\end{array}\right)\right]^{h\left(\frac{\omega_{n-1}+\omega_n}{W_n}\right)},\\
&\le \prod_{j=1}^{n-2}\left[\Psi^*\left(z_j,\beta\right)\right]^{h\left(\frac{\omega_j}{W_n}\right)}\\
&\quad\times\left[\Psi^*\left(\begin{array}{c}\frac{\omega_{n-1}}{\omega_{n-1}+\omega_n}z_{n-1}\\ +\frac{\omega_n}{\omega_{n-1}+\omega_n}z_n, \beta\end{array}\right)\right]^{h\left(\frac{\omega_{n-1}+\omega_n}{W_n}\right)},\\
&\le \prod_{j=1}^{n-2}\left[\Psi_*\left(z_j,\beta\right)\right]^{h\left(\frac{\omega_j}{W_n}\right)}\times\left[\Psi_*\left(z_{n-1},\beta\right)\right]^{h\left(\frac{\omega_{n-1}}{W_n}\right)}\\
&\quad\times\left[\Psi_*\left(z_n,\beta\right)\right]^{h\left(\frac{\omega_n}{W_n}\right)},\\
&\le \prod_{j=1}^{n-2}\left[\Psi^*\left(z_j,\beta\right)\right]^{h\left(\frac{\omega_j}{W_n}\right)}\times\left[\Psi^*\left(z_{n-1},\beta\right)\right]^{h\left(\frac{\omega_{n-1}}{W_n}\right)}\\
&\quad\times\left[\Psi^*\left(z_n,\beta\right)\right]^{h\left(\frac{\omega_n}{W_n}\right)},\\
&= \prod_{j=1}^{n}\left[\Psi_*\left(z_j,\beta\right)\right]^{h\left(\frac{\omega_j}{W_n}\right)},\\
&= \prod_{j=1}^{n}\left[\Psi^*\left(z_j,\beta\right)\right]^{h\left(\frac{\omega_j}{W_n}\right)},
\end{aligned}$$

From which, we have

$$\begin{aligned}
&\left[\Psi_*\left(\frac{1}{W_n}\sum_{j=1}^{n}\omega_j z_j,\ \beta\right),\ \Psi^*\left(\frac{1}{W_n}\sum_{j=1}^{n}\omega_j z_j,\ \beta\right)\right]\\
&\le_I \left[\prod_{j=1}^{n}\left[\Psi_*\left(z_j,\ \beta\right)\right]^{h\left(\frac{\omega_j}{W_n}\right)},\ \prod_{j=1}^{n}\left[\Psi^*\left(z_j,\ \beta\right)\right]^{h\left(\frac{\omega_j}{W_n}\right)}\right],
\end{aligned}$$

that is,

$$\Psi\left(\frac{1}{W_n}\sum_{j=1}^{n}\omega_j z_j\right)\preccurlyeq\prod_{j=1}^{n}\left[\Psi\left(z_j\right)\right]^{h\left(\frac{\omega_j}{W_n}\right)},$$

and the result follows.

If $\omega_1=\omega_2=\omega_3=\cdots=\omega_k=1$, then Theorem 3.8 reduces to the following:

Corollary 3.9 *Let* $u_j\in[u,v]$, $\left(\begin{array}{c}j=1,2,3,\ldots k,\\ k\ge 2\end{array}\right)$ *and* $\Psi:[u,v]\to\mathbb{F}_0$ *be a log-h-convex FIVF with non-negative real valued function* $h:[0,1]\to\mathbb{R}$. *For all* $\beta\in[0,1]$, $\Psi_\beta: K\subset\mathbb{R}\to\mathcal{K}_C^+\subset\mathcal{K}_C$ *represent the family of IVFs through* β*-levels. If h is multiplicative function, then*

$$\Psi\left(\frac{1}{k}\sum_{j=1}^{k}z_j\right)\preccurlyeq\prod_{j=1}^{k}\left[\Psi\left(z_j\right)\right]^{h\left(\frac{1}{k}\right)},\tag{42}$$

If Ψ is a log-h-concave, then (42) is reversed.

Theorem 3.10 *Let* $\omega_j\in\mathbb{R}^+$, $u_j\in[u,v]$, $(j=1,2,3,\ldots k, k\ge 2)$ *and* $\Psi:[u,v]\to\mathbb{F}_0$ *be a log-h-convex FIVF with non-negative real valued function* $h:[0,1]\to\mathbb{R}$. *For all* $\beta\in[0,1]$, $\Psi_\beta: K\subset\mathbb{R}\to\mathcal{K}_C^+\subset\mathcal{K}_C$ *represent the family of IVFs through* β*-levels. If* $(L,U)\subseteq[u,v]$ *h is multiplicative function then,*

$$\prod_{j=1}^{k}\left[\Psi\left(z_j\right)\right]^{h\left(\frac{\omega_j}{W_k}\right)}\preccurlyeq\prod_{j=1}^{k}\left([\Psi(L)]^{h\left(\frac{U-z_j}{U-L}\right)h\left(\frac{\omega_j}{W_k}\right)}\times[\Psi(U)]^{h\left(\frac{z_j-L}{M-L}\right)h\left(\frac{\omega_j}{W_k}\right)}\right),\tag{43}$$

If Ψ *is* log-*h*-*concave FIVF, then*

$$\prod_{j=1}^{k}\left[\Psi\left(z_j\right)\right]^{h\left(\frac{\omega_j}{W_k}\right)}\succcurlyeq\prod_{j=1}^{k}\left([\Psi(L)]^{h\left(\frac{U-z_j}{U-L}\right)h\left(\frac{\omega_j}{W_k}\right)}\times[\Psi(U)]^{h\left(\frac{z_j-L}{M-L}\right)h\left(\frac{\omega_j}{W_k}\right)}\right),$$

where $W_k=\sum_{j=1}^{k}\omega_j$. *If* Ψ *is* log-*h*-*concave, then* (43) *is reversed.*

Proof Consider $=z_1$, $z_j=z_2$, $(j=1,2,3,\ldots k)$, $U=z_3$ in (43). Then, for each $\beta\in[0,1]$, then by Theorem 2.13, we have

$$\begin{aligned}
\Psi_*\left(z_j,\beta\right)&\le\left[\Psi_*(L,\beta)\right]^{h\left(\frac{U-z_j}{U-L}\right)}\times\left[\Psi_*(U,\beta)\right]^{h\left(\frac{z_j-L}{M-L}\right)},\\
\Psi^*\left(z_j,\beta\right)&\le\left[\Psi^*(L,\beta)\right]^{h\left(\frac{U-z_j}{U-L}\right)}\times\left[\Psi^*(U,\beta)\right]^{h\left(\frac{z_j-L}{M-L}\right)}.
\end{aligned}$$

Above inequality can be written as,

$$\begin{aligned}
\Psi_*\left(z_j,\beta\right)^{h\left(\frac{\omega_j}{W_k}\right)}&\le\left[\Psi_*(L,\beta)\right]^{h\left(\frac{U-z_j}{U-L}\right)h\left(\frac{\omega_j}{W_k}\right)}\\
&\quad\times\left[\Psi_*(U,\beta)\right]^{h\left(\frac{z_j-L}{M-L}\right)h\left(\frac{\omega_j}{W_k}\right)}\\
\Psi^*\left(z_j,\beta\right)^{h\left(\frac{\omega_j}{W_k}\right)}&\le\left[\Psi^*(L,\beta)\right]^{h\left(\frac{U-z_j}{U-L}\right)h\left(\frac{\omega_j}{W_k}\right)}\\
&\quad\times\left[\Psi^*(U,\beta)\right]^{h\left(\frac{z_j-L}{M-L}\right)h\left(\frac{\omega_j}{W_k}\right)}.
\end{aligned}\tag{44}$$

Taking multiplication of all inequalities (44) for $j = 1, 2, 3, \ldots k$, we have

$$\prod_{j=1}^{k} \Psi_*\left(z_j, \beta\right)^{h\left(\frac{\omega_j}{W_k}\right)} \leq \prod_{j=1}^{k} \left(\begin{array}{c} \left[\Psi_*(L, \beta)\right]^{h\left(\frac{U-z_j}{U-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \\ \times\left[\Psi_*(U, \beta)\right]^{h\left(\frac{z_j-L}{M-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \end{array} \right),$$

$$\prod_{j=1}^{k} \Psi^*\left(z_j, \beta\right)^{h\left(\frac{\omega_j}{W_k}\right)} \leq \prod_{j=1}^{k} \left(\begin{array}{c} \left[\Psi^*(L, \beta)\right]^{h\left(\frac{U-z_j}{U-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \\ \times\left[\Psi^*(U, \beta)\right]^{h\left(\frac{z_j-L}{M-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \end{array} \right).$$

that is

$$\begin{aligned} &\prod_{j=1}^{k} \Psi_\beta\left(z_j\right)^{h\left(\frac{\omega_j}{W_k}\right)} \\ &= \left[\prod_{j=1}^{k} \Psi_*\left(z_j, \beta\right)^{h\left(\frac{\omega_j}{W_k}\right)}, \prod_{j=1}^{k} \Psi^*\left(z_j, \beta\right)^{h\left(\frac{\omega_j}{W_k}\right)} \right] \\ &\leq_I \left[\begin{array}{c} \prod_{j=1}^{k} \left(\begin{array}{c} \left[\Psi_*(L, \beta)\right]^{h\left(\frac{U-z_j}{U-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \\ \left[\Psi_*(U, \beta)\right]^{h\left(\frac{z_j-L}{M-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \end{array} \right), \\ \prod_{j=1}^{k} \left(\begin{array}{c} \left[\Psi^*(L, \beta)\right]^{h\left(\frac{U-z_j}{U-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \\ \left[\Psi^*(U, \beta)\right]^{h\left(\frac{z_j-L}{M-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \end{array} \right) \end{array} \right], \\ &\leq_I \left[\begin{array}{c} \prod_{j=1}^{k} \left(\begin{array}{c} \left[\Psi_*(L, \beta)\right]^{h\left(\frac{U-z_j}{U-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \\ \left[\Psi_*(U, \beta)\right]^{h\left(\frac{z_j-L}{M-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \end{array} \right), \\ \prod_{j=1}^{k} \left(\begin{array}{c} \left[\Psi^*(L, \beta)\right]^{h\left(\frac{U-z_j}{U-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \\ \left[\Psi^*(U, \beta)\right]^{h\left(\frac{z_j-L}{M-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \end{array} \right) \end{array} \right], \\ &\leq_I \prod_{j=1}^{k} \left(\left[\begin{array}{c} \left[\Psi_*(L, \beta)\right]^{h\left(\frac{U-z_j}{U-L}\right) h\left(\frac{\omega_j}{W_k}\right)}, \\ \left[\Psi^*(L, \beta)\right]^{h\left(\frac{U-z_j}{U-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \end{array} \right] \right) \\ &\times \prod_{j=1}^{k} \left(\left[\begin{array}{c} \left[\Psi_*(U, \beta)\right]^{h\left(\frac{z_j-L}{M-L}\right) h\left(\frac{\omega_j}{W_k}\right)}, \\ \left[\Psi^*(U)\right]^{h\left(\frac{z_j-L}{M-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \end{array} \right] \right) \\ &= \prod_{j=1}^{k} \left[\Psi_\beta(L)\right]^{h\left(\frac{U-z_j}{U-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \\ &\times \prod_{j=1}^{k} \left[\Psi_\beta(U)\right]^{h\left(\frac{z_j-L}{M-L}\right) h\left(\frac{\omega_j}{W_k}\right)}. \end{aligned}$$

Thus,

$$\begin{aligned} &\prod_{j=1}^{k} \left[\Psi\left(z_j\right)\right]^{h\left(\frac{\omega_j}{W_k}\right)} \\ &\preccurlyeq \prod_{j=1}^{k} \left([\Psi(L)]^{h\left(\frac{U-z_j}{U-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \times [\Psi(U)]^{h\left(\frac{z_j-L}{M-L}\right) h\left(\frac{\omega_j}{W_k}\right)} \right), \end{aligned}$$

this completes the proof.

Remark 3.1.1 If $\Psi_*(\mathcal{u}, \beta) = \Psi^*(\nu, \beta)$ with $\beta = 1$, then from Theorem 3.8 and Theorem 3, we obtain result for h-convex function, see [23].

If $\Psi_*(\mathcal{u}, \beta) = \Psi^*(\nu, \beta)$ with $\beta = 1$ and $h(\xi) = \xi^s$, then from Theorem 3.8 and Theorem 3, we obtain result for s -convex FIVF, see [43].

If $\Psi_*(\mathcal{u}, \beta) = \Psi^*(\nu, \beta)$ with $\beta = 1$ and $h(\xi) = \xi$, then from Theorem 3.8 and Theorem 3, we obtain result for convex FIVF, see [23].

4 Conclusion and Future Plan

Hermite–Hadamard and Jensen's inequalities are hold for this new class of convex FIVFs. Moreover, we have discussed some special cases which can be obtained by main results. In future, we intend to discuss generalized log-h -convex functions. We hope that this notion will assist other authors in fulfilling their tasks in various disciplines of study.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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