



A Hybrid Picture Fuzzy Similarity Measure and Improved VIKOR Method

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Abstract

Picture fuzzy set (PFS) can intuitively express the answers of “yes”, “neutral”, “no” and “reject”, which have strong advantages in solving uncertain information. The similarity measure is an effective tool to determine the relationship between two picture fuzzy sets (PFSs). In this paper, we propose a hybrid picture fuzzy (PF) similarity measure which combines the Hamming distance and the transformed tetrahedral centroid distance and verifies that it satisfies the four properties of the similarity measure. The proposed and existing picture fuzzy similarity measures are compared and investigated through numerical examples and some applications of pattern recognition. The results show that the proposed similarity measure not only produces no unreasonable results, but also overcomes the shortcomings of the existing similarity measures. Furthermore, we investigate an improved VIKOR method based on the proposed similarity measure of PFS. Finally, through an example, several multi-attribute decision-making (MADM) methods are compared and analyzed to illustrate the effectiveness and practicability of the improved VIKOR method.

Keywords Picture fuzzy set · Similarity measure · Pattern recognition · Picture fuzzy VIKOR method

Abbreviations

FS	Fuzzy set
IFS	Intuitionistic fuzzy set
PF	Picture fuzzy
PFS	Picture fuzzy set
PFSs	Picture fuzzy sets
MADM	Multi-attribute decision-making
PFWG	Picture fuzzy weighted geometric
PFPN	Picture fuzzy normalized projection

1 Introduction

Zadeh [1] introduced the concept of fuzzy set (FS) in 1965, a fuzzy set is mainly understood and represented by membership degree function. The emergence of fuzzy sets can handle the uncertain information that occurs in the real world. Fuzzy integrodifferential equations theory has been studied to adapt to model various phenomena under uncertainty, thereby solving more complex engineering and applied science problems [2–5]. Scholars have promoted and developed fuzzy set theory, among which the intuitionistic fuzzy set (IFS) proposed by Atanassov [6] in 1986 is an effective extension of fuzzy set theory. Intuitionistic fuzzy sets are composed of membership and non-membership degree functions and can overcome some disadvantages of fuzzy sets. Because of the effectiveness and superiority of intuitionistic fuzzy sets, they have been broadly used in cluster analysis [7–11], pattern recognition [12–14], group decision-making [11, 15, 16] and other fields. Although intuitionistic fuzzy sets have greater advantages than fuzzy sets in expressing uncertain information, there are still some situations that cannot be described by intuitionistic fuzzy sets in real life, such as human voting, medical diagnosis and other issues. In the voting model, there are four results when voters vote for

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candidates: “vote for”, “abstain”, “vote against” and “refuse to vote”. Among medically diagnosed problems: symptoms of cough and headache may have little effect on chest and stomach illnesses. Likewise, symptoms of chest and stomach pain may have a neutral effect on disorders of the head and lungs. The neutral attitude of voter abstention, and the neutral influence of some symptoms on specific diseases, so we can see that the degree of neutrality plays a significant role in the decision making process, which cannot be solved by intuitionistic fuzzy sets. To address this type of problem, Cuong and Kreinovich [17] proposed a new generalization of fuzzy sets and intuitionistic fuzzy sets in 2013, called picture fuzzy sets. The picture fuzzy set consists of three functions: membership degree function, neutral degree function and non-membership degree function. In [17, 18], Cuong and Kreinovich gave some basic operations and relations of picture fuzzy sets.

When using picture fuzzy sets to solve practical problems such as pattern recognition, cluster analysis, information retrieval, image processing, medical diagnosis, and decision-making, it is necessary to measure the connection and difference between two picture fuzzy sets. Similarity measures and distance measures are effective tools to determine the degree of connection and difference between two sets or objects. Cuong [17] first gave the Hamming distance and Euclidean distance of picture fuzzy sets in 2013. Wei [19] proposed eight kinds of picture fuzzy similarity measures based on cosine function, and proved the effectiveness and practicability of the proposed similarity measures by applying it to an example of selecting the optimal production. Similarity measures of cosine, weighted cosine, set theory, weighted set theory, gray and weighted gray were proposed by Wei [20] in the picture fuzzy environment, and were effectively applied to building material identification and mineral field identification. Wei and Gao [21] proposed picture fuzzy generalized Dice similarity measure, and proved the effectiveness of the picture fuzzy generalized Dice similarity measure in the application of pattern recognition through the selection of building materials. In [22], Singh et al. gave a geometric interpretation of picture fuzzy sets and proposed several distance and similarity measures for picture fuzzy sets, which were applied in flood disaster risk analysis. The picture fuzzy similarity measure composed of three functions was proposed by Luo et al. [23] and applied to medical diagnosis. Ganie and Singh [24] studied a picture fuzzy similarity measure based on the upper and lower bounds of membership degree, non-membership degree, neutral degree and rejection degree, and proposed a new method of picture fuzzy inferior ratio, which was applied in multi-attribute decision-making. Singh and Ganie [25] also continued to study some similarity measures that can distinguish highly similar picture fuzzy sets, and developed a

maximum spanning tree clustering method for picture fuzzy. Bi-parametric picture fuzzy similarity and distance measures based on the tetrahedral centroid were proposed by Khan [26] and applied to medical diagnosis and pattern recognition. There are also many studies and applications on the similarity and distance measures of picture fuzzy sets in the literature [27–33].

It is of great significance to deal with multi-attribute decision making problems with picture fuzzy sets as the background. Wang [34] et al. applied the developed picture fuzzy geometric operator to multi-attribute decision making problems, and verified the practicability of the method through an example. In [35], the authors build a projection model to measure the similarity of each scheme to the ideal point of picture fuzzy set, rank the given options according to the projection model, and then select the most ideal option. Due to the advantages of VIKOR method proposed by Opricovic [36], scholars are attracted to study and apply it to decision-making problems. Wang and Zhang et al. [37] proposed the picture fuzzy normalized projection (PFNP) model, which overcomes the limitations of [35], and combined the PFNP model with the VIKOR method to construct a picture fuzzy normalization based Projected VIKOR method, applied to multiple attribute decision problems. Yue [38] applied the proposed picture fuzzy normalization projection and extended VIKOR method to software reliability evaluation. Tian et al. [33] proposed a corresponding WET-PPP project sustainability evaluation method based on the picture fuzzy similarity VIKOR method. In [39], the PFNs algorithm operator is used, and the VIKOR method is applied to evaluate citizens' satisfaction with municipal services.

Ganie and Singh [24] studied the existing PF similarity measure [19–22] and found that there was a counterintuitive situation. First of all, in the process of research and learning, we found that some PF similarity measures proposed in the past two years [23–26] [33] cannot distinguish the similarity degree between different PFSs in some cases. And when dealing with pattern recognition problems, the problem of classification failure often occurs. Second, Khan et al. [26] transformed PFS into a tetrahedron model and constructed a bi-parametric similarity measure of PFS based on the distance between the centroids of two tetrahedrons. However, the similarity measure formula given at the end changes the sign in two directions, then there are two directions that are not the distance between the centroids. To overcome the above existing problems, we propose a PF similarity measure that mixes Hamming distance and transformed tetrahedral centroid distance. When we carefully study the extended VIKOR method based on picture fuzzy similarity by Tian et al. [33] their method only considers group utility and individual regret that are close to the positive ideal, and obtains VIKOR

values that are only close to the positive ideal. Therefore, based on the proposed picture fuzzy similarity measure and the extended VIKOR method, we investigate an improved VIKOR method that considers VIKOR values close to the positive ideal and negative ideal. Finally, TOPSIS method is used to calculate the relative closeness coefficient of candidate schemes and sort them.

The main contributions of this paper are as follows:

1. We propose a hybrid picture fuzzy similarity measure which combines the Hamming distance and the transformed tetrahedral centroid distance, and verifies that it satisfies the four properties of similarity measure.
2. Numerical examples illustrate the rationality of the proposed picture fuzzy similarity measure, overcoming the counter-intuitive situation of existing picture fuzzy similarity measures.
3. The superiority of the new picture fuzzy similarity measure is further verified by pattern recognition.
4. Research an improved VIKOR method based on the proposed picture fuzzy similarity measure, and illustrate its practicability and effectiveness through comparative analysis of examples.

The remainder of this article is structured as follows: in the second part, we review some basic concepts and properties of picture fuzzy sets, give some existing picture fuzzy similarity measures. In the third part, we propose and verify a new picture similarity measure and a weighted picture fuzzy similarity measure. In the fourth part, we apply the proposed and existing PF similarity measure to some instances and pattern recognition, compare the results and illustrate the advantages of the proposed similarity measure. In the fifth part, the algorithm steps of the improved VIKOR method are given and applied to the examples in the literature [34] for comparative analysis. In the sixth part, it summarizes the full text and looks forward to the future.

2 Preliminaries

In this section, we will briefly give the basic definition, operation of picture fuzzy set and some important PF similarity measures.

2.1 Basic Definitions

Definition 1 [17] Let $X = \{x_1, x_2, \dots, x_n\}$ be a universe of discourse, a PFS B in X is defined as $B = \{\langle x_k, m_B(x_k), \eta_B(x_k), v_B(x_k) \rangle | x_k \in X\}$, where $m_B(x_k)$, $\eta_B(x_k)$ and $v_B(x_k)$ represents the membership degree,

neutral degree and non-membership degree, respectively, of the element $x_k \in X$ in the set B such that $0 \leq m_B(x_k) + \eta_B(x_k) + v_B(x_k) \leq 1$. The refusal degree of element x_k belonging to the PFS B is denoted by $\rho_B(x_k) = 1 - m_B(x_k) - \eta_B(x_k) - v_B(x_k)$. For the sake of simplicity, the triad $(m_B(x_k), \eta_B(x_k), v_B(x_k))$ represents picture fuzzy value.

Definition 2 [18] For any two PFSs B and C in X , the operations of complement, equality, inclusion, union, and intersection are defined as:

- (1) $(B)^C = \{\langle x_k, v_B(x_k), \eta_B(x_k), m_B(x_k) \rangle | x_k \in X, k = 1, 2, \dots, n\}$
- (2) $B = C$ if and only if, $m_B(x_k) = m_C(x_k), \eta_B(x_k) = \eta_C(x_k), v_B(x_k) = v_C(x_k), \forall x_k \in X$.
- (3) $B \subseteq C$ if and only if, $m_B(x_k) \leq m_C(x_k), \eta_B(x_k) \leq \eta_C(x_k), v_B(x_k) \geq v_C(x_k), \forall x_k \in X$.
- (4) $B \cup C = \{\langle x_k, \max(m_B(x_k), m_C(x_k)), \min(\eta_B(x_k), \eta_C(x_k)), \min(v_B(x_k), v_C(x_k)) \rangle | x_k \in X\}$.
- (5) $B \cap C = \{\langle x_k, \min(m_B(x_k), m_C(x_k)), \max(\eta_B(x_k), \eta_C(x_k)), \max(v_B(x_k), v_C(x_k)) \rangle | x_k \in X\}$.

Definition 3 [20] Let B, C and D be PFSs defined on the universe of discourse X . A function $S(B, C)$ is called a PF similarity measure if it satisfies:

- (1) $0 \leq S(B, C) \leq 1$;
- (2) $S(B, C) = S(C, B)$;
- (3) $S(B, C) = 1$, if and only if $B = C$;
- (4) If $B \subseteq C \subseteq D$, then $S(B, D) \leq S(B, C)$ and $S(B, D) \leq S(C, D)$.

2.2 The Existing PF Similarity Measures

Next, we write the existing PF similarity measure in [19–26] [33]. Some of the shortcomings and limitations of these PF similarity measures will be illustrated by comparative analysis in Sect 4.

PF similarity measures $S_{W1}, S_{W2}, S_{W3}, S_{W4}$ [19]:

$$S_{W1}(B, C) = \frac{1}{n} \sum_{k=1}^n \cos \left[\frac{\pi}{2} \left(|m_B(x_k) - m_C(x_k)| \vee |\eta_B(x_k) - \eta_C(x_k)| \vee |v_B(x_k) - v_C(x_k)| \right) \right]; \quad (1)$$

$$S_{W2}(B, C) = \frac{1}{n} \sum_{k=1}^n \cos \left[\frac{\pi}{2} \left(|m_B(x_k) - m_C(x_k)| \vee |\eta_B(x_k) - \eta_C(x_k)| \vee |v_B(x_k) - v_C(x_k)| \vee |\rho_B(x_k) - \rho_C(x_k)| \right) \right]; \quad (2)$$

$$S_{W3}(B, C) = \frac{1}{n} \sum_{k=1}^n \cos \left[\frac{\pi}{4} \left(|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| \right) \right]; \quad (3)$$

$$S_{W4}(B, C) = \frac{1}{n} \sum_{k=1}^n \cot \left[\frac{\pi}{4} + \frac{\pi}{4} \left(|m_B(x_k) - m_C(x_k)| \vee |\eta_B(x_k) - \eta_C(x_k)| \right) \right]. \quad (4)$$

PF similarity measures S_{W5} [20]:

$$S_{W5}(B, C) = \frac{1}{n} \sum_{k=1}^n \frac{m_B(x_k)m_C(x_k) + \eta_B(x_k)\eta_C(x_k) + v_B(x_k)v_C(x_k)}{\max \left((m_B(x_k))^2 + (\eta_B(x_k))^2 + (v_B(x_k))^2, (m_C(x_k))^2 + (\eta_C(x_k))^2 + (v_C(x_k))^2 \right)}. \quad (5)$$

PF similarity measures S_{WG1}, S_{WG2} [21]:

$$S_{WG1}(B, C) = \frac{1}{n} \sum_{k=1}^n \frac{2(m_B(x_k)m_C(x_k) + \eta_B(x_k)\eta_C(x_k) + v_B(x_k)v_C(x_k))}{\left((m_B(x_k))^2 + (\eta_B(x_k))^2 + (v_B(x_k))^2 \right) + \left((m_C(x_k))^2 + (\eta_C(x_k))^2 + (v_C(x_k))^2 \right)}; \quad (6)$$

$$S_{WG2}(B, C) = \frac{1}{n} \sum_{k=1}^n \frac{2(m_B(x_k)m_C(x_k) + \eta_B(x_k)\eta_C(x_k) + v_B(x_k)v_C(x_k) + \rho_B(x_k)\rho_C(x_k))}{(m_B^2(x_k) + \eta_B^2(x_k) + v_B^2(x_k) + \rho_B^2(x_k)) + (m_C^2(x_k) + \eta_C^2(x_k) + v_C^2(x_k) + \rho_C^2(x_k))}. \quad (7)$$

PF similarity measures $S_{SMK1}, S_{SMK2}, S_{SMK3}$ [22]:

$$S_{SMK1}(B, C) = 1 - \frac{1}{4n} \sum_{k=1}^n \left(|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| \right); \quad (8)$$

$$S_{SMK2}(B, C) = 1 - \frac{1}{4n} \sum_{k=1}^n \max \left(|m_B(x_k) - m_C(x_k)|, |\eta_B(x_k) - \eta_C(x_k)|, |v_B(x_k) - v_C(x_k)|, |\rho_B(x_k) - \rho_C(x_k)| \right); \quad (9)$$

$$S_{SMK3}(B, C) = \frac{1}{4n} \sum_{k=1}^n \frac{\min(|m_B(x_k) - m_C(x_k)|, |\eta_B(x_k) - \eta_C(x_k)|, |v_B(x_k) - v_C(x_k)|, |\rho_B(x_k) - \rho_C(x_k)|)}{\max(|m_B(x_k) - m_C(x_k)|, |\eta_B(x_k) - \eta_C(x_k)|, |v_B(x_k) - v_C(x_k)|, |\rho_B(x_k) - \rho_C(x_k)|)}. \quad (10)$$

PF similarity measure S_{GS1} [24]:

$$S_{GS1}(B, C) = \frac{1}{4n} \sum_{k=1}^n \left[\begin{aligned} & 3\sqrt{m_B(x_k)m_C(x_k)} + 3\sqrt{\eta_B(x_k)\eta_C(x_k)} \\ & + 3\sqrt{v_B(x_k)v_C(x_k)} + \sqrt{\rho_B(x_k)\rho_C(x_k)} \\ & + \sqrt{(1 - m_B(x_k) - \eta_B(x_k)) \times (1 - m_C(x_k) - \eta_C(x_k))} \\ & + \sqrt{(1 - m_B(x_k) - v_B(x_k)) \times (1 - m_C(x_k) - v_C(x_k))} \\ & + \sqrt{(1 - v_B(x_k) - \eta_B(x_k)) \times (1 - v_C(x_k) - \eta_C(x_k))} \end{aligned} \right] \quad (11)$$

PF similarity measures $S_{GS2}, S_{GS3}, S_{GS4}, S_{GS5}$ [25]:

$$S_{GS2}(B, C) = \frac{1}{n} \sum_{k=1}^n \left\{ 2^{[1 - (|m_B(x_k) - m_C(x_k)| \vee |\eta_B(x_k) - \eta_C(x_k)| \vee |v_B(x_k) - v_C(x_k)|)]} - 1 \right\};$$

$$S_{GS4}(B, C) = \frac{1}{n} \sum_{k=1}^n \left\{ 2^{[1 - (|m_B(x_k) - m_C(x_k)| \vee |\eta_B(x_k) - \eta_C(x_k)| \vee |v_B(x_k) - v_C(x_k)| \vee |\rho_B(x_k) - \rho_C(x_k)|)]} - 1 \right\}; \quad (14)$$

$$S_{GSS}(B, C) = \frac{1}{n} \sum_{k=1}^n \left\{ 2^{[1 - \frac{1}{2}(|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)| + |\rho_B(x_k) - \rho_C(x_k)|)]} - 1 \right\}. \quad (15)$$

PF similarity measure S_{LZ} [23]:

$$S_{LZ}(B, C) = \frac{1}{3n} \sum_{k=1}^n \left[\begin{aligned} &2\sqrt{m_B(x_k)m_C(x_k)} + 2\sqrt{\eta_B(x_k)\eta_C(x_k)} + 2\sqrt{v_B(x_k)v_C(x_k)} \\ &+ \sqrt{(1 - m_B(x_k) - \eta_B(x_k)) \times (1 - m_C(x_k) - \eta_C(x_k))} \\ &+ \sqrt{(1 - m_B(x_k) - v_B(x_k)) \times (1 - m_C(x_k) - v_C(x_k))} \\ &+ \sqrt{(1 - v_B(x_k) - \eta_B(x_k)) \times (1 - v_C(x_k) - \eta_C(x_k))} \end{aligned} \right]. \quad (16)$$

PF similarity measure S_{KK} [26]:

$$S_{KK}(B, C) = 1 - \left[\frac{1}{3n(t+1)^p} \sum_{k=1}^n \left(\begin{aligned} &|t(m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k)) - (v_B(x_k) - v_C(x_k))|^p \\ &+ |t(\eta_B(x_k) - \eta_C(x_k)) - (m_B(x_k) - m_C(x_k)) + (v_B(x_k) - v_C(x_k))|^p \\ &+ |t(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) + (\eta_B(x_k) - \eta_C(x_k))|^p \end{aligned} \right) \right]^{\frac{1}{p}}, \quad (17)$$

Where $t = 3, 4, \dots$ and $p = 1, 2, 3, \dots$.

PF similarity measure S_{TP} [33]:

$$S_{TP}(B, C) = 1 - \left[\frac{1}{6n(t+1)^p} \sum_{k=1}^n \left(\begin{aligned} &|t(m_B(x_k) - m_C(x_k)) - (v_B(x_k) - v_C(x_k))|^p \\ &+ |t(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k))|^p \\ &+ |t(m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|^p \\ &+ |t(\eta_B(x_k) - \eta_C(x_k)) - (m_B(x_k) - m_C(x_k))|^p \\ &+ |t(\eta_B(x_k) - \eta_C(x_k)) - (v_B(x_k) - v_C(x_k))|^p \\ &+ |t(v_B(x_k) - v_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|^p \end{aligned} \right) \right]^{\frac{1}{p}}, \quad (18)$$

Where $t = 2, 3, 4, \dots$ and $p = 1, 2, 3, \dots$.

3 A Hybrid Similarity Measures for PFSs

3.1 Background

Let B and C be two PFSs in universe of discourse X , where $B = \{ \langle x_k, m_B(x_k), \eta_B(x_k), v_B(x_k) \rangle \}$ and $C = \{ \langle x_k, m_C(x_k), \eta_C(x_k), v_C(x_k) \rangle \}$.

Due to $0 \leq m_B(x_k) + \eta_B(x_k) + v_B(x_k) \leq 1$ and $\rho_B(x_k) = 1 - m_B(x_k) - \eta_B(x_k) - v_B(x_k)$, the membership degree $m_B(x_k)$ can be taken up to $m_B(x_k) + \rho_B(x_k)$. In the same way, the neutral degree $\eta_B(x_k)$ can be taken up to $\eta_B(x_k) + \rho_B(x_k)$, the non-membership degree $v_B(x_k)$ can be taken up to $v_B(x_k) + \rho_B(x_k)$. We establish a spatial Cartesian coordinate system with membership, neutrality and

non-membership as three directions. Take points $B_1(m_B(x_k), \eta_B(x_k), v_B(x_k))$, $B_2(m_B(x_k) + \rho_B(x_k), \eta_B(x_k), v_B(x_k))$, $B_3(m_B(x_k), \eta_B(x_k) + \rho_B(x_k), v_B(x_k))$, $B_4(m_B(x_k), \eta_B(x_k), v_B(x_k) + \rho_B(x_k))$, and the tetrahedron obtained by connecting the four points is all the value ranges of PFS B . Since the centroid of the tetrahedron has the largest amount of information, the centroid of the tetrahedron is taken as $(m_B(x_k) + \frac{\rho_B(x_k)}{4}, \eta_B(x_k) + \frac{\rho_B(x_k)}{4}, v_B(x_k) + \frac{\rho_B(x_k)}{4})$. Since $\rho_B(x_k) = 1 - m_B(x_k) - \eta_B(x_k) - v_B(x_k)$, therefore, the centroid can be expressed as $(\frac{1+3m_B(x_k)-\eta_B(x_k)-v_B(x_k)}{4},$

$\frac{1+3\eta_B(x_k)-m_B(x_k)-v_B(x_k)}{4}, \frac{1+3v_B(x_k)-m_B(x_k)-\eta_B(x_k)}{4})$. Similarly, the tetrahedral centroid formed by the membership degree, neutral degree and non-membership degree of PFSs C is expressed as $(\frac{1+3m_C(x_k)-\eta_C(x_k)-v_C(x_k)}{4}, \frac{1+3\eta_C(x_k)-m_C(x_k)-v_C(x_k)}{4}, \frac{1+3v_C(x_k)-m_C(x_k)-\eta_C(x_k)}{4})$.

The distance between the two centroids in the membership degree direction is $|\frac{1+3m_B(x_k)-\eta_B(x_k)-v_B(x_k)}{4} - \frac{1+3m_C(x_k)-\eta_C(x_k)-v_C(x_k)}{4}| = |\frac{3(m_B(x_k)-m_C(x_k))}{4} - (\eta_B(x_k) - \eta_C(x_k)) - (v_B(x_k) - v_C(x_k))|/4$. The distance between the two centroids in the neutral degree direction is $|\frac{1+3\eta_B(x_k)-m_B(x_k)-v_B(x_k)}{4} - \frac{1+3\eta_C(x_k)-m_C(x_k)-v_C(x_k)}{4}| = |\frac{3(\eta_B(x_k)-\eta_C(x_k))-(m_B(x_k)-m_C(x_k))}{4} - (v_B(x_k) - v_C(x_k))|/4$. The distance between the two centroids in the non-membership

degree direction is $\left| \frac{1+3v_B(x_k)-m_B(x_k)-\eta_B(x_k)}{4} - \frac{1+3v_C(x_k)-m_C(x_k)-\eta_C(x_k)}{4} \right| = \left| 3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k)) \right| 4$.

Khan et al. [26] constructed a similarity measure S_{KK} for PFSs based on the distance between the two tetrahedral centroid. However, the similarity measure formula (17) given at the end changes the sign in the direction of neutral and non-membership degree, then these two directions are not the distance between the centroids.

When Ganie and Singh [24] studied PF similarity measure based on Hamming distance [22], they found some unreasonable results in calculating different similarity measure. In addition, the application in the pattern recognition problem will have unclassifiable results. Due to the shortcomings of the PF similarity measure based on the tetrahedral centroid distance proposed by Khan et al and the Hamming distance proposed by Singh et al, we next consider a hybrid PF similarity measure which combines the Hamming distance and the transformed tetrahedral centroid distance.

3.2 PF Similarity Measure Combines the Hamming Distance and the Transformed Tetrahedral Centroid Distance

In this section, we define a new PF similarity measure and verify that definition 3 is satisfied.

Let B and C be two PFSs in universe of discourse X , where $B = \{ \langle x_k, m_B(x_k), \eta_B(x_k), v_B(x_k) \rangle \}$ and $C = \{ \langle x_k, m_C(x_k), \eta_C(x_k), v_C(x_k) \rangle \}$. We introduce a hybrid PF similarity measure which combines the Hamming distance and the transformed tetrahedral centroid distance between the PFSs B and C as

$\left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right)$ denotes the geometric mean of rejection degree as the weight.

Theorem 1 $S_L(B, C)$ is a PF similarity measure between the two PFSs B and C , due to satisfying the four properties of definition 3.

Proof (1) $0 \leq S_L(B, C) \leq 1$

We can write the following equations:

$$|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))| = |(3v_B(x_k) - m_B(x_k) - \eta_B(x_k)) - (3v_C(x_k) - m_C(x_k) - \eta_C(x_k))|.$$

Since from the definition 1 of PFS, we have

$$0 \leq m_B(x_k) \leq 1, 0 \leq \eta_B(x_k) \leq 1, 0 \leq v_B(x_k) \leq 1, 0 \leq m_C(x_k) \leq 1, 0 \leq \eta_C(x_k) \leq 1, 0 \leq v_C(x_k) \leq 1, 0 \leq m_B(x_k) + \eta_B(x_k) + v_B(x_k) \leq 1, 0 \leq m_C(x_k) + \eta_C(x_k) + v_C(x_k) \leq 1.$$

Therefore, we have the following inequalities:

$$\begin{aligned} -1 &\leq 3v_B(x_k) - m_B(x_k) - \eta_B(x_k) \leq 3; \\ -3 &\leq -(3v_C(x_k) - m_C(x_k) - \eta_C(x_k)) \leq 1; \\ 0 &\leq |m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)| \leq 2, \end{aligned}$$

then we have

$$-4 \leq (3v_B(x_k) - m_B(x_k) - \eta_B(x_k)) - (3v_C(x_k) - m_C(x_k) - \eta_C(x_k)) \leq 4.$$

So, we have the following inequalities:

$$0 \leq \frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \leq 1;$$

$$0 \leq \frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4} \leq 1.$$

Due to $\rho_B(x_k) = 1 - m_B(x_k) - \eta_B(x_k) - v_B(x_k)$ and $\rho_C(x_k) = 1 - m_C(x_k) - \eta_C(x_k) - v_C(x_k)$, we obtain

$$0 \leq \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \leq 1, \text{ furthermore, we have:}$$

$$0 \leq \frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right) \leq 1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2};$$

$$S_L(B, C) = 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right) + \frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right]. \quad (19)$$

Where $\frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4}$ denotes the distance of the two centroids in the non-membership degree direction. Where $\frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2}$ denotes the Hamming distance base on membership degree, neutral degree and non-membership degree. Where $\frac{\rho_B(x_k) + \rho_C(x_k)}{2}$ and

$$0 \leq \frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \leq \frac{\rho_B(x_k) + \rho_C(x_k)}{2}.$$

It means that

$$0 \leq \frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2}\right) + \frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \leq 1.$$

Then we have the following inequality:

$$0 \leq \frac{1}{n} \sum_{k=1}^n \left[\frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2}\right) + \frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right] \leq 1$$

Finally, $0 \leq S_L(B, C) \leq 1$.

(2) $S_L(B, C) = S_L(C, B)$

$$\begin{aligned} S_L(C, B) &= 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{|m_C(x_k) - m_B(x_k)| + |\eta_C(x_k) - \eta_B(x_k)| + |v_C(x_k) - v_B(x_k)|}{2} \times \left(1 - \frac{\rho_C(x_k) + \rho_B(x_k)}{2}\right) + \frac{|3(v_C(x_k) - v_B(x_k)) - (m_C(x_k) - m_B(x_k)) - (\eta_C(x_k) - \eta_B(x_k))|}{4} \times \frac{\rho_C(x_k) + \rho_B(x_k)}{2} \right] \\ &= 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2}\right) + \frac{|-\{3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))\}|}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right] \\ &= 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2}\right) + \frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right] \end{aligned}$$

$= S_L(B, C)$.

(3) $S_L(B, C) = 1$, if and only if $B = C$.

“ $\Rightarrow$ ” If $S_L(B, C) = 1$, according to formula (19), we have:

$$\frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2}\right) + \frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} = 0$$

If $\frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2}\right) = 0$, then we can get $m_B(x_k) = m_C(x_k)$, $\eta_B(x_k) = \eta_C(x_k)$, $v_B(x_k) = v_C(x_k)$ mean $B = C$ or $\frac{\rho_B(x_k) + \rho_C(x_k)}{2} = 1$.

If $\frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} = 0$, then we can get $3v_B(x_k) - m_B(x_k) - \eta_B(x_k) = 3v_C(x_k) - m_C(x_k) - \eta_C(x_k)$ or $\frac{\rho_B(x_k) + \rho_C(x_k)}{2} = 0$.

$\eta_B(x_k) = 3v_C(x_k) - m_C(x_k) - \eta_C(x_k)$ or $\frac{\rho_B(x_k) + \rho_C(x_k)}{2} = 0$.

Because $\frac{\rho_B(x_k) + \rho_C(x_k)}{2} = 1$ and $\frac{\rho_B(x_k) + \rho_C(x_k)}{2} = 0$ are disjoint events, we talk about the following one case: If $3v_B(x_k) - m_B(x_k) - \eta_B(x_k) = 3v_C(x_k) - m_C(x_k) - \eta_C(x_k)$ and $\frac{\rho_B(x_k) + \rho_C(x_k)}{2} = 1$, then we can get $\rho_B(x_k) = \rho_C(x_k) = 1$, it means that $v_B(x_k) = m_B(x_k) = \eta_B(x_k) = v_C(x_k) = m_C(x_k) = \eta_C(x_k) = 0$. Therefore, we have $B = C$.

Accordingly, we have proved that if $S_L(B, C) = 1$ then $B = C$.

“ $\Leftarrow$ ” If $B = C$, according to definition 2, we have: $m_B(x_k) = m_C(x_k)$, $\eta_B(x_k) = \eta_C(x_k)$, $v_B(x_k) = v_C(x_k)$, $\rho_B(x_k) = \rho_C(x_k)$. According to formula (19), we have $S_L(B, C) = 1$.

(4) If $B \subseteq C \subseteq D$, then $S_L(B, C) \geq S_L(B, D)$ and $S_L(C, D) \geq S_L(B, D)$.

According to formula (19), we know that:

$$S_L(B, C) = 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right) \right. \\ \left. + \frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right];$$

$$S_L(B, D) = 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{|m_B(x_k) - m_D(x_k)| + |\eta_B(x_k) - \eta_D(x_k)| + |v_B(x_k) - v_D(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_D(x_k)}{2} \right) \right. \\ \left. + \frac{|3(v_B(x_k) - v_D(x_k)) - (m_B(x_k) - m_D(x_k)) - (\eta_B(x_k) - \eta_D(x_k))|}{4} \times \frac{\rho_B(x_k) + \rho_D(x_k)}{2} \right].$$

We can write the following equations:

$$|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))| \\ = |(3v_B(x_k) - m_B(x_k) - \eta_B(x_k)) - (3v_C(x_k) - m_C(x_k) - \eta_C(x_k))|;$$

$$|3(v_B(x_k) - v_D(x_k)) - (m_B(x_k) - m_D(x_k)) - (\eta_B(x_k) - \eta_D(x_k))| \\ = |(3v_B(x_k) - m_B(x_k) - \eta_B(x_k)) - (3v_D(x_k) - m_D(x_k) - \eta_D(x_k))|.$$

If $B \subseteq C \subseteq D$, according to definition 2, we have:

$$0 \leq m_B(x_k) \leq m_C(x_k) \leq m_D(x_k) \leq 1, 0 \leq \eta_B(x_k) \leq \eta_C(x_k) \leq \eta_D(x_k) \leq 1 \text{ and } 1 \geq v_B(x_k) \geq v_C(x_k) \geq v_D(x_k) \geq 0.$$

Therefore, we have the following inequalities:

$$3v_B(x_k) - m_B(x_k) - \eta_B(x_k) \geq 3v_C(x_k) - m_C(x_k) - \eta_C(x_k) \\ \geq 3v_D(x_k) - m_D(x_k) - \eta_D(x_k).$$

Hence, we have

$$S_L(B, C) = 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{(m_C(x_k) - m_B(x_k)) + (\eta_C(x_k) - \eta_B(x_k)) + (v_B(x_k) - v_C(x_k))}{2} - \right. \\ \left. \frac{(m_C(x_k) - m_B(x_k)) + (\eta_C(x_k) - \eta_B(x_k)) + (v_B(x_k) - v_C(x_k))}{2} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} + \right. \\ \left. \frac{3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right] \\ = 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{(m_C(x_k) - m_B(x_k)) + (\eta_C(x_k) - \eta_B(x_k)) + (v_B(x_k) - v_C(x_k))}{2} - \right. \\ \left. \frac{(m_C(x_k) - m_B(x_k)) + (\eta_C(x_k) - \eta_B(x_k)) - (v_B(x_k) - v_C(x_k))}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right] \\ = 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{(m_C(x_k) - m_B(x_k)) + (\eta_C(x_k) - \eta_B(x_k)) + (v_B(x_k) - v_C(x_k))}{2} - \right. \\ \left. \frac{(m_C(x_k) + \eta_C(x_k) + v_C(x_k)) - (m_B(x_k) + \eta_B(x_k) + v_B(x_k))}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right] \\ = 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{(m_C(x_k) - m_B(x_k)) + (\eta_C(x_k) - \eta_B(x_k)) + (v_B(x_k) - v_C(x_k))}{2} \right. \\ \left. - \frac{(1 - \rho_C(x_k)) - (1 - \rho_B(x_k))}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right] \\ = 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{(m_C(x_k) - m_B(x_k)) + (\eta_C(x_k) - \eta_B(x_k)) + (v_B(x_k) - v_C(x_k))}{2} - \frac{\rho_B(x_k)^2 - \rho_C(x_k)^2}{8} \right].$$

Similarly, we can obtain:

$$S_L(B, D) = 1 - \frac{1}{n} \sum_{k=1}^n \left[\frac{(m_D(x_k) - m_B(x_k)) + (\eta_D(x_k) - \eta_B(x_k)) + (v_B(x_k) - v_D(x_k))}{2} - \frac{\rho_B(x_k)^2 - \rho_D(x_k)^2}{8} \right].$$

Accordingly, we have

$$\begin{aligned}
& S_L(B, C) - S_L(B, D) \\
&= \frac{1}{n} \sum_{k=1}^n \left[\frac{(m_D(x_k) - m_C(x_k)) + (\eta_D(x_k) - \eta_C(x_k)) + (v_C(x_k) - v_D(x_k))}{2} + \frac{\rho_D(x_k)^2 - \rho_C(x_k)^2}{8} \right] \\
&= \frac{1}{n} \sum_{k=1}^n \left[\frac{4[(m_D(x_k) - m_C(x_k)) + (\eta_D(x_k) - \eta_C(x_k)) + (v_C(x_k) - v_D(x_k))]}{8} \right. \\
&\quad \left. + \frac{(\rho_D(x_k) - \rho_C(x_k))(\rho_D(x_k) + \rho_C(x_k))}{8} \right] \\
&= \frac{1}{8n} \sum_{k=1}^n \left\{ \begin{aligned} & 4[(m_D(x_k) - m_C(x_k)) + (\eta_D(x_k) - \eta_C(x_k)) + (v_C(x_k) - v_D(x_k))] + \\ & [(v_C(x_k) - v_D(x_k)) - (m_D(x_k) - m_C(x_k)) - (\eta_D(x_k) - \eta_C(x_k))] \times (\rho_D(x_k) + \rho_C(x_k)) \end{aligned} \right\} \\
&= \frac{1}{8n} \sum_{k=1}^n \left\{ \begin{aligned} & [(m_D(x_k) - m_C(x_k)) + (\eta_D(x_k) - \eta_C(x_k))] \times (4 - \rho_D(x_k) - \rho_C(x_k)) \\ & + (v_C(x_k) - v_D(x_k)) \times (4 + \rho_D(x_k) + \rho_C(x_k)) \end{aligned} \right\}.
\end{aligned}$$

Because of $m_C(x_k) \leq m_D(x_k)$, $\eta_C(x_k) \leq \eta_D(x_k)$, $v_C(x_k) \geq v_D(x_k)$ and $0 \leq \rho_C(x_k) + \rho_D(x_k) \leq 2$, we can get $S_L(B, C) - S_L(B, D) \geq 0$, so $S_L(B, C) \geq S_L(B, D)$.

Similarly, we can prove $S_L(C, D) \geq S_L(B, D)$.

If we consider the weights w_k of x_k such that $w_k \in [0, 1]$ and $\sum_{k=1}^n w_k = 1$, then the PF weighted similarity measure between the PFSs B and C can be defined as

4 Experiments and Analysis

In this section, to verify the rationality and superiority of our proposed PF similarity measure in practical cases, the proposed and existing PF similarity measure formulas (1)–(19) are compared and analyzed through numerical examples and pattern recognition.

$$S_{WL}(B, C) = 1 - \sum_{k=1}^n w_k \left[\frac{|m_B(x_k) - m_C(x_k)| + |\eta_B(x_k) - \eta_C(x_k)| + |v_B(x_k) - v_C(x_k)|}{2} \times \left(1 - \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right) \right. \\
\left. + \frac{|3(v_B(x_k) - v_C(x_k)) - (m_B(x_k) - m_C(x_k)) - (\eta_B(x_k) - \eta_C(x_k))|}{4} \times \frac{\rho_B(x_k) + \rho_C(x_k)}{2} \right]. \quad (20)$$

Theorem 2 $S_{WL}(B, C)$ is a PF similarity measure between the two PFSs B and C .

Proof According to the above proof process and definition 3, it is straightforward.

4.1 Numerical Comparisons

In the previous section, the newly proposed PF similarity measure combines the Hamming distance and the transformed tetrahedral centroid distance. Next, we use an example to illustrate the counterintuitive situation of the PF similarity measure proposed by Khan et al. [26] based on the transformed tetrahedral centroid, and the insufficiency

Table 1 Comparison results of PF similarity measure

Similarity measures	(B, D)	(C, D)
S_{W1}	0.707	0
S_{W2}	0	0
S_{W3}	0	0
S_{W4}	0	0
S_{W5}	0	0
S_{WG1}	0	0
S_{WG2}	0	0
S_{SMK1}	0.5	0.5
S_{SMK2}	0.75	0.75
S_{SMK3}	0	0
S_{GS1}	0.354	0.25
S_{GS2}	0.414	0
S_{GS3}	0.414	0.414
S_{GS4}	0	0
S_{GS5}	0	0
S_{LZ}	0.471	0.333
S_{KK}	0.833	0.583
S_{TP}	0.778	0.667
$S_L(\text{proposed})$	0.625	0.375

Bold fonts in tables indicate unconscionable results. $t = 3, p = 1$ in S_{KK} . $t = 2, p = 1$ in S_{TP}

Table 3 The results of different similarity measures for pattern recognition

Similarity measures	(B_1, C)	(B_2, C)	(B_3, C)	Classification results
S_{W1}	0.951	0.881	0.875	B_1
S_{W2}	0.951	0.881	0.875	B_1
S_{W3}	0.951	0.857	0.81	B_1
S_{W4}	0.726	0.674	0.678	B_1
S_{W5}	0.819	0.722	0.686	B_1
S_{WG1}	0.926	0.766	0.754	B_1
S_{WG2}	0.912	0.778	0.75	B_1
S_{SMK1}	0.9	0.85	0.83	B_1
S_{SMK2}	0.95	0.935	0.935	B_1
S_{SMK3}	0	0.083	0.087	B_3
S_{GS1}	0.944	0.922	0.904	B_1
S_{GS2}	0.741	0.683	0.685	B_1
S_{GS3}	0.821	0.66	0.651	B_1
S_{GS4}	0.741	0.683	0.685	B_1
S_{GS5}	0.741	0.637	0.599	B_1
S_{LZ}	0.952	0.926	0.888	B_1
S_{KK}	0.851	0.813	0.772	B_1
S_{TP}	0.9	0.844	0.838	B_1
$S_L(\text{proposed})$	0.858	0.737	0.729	B_1

$t = 3, p = 1$ in S_{KK} . $t = 2, p = 1$ in S_{TP}

Table 2 Comparison results of PF similarity measure

Similarity measures	(B, C)	(C, D)	(B, D)	Ordering
S_{W1}	0.988	0.988	0.988	$S(B, C) = S(C, D) = S(B, D)$
S_{W2}	0.988	0.988	0.988	$S(B, C) = S(C, D) = S(B, D)$
S_{W3}	0.988	0.988	0.988	$S(B, C) = S(C, D) = S(B, D)$
S_{W4}	0.854	0.854	0.854	$S(B, C) = S(C, D) = S(B, D)$
S_{W5}	0.833	0.933	0.852	$S(C, D) > S(B, D) > S(B, C)$
S_{WG1}	0.98	0.982	0.958	$S(C, D) > S(B, C) > S(B, D)$
S_{WG2}	0.969	0.971	0.97	$S(C, D) > S(B, D) > S(B, C)$
S_{SMK1}	0.95	0.95	0.95	$S(B, C) = S(C, D) = S(B, D)$
S_{SMK2}	0.975	0.975	0.975	$S(B, C) = S(C, D) = S(B, D)$
S_{SMK3}	0	0	0	$S(B, C) = S(C, D) = S(B, D)$
S_{GS1}	0.995	0.991	0.99	$S(B, C) > S(C, D) > S(B, D)$
S_{GS2}	0.866	0.866	0.866	$S(B, C) = S(C, D) = S(B, D)$
S_{GS3}	0.932	0.932	0.866	$S(B, C) = S(C, D) > S(B, D)$
S_{GS4}	0.866	0.866	0.866	$S(B, C) = S(C, D) = S(B, D)$
S_{GS5}	0.866	0.866	0.866	$S(B, C) = S(C, D) = S(B, D)$
S_{LZ}	0.996	0.993	0.991	$S(B, C) > S(C, D) > S(B, D)$
S_{KK}	0.958	0.958	0.917	$S(B, C) = S(C, D) > S(B, D)$
S_{TP}	0.967	0.967	0.933	$S(B, C) = S(C, D) > S(B, D)$
$S_L(\text{proposed})$	0.956	0.944	0.9	$S(B, C) > S(C, D) > S(B, D)$

Bold fonts in tables indicate unconscionable results. $t = 3, p = 1$ in S_{KK} . $t = 2, p = 1$ in S_{TP}

Table 4 The results of different similarity measures for pattern recognition

Similarity measures	(B_1, C)	(B_2, C)	Classification results
S_{W1}	0	0	Not Classified
S_{W2}	0	0	Not Classified
S_{W3}	0	0	Not Classified
S_{W4}	0	0	Not Classified
S_{W5}	0	0	Not Classified
S_{WG1}	0	0	Not Classified
S_{WG2}	0	0	Not Classified
S_{SMK1}	0.5	0.5	Not Classified
S_{SMK2}	0.75	0.75	Not Classified
S_{SMK3}	0.05	0.025	B_1
S_{GS1}	0.177	0.177	Not Classified
S_{GS2}	0	0	Not Classified
S_{GS3}	0.189	0.189	Not Classified
S_{GS4}	0	0	Not Classified
S_{GS5}	0	0	Not Classified
S_{LZ}	0.236	0.236	Not Classified
S_{KK}	0.625	0.692	B_2
S_{TP}	0.556	0.533	B_1
$S_L(\text{proposed})$	0.331	0.306	B_1

Bold fonts in tables indicate unconscionable results. $t = 3, p = 1$ in S_{KK} . $t = 2, p = 1$ in S_{TP}

of the PF similarity measure proposed by Singh et al. [22] based on the Hamming distance.

Example 1 Let us consider the following three PFSs in the universe of discourse $X = \{x\}$, where $B = \{\langle 0.1, 0.1, 0.4 \rangle\}$, $C = \{\langle 0.1, 0.6, 0.3 \rangle\}$ and $D = \{\langle 0.1, 0.6, 0.2 \rangle\}$. By definition 2(3), the relationship among PFSs B , C and D is obtained as $B \subseteq C \subseteq D$. It can be obtained from definition 3(4): $S(B, C) \geq S(B, D)$. Using the PF similarity measure formula (8) based on the Hamming distance, we calculate $S_{SMK1}(B, C) = 0.75$ and $S_{SMK1}(B, D) = 0.75$. Using the PF similarity measure formula (17) based on the transformed tetrahedral centroid distance, we calculate $S_{KK}(B, C) = 0.833$ and $S_{KK}(B, D) = 0.858$. Using our proposed PF similarity measure formula (19) that combines the Hamming distance and the transformed tetrahedral centroid distance, we calculate $S_L(B, C) = 0.72$ and $S_L(B, D) = 0.669$.

From the above calculation results, we can see that the similarity measure S_{KK} based on the tetrahedral transformation centroid distance violates Definition 3(4), which is obviously counter-intuitive. Although the similarity measure S_{SMK1} based on Hamming distance does not violate the basic definition, it will be difficult to attribute PFS B to C or D if it is applied to pattern recognition problems. And our proposed similarity measure S_L not only has no

unreasonable results, but also clearly sees that PFSs B and C are more similar. S_L is proposed by combining the Hamming distance and the transformed tetrahedral centroid distance. It not only combines the advantages of the two distances but also overcomes their shortcomings.

The next two examples further illustrate some of the advantages of our proposed PF similarity measure through comparative analysis with existing PF similarity measures.

Example 2 Let us consider the following three PFSs in the universe of discourse $X = \{x\}$, where $B = \{\langle 0.0, 0.5, 0.0 \rangle\}$, $C = \{\langle 0.0, 0.0, 1.0 \rangle\}$ and $D = \{\langle 0.0, 0.0, 0.0 \rangle\}$. Apparently PFS D is closer to B than to C . So we can get that the similarity measure between B and D is higher than that between C and D , that is, $S(B, D) > S(C, D)$. Table 1 is a detailed comparison of the results of calculating similarity measure data using formulas (1)–(19).

In Table 1, it can be clearly seen that the PF similarity measures S_{W2} , S_{W3} , S_{W4} , S_{W5} , S_{WG1} , S_{WG2} , S_{SMK1} , S_{SMK2} , S_{SMK3} , S_{GS3} , S_{GS4} and S_{GS5} appear unreasonable cases, that is, $S(B, D) = S(C, D)$. This is the same as Contrary to our above analysis, it is obviously impossible to distinguish the similarity of PFSs B , C and D , respectively. However, there is no unreasonable situation between our proposed similarity measure S_L and the existing similarity measures S_{W1} , S_{GS1} , S_{GS2} , S_{LZ} , S_{KK} and S_{TP} , which shows that our proposed similarity measure is effective in the instance.

Example 3 Let us consider the following three PFSs in the universe of discourse $X = \{x\}$, where $B = \{\langle 0.4, 0.1, 0.2 \rangle\}$, $C = \{\langle 0.5, 0.1, 0.2 \rangle\}$ and $D = \{\langle 0.5, 0.1, 0.1 \rangle\}$. By definition 2(3), the relationship among PFSs B , C and D is obtained as $B \subseteq C \subseteq D$. Therefore, according to definition 3(4), the similarity measure between these three PFSs should follow the following size relationships $S(B, C) \geq S(B, D)$ and $S(C, D) \geq S(B, D)$. Table 2 is a detailed comparison of the results of calculating similarity measure data using formulas (1)–(19).

In Table 2, it can be clearly seen that the PF similarity measures S_{W5} and S_{GS2} appear counter-intuitive, that is, $S(B, C) < S(B, D)$. This violates the fourth property of similarity measure definition 3. We observe that the PF similarity measures S_{W1} , S_{W2} , S_{W3} , S_{W4} , S_{SMK1} , S_{SMK2} , S_{SMK3} , S_{GS2} , S_{GS4} and S_{GS5} all show the result of $S(B, C) = S(C, D) = S(B, D)$, so none of them can distinguish between the three PFSs degree of similarity. The PF similarity measures S_{GS3} , S_{KK} and S_{TP} showed the result of $S(B, C) = S(C, D)$, so none of them could distinguish the similarity degree between C and different PFSs B and D . The above-mentioned thirteen similarity measure formulas all appear in weak order. If they are applied to decision-

Table 5 The results of different similarity measures for pattern recognition

Similarity measures	(B_1, C)	(B_2, C)	(B_3, C)	Classification results
S_{W1}	0.975	0.975	0.88	Not Classified
S_{W2}	0.976	0.976	0.666	Not Classified
S_{W3}	0.943	0.943	0.639	Not Classified
S_{W4}	0.812	0.812	0.473	Not Classified
S_{W5}	0.828	0.841	0.466	B_2
S_{WG1}	0.925	0.956	0.539	B_2
S_{WG2}	0.922	0.927	0.55	B_2
S_{SMK1}	0.9	0.9	0.75	Not Classified
S_{SMK2}	0.967	0.967	0.883	Not Classified
S_{SMK3}	0.875	0.875	0.046	Not Classified
S_{GS1}	0.953	0.974	0.81	B_2
S_{GS2}	0.824	0.824	0.627	Not Classified
S_{GS3}	0.805	0.825	0.612	B_2
S_{GS4}	0.824	0.824	0.479	Not Classified
S_{GS5}	0.744	0.744	0.443	Not Classified
S_{LZ}	0.981	0.984	0.839	B_2
S_{KK}	0.897	0.911	0.808	B_2
S_{TP}	0.919	0.93	0.867	B_2
S_L (proposed)	0.86	0.871	0.772	B_2

Bold fonts in tables indicate unconscionable results. $t = 3, p = 1$ in S_{KK} . $t = 2, p = 1$ in S_{TP}

making problems, there may be a problem that the best solution cannot be selected. However, our proposed PF similarity measure S_L and the existing S_{WG1} , S_{GS1} , S_{LZ} not only do not violate Definition 3(4), but also produce strong ranking results. The ranking result of S_L is the same as the ranking order of S_{GS1} and S_{LZ} , which is $S(B, C) > S(C, D) > S(B, D)$. From this result, the similarity measure we propose is stronger than most of the existing ones, and it is in line with the sorting problem that occurs in actual decision-making.

From the comparative analysis of the experimental results of the first three examples, it is not difficult to find that most of the existing PF similarity measures are counter-intuitive, unreasonable, and unable to be sorted in practice. Only a few PF similarity measures can handle the above problem and achieve satisfactory results, including our proposed similarity measure formula S_L . Next, we apply these formulations (1)-(19) to a pattern recognition problem to test the practicability of formulation S_L .

4.2 Application in Pattern Recognition

Pattern recognition is the problem of classifying an unknown pattern into known patterns. Due to its practi-

cality, it has been applied to practical problems such as medical diagnosis and corporate decision-making. In solving the problem of pattern recognition in PF environment, many measurement tools can be used to classify unknown patterns, such as distance measure, accuracy measure, divergence measure, similarity measure and correlation measure. We employ the similarity measure as a tool to solve the pattern recognition problem for the PF similarity measure formula (19) provided in this research, and we apply the PF similarity measure (1-19) to the pattern recognition problem to compare and analyze its benefits and drawbacks.

In PF environment, using similarity measure to solve pattern recognition problem is formulated as follows:

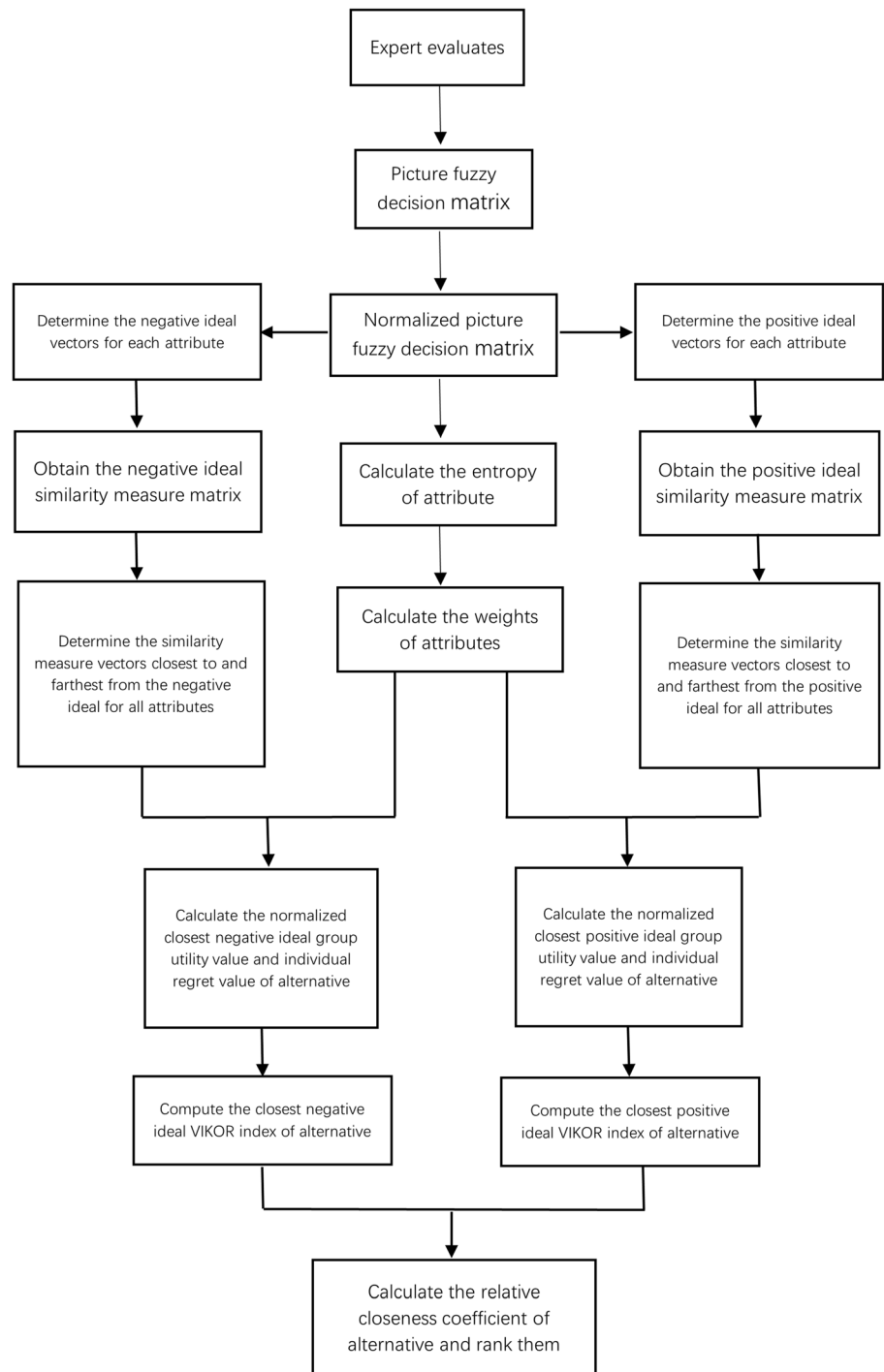
Let $B_i (i = 1, 2, \dots, n)$ be some known patterns and C be an unknown pattern, where B_i and C are given in the form of PFSs as follows:

$$B_i = \{ \langle x_k, m_{B_i}(x_k), \eta_{B_i}(x_k), \nu_{B_i}(x_k) \rangle | x_k \in X, k = 1, 2, \dots, n \};$$

$$C = \{ \langle x_k, m_C(x_k), \eta_C(x_k), \nu_C(x_k) \rangle | x_k \in X \}.$$

To classify the unknown pattern C into one of the known patterns $B_i (i = 1, 2, \dots, n)$, the similarity measure formula (1-19) is used to calculate the similarity degree between and each unknown patterns, namely $S(B_i, C)$. The calculated maximum similarity value $S(B_{i^*}, C)$ represents C

Fig. 1 Flowchart of the improved VIKOR method based on the proposed PFS similarity measure



being classified into B_{i^*} , where $i^* = \arg \max \{S(B_i, C)\}, i = 1, 2, \dots, n$.

Example 4 Let us consider the following four PFSs in the universe of discourse $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $B_1 = \{\langle 0.4, 0.1, 0.3 \rangle, \langle 0.5, 0.2, 0.3 \rangle, \langle 0.4, 0.0, 0.3 \rangle, \langle 0.7, 0.2, 0.0 \rangle, \langle 0.6, 0.1, 0.1 \rangle\}$, $B_2 = \{\langle 0.7, 0.1, 0.1 \rangle, \langle 0.2, 0.4, 0.3 \rangle, \langle 0.2, 0.5, 0.1 \rangle, \langle 0.1, 0.2, 0.5 \rangle, \langle 0.3, 0.3, 0.3 \rangle\}$, $B_3 = \{\langle 0.1, 0.4, 0.3 \rangle, \langle 0.4, 0.1, 0.3 \rangle, \langle 0.3, 0.2,$

$0.4 \rangle, \langle 0.2, 0.3, 0.5 \rangle, \langle 0.5, 0.1, 0.3 \rangle\}$, $C = \{\langle 0.6, 0.1, 0.2 \rangle, \langle 0.3, 0.2, 0.4 \rangle, \langle 0.4, 0.2, 0.3 \rangle, \langle 0.7, 0.0, 0.1 \rangle, \langle 0.4, 0.2, 0.2 \rangle\}$ from literature [40]. Now our purpose is to classify unknown pattern C into known patterns B_1, B_2, B_3 , and calculate the similarity measure of C and each known pattern, respectively. C belongs to the pattern with the largest similarity measure. The classification results of different similarity measures are shown in Table 3.

Table 6 Comparisons of the different methods with the example 7 [34]

Method	Preference orders of example 5
PFWG operator [34]	$A_4 > A_3 > A_1 > A_5 > A_2$
Extant projection model of PFSs [35]	$A_4 > A_1 > A_3 > A_5 > A_2$
PFNP-VIKOR method [37]	$A_4 > A_3 > A_1 > A_5 > A_2$
VIKOR method in the PF environment [33]	$A_4 > A_5 > A_2 > A_1 > A_3$
Improved VIKOR method	$A_4 > A_3 > A_1 > A_5 > A_2$

In Table 3, we can clearly see that the existing and our proposed PF similarity measure classifies C to known patterns B_1 , except that the PF similarity measure S_{SMK3} has different classification results. This result shows that the application of our proposed similarity measure S_L is effective on pattern recognition problems.

Example 5 Let us consider the following three PFSs in the universe of discourse $X = \{x\}$, where $B_1 = \{\langle 0.2, 0.0, 0.3 \rangle\}$, $B_2 = \{\langle 0.1, 0.0, 0.4 \rangle\}$ and $C = \{\langle 0.0, 1.0, 0.0 \rangle\}$. Now our purpose is to classify unknown pattern C into known patterns B_1 , B_2 , and calculate the similarity measure of C and each known pattern, respectively. C belongs to the pattern with the largest similarity measure. The classification results of different similarity measures are shown in Table 4.

In Table 4, we can clearly observe that the PF similarity measures S_{W1} , S_{W2} , S_{W3} , S_{W4} , S_{W5} , S_{WG1} , S_{WG2} , S_{SMK1} , S_{SMK2} , S_{GS1} , S_{GS2} , S_{GS3} , S_{GS4} , S_{GS5} and S_{LZ} all have failed classifications. While the PF similarity measures S_{SMK3} , S_{KK} , S_{TP} and S_L all successfully classified the unknown pattern C , but most similarity measures classified C to B_1 . From the above situation, we can see that our proposed similarity measure is not only successful in classification, but also the classification results are consistent with most similarity measures. This shows that the PF similarity measure S_L has a superior function in the pattern recognition problem.

Example 6 Let us consider the following four PFSs in the universe of discourse $X = \{x_1, x_2, x_3\}$, where $B_1 = \{\langle 0.5, 0.1, 0.1 \rangle, \langle 0.3, 0.1, 0.3 \rangle, \langle 0.3, 0.1, 0.4 \rangle\}$, $B_2 = \{\langle 0.4, 0.3, 0.2 \rangle, \langle 0.3, 0.2, 0.5 \rangle, \langle 0.4, 0.1, 0.3 \rangle\}$, $B_3 = \{\langle 0.2, 0.4, 0.3 \rangle, \langle 0.1, 0.4, 0.4 \rangle, \langle 0.0, 0.0, 0.0 \rangle\}$ and $C = \{\langle 0.3, 0.2, 0.3 \rangle, \langle 0.4, 0.1, 0.3 \rangle, \langle 0.4, 0.2, 0.3 \rangle\}$. Now our purpose is to classify unknown pattern C into known patterns B_1 , B_2 , B_3 , and calculate the similarity measure of C and each known pattern, respectively. C belongs to the

pattern with the largest similarity measure. The classification results of different similarity measures are shown in Table 5.

In Table 5, we clearly see that the unknown pattern C are classified into B_2 according to our proposed similarity measure S_L . These existing similarity measures S_{W5} , S_{WG1} , S_{WG2} , S_{GS1} , S_{GS3} , S_{LZ} , S_{KK} and S_{TP} are consistent with the proposed S_L classification results. Since the similarity measures S_{W1} , S_{W2} , S_{W3} , S_{W4} , S_{SMK1} , S_{SMK2} , S_{SMK3} , S_{GS2} , S_{GS4} and S_{GS5} respectively calculate the known patterns B_1 and B_2 with the same similarity value as the unknown pattern C , these existing similarity measures cannot classify the unknown pattern C . Therefore, the similarity measure S_L proposed by us can not only correctly classify unknown patterns, but also solve some pattern recognition problems that cannot be classified by existing similarity measures.

From the comparative analysis of the experimental results of different similarity measures applied to pattern recognition in Table 3, Table 4 and Table 5, it can be found that most of the existing PF similarity measures have classification failures. However, only the PF similarity measures S_{TP} and S_L can well address the above shortcomings and achieve the same classification results as most existing similarity measures. It can be seen that our proposed PF similarity measure is more effective than most existing PF similarity measures in classifying the best attribution patterns in pattern recognition problems.

In the six examples given above, applying the existing PF similarity measure formula (1)–(18) and our proposed PF similarity measure formula (19), we can clearly see that the similarity measure S_L has the following advantages:

1. Our proposed PF similarity measure combines Hamming distance and transformed tetrahedral centroid distance. While retaining the advantages of the two distances, it also overcomes the shortcomings of the PF similarity measure proposed by the two distances.
2. Most PF similarity measures cannot distinguish the similarity measures between different PFSs, while our proposed similarity measure not only can distinguish, but also achieves more reasonable results compared with a small number of similar measures that can distinguish.
3. For some highly similar but different PFSs, there is inclusion relationship. Most PF similarity measures cannot distinguish the degree of connection between these PFSs, and even violate the similarity measure definition 3 (3). Our proposed PF similarity measure can perfectly solve the above existing problems, indicating that it has better compatibility.

4. In the pattern recognition problem, the PF similarity measure proposed by us can maintain the same classification results as most PF similarity measures in the examples proposed by other literature, indicating that it has good consistency.
5. In pattern recognition problems, it can be seen from many examples that most of the existing PF similarity measures have failed in classification, while our proposed PF similarity measure can accurately classify. It can be seen from the above situation that S_L has more obvious advantages in pattern recognition than the existing similarity measure.
6. From the above six examples, it can be clearly seen that the existing PF similarity measures has some unreasonable or even wrong results, while our proposed PF similarity measure can handle all cases correctly, and its stability is better.

Based on the advantages of the PF similarity measure S_L , we next apply it to multi-attribute decision-making problems and propose an improved VIKOR method.

5 An Improved VIKOR Method Based on the Proposed Similarity Measure of PFS

In the PF environment, we propose an improved VIKOR method based on the similarity measure S_L to solve the MADM problem. In [33], Tian et al. proposed a VIKOR method in the PF environment, which only considers group utility and individual regret that are close to the positive ideal, and then obtains VIKOR values for the alternatives that are only close to the positive ideal. We improve the VIKOR method based on Tian et al. [33], which obtains an optimal alternative that considers not only the closest to the positive ideal solution, but also the farthest from the negative ideal solution. The proposed similarity measure S_L is used as a tool to calculate the group utility and individual regret values closest to the positive and negative ideals, and then the VIKOR values of the alternatives closest to the positive and negative ideals are obtained. Finally, the concept of the TOPSIS method is used to calculate the relative closeness coefficient of each alternative. The smaller the relative closeness coefficient, the better the order of the alternatives.

To verify the effectiveness and practicability of the improved VIKOR method, this method is applied to the MADM problem of literature [34], and the prioritization of the obtained alternatives is compared and analyzed with the decision method of literature [33–35] [37].

Below we will give the specific steps and flowchart to improve the VIKOR method.

5.1 Algorithmic Steps of the Improved VIKOR Method

Suppose the expert evaluates m alternatives based on n attributes, where $m = \{A_1, A_2, \dots, A_m\}$, $n = \{C_1, C_2, \dots, C_n\}$. Let $B = (b_{ij})_{m \times n}$ ($0 \leq i \leq m, 0 \leq j \leq n$) be the PF decision matrix obtained from the expert evaluation results as follows:

$$B = (b_{ij})_{m \times n} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$

where $b_{ij} = (m_{ij}, \eta_{ij}, v_{ij})$ is an evaluation value expressed by picture fuzzy values. m_{ij}, η_{ij} and v_{ij} represent the membership degree, neutral degree and non-membership degree of the alternative A_i satisfies attribute C_j . The rows of matrix B represent alternatives, and the columns represent attributes.

Step1: Constructing normalized picture fuzzy decision matrix.

In the decision matrix B , there are two types of attributes, namely benefit attribute and cost attribute. To make the following calculation simple, we uniformly transform the cost attribute into the benefit attribute according to Definition 2 (1). Therefore, the normalized picture fuzzy decision matrix $D = (d_{ij})_{m \times n}$ is shown below, where the rows represent alternatives and the columns represent attributes:

$$D = (d_{ij})_{m \times n} = \begin{cases} b_{ij} = (m_{ij}, \eta_{ij}, v_{ij}), C_j \text{ is a benefit attribute;} \\ b_{ij}^C = (v_{ij}, \eta_{ij}, m_{ij}), C_j \text{ is a cost attribute.} \end{cases} \quad (21)$$

Step2: Calculate the weights of attributes.

In the multi-attribute decision making problem, there is not much difference in the evaluation value of the alternatives under the same attribute, so we think that this attribute is not important here; On the contrary, under the same attribute, the evaluation value of the alternatives is greatly different, so we consider this attribute to be more important. Therefore, it is necessary to calculate the weight of each attribute to express its importance, and here we use picture fuzzy entropy [37] to calculate the attribute weights. The weight w_j of attribute C_j is given by the following formula:

$$w_j = \frac{1 - E_j}{n - \sum_{j=1}^n E_j}, \text{ where } w_j \in [0, 1], \text{ and } \sum_{j=1}^n w_j = 1. \quad (22)$$

Here E_j stands for picture fuzzy entropy [37] of attribute C_j :

$$E_j = 1 - \frac{1}{2m} \sum_{i=1}^m (|m_{ij} - \eta_{ij}| + |m_{ij} - v_{ij}| + |\eta_{ij} - v_{ij}|). \quad (23)$$

Step3: Determine the positive and negative ideal vectors for each attribute.

According to definitions 2(4) and (5), construct positive ideal value picture fuzzy vector F^+ and negative ideal value picture fuzzy vector F^- , respectively, as follows:

$$\begin{aligned} F^+ &= [f_1^+ \quad f_2^+ \quad \cdots \quad f_n^+], \\ F^- &= [f_1^- \quad f_2^- \quad \cdots \quad f_n^-], \\ \text{where } f_j^+ &= \left(\max_i m_{ij}, \min_i \eta_{ij}, \min_i v_{ij} \right), \\ f_j^- &= \left(\min_i m_{ij}, \min_i \eta_{ij}, \max_i v_{ij} \right). \end{aligned} \quad (24)$$

Step4: Obtain the positive ideal similarity measure matrix H^+ and the negative ideal similarity measure matrix H^- .

Based on the similarity measure formula (19), calculate the similarity measure of F^+ and D under each attribute, calculate the similarity measure of F^- and D under each attribute, and obtain the positive ideal similarity measure matrix H^+ and the negative ideal similarity measure matrix H^- .

$$\begin{aligned} H^+ &= (h_{ij}^+)_{m \times n}, \quad H^- = (h_{ij}^-)_{m \times n}, \\ \text{where } h_{ij}^+ &= S_L(d_{ij}, f_j^+), \quad h_{ij}^- = S_L(d_{ij}, f_j^-). \end{aligned} \quad (25)$$

Step5: Determine the similarity measure vectors P^+ and P^- closest to and farthest from the positive ideal for all attributes; Determine the similarity measure vectors K^+ and K^- closest to and farthest from the negative ideal for all attributes.

$$\begin{aligned} P^+ &= [p_1^+ \quad p_2^+ \quad \cdots \quad p_n^+], \quad P^- = [p_1^- \quad p_2^- \quad \cdots \quad p_n^-]; \\ K^+ &= [k_1^+ \quad k_2^+ \quad \cdots \quad k_n^+], \quad K^- = [k_1^- \quad k_2^- \quad \cdots \quad k_n^-]. \\ \text{Where } p_j^+ &= \max_i h_{ij}^+, \quad p_j^- = \min_i h_{ij}^+; \quad k_j^+ = \max_i h_{ij}^-, \quad k_j^- = \min_i h_{ij}^-. \end{aligned} \quad (26)$$

Step6: Calculate the normalized closest positive ideal group utility value PS_i and the normalized closest positive ideal individual regret value PR_i of alternative A_i , respectively, shown as follows:

$$PS_i = \sum_{j=1}^n w_j \frac{p_j^+ - h_{ij}^+}{p_j^+ - p_j^-}, \quad PR_i = \max_j \left(w_j \frac{p_j^+ - h_{ij}^+}{p_j^+ - p_j^-} \right). \quad (27)$$

Calculate the normalized closest negative ideal group utility value KS_i and the normalized closest negative ideal individual regret value KR_i of alternative A_i , respectively, shown as follows:

$$KS_i = \sum_{j=1}^n w_j \frac{k_j^+ - h_{ij}^-}{k_j^+ - k_j^-}, \quad KR_i = \max_j \left(w_j \frac{k_j^+ - h_{ij}^-}{k_j^+ - k_j^-} \right). \quad (28)$$

Step7: Compute the closest positive ideal VIKOR index Q_i^+ and the closest negative ideal VIKOR index Q_i^- of alternative A_i , respectively, shown as follows:

$$\begin{aligned} Q_i^+ &= \lambda \frac{PS_i - PS^*}{PS^- - PS^*} + (1 - \lambda) \frac{PR_i - PR^*}{PR^- - PR^*}, \\ Q_i^- &= \lambda \frac{KS_i - KS^*}{KS^- - KS^*} + (1 - \lambda) \frac{KR_i - KR^*}{KR^- - KR^*}. \end{aligned}$$

$$\text{Where } PS^- = \max_i PS_i, \quad PS^* = \min_i PS_i, \quad (29)$$

$$PR^- = \max_i PR_i, \quad PR^* = \min_i PR_i.$$

$$\text{Where } KS^- = \max_i KS_i, \quad KS^* = \min_i KS_i,$$

$$KR^- = \max_i KR_i, \quad KR^* = \min_i KR_i.$$

The values of λ and $1 - \lambda$ are the weight of the strategy of “the majority of attribute” and “the individual regret”, respectively. In general, the value of $\lambda = 0.5$ is adopted.

Step8: Calculate the relative closeness coefficient DQ_i of alternative A_i , shown as follows:

$$DQ_i = \frac{Q_i^+}{Q_i^+ + Q_i^-}. \quad (30)$$

The smaller the degree of relative closeness DQ_i , the better the preference order of alternative A_i , where $1 \leq i \leq m$.

To understand the steps of improving VIKOR method given above, the flowchart is shown in Figure 1.

5.2 Experimental Results and Comparison

Next, to demonstrate the effectiveness of the improved VIKOR method, we use numerical examples from [34] for verification to obtain the preference order of the alternatives.

Example 7 Suppose the expert evaluates five alternatives based on four attributes, where five alternatives $= \{A_1, A_2, A_3, A_4, A_5\}$, four attributes $= \{C_1, C_2, C_3, C_4\}$. PF decision matrix $B = (b_{ij})_{5 \times 4}$ of experts evaluating alternatives based on attributes is obtained from the literature [34] as follows:

$$B = \begin{bmatrix} (0.2, 0.3, 0.1) & (0.7, 0.1, 0.1) & (0.1, 0.2, 0.6) & (0.4, 0.1, 0.2) \\ (0.4, 0.2, 0.3) & (0.1, 0.6, 0.1) & (0.3, 0.2, 0.4) & (0.3, 0.1, 0.4) \\ (0.2, 0.5, 0.1) & (0.6, 0.1, 0.1) & (0.5, 0.1, 0.2) & (0.5, 0.1, 0.3) \\ (0.2, 0.3, 0.1) & (0.6, 0.2, 0.1) & (0.5, 0.3, 0.2) & (0.5, 0.0, 0.3) \\ (0.6, 0.1, 0.2) & (0.4, 0.2, 0.3) & (0.6, 0.1, 0.2) & (0.3, 0.4, 0.2) \end{bmatrix},$$

where the rows represent alternatives and the columns represent attributes.

Step1: In the literature [34], we know that attributes are all benefit attributes, and the normalized picture fuzzy decision matrix $D = B$ is obtained by formula (21).

Step2: Using formula (22) and (23), the weight of all attributes are calculated as $w_1 = 0.2027$, $w_2 = 0.3108$, $w_3 = 0.2568$ and $w_4 = 0.2297$.

Step3: Using formula (24), write the positive and negative ideal vectors, respectively, as follows:

$$F^+ = [(0.6, 0.1, 0.1) \quad (0.7, 0.1, 0.1) \quad (0.6, 0.1, 0.2) \quad (0.5, 0.0, 0.2)];$$

$$F^- = [(0.2, 0.1, 0.3) \quad (0.1, 0.1, 0.3) \quad (0.1, 0.1, 0.6) \quad (0.3, 0.0, 0.4)].$$

Step4: Through our proposed similarity measure formula (19), calculate the similarity measure of F^+ and D under each attribute, calculate the similarity measure of F^- and D under each attribute. Using formula (25), obtain the positive ideal similarity measure matrix H^+ and the negative ideal similarity measure matrix H^- as follows:

$$H^+ = \begin{bmatrix} 0.775 & 1 & 0.51 & 0.93 \\ 0.76125 & 0.52875 & 0.71 & 0.76875 \\ 0.68 & 0.95375 & 0.95375 & 0.91 \\ 0.775 & 0.91 & 0.85625 & 0.94375 \\ 0.94625 & 0.71 & 1 & 0.75 \end{bmatrix};$$

$$H^- = \begin{bmatrix} 0.8 & 0.63 & 0.95375 & 0.8 \\ 0.86875 & 0.67625 & 0.75375 & 0.95625 \\ 0.715 & 0.67625 & 0.6 & 0.81 \\ 0.8 & 0.63 & 0.505 & 0.85625 \\ 0.76875 & 0.83 & 0.55375 & 0.71 \end{bmatrix}.$$

Step5: Using formula (26), write the similarity measure vectors P^+ and P^- closest to and farthest from the positive ideal for all attributes, respectively, as follows:

$$P^+ = [0.94625 \quad 1 \quad 1 \quad 0.94375],$$

$$P^- = [0.68 \quad 0.52875 \quad 0.51 \quad 0.75].$$

Using formula (26), write the similarity measure vectors K^+ and K^- closest to and farthest from the negative ideal for all attributes, respectively, as follows:

$$K^+ = [0.86875 \quad 0.83 \quad 0.95375 \quad 0.95625],$$

$$K^- = [0.715 \quad 0.63 \quad 0.505 \quad 0.71].$$

Step6: Using formula (27), calculate the normalized closest positive ideal group utility value PS_i and the normalized closest positive ideal individual regret value PR_i of alternative A_i , respectively, shown as follows:

$$PS_1 = 0.4035, PS_2 = 0.8111, PS_3 = 0.2975,$$

$$PS_4 = 0.2651, PS_5 = 0.4210;$$

$$PR_1 = 0.2568, PR_2 = 0.3108, PR_3 = 0.2027,$$

$$PR_4 = 0.1304, PR_5 = 0.2297.$$

Using formula (28), calculate the normalized closest negative ideal group utility value KS_i and the normalized closest negative ideal individual regret value KR_i of alternative A_i , respectively, shown as follows:

$$KS_1 = 0.5472, KS_2 = 0.3534, KS_3 = 0.7805,$$

$$KS_4 = 0.7515, KS_5 = 0.5904;$$

$$KR_1 = 0.3108, KR_2 = 0.2389, KR_3 = 0.2389,$$

$$KR_4 = 0.3108, KR_5 = 0.2297.$$

Step7: Using formula (29), compute the closest positive ideal VIKOR index Q_i^+ and the closest negative ideal VIKOR index Q_i^- of alternative A_i , respectively, shown as follows:

$$Q_1^+ = 0.4771, Q_2^+ = 1, Q_3^+ = 0.2301,$$

$$Q_4^+ = 0, Q_5^+ = 0.418;$$

$$Q_1^- = 0.7269, Q_2^- = 0.0567,$$

$$Q_3^- = 0.5567, Q_4^- = 0.9661, Q_5^- = 0.2775.$$

Step8: Using formula (30), calculate the degree of relative closeness DQ_i of alternative A_i , shown as follows:

$$DQ_1 = 0.3963, DQ_2 = 0.9463,$$

$$DQ_3 = 0.2925, DQ_4 = 0, DQ_5 = 0.601.$$

Because $DQ_4 < DQ_3 < DQ_1 < DQ_5 < DQ_2$, the preference order of the alternatives A_1, A_2, A_3, A_4 and A_5 is: $A_4 > A_3 > A_1 > A_5 > A_2$.

Tian et al. [33] used their own similarity measure (18) combined with the traditional VIKOR method, which is widely used in multi-attribute decision-making problems in PF environments. In [34], Wang et al. adopted the developed picture fuzzy weighted geometric (PFWG) operator for multi-attribute decision-making. Wei et al. [35] ranked the given options according to an extended projection model of picture fuzzy set, and then selected the most ideal option. In the literature [37], the author proposed the picture fuzzy normalized projection (PFPN) model, combined the PFPN model with the VIKOR method, constructed a VIKOR method based on picture fuzzy normalized projection and applied it in multi-attribute decision-making. We apply these methods to example 7 and obtain the prioritization of alternatives in Table 6.

It can be seen from Table 6 that the best alternatives obtained by the five multi-attribute decision-making

methods are consistent. And the improved VIKOR method, the PFWG operator method and the PFNP-VIKOR method have the same ordering method for the alternatives. Among the extant projection model methods, only the second and third alternatives are ranked differently, and the ranking results of other alternatives are consistent with other methods. In the VIKOR method in the PF environment, the ranking of all the alternatives except the best alternative is inconsistent. If the first two best alternatives are selected in the actual decision-making problem, the VIKOR method in the PF environment will have a different choice than other schemes. Therefore, the improved VIKOR method based on PF similarity measure S_L is not only effective in multi-attribute decision-making problems, but also better than the traditional VIKOR method in PF environment proposed by Tian et al. [33].

6 Conclusion

This paper proposes a hybrid PF similarity measure combining Hamming distance and transformed tetrahedron centroid distance, and proves that it satisfies the property of similarity measure. We find that the hybrid PF similarity measure can not only combine the advantages of the two distances, but also overcome the shortcomings of the PF similarity measure proposed by the two distances. The existing and proposed PF similarity measures are applied to numerical examples and pattern recognition problems. The experimental results show that the proposed PF similarity measures have more advantages than the existing ones. Based on the hybrid PF similarity measure, we study the application of an improved VIKOR method in PF environment, and compare it with other MADM methods through examples. The results verify the practicability and effectiveness of the improved VIKOR method. In addition, we do not give a geometric image explanation of the hybrid PF similarity measure, which is missing some confidence. Future research work will be carried out in the following areas:

1. The hybrid PF similarity measure is applied to medical diagnosis, cluster analysis and image processing.
2. Like Rong et al. [41, 42] applied the proposed hybrid PF similarity measure to the MARCOS method for multi-criterion group decision-making.
3. Extend the improved VIKOR method to interval picture fuzzy set, generalized picture fuzzy soft set and other fields.

Author Contributions ZC conceived the study and helped to review the manuscript. LL drafted the manuscript, studied the picture fuzzy similarity measure based on Hamming distance and transformed

tetrahedron centroid distance, and implemented its application. XJ and LL proved the theorems and properties in the manuscript. XJ and LL consulted relevant literature and research status at home and abroad. All authors read and approved the final manuscript.

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Consent for Publication I would like to declare on behalf of my co-authors that the work described was original research that has not been published previously, and is not under consideration for publication elsewhere, in whole or in part. All the authors listed have approved the manuscript that is enclosed.

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